Core Equivalence Theorem with Production*

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Abstract

In production economies, the extent to which non-equilibria are blocked depends on the allocation of control rights among shareholders, because a blocking coalition's resources are affected by the firms it jointly owns with outsiders. We formulate two notions of blocking. Corporate control is allocated deterministically (majority shares, unanimity, etc.) in one notion and stochastically (partnership dissolution) in the other. Based on each blocking notion, we prove an analog of the Debreu-Scarf theorem for replica production economies, and we prove that the class of economies for core convergence is expanded if the blocking notion is switched from deterministic to stochastic.

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1 Introduction

Built upon Edgeworth's [6] elegant idea that a prediction of decentralized markets should not be blocked by any multilateral deviation from the predicted outcome, the core equivalence theorem has been interpreted as a justification for the price-taking assumption inherent in Walras (competitive) equilibrium. But the theorem is mostly based on exchange economies. This paper examines the extent to which core equivalence can be true in production economies with private ownership. The introduction of production is nontrivial. Without production, multilateral deviation from a status quo can be viewed simply as forming a self-sufficient blocking coalition such that each member is better-off than the status quo. With production, by contrast, it is not clear what it means for a coalition to be "self-sufficient." Can the coalition change the action of a firm not entirely owned by the members of the coalition? If Yes, what conditions does it need to respect for the shareholders outside the coalition? If No, does the coalition still have to carry out the part of the status quo production plan assigned to its members who are shareholders of the firm?

For example, suppose a blocking coalition owns only 1/3 of a firm but is endowed with a good needed by outsiders. If the coalition blocks the supply of the good, the outsiders cannot obtain feasible consumptions unless they change the action of the firm, which they can control via majority shares. Once the firm's action is changed, the blocking coalition either carries out 1/3 of the new action or forfeits its shares of the firm if default is allowed. Either way would change the resources within the coalition.

In this paper, we find that the scope of the core equivalence theorem depends on the particular nature of corporate control rights that the notion of blocking captures:

If we interpret the *corporate shares* in the Arrow-Debreu model as shareholders' control rights of the firms and allow a coalition to operate a firm it controls without respect to the outsiders of the coalition, core equivalence (Proposition 2) holds for a general class of replica economies with preference- and endowment-assumptions similar to those in Debreu and Scarf [5]. This notion of blocking we call *inconsiderate blocking*.

If we strengthen the above notion by requiring that a coalition's blocking plan should allow the outsiders to have feasible consumptions, core equivalence still holds for a general class of replica economies (Theorem 1), but this class is smaller than the one for Proposition 2 due to an additional assumption, which is shown indispensable by our Example 1.

For a general class of blocking notions that interpret corporate shares as *deterministic* control rights and respect the feasibility for the outsiders of a coalition, Proposition 3 says that the class of economies where core equivalence holds cannot be expanded beyond Theorem 1. Here control right is deterministic in the sense that the assignment of corporate shares alone determines whether a coalition can unilaterally change the action of a firm.

For a notion of blocking that interprets corporate shares as a particular kind of *stochastic* control rights and respect the feasibility for the outsiders of a coalition, Theorem 2 obtains core convergence for a much larger class of economies. Not only is the additional assumption for Theorem 1 unnecessary, the replica structure of the economies are also dispensable.[1]

We will review the related works in §5. Haller [10] is the only earlier contribution to core equivalence that considers production and the ramifications of corporate control rights. Haller's notion of blocking does not take into account the constraint that the outsiders of a coalition need to have feasible consumptions, while the notions of blocking which our theorems are based on take that feasibility constraint into account.

In addition to this main difference, there are other substantial differences between Haller's work and its counterpart in our paper. Haller's notion of blocking is a precedent of one of our intermediary notions of blocking, the aforementioned *inconsiderate blocking*. However, the two notions are different. As we will show, in his Example 2, an allocation cannot be Haller-blocked but can be inconsiderately blocked. Furthermore, Haller's core convergence theorem says that if a non-equilibrium allocation satisfies an interior condition and a liquidity condition, then the allocation is Haller-blocked. By contrast, our Proposition 2, core convergence based on inconsiderate blocking, says that *any* non-equilibrium allocation is inconsiderately blocked. Appendix E has detailed comparison.

2 The Model

2.1 The Primitives

There are a finite set I of individuals, a finite set J of firms, and a finite number l of goods. Let i be the index for individuals and j for firms. Let θ_{ij} be i's share of firm j, with

[1] Since Xiong and Zheng [13] have proved an analog of Anderson's [3] theorem based on stochastic blocking, the scope of core convergence given stochastic control rights goes beyond replica economies.

$\sum_{i\in I}\theta_{ij} = 1$ for all j. Let $\mathbb{R}^l_+$ be the consumption set of each individual, $\succeq_i$ individual i's preference relation on $\mathbb{R}^l_+$, e_i ($\in \mathbb{R}^l_+$) his endowment, and Y_j ($\subseteq \mathbb{R}^l$) the production set of firm j. An allocation is denoted by $(x,y) := ((x_i)_{i\in I}, (y_j)_{j\in J})$, meaning that individual i consumes the bundle x_i and firm j's production plan is y_j. An allocation (x,y) is *feasible* if $x_i \in \mathbb{R}^l_+$ and $y_j \in Y_j$ for each individual i and each firm j and $\sum_{i\in I} x_i = \sum_{i\in I} e_i + \sum_{j\in J} y_j$. A *coalition* S is a nonempty set of individuals.[2]

We make the following **assumptions** throughout the paper without mentioning them in our theorems or lemmas. Every individual i's preference relation is strongly monotone, strictly convex (convex and strongly convex), and *lower* semicontinuous on $\mathbb{R}^l_+$ and is represented by a utility function u_i. The total endowment $\sum_{i\in I} e_i \gg \mathbf{0}$, i.e., it has positive quantity of each good. Every individual's endowment is a point in $\mathbb{R}^l_+ \setminus \{\mathbf{0}\}$ (i.e., $e_i \gneqq \mathbf{0}$ for all $i \in I$). For every firm j, $\mathbf{0} \in Y_j$ and Y_j is convex. These assumptions are usual in the general equilibrium literature. To ensure the equal treatment property for firms, we assume that, for each firm j, the production possibility frontier is a strictly concave surface in $\mathbb{R}^l$.

2.2 Replica Economies

A *replica economy* of size r, denoted by $\mathcal{E}^r$, consists of r *units*, each of which has exactly the same composition of individuals, firms, endowments, and corporate ownership. We name an individual by an integer-pair $i := (i_1, i_2)$, and a firm by $j := (j_1, j_2)$, meaning—

i_1 := the type of the individual (exactly one such individual in each unit);

i_2 := the unit to which the individual belongs;

j_1 := the type of the firm (exactly one such firm in each unit);

j_2 := the unit to which the firm belongs.

Individuals or firms of the same type have the same characteristics:

$$\begin{aligned}
u_{(i_1,i_2)} &= u_{(i_1,i_2')} &&=: u_{i_1};\\
e_{(i_1,i_2)} &= e_{(i_1,i_2')} &&=: e_{i_1};\\
Y_{(j_1,j_2)} &= Y_{(j_1,j_2')} &&=: Y_{j_1};\\
\theta_{(i_1,k),(j_1,k)} &= \theta_{(i_1,k'),(j_1,k')} &&=: \theta_{i_1 j_1}.
\end{aligned}$$

[2]Note that a firm in this model is not a player but is a joint venture of its shareholders. Formulating firms as players would require specification of their objectives. Profits would be an inadequate objective because they depend on market prices, not taken as given a priori in the context of the core equivalence theorem.

Individuals in one unit have zero share of firms in other units:

$$\theta_{(i_1,i_2),(j_1,j_2)} = \begin{cases} \theta_{i_1 j_1} & \text{if } i_2 = j_2 \\ 0 & \text{if } i_2 \neq j_2. \end{cases}$$

The following notations will be useful:

$$\begin{aligned} I_1 &:= \text{the index set for individual-types;} \\ J_1 &:= \text{the index set for firm-types;} \\ I_2 &:= J_2 := \{1, \ldots, r\}; \\ \text{Proj}_2(S) &:= \text{the set of units each of which contains some member of } S. \end{aligned}$$

An allocation $((x_i)_{i \in I}, (y_j)_{j \in J})$ has the *equal treatment property (ETP)* if individuals and firms of the same type have the same consumption and production plan, i.e.,

$$\begin{aligned} x_{(i_1,i_2)} &= x_{(i_1,i_2')} &=: x_{i_1}; \\ y_{(j_1,j_2)} &= y_{(j_1,j_2')} &=: y_{j_1}. \end{aligned}$$

If it has this property, an allocation $((x_i)_{i \in I}, (y_j)_{j \in J})$ is identified with its type-representation $((x_{i_1})_{i_1 \in I_1}, (y_{j_1})_{j_1 \in J_1})$ that specifies for each individual-type a consumption bundle and for each firm-type a production plan, so it correspond to elements of the same set, $\left(\mathbb{R}^l_+\right)^{I_1} \times \prod_{j_1 \in J_1} Y_{j_1}$, regardless of the size of the economy.

By the assumption that the total endowment $\sum_{i \in I} e_i \gg \mathbf{0}$ in the replica economy, the total endowment in each unit is also strictly positive: $\sum_{i_1 \in I_1} e_{i_1} = \frac{1}{r} \sum_{i \in I} e_i \gg \mathbf{0}$.

2.3 The Interpretation of Corporate Shares

As the resources for a blocking coalition partially come from the firms that they share with outsiders, the notion of blocking depends on (i) how a firm allocates its tasks and products and (ii) how a firm makes decisions. As we consider a purely private ownership economy, both issues should be settled according to the allocation of shares, $((\theta_{ij})_{i \in I})_{j \in J}$.

For issue (i), we assume that shareholders are bonded by their shares unless default is allowed: Once a firm j's production bundle say $y_j := ((y_j)_1, \ldots, (y_j)_l)$ has been determined, each shareholder i is responsible to carry out a fraction θ_{ij} of the plan, contributing (resp. receiving) $\theta_{ij}|(y_j)_k|$ units of good k if $(y_j)_k < 0$ (resp. if $(y_j)_k > 0$). However, if *default* is allowed, an individual can default on his share of the firm so that his net receipt from the firm is null; if he defaults on a firm, all his share of the firm is distributed to other shareholders.

For issue (ii), we assume that there is a systematic *corporate decision rule* that determines, for every assignment $(\theta_{ij})_{i,j}$ of shares and every coalition S, the set $\tilde{J}(S)$ of all the firms that switch to the proposal of coalition S if S is blocking the status quo. We say a firms j is *controlled* by coalition S if and only if $j \in \tilde{J}(S)$. If the set $\tilde{J}(S)$ is uniquely determined by $(S, (\theta_{ij})_{i,j})$, the corporate decision rule is *deterministic*. In this case, we make the innocuous assumption that

$$\sum_{i \in S} \theta_{ij} = 1 \Rightarrow j \in \tilde{J}(S); \quad \sum_{i \in S} \theta_{ij} = 0 \Rightarrow j \notin \tilde{J}(S). \tag{1}$$

If the set $\tilde{J}(S)$ is stochastic even after S and $(\theta_{ij})_{i,j}$ are determined, the corporate decision rule is *stochastic*.

An example for deterministic corporate decision is that every firm holds a referendum with all its shareholders and it switches to the blocking coalition's proposal if and only if the proposal gets a vote greater than a predetermined threshold (with the assumption that all shareholders outside the coalition oppose the proposal). An example for stochastic corporate decision is that opposing parties lobby the management of the firm and the firm's decision is determined by a lottery depending on each party's influence. We will further describe stochastic rules in §4.

3 Deterministic Core Equivalence

A coalition considering deviation from the status quo must confront the following questions: How will each firm behave? Which firm's production can be altered by this coalition? How will the firms' actions affect the deviation plan of the coalition? To answer these questions, we borrow an idea from Nash equilibrium: a deviating player expects others to stick to the hypothetical equilibrium. Hence we assume within this section that a blocking coalition considers changes only in the firms that they can control and expects the other firms to stick to the status quo. With deterministic corporate governance, the set $\tilde{J}(S)$ of firms controlled by coalition S is predetermined. That leads to the following definition:

Definition 1 (inconsiderate blocking) *A feasible allocation $((x_i)_{i \in I}, (y_j)_{j \in J})$ is said to be* inconsiderately blocked *by a coalition S if:*

a. for every $i \in S$ there exists an $x_i' \in \mathbb{R}_+^l$ such that $u_i(x_i') > u_i(x_i)$, and

b. for every $j \in \tilde{J}(S)$ there exists a $y_j' \in Y_j$ such that

$$\sum_{i \in S} x_i' = \sum_{i \in S} e_i + \sum_{j \in \tilde{J}(S)} \sum_{i \in S} \theta_{ij} y_j' + \sum_{j \notin \tilde{J}(S)} \sum_{i \in S} \theta_{ij} y_j. \tag{2}$$

Eq. (2) says: the resources within a blocking coalition consist of the endowments of its members and a fraction of the production plan of every firm; the coalition can change the action of any firm it controls but expects the action of any other firm to be fixed at the status quo. Note that this definition implicitly prohibits a coalition from defaulting on its obligation of carrying out the plans of the firms it cannot control.

The above notion of blocking is called *inconsiderate* because such a blocking coalition gives no regard to whether the consequence of its blocking is feasible at all for outsiders. With $\mathbb{R}_+^l$ being the sum of ousiders' consumption sets, it is possible that

$$\sum_{i \notin S} e_i + \sum_{j \in \tilde{J}(S)} \sum_{i \notin S} \theta_{ij} y_j' + \sum_{j \notin \tilde{J}(S)} \sum_{i \notin S} \theta_{ij} y_j \notin \mathbb{R}_+^l;$$

then the outsiders cannot carry out their shares of the production plans, hence it is physically infeasible for the coalition to have the resources described by the right-hand side of Eq. (2). This is the *interdependency problem*. Hence we need a better definition of blocking:

Definition 2 (blocking) *A feasible allocation $((x_i)_{i \in I}, (y_j)_{j \in J})$ is* blocked *by a coalition S if: (a) condition a of Definition 1 is satisfied, and (b) for every $j \in \tilde{J}(S)$ there exists a $y_j' \in Y_j$ such that Eq. (2) is satisfied and*

$$\sum_{i \notin S} e_i + \sum_{j \in \tilde{J}(S)} \sum_{i \notin S} \theta_{ij} y_j' + \sum_{j \notin \tilde{J}(S)} \sum_{i \notin S} \theta_{ij} y_j \in \mathbb{R}_+^l. \tag{3}$$

The *core* given any notion of blocking is the set of feasible allocations that are not blocked according to that notion of blocking. Given replica economy $\mathcal{E}^r$ of size r, let C^r_{incon} denote the core given inconsiderate blocking, and C^r the core given the blocking notion of Definition 2. Obviously, $C^r_{\text{incon}} \subseteq C^r$.

Lemma 1 *A Walras equilibrium allocation can neither be blocked nor inconsiderately blocked.*

Proof Mimic the proof of the first welfare theorem with the blocking coalition. ■

Lemma 2 (equal treatment) *For any $r = 1, 2, \ldots$, any allocation in the core C^r has the equal treatment property (ETP). (Since $C^r_{\text{incon}} \subseteq C^r$, ETP for C^r_{incon} is also satisfied.)*

Proof First, we show the ETP for firms: Since an element of the core C^r is Pareto efficient (Eq. (3) is vacuously satisfied for the grand coalition), by a modified version of the second welfare theorem (Corollary 1 in Appendix A), there exists a price vector $p \gg \mathbf{0}$ at which each firm's production plan is profit-maximizing. Then the strict concavity of the production possibility frontier implies that such production plan is unique for each type of firms.

Thus, at any core allocation (x, y), individuals of the same type i_1 have the same production bundle $\sum_{j_1 \in J_1} \theta_{i_1 j_1} y_{j_1}$ and hence the same bundle of resources $e_{i_1} + \sum_{j_1} \theta_{i_1 j_1} y_{j_1}$.

Then the ETP for consumers is proved as follows: If allocation (x, y) does not have this property, then the well-known "underdog coalition," consisting of a least favored individual of each type, blocks (x, y) by the plan of keeping production unchanged and having each member of type i_1 consume $\frac{1}{r} \sum_{i_2=1}^{r} x_{(i_1, i_2)}$. As in the standard proof (e.g. [12, prop. 18.D.2]), this plan is an improvement for the coalition members; it is also feasible within the coalition:

$$\sum_{i \in S} x'_i = \frac{1}{r} \sum_{i_1 \in I_1} \sum_{i_2=1}^{r} x_{(i_1, i_2)} = \sum_{i_1 \in I_1} e_{i_1} + \sum_{i_1 \in I_1} \sum_{j_1 \in J_1} \theta_{i_1 j_1} y_{j_1} = \sum_{i \in S} e_i + \sum_{i \in S} \sum_{j \in J} \theta_{ij} y_j. \qquad (4)$$

To complete the proof, we need to verify Eq. (3), i.e., this plan is feasible for outsiders. Given this plan, the total resource for outsiders is equal to

$$(r-1) \left[\sum_{i_1 \in I_1} e_{i_1} + \sum_{i_1 \in I_1} \sum_{j_1 \in J_1} \theta_{i_1 j_1} y_{j_1} \right] = (r-1) \sum_{i \in S} x'_i,$$

where the equality follows from Eq. (4). Since $r - 1 \geq 0$ and since $\sum_{i \in S} x'_i \in \mathbb{R}^l_+$, the total resource for outsiders also belongs to $\mathbb{R}^l_+$. Hence Eq. (3) is satisfied, as desired. ■

With ETP, the core allocations in economies of different sizes can be identified with the element of the same space, $\left(\mathbb{R}^l_+\right)^{I_1} \times \prod_{j_1 \in J_1} Y_{j_1}$. That is also true for Walras equilibrium allocations, as they are in both kinds of core by Lemma 1. Represented by elements in $\left(\mathbb{R}^l_+\right)^{I_1} \times \prod_{j_1 \in J_1} Y_{j_1}$, the set of Walras equilibrium allocations is preserved by replication of the economy. Hence denote this set of equilibrium allocations by $\mathbb{W}$.

Obviously, $C^1_{\text{incon}} \supseteq C^2_{\text{incon}} \supseteq C^3_{\text{incon}} \supseteq \ldots$ and $C^1 \supseteq C^2 \supseteq C^3 \supseteq \ldots$. The main result of this section is that these decreasing sequences shrink to the set $\mathbb{W}$ of equilibrium allocations.

3.1 Inconsiderate Core Equivalence

Recall that our notion of blocking takes into account the feasibility condition for the outsiders of a coalition, while inconsiderate blocking does not. To pin down the impact of this feasibility constraint, we start by establishing core convergence based on inconsiderate blocking, which will be contrasted with the core convergence theorem proved in the next subsection. The propositions proved here will be useful for our main theorems.

To get an intuition for core convergence with inconsiderate blocking, assume within this paragraph that the utility functions are all continuously differentiable and consider a non-equilibrium allocation where everyone's consumption is strictly positive. We show that this allocation is inconsiderately blocked: First, if the allocation is not Pareto efficient, it is blocked by the grand coalition. Second, if the allocation is Pareto efficient, by the second welfare theorem (Corollary 1), it is a price equilibrium with transfers under some price $p \gg \mathbf{0}$. Then the conclusion follows from a similar argument as Mas-Colell, Whinston and Green [12, p658]. Just replace $x_1 - e_1$ in their proof by $x_1 - e_1 - \sum_{j_1} \theta_{1j_1} y_{j_1}$, where j_1 indexes the types of firms and y_{j_1} is a type-j_1 firm's production plan in the status quo allocation.

To prove inconsiderate core convergence for the general case without assuming differentiability or interior allocations, we start with an intermediary notion of blocking, where a coalition's blocking plan is based on the wishful thinking that it can change the action of any firm, whether they have the control right or not:

Definition 3 (wishful blocking) *A feasible allocation $((x_i)_i, (y_j)_{j\in J})$ is* wishfully blocked *by a coalition S if:*

a. for every $i \in S$ there exists an $x'_i \in \mathbb{R}^l_+$ such that $u_i(x'_i) > u_i(x_i)$, and

b. for every $j \in J$ there exists a $y'_j \in Y_j$ such that

$$\sum_{i\in S} x'_i = \sum_{i\in S} e_i + \sum_{j\in J}\sum_{i\in S} \theta_{ij} y'_j. \tag{5}$$

Let C^r_{wish} denote the core under the wishful blocking in the r-replica economy. Wishful-core allocations have the equal treatment property, because the proof of Lemma 2 is applicable here. Thus, $C^1_{\text{wish}}, C^2_{\text{wish}}, C^3_{\text{wish}}, \ldots$ constitute a nested sequence of subsets of the same space which also contains the set $\mathbb{W}$ of Walras equilibrium allocations.

Proposition 1 $\mathbb{W} = \cap_{r=1}^{\infty} C_{\text{wish}}^{r}$.

Proof Mimic Debreu and Scarf [5]. See Appendix B. ■

By Proposition 1, a non-equilibrium allocation is wishfully blocked in sufficiently large economies. Is it also inconsiderately blocked? The answer is Yes if the economy is large enough. To see the intuition, suppose an individual i prefers a consumption x_i' to the status quo x_i, but to consume x_i' he needs to change a firm j's action into y_j' from the status quo y_j. The problem is that his share θ_{ij} of firm j is too small for him to make that change. Hence i only wishfully blocks the status quo. To pass that into inconsiderate blocking, make another copy of the economy and label it by $\mathcal{E}'$. Form a coalition consisting of this i and everyone in $\mathcal{E}'$. Then they can inconsiderately block the status quo: i receives the status quo production bundle $\theta_{ij}y_j$ assigned to him from firm j; the counterpart of i in $\mathcal{E}'$, however, receives the bundle $\theta_{ij}y_j'$ that i wishes for—the counterpart can do that because all the firms in $\mathcal{E}'$ are controlled by the enlarged coalition; then the counterpart swaps the bundle $\theta_{ij}y_j'$ with i's bundle $\theta_{ij}y_j$. With this plan, i gets to consume his perferred x_i', and everyone in $\mathcal{E}'$ can still get her status quo consumption. Thus, i can make every coalition member better-off by distributing part of his preferred bundle to the other members. Hence we have—

Proposition 2 (inconsiderate core equivalence) $\mathbb{W} = \cap_{r=1}^{\infty} C_{\text{incon}}^{r}$.

Proof See Appendix B. ■

Note that Proposition 2 would be the counterpart of the Debreu-Scarf theorem in production economies if we ignore the feasibility constraint for the outsiders of a coalition.

3.2 Considerate Core Equivalence

The only hurdle between inconsiderate versus considerate core convergence is that the latter needs to take into account the constraint that the outsiders of a blocking coalition have feasible consumptions. The main idea to overcome this hurdle is embodied by the following Lemma 3, which says that a blocking coalition's disturbance on outsiders diminishes as the replica economy enlarges.

An allocation (x, y) is *edgy* if the aggregate consumption bundle $\sum_{i \in I} x_i$ lies on $\partial\mathbb{R}_+^l$, the boundary of the consumption set $\mathbb{R}_+^l$. If (x, y) has the equal treatment property, then we can write it as $((x_{i_1})_{i_1\in I_1}, (y_{j_1})_{j_1\in J_1})$, and it is edgy if only if $\sum_{i_1\in I_1} x_{i_1} \in \partial\mathbb{R}^l_+$. Note that, if an allocation with equal treatment property is feasible and not edgy, then

$$\sum_{i_1\in I_1} e_{i_1} + \sum_{i_1\in I_1}\sum_{j_1\in J_1} \theta_{i_1 j_1} y_{j_1} \gg \mathbf{0}. \tag{6}$$

Lemma 3 (diminishing externality of blocking coalitions) *For any feasible allocation with equal treatment property, if it is inconsiderately blocked and is not edgy, then it is blocked (in the sense of Definition 2) for all replica economies with sufficiently large size.*

Proof Suppose allocation (x, y), as specified by the hypothesis, is inconsiderately blocked by coalition S with a plan $((x'_i)_{i\in S}, (y'_j)_{j\in\tilde{J}(S)})$. It suffices to prove Eq. (3) for sufficiently large size r. (Inconsiderate blocking is preserved when the replica economy enlarges.) Let

$$r^* := \max\{i_2 : (i_1, i_2) \in S \text{ for some } i_1\}.$$

Thus, for any $j_2 = r^* + 1, \ldots, r$ and for any firm-type j_1, none of the coalition members hold any share of firm (j_1, j_2). Hence the left-hand side of Eq. (3) is equal to a fixed vector plus

$$(r - r^*)\left[\sum_{i_1\in I_1} e_{i_1} + \sum_{i_1\in I_1}\sum_{j_1\in J_1} \theta_{i_1 j_1} y_{j_1}\right]. \tag{7}$$

As the allocation is feasible and not edgy, the vector in the square bracket $[\cdots]$ is positive in every component by (6); since r^* is unchanged as r enlarges, the vector (7) goes to infinity in every component when $r \to \infty$. Thus, the left-hand side of Eq. (3) belongs to $\mathbb{R}^l_+$ when r is sufficiently large, as desired. ■

To ensure the non-edgyness for Lemma 3, within the rest of this subsection we assume—

Assumption 1 (boundary aversion) *Every individual strictly prefers any interior point of $\mathbb{R}^l_+$ to any boundary point of $\mathbb{R}^l_+$.*

Lemma 4 *Suppose Assumption 1. Then any element in the core C^r is not edgy.*

Proof If an allocation (x, y) in the core C^r is edgy, then $x_i \in \partial\mathbb{R}^l_+$ for all individuals $i \in I$ (since $x_i \in \mathbb{R}^l_+$ for all individuals i). Then the grand coalition I blocks the allocation by having every firm produce $\mathbf{0}$ and equally dividing the total endowment $\sum_{i\in I} e_i$ among all individuals. Since $\sum_{i\in I} e_i \gg \mathbf{0}$ by assumption, everyone is better-off by the assumption of boundary aversion. The outsiders' feasibility condition Eq. (3) is vacuously satisfied. ■

Theorem 1 (considerate core equivalence) *If Assumption 1 holds, then* $\mathbb{W} = \cap_{r=1}^{\infty} C^r$.

Proof Lemma 1 implies $\mathbb{W} \subseteq \cap_{r=1}^{\infty} C^r$. For the converse, pick any allocation in $\cap_{r=1}^{\infty} C^r$. Then the allocation is not edgy (Lemma 4 and Assumption 1) and consequently it is not inconsiderately blocked no matter how large the replica economy is (for otherwise it is blocked for any sufficiently large r, by Lemma 3 and non-edginess). Then Proposition 2 implies that the allocation is an equilibrium allocation. ■

Except Assumption 1, the preference- and endowment- assumptions of Theorem 1 are similar to those in Debreu and Scarf [5] (detailed in §5). Since Assumption 1 is shown indispensable in §3.3 and is indispensable even for other deterministic notions of blocking by the impossibility result in §3.4, Theorem 1 is a general version of the production-counterpart of the Debreu-Scarf theorem, given deterministic allocation of corporate controls.

3.3 The Indispensability of Boundary Aversion

Theorem 1 is based on a notion of blocking that simply adds to the notion of inconsiderate blocking a requirement that a blocking plan should allow feasible consumptions for those outside the coalition. Compared to Proposition 2, the core equivalence result based on inconsiderate blocking, Theorem 1 uses an arguably stronger set of assumptions and the main difference is Assumption 1, boundary aversion. Then is the restriction to boundary aversion precisely the cost we have to pay in order to add production to the core equivalence analysis, or can there be a stronger core equivalence result without this restriction?

We argue that the answer is the former, that boundary aversion is indispensable. Our reasoning is two-fold. First, we present in this subsection an example showing that boundary aversion is indispensable for Theorem 1. In this example, preferences are not boundary-averse, and there is a non-equilibrium that cannot be blocked, though it is inconsiderately blocked. Second, we go further in the next subsection to argue that boundary aversion is indispensable even if we change the other requirements commonly used by "blocking" and "inconsiderate blocking" such as the possibility for default.[3]

[3] As our intention is just to identify the effect of production to core equivalence, we do not argue whether boundary aversion is a strong or weak assumption. Nevertheless, we should point out that the assumption is incompatible with strongly monotone and upper semicontinuous preferences. (Note however that Theorem 1 does not need upper semicontinuity.) This assumption also rules out quasilinear preferences. (William Zame pointed out that it also rules out commodities that serve purely as intermediary products.)

Example 1 (edgy core) *There are two goods, two individual-types, and one firm-type. In each unit, the endowment for the individual of type-1 is* $e_1 = (1, 0)$*, and that for type-2 is* $e_2 = (0, 1)$*. For each type, the utility from consumption bundle* (x_{i1}, x_{i2}) *is*

$$u(x_{i1}, x_{i2}) := x_{i1} + 10\sqrt{x_{i2}}.$$

Each firm is equally shared by a type-1 and a type-2 individuals, its decision is determined by unanimity rule ($(j_1, j_2) \in \tilde{J}(S) \Leftrightarrow \{(1, j_2), (2, j_2)\} \subseteq S$)*, and its production set is*

$$Y := \left\{(y_{j1}, y_{j2}) \in \mathbb{R}^2 : y_{j2} \leq \sqrt{-y_{j1}}; y_{j1} \leq 0\right\}.$$

In any replica of the economy, there is a unique Walras equilibrium allocation:

$$x_1^* = (0, 3/4)\,; \quad x_2^* = (0, 5/4)\,; \quad y^* = (-1, 1); \quad p^* = (1, 2).$$

But the following allocation belongs to the core in any replica of the economy:

$$x_1^o = (0, 1/2)\,; \quad x_2^o = (0, 3/2)\,; \quad y^o = (-1, 1). \tag{8}$$

However, it is inconsiderately blocked in sufficiently large economies, by Proposition 2.

To derive the uniqueness of the equilibrium, note that an individual's consumption of good 2 cannot exceed 2 units. This follows from the equal treatment property of equilibrium and the fact that there are at most two units of good 2 in each unit economy, due to the technology and endowments. Thus, the marginal rate of substitution, in absolute value, is at most $\sqrt{2}/5$. Since the slope of the production possibility frontier (the northeastern boundary of the set $\mathbb{R}^2_+ \cap (e_1 + e_2 + Y)$), in absolute value, is at least $1/2$, any Pareto efficient allocation would use up all the good 1 to produce good 2, i.e., an efficient production plan is necessarily $(-1, 1)$. By the first welfare theorem, the equilibrium price ratio is $p_1^*/p_2^* = 1/2$, the slope of the frontier at $(-1, 1)$. Then the equilibrium consumption bundles are easily derived.

Let us prove that the allocation (8) belongs to the core in any replica economy. First, note that this allocation constitutes a price equilibrium with transfers, with price $p^o = (1, 2)$ and the wealth given to a type-i_1 individual being $w_{i_1}^o := p^o \cdot x_{i_1}^o$. Under this price, if there were no transfer, a type-i_1 individual's wealth would be

$$w_{i_1}^* := p^o \cdot e_{i_1} + \sum_{j \in J} \theta_{ij} p^o \cdot y_j^o = p^o \cdot e_{i_1} + (1/2) p^o \cdot y^o. \tag{9}$$

Note:

$$w_1^o + w_2^o = w_1^* + w_2^*; \tag{10}$$

$$w_2^o > w_2^*. \tag{11}$$

Suppose the allocation (8) is blocked by a coalition S with blocking plan $((x_i')_{i\in S}, (y_j')_{j\in \tilde{J}(S)})$. The coalition necessarily consists of (i) n unit economies (i.e., n pairs of type-1 and type-2 individuals from the same unit), (ii) g type-1 individuals from g separate units (where the type-2 individuals are not in S), and (iii) h type-2 individuals from h units (where the type-1 individuals are not in S). We claim that $g \le h$ by the outsiders' feasibility condition (3): There are $g + h$ firms that the coalition shares with outsiders and these firms all stick to the production plan $y^o = (-1, 1)$ (recall the unanimity rule in this example); hence the total resource available to the outsiders is equal to a multiple of $x_1^o + x_2^o = (0, 2)$ plus the vector

$$g(0,1) + h(1,0) + (g+h)(1/2)(-1,1) = ((h-g)/2, g + (g+h)/2),$$

which would be infeasible unless $h \ge g$. Since the allocation (8) is an equilibrium with transfers under the price p^o and wealth $(w_i^o)_{i\in I}$,

$$\sum_{i\in S} p^o \cdot x_i' > \sum_{i\in S} w_i^o. \tag{12}$$

By the above characterization of S in terms (n, g, h),

$$\begin{aligned}
\sum_{i\in S} w_i^o &= nw_1^o + nw_2^o + gw_1^o + hw_2^o \\
&= n(w_1^o + w_2^o) + g(w_1^o + w_2^o) + (h-g)w_2^o \\
&\ge n(w_1^* + w_2^*) + g(w_1^* + w_2^*) + (h-g)w_2^* \\
&= \sum_{i\in S} w_i^*,
\end{aligned}$$

where the inequality follows from (10) and (11) and the fact that $h \ge g$. By Eq. (9),

$$\sum_{i\in S} w_i^* = \sum_{i\in S}\left[p^o \cdot e_i + \sum_{j\in J}\theta_{ij} p^o \cdot y^o\right] \ge p^o \cdot \sum_{i\in S}\left[e_i + \sum_{j\in \tilde{J}(S)}\theta_{ij} y_j' + \sum_{j\notin \tilde{J}(S)}\theta_{ij} y^o\right] = p^o \cdot \sum_{i\in S} x_i',$$

where the inequality is due to profit maximization at the price equilibrium with transfers. Thus, we have

$$\sum_{i\in S} w_i^o \ge \sum_{i\in S} w_i^* \ge p^o \cdot \sum_{i\in S} x_i',$$

which contradicts (12). It follows that the allocation (8) cannot be blocked.

Although this non-equilibrium is never blocked, it is inconsiderately blocked in large enough economies by Proposition 2. (Independent of Proposition 2, Appendix C constructs inconsiderate blocking plans for all non-equilibria in this example.) Thus, the example illustrates the difference caused by the constraint that a blocking plan be feasible for the outsiders of the coalition.

3.4 Other Notions of Deterministic Blocking

As we want to pin down the effect of adding production to the core equivalence analysis, let us carry on the investigation further. The previous subsection identifies the difference due to the feasibility conditions that a blocking coalition needs to respect in production economies. That identification is based on a comparison between the core convergence results based on "blocking" versus "inconsiderate blocking." However, the two notions of blocking share some common conditions. In particular, both notions rule out the possibility for a shareholder to default on his corporate shares. Then does the difference identified by the previous subsection diminish if the notion of blocking is modified in some reasonable manner?

We argue that the answer is No as long as corporate control rights captured in the notion of blocking is deterministic. In some sense, there is a predicament of deterministic control rights: To disentangle the interdependency between a blocking coalition and the outsiders who share the same firms with some of the coalition members, allowing default may be a natural solution. Yet, if default is allowed, a coalition who ex ante knows it can control a firm can take over outsiders' shares for free. A coalition has that knowledge ex ante because control rights are deterministic.

3.4.1 The Axioms for Deterministic Blocking

Instead of the requirement that a blocking plan should be feasible for the outsiders of a coalition, one could disentangle the interdependency between a blocking coalition and its outsiders by allowing shareholders to default on their shares. Then the outsiders of a coalition can at least retain their endowments and secure a feasible allocation among them. If a blocking coalition expects the outsiders to adjust via default, it needs only to find a blocking plan feasible within the coalition. Hence suppose—

Axiom 1 (default) *For any firm, any individual can choose to default on all his shares of the firm (so that he gives* $\mathbf{0}$ *to and gets* $\mathbf{0}$ *from the firm). If he does so, his shares are divided among other individuals in the proportion of their previous shares of the firm.*

Then the outsiders' feasibility condition (3) is not needed for a blocking notion.

However, merely allowing default would make core equivalence easier to fail, because even a Walras equilibrium may be blocked with default. For instance, suppose that 51% of a firm is owned by Ms. Big and the other 49% is owned by Mr. Small, and the firm is ruled by majority of share. Then Ms. Big can choose a production plan such that it is infeasible for Mr. Small to carry out his share of that plan with his individual endowment, so he has to default and the firm becomes entirely owned by Ms. Big. Thus, a transfer of properties occurs and results in an outcome different from any Walras equilibrium.

Thus, if one still wishes to expand the scope of core equivalence, the other conditions required by the blocking notions in previous subsections need to be modified.

Recall that $\tilde{J}(S)$ is the set of firms controlled by the blocking coalition S. Let Θ_j denote the set of shareholders of firm j: $\Theta_j := \{i \in I : \theta_{ij} > 0\}$. Let us assume:

Axiom 2 (triviality) $\tilde{J}(\varnothing) = \varnothing$.

Axiom 3 (unanimity) $j \in \tilde{J}(\Theta_j)$.

Axiom 4 (irrelevance of non-shareholders) $j \in \tilde{J}(S) \Longleftrightarrow j \in \tilde{J}(S \cap \Theta_j)$.

All three axioms are self-evident. Axiom 4 says that adding non-shareholders of a firm to the coalition does not affect whether the firm is controlled by the coalition or not. Coupled with Axiom 2, it implies that non-shareholders of a firm cannot control the firm.

The notions of blocking in the previous subsections assume that the firms outside $\tilde{J}(S)$ cannot deviate from the status quo. Now we relax that assumption.

Let $\tilde{K}(S)$ be the set of all the firms that are not controlled by the coalition S and whose productions may be changed in response to the blocking coalition—either to restore outsiders' feasibility or to retaliate the coalition—and such changes are expected by the coalition when it plots the blocking plan. Call $\tilde{K}$ *response coordination*.

We shall restrict the response coordination by the next axiom. Recall that $\mathrm{Proj}_2(S)$ is the set of units each of which contains some members of the set S. Given a blocking coalition S, the units in $\text{Proj}_2(S)$ are called *involved* with S. The other units are called *uninvolved* with S. The next axiom implies that only firms in the involved units may change their production plans in response to the blocking coalition.

Axiom 5 (irrelevance of uninvolved units) $\tilde{K}(S) = (J_1 \times \text{Proj}_2(S)) \setminus \tilde{J}(S)$.

Without the "$\subseteq$" part of Axiom 5, the feasibility problem for outsiders could be resolved trivially by altering production in the uninvolved units. To see this, suppose in an N-replica economy, an allocation (x, y) is inconsiderately blocked by coalition S with a plan $((x'_i)_{i\in S}, (y'_j)_{j\in\tilde{J}(S)})$ that is not feasible for outsiders:

$$\sum_{i\notin S} e_i + \sum_{j\in\tilde{J}(S)} \sum_{i\notin S} \theta_{ij} y'_j + \sum_{j\notin\tilde{J}(S)} \sum_{i\notin S} \theta_{ij} y_j =: v \notin \mathbb{R}^l_+.$$

Consider the $(N+m)-$replica economy and the same coalition S with the same blocking plan. Let all firms in the uninvolved units produce $\mathbf{0}$. Then the total resource for the outsiders is $v + m\sum_{i_1} e_{i_1}$. Since $\sum_{i_1} e_{i_1} \gg \mathbf{0}$, for all sufficiently large m, $v + m\sum_{i_1} e_{i_1}$ is positive in each coordinate and hence S can "block" (x, y) in the $(N+m)-$replica economy.

However, it is not reasonable to expect that an uninvolved unit would change its action in response to the blocking coalition. That is because an uninvolved unit can completely insulate itself from the effect of the blocking coalition by sticking to any core allocation—due to its equal treatment property, any core allocation can be achieved within the unit, as each unit is self-sufficient. Whereas, the non-coalition members in the involved units are directly affected by the blocking plan, so it is more reasonable to expect that they would adjust productions to recover their feasibility condition instead of hoping the people in the uninvolved units to do that for them. Thus, we postulate the "$\subseteq$" part of Axiom 5.

The "$\supseteq$" part of Axiom 5 says that a firm in an involved unit belongs to $\tilde{K}(S)$ if it does not belong to $\tilde{J}(S)$. This sounds reasonable because the non-coalition members in the involved units are directly affected by the blocking coalition, so they would adjust the plans for the firms under their control.

To specify the response of outsiders, the next axiom assumes that they always try to deter any blocking coalition that excludes them if such deterrence plan exists.

Axiom 6 (aversion to being left behind) *The outsiders of S choose actions for the firms in $\tilde{K}(S)$ so that S cannot block the status quo, if such actions exist.*

Axiom 6 is needed only if a blocking coalition does not include everyone. The outsider who cannot be included is worse-off if the blocking plan is implemented. Naturally such an individual would deter the blocking plan if he can. Hence this axiom is reasonable.

3.4.2 An Impossibility Result

If we are free to pick any notion of blocking that satisfies Axioms 1–6, can the class of economies where core equivalence holds be expanded beyond those in Theorem 1? The next proposition says No.

To see the intuition, imagine a situation where there is an individual-type who owns some share of a firm and is endowed with all the inputs needed by the firm. If this monopolist type is the decision maker of the firm, he can freely take over the stocks from the other shareholders by telling them: "I am having the firm take an action infeasible for you unless you default and give all your shares to me. Your default cannot undermine my plan because you are endowed with no input for the firm." To avoid this problem, one has to exclude the monopolist from decision making. But then the core would allow a transfer from the monopolist to the other shareholders: Having no control of the firm, the monopolist alone cannot block this transfer; nor can he block it with some other shareholders, who gain from this transfer. Thus, one way or another, someone is victimized.

Proposition 3 (impossibility) *There exists a replica economy $\mathcal{E}^r$ that satisfies all the assumptions of Theorem 1 except the assumption of boundary aversion and yet, for any deterministic corporate decision rule $\tilde{J}$ and response coordination $\tilde{K}$, given Axioms 1–6, the core does not converge to Walras equilibrium allocations as $r \to \infty$.*

Proof We shall prove that Example 1 in §3.3 is such a replica economy. The unique Walrasian equilibrium allocation is:

$$x_1^* = (0, 3/4)\,; \quad x_2^* = (0, 5/4)\,; \quad y^* = (-1, 1); \quad p^* = (1, 2).$$

Pick any $\tilde{J}$ and $\tilde{K}$, given Axioms 1–6. There are only two cases:

i. Some coalition S contains the type-1 individual $(1, k)$ in some unit k but does not contain the other individual $(2, k)$ in this unit and yet S controls the firm in unit k.

ii. If coalition S contains the type-1 individual $(1, k)$ in some unit k but does not contain the type-2 individual $(2, k)$ in this unit, then S does not control the firm in unit k.

In case (i), we show that the conclusion of the proposition follows. By the axiom of irrelevance of non-shareholders, the firm in case (i) is also controlled by the singleton coalition $\{(1, k)\}$. With the blocking plan of consuming the bundle $(0, 1)$ and having the firm produce $(-1, 1)$, this type-1 individual blocks a neighborhood of the Walras equilibrium allocation. Hence the core is bounded away from the equilibrium allocation for any size of the economy. (By the axiom of irrelevance of uninvolved units, only the action of the firm in the involved unit is changeable but this firm is already controlled by the blocking individual. Thus, when this type-1 individual chooses $(-1, 1)$ for the firm in unit k, individual $(2, k)$ must default to have a feasible consumption, leaving the entire output $(-1, 1)$ to the type-1 individual, so his blocking plan is feasible.)

In case (ii), we claim: *The allocation where every type-1 individual consumes* $x_1^o = (0, 1/2)$ *and every type-2 individual consumes* $x_2^o = (0, 3/2)$ *cannot be blocked.* If this claim is true, then the conclusion of the proposition follows.

It suffices to prove that the above allocation cannot be blocked under an additional *shut-down assumption*: for any blocking coalition S, any firm j that belongs to $\tilde{K}(S)$ (controlled by outsiders of S) produces $\mathbf{0}$. If the allocation cannot be blocked under this additional assumption, then the axiom of aversion to being left behind implies that the firms in $\tilde{K}(S)$ do produce in such a way that deters blocking.

Note that $[x_1^o = (0, 1/2)\,, x_2^o = (0, 3/2)\,, y^o = (-1, 1), p^o = (1, 2)]$ is a price equilibrium with the endowments reallocated into: $e_1^o = (1/2, 0)\,; \quad e_2^o = (1/2, 1)$.

Suppose (x_1^o, x_2^o) is blocked by a coalition S with a blocking plan where coalition member i consumes x_i'. As (x_1^o, x_2^o) is a price equilibrium with transfer, we have $p^o \cdot \sum_{i \in S} x_i' > p^o \cdot \sum_{i \in S} x_i^o$. Claim: To reach a contradiction, it suffices to prove that

$$\sum_{i \in S} p^o \cdot x_i^o \geq \sum_{i \in S} p^o \cdot e_i + \sum_{i \in S} \sum_{j \in \tilde{J}(S)} \theta_{ij}^* p^o \cdot y_j^o, \tag{13}$$

where $\theta_{ij}^* := 1$ if the outsiders default on firm j and otherwise $\theta_{ij}^* := \theta_{ij}$. If (13) is true, then

$$\sum_{i \in S} p^o \cdot x_i^o \geq \sum_{i \in S} p^o \cdot e_i + \sum_{i \in S} \sum_{j \in \tilde{J}(S)} \theta_{ij}^* p^o \cdot y_j' = p^o \cdot \sum_{i \in S} x_i',$$

where the inequality uses (13) and the fact that y_j^o is a profit maximizer at p^o, and the equality uses the shutdown assumption that the coalition S gets $\mathbf{0}$ from any firm outside $\tilde{J}(S)$, hence we reach a desired contradiction.

To prove (13), partition the coalition S into three parts: (i) n unit economies (i.e., n pairs of type-1 and type-2 individuals in the same unit), (ii) g type-1 individuals from g separate units (where the type-2 individuals are not in S), and (iii) h type-2 individuals from h units (where the type-1 individuals are not in S), such that n, g, and h are nonnegative integers, with $n + g + h > 0$. For each part, we will show that its marginal contribution to the right-hand side of (13) cannot exceed its marginal contribution to the left-hand side.

Adding a unit economy to the coalition always adds the same number to the left- and right-hand sides of (13): Its marginal contribution to the left is equal to $p^o \cdot (x_1^o + x_2^o)$, and its marginal contribution to the right is equal to $p^o \cdot (e_1 + e_2) + p^o \cdot y_j^o$ (here the entire profit of the unit is counted because the firm in that unit remains controlled by the unit, by the axiom of unanimity). By the definition of $(x^o, y^o; p^o)$, $p^o \cdot (x_1^o + x_2^o) = p^o \cdot (e_1 + e_2) + p^o \cdot y_j^o$.

For each type-1 individual in the g units (in which only the type-1 individuals join S), his marginal contribution to the left-hand side of (13) is equal to $(1, 2) \cdot (0, 1/2) = 1$. His marginal contribution to the right-hand side is equal to the wealth of his endowment—his dividend is not counted in (13) because the firm in his unit does not belong to $\tilde{J}(S)$—and the wealth of his endowment is equal to $(1, 2) \cdot (1, 0) = 1$. Thus, any type-1 individual in the g units has equal marginal contribution on both sides of (13).

For each type-2 individual in the h units, his marginal contribution to the left-hand side of (13) is equal to $(1, 2) \cdot (0, 3/2) = 3$. By contrast, his marginal contribution to the right-hand side is at most 3: the wealth of his endowment is equal to $(1, 2) \cdot (0, 1) = 2$; the profit of the firm in his unit is at most $p^o \cdot y_j^o = (1, 2) \cdot (-1, 1) = 1$. Thus, any type-2 individual in the h units contributes no less to the left-hand side of (13) than to the right-hand side.

Combining the above calculations, we see that inequality (13) is true, as desired. ■

4 Stochastic Core Equivalence

The previous section demonstrates that the scope of core equivalence with production is restricted by boundary aversion as long as the notion of blocking is based on deterministic corporate control. Can this scope be different if the corporate control rights are allocated in some stochastic way? The answer is Yes.

To see why stochastic corporate control makes a difference, recall Example 1. We have seen that the status quo allocation $[x_1^o = (0, 1/2), x_2^o = (0, 3/2), y^o = (-1, 1)]$ cannot be deterministically blocked, whether the blocking notion is Definition 2 or one of those in §3.4. Now suppose that the type-1 shareholder can dissolve his partnership in the firm. Suppose further that the mechanism of partnership dissolution is that the technology embodied by the firm, being an indivisible object, has to be awarded to one of the two shareholders, and the probability with which the type-1 shareholder wins is some parameter $\hat{\theta}_1 \in [0..1]$ (not necessarily equal to his profit share θ_1). Suppose the winner of the technology has complete discretion of what to do with it and complete ownership of its output. Then a 3-member coalition consisting of a unit economy and a single type-1 individual from another unit would rather deviate from the status quo to the following plan:

i. The single type-1 person dissolves his partnership from the firm of his unit.

ii. If the firm is awarded to him, the coalition controls two firms and produces $(-1, 1)$ in each firm; the type-2 coalition member consumes $(0, 3/2)$ as in the status quo, and each of the type-1 member consumes $(0, 3/4)$.

iii. If the firm is not awarded to the single type-1 person, the coalition controls one firm and produces $(-2, \sqrt{2})$; the type-2 member consumes $(0, 3/2)$ as in the status quo, and each type-1 member consumes $(0, (\sqrt{2} - 1/2)/2)$.

This plan is feasible for the coalition at each possible state, since the total endowment of the coalition is $(2, 1)$. With this plan, each type-1 coalition member is better-off if the probability $\hat{\theta}_1$ of winning the firm is greater than 0.17:

$$\hat{\theta}_1 > 0.17 \quad \Rightarrow \quad \hat{\theta} 10\sqrt{3/4} + (1 - \hat{\theta}) 10\sqrt{(\sqrt{2} - 1/2)/2} > 10\sqrt{1/2}.$$

Then they can make the type-2 member also better-off by giving him a small positive bundle.

4.1 Partnership Dissolution and Stochastic Blocking

Let us formalize the above idea with the following setup. As in the previous section, an individual i's *corporate share* of firm j, θ_{ij}, determines the fraction of the firm's production plan that i is obligated and entitled to carry out, unless his partnership in the firm is dissolved. If i dissolves his partnership in firm j, the technology of the firm, being indivisible, is allocated to either i or someone else, and is allocated to i with probability $\hat{\theta}_{ij}$. Here the *winning probability* $\hat{\theta}_{ij} \in [0..1]$ is exogenous and may be different from the corporate share θ_{ij}. After partnership dissolution, the winner of the technology has complete discretion of what to do with it and complete ownership of its net output.

In this setup, there are two ways for a coalition to deviate from a status quo allocation. First, the coalition may come up with an alternative allocation within itself with its own endowments and the part of the status quo production bundles received by its members, provided that this blocking plan allows feasible consumptions for those outside the coalition. Second, the members of the coalition may dissolve their partnerships in the firms where they hold shares, and the coalition comes up with plans contingent on the allocation outcome of the technologies of the dissolved firms.

Formally, for any individual $i \in I$ and any firm $j \in J$, define

$$\mathbf{z}_j^i := \begin{cases} 1 & \text{if individual } i \text{ wins firm } j \text{ in case of partnership dissolution} \\ 0 & \text{else.} \end{cases} \tag{14}$$

Assumption 2 *The random vector* $(\mathbf{z}_j^i)_{i\in I}$ *is independent across firms* j*. For each firm* j*,* $\text{Prob}(\mathbf{z}_j^i = 1) = \hat{\theta}_{ij}$*. Individuals of the same type have the same winning probabilities for the same type of firms:* $\forall i_1 \in I_1$*,* $\forall j_1 \in J_1$*,* $\forall k$*,* $\forall k'$ *with* $k \neq k'$*,* $\hat{\theta}_{(i_1,k),(j_1,k)} = \hat{\theta}_{(i_1,k'),(j_1,k')} =: \hat{\theta}_{i_1 j_1}$ *and* $\hat{\theta}_{(i_1,k),(j_1,k')} = 0$*.*

A coalition wins a firm j if and only if one of its members wins j in the partnership dissolution of j. If coalition S wins firm j, it can pick any action in Y_j for the firm and carries out the action all by itself. If S loses firm j, the outsiders solely own the output of the firm, so j's production plan from the perspective of S is equal to $\mathbf{0}$.

A blocking plan $((x_i')_{i\in S}, (y_j')_{j\in J})$ for coalition S is called *noncontingent* if it is independent of the realization of $((\mathbf{z}_j^i)_{j\in J})_{i\in I}$. Given a status quo allocation (x, y), a noncontingent blocking plan is *nondisturbing* if (i) $y_j' = y_j$ for every firm $j \in J$, and (ii) the outsiders' feasibility condition (3) is satisfied. Else the blocking plan is called *disturbing*. The first route of deviation, as mentioned above, is to come up with a nondisturbing blocking plan.

A *contingent blocking plan* $\left((\mathbf{x}_i')_{i\in S}, (\mathbf{y}_j')_{j\in J}\right)$ is a mapping from a realization of $((\mathbf{z}_j^i)_{j\in J})_{i\in I}$ to a consumption-production plan $((x_i')_{i\in S}, (y_j')_{j\in J})$ such that, for every realization of $((\mathbf{z}_j^i)_{j\in J})_{i\in I}$,

$x'_i \in \mathbb{R}^l_+$ for all $i \in S$, $y'_j \in Y_j$ for any firm j, and $y'_j = \mathbf{0}$ if firm j is not won by S. As mentioned above, the second route of deviation is to find a contingent blocking plan.

Abusing notations, let e_i also denote the mapping that constantly associates individual i's endowment to every realization of $((\mathbf{z}^i_j)_{j\in J})_{i\in I}$. Let $\mathbb{E}$ denote expected values.

Assumption 3 *Every individual i is von Neumann Morgenstern rational.*

Definition 4 (stochastic blocking) *A feasible allocation (x, y) is* stochastically blocked *by a coalition S if:*

a. either S considerately blocks (x, y) with a nondisturbing noncontingent blocking plan;

b. or there is a contingent blocking plan $\big((\mathbf{x}'_i)_{i\in S}, (\mathbf{y}'_j)_{j\in J}\big)$ such that $\mathbb{E}[u_i(\mathbf{x}'_i)] > u_i(x_i)$ for all $i \in S$ and (for every realization of $(\mathbf{z}^i_j)_{j\in J}$)

$$\sum_{i\in S} \mathbf{x}'_i = \sum_{i\in S} e_i + \sum_{j\in J} \mathbf{y}'_j. \tag{15}$$

Thus, if the blocking plan does not involve partnership dissolution, it needs to be nondisturbing and the resources within the coalition are the coalition's total endowment and its share of the status quo production plans measured by the corporate shares θ_{ij}. If the blocking plan involves partnership dissolution, the resources within the coalition are stochastic and ex post consist of its total endowments plus the entire technology of every firm won by the coalition, with winning probability determined by $\hat{\theta}_{ij}$.

Note that the outsiders of a blocking coalition can always come up with a feasible consumption plan: either the blocking plan is nondisturbing, or the coalition and its outsiders are disentangled via partnership dissolution.

4.2 Stochastic Core Convergence

Let C^r_{stoch} denote the *stochastic core* defined by stochastic blocking in the replica economy $\mathcal{E}^r$.

Given winning probabilities $\hat{\theta}_{ij}$, a *virtual Walras equilibrium* is a Walras equilibrium such that the profit shares are $(\hat{\theta}_{ij})_{i,j}$ instead of $(\theta_{ij})_{i,j}$. Let $\mathbb{W}^r(\hat{\theta})$ denote the set of virtual Walras equilibrium allocations in the replica economy $\mathcal{E}^r$.

Lemma 5 (equal treatment) *For any $r = 1, 2, \ldots,$ any allocation in C^r_{stoch} or $\mathbb{W}^r(\hat{\theta})$ has the equal treatment property (ETP). Furthermore, $((x_{i_1})_{i_1 \in I_1}, (y_{j_1})_{j_1 \in J_1})$ is the type-representation of an element of $\mathbb{W}^r(\hat{\theta})$ if and only if it is the type-representation of an element of $\mathbb{W}^{r'}(\hat{\theta})$, for all r, r'.*

Proof The ETP of any stochastic core allocation follows from the proof of Lemma 2, as the blocking plan there is nondisturbing. The ETP of any virtual Walras equilibrium allocation follows from the fact that the solution for every firm-type's optimization problem is unique due to strictly concave production frontiers, and the one for every consumer-type is unique due to strictly convex preferences. The set of virtual equilibrium allocations are identical across r because every agent's optimization problem is identical across r. ■

Thus, as in the deterministic case, the cores $C^1_{\text{stoch}}, C^2_{\text{stoch}}, \ldots$ of replica economies of various sizes can be regarded as subsets of the same space, $(\mathbb{R}^l)^{I_1 \cup J_1}$, and clearly

$$C^1_{\text{stoch}} \supseteq C^2_{\text{stoch}} \supseteq \cdots \supseteq C^r_{\text{stoch}} \supseteq \cdots .$$

The sets of virtual equilibrium allocations, $\mathbb{W}^1(\hat{\theta}), \mathbb{W}^2(\hat{\theta}), \ldots$, can also be viewed as subsets of this space and furthermore they can be viewed as the same subset

$$\mathbb{W}(\hat{\theta}) := \mathbb{W}^1(\hat{\theta}) = \cdots = \mathbb{W}^r(\hat{\theta}) = \cdots .$$

The main result of this section is that $\mathbb{W}(\hat{\theta}) \supseteq \cap_{r=1}^{\infty} C^r_{\text{stoch}}$. To see the intuition, suppose an individual i slightly prefers a consumption x'_i to the status quo x_i, but to consume x'_i he needs to change a firm j's action into y'_j:

$$u_i\left(e_i + \hat{\theta}_{ij} y'_j\right) = u_i(x'_i) = u_i(x_i) + \epsilon. \tag{16}$$

The problem is that if he alone dissolves his partnership in firm j then with probability $1 - \hat{\theta}_{ij}$ he will lose and do much worse than the status quo. To resolve this problem, suppose there are n copies of the economy. In each copy, the counterpart of i dissolves partnership in the counterpart of firm j. If the firm is won (with probability $\hat{\theta}_{ij}$), it carries out the new production plan; else the firm means $\mathbf{0}$ to this coalition. Then the resources within the n clones, in expected value, are equal to $e_i + \hat{\theta}_{ij} y'_j = x'_i$ per capita. By the law of large numbers, if n is sufficiently large, there is a sufficiently large probability with which the ex post bundle $\mathbf{x}'_i$ for each clone is sufficiently close to the the preferred target x'_i.

Lemma 6 *If an allocation (x, y) has the equal treatment property and is wishfully blocked in a replica economy $\mathcal{E}^m$, then it is wishfully blocked in $\mathcal{E}^{m+1}$ by a blocking plan $(\hat{x}, \hat{y})$ such that $\hat{x}_i$ is interior to the consumption set $\mathbb{R}^l_+$ for all coalition members i.*

Proof See Appendix D. ■

Theorem 2 (stochastic core convergence) *If Assumptions 2–3 hold, $\mathbb{W}(\hat{\theta}) \supseteq \cap_{r=1}^{\infty} C^r_{\text{stoch}}$.*

Proof It suffices to prove that any feasible allocation (x, y) with ETP is stochastically blocked in some replica economy if $(x, y) \notin \mathbb{W}(\hat{\theta})$. By Proposition 1, with the corporate shares θ_{ij} there replaced by winning probabilities $\hat{\theta}_{ij}$ here, (x, y) is wishfully blocked by some coalition S in some $\mathcal{E}^r$ with a noncontingent blocking plan

$$(x', y') := \left((x'_i)_{i\in S}, (y'_j)_{j\in J}\right).$$

By Lemma 6, assume without loss that the consumption bundle x'_i in this blocking plan is interior to $\mathbb{R}^l_+$ for each $i \in S$. To avoid triviality, suppose that the blocking plan is disturbing.

For each firm j, denote

$$\mathbf{z}_j := \max_{i\in S} \mathbf{z}^i_j = \begin{cases} 1 \quad \text{(i.e., } S \text{ wins firm } j) \quad \text{with probability} \quad \sum_{i\in S} \hat{\theta}_{ij} \\ 0 \quad \text{(i.e., } S \text{ loses firm } j) \quad \text{with probability} \quad 1 - \sum_{i\in S} \hat{\theta}_{ij}. \end{cases}$$

Consider n copies of this economy $\mathcal{E}^r$ and label them as $\mathcal{E}^r_1, \ldots, \mathcal{E}^r_n$. Label the copies of the blocking coalition S by $S_1, \ldots, S_n$, with S_k being the one in $\mathcal{E}^r_k$; as $(\mathbf{z}_j)_{j\in J}$ denotes the random vector for S, $(\mathbf{z}_{j,k})_{j\in J}$ denotes the random vetor for the kth copy S_k. The replica economy $\mathcal{E}^{nr}$ consisting of the n copies $\mathcal{E}^r_1, \ldots, \mathcal{E}^r_n$. Abusing notations, let I be the index set of individuals in the original $\mathcal{E}^r$, and J the index set of firms in the original $\mathcal{E}^r$; thus, in $\mathcal{E}^{nr}$, an individual (resp. firm) that belongs to the kth copy is indexed by (i, k) (resp. (j, k)).

By Assumption 2, $\mathbb{E}\left[\mathbf{z}_j y'_j\right] = \sum_{i\in S} \hat{\theta}_{ij} y'_j$ for each $j \in J$. Thus, as the random variables $\mathbf{z}_{j,1}, \ldots, \mathbf{z}_{j,n}$ are i.i.d., the law of large number implies that $\frac{1}{n}\sum_{k=1}^{n} \mathbf{z}_{j,k} y'_j \to \sum_{i\in S} \hat{\theta}_{ij} y'_j$ in probability as $n \to \infty$. Thus, there exists a sufficiently large n for which the event

$$\Lambda := \left\{ \left\| \frac{1}{n}\sum_{k=1}^{n} \mathbf{z}_{j,k} y'_j - \sum_{i\in S} \hat{\theta}_{ij} y'_j \right\| < \epsilon, \forall j \in J \right\}$$

has a probability greater than α, where ϵ and α are chosen so that, for all $i \in S$, any point x''_i within $\epsilon|J|$-distance from x'_i still belongs to $\mathbb{R}^l_+$ and

$$\alpha u_i(x''_i) + (1 - \alpha) u_i(\mathbf{0}) > u_i(x_i);$$

Such ϵ and α exist because $u_i(x'_i) > u_i(x_i)$, u_i is lower semicontinuous, and x'_i is interior to $\mathbb{R}^l_+$.

We claim that the coalition $S^n := \cup_{k=1}^n S_k$ stochastically blocks the allocation (x, y). The contingent blocking plan is: (i) in the event Λ, the consumption bundle for every coalition member $(i, k) \in S \times \{1, \dots, n\}$ is

$$\mathbf{x}''_{i,k} := x'_i + \frac{1}{|S|} \sum_{j \in J} \left[\frac{1}{n} \sum_{k=1}^{n} \mathbf{z}_{j,k} y'_j - \sum_{i \in S} \hat{\theta}_{ij} y'_j \right], \tag{17}$$

and the production for firm $(j, k) \in J \times \{1, \dots, n\}$ is (from the coalition's perspective)

$$\mathbf{y}''_{j,k} := \mathbf{z}_{j,k} y'_j = \begin{cases} y'_j & \text{if } \mathbf{z}_{j,k} = 1 \\ \mathbf{0} & \text{if } \mathbf{z}_{j,k} = 0; \end{cases} \tag{18}$$

(ii) in the event of "not Λ," let

$$\mathbf{x}''_{i,k} := e_i; \ \mathbf{y}''_{j,k} := \mathbf{0}. \tag{19}$$

This contingent blocking plan is feasible (i.e., Eq. (15) is satisfied): This is a direct consequence of (19) in the event "not Λ"; in the event Λ,

$$\sum_{k=1}^{n} \sum_{i \in S} \mathbf{x}''_{i,k} = n \sum_{i \in S} x'_i + \sum_{j \in J} \sum_{k=1}^{n} \mathbf{z}_{j,k} y'_j - n \sum_{j \in J} \sum_{i \in S} \hat{\theta}_{ij} y'_j = n \sum_{i \in S} e_i + \sum_{j \in J} \sum_{k=1}^{n} \mathbf{y}''_j,$$

where the first equality follows from (17) and the second follows from (5) and (18).

By the choice of ϵ, every possible realization of $\mathbf{x}''_i$, conditional on the event Λ, is at most $\epsilon|J|$ from x'_i and hence belongs to $\mathbb{R}^l_+$; by the choice of α, $\mathbb{E}\left[u_i(\mathbf{x}''_i)\right] > u_i(x_i)$. Thus, Definition 4 is satisfied, as claimed. ■

4.3 Interpretations and Conclusion

Theorem 2 can be interpreted in two ways. First, if shareholders can dissolve their partnership and a firm's technology after dissolution is allocated stochastically among shareholders, the core converges to virtual equilibria, which differ from Walras equilibria only in the sense that an individual's profit share from a firm is replaced by his winning probability in partnership dissolution. Thus, if core is taken to be the solution concept for decentralized markets, this interpretation means that, the bigger an economy, the more its outcomes approximate a kind of virtual equilibria where a shareholder's profit share from a firm is *derived* from his winning probability in the off-path event of partnership dissolution.

The second interpretation is that the core converges to the actual Walras equilibria if the winning probabilities $(\hat{\theta}_{ij})_{i,j}$ in partnership dissolution coincide with the corporate shares $(\theta_{ij})_{i,j}$ that determine one's share of a firm's production plan. In this case, Walras equilibria belong to the core, as the next Remark 1 shows. Thus, if corporate control is stochastically allocated according to corporate shares, the class of economies where core equivalence holds is expanded beyond the one for deterministic corporate control.

Remark 1 *If Assumptions 2–3 hold, if $(\hat{\theta}_{ij})_{i,j} = (\theta_{ij})_{i,j}$, and if everyone's utility function is concave, then any Walras equilibrium allocation cannot be stochastically blocked.*

Proof Let $(x^*, y^*; p^*)$ be a Walras equilibrium. Suppose it is stochastically blocked by a coalition S. If the blocking plan is nondisturbing, with noncontingent consumption plan $(x'_i)_{i \in S}$ such that

$$\sum_{i \in S} x'_i = \sum_{i \in S} e_i + \sum_{i \in S} \sum_{j \in J} \theta_{ij} y_j.$$

Then one can prove a contradiction by mimicking the proof of the first welfare theorem.

Now suppose that the blocking plan is disturbing and the associated contingent blocking plan is $(\mathbf{x}', \mathbf{y}')$. By definition of contingency blocking plan,

$$\mathbb{E}\left[\sum_{j \in J} \mathbf{y}'_j\right] = \sum_{j \in J} \sum_{i \in S} \hat{\theta}_{ij} \mathbb{E}\left[\mathbf{y}'_j \mid \mathbf{z}_j = 1\right].$$

Then Eq. (15) implies that

$$\sum_{i \in S} \mathbb{E}\left[\mathbf{x}'_i\right] = \sum_{i \in S} e_i + \sum_{j \in J} \sum_{i \in S} \hat{\theta}_{ij} \mathbb{E}\left[\mathbf{y}'_j \mid \mathbf{z}_j = 1\right]. \tag{20}$$

By concavity of u_i, $\mathbb{E}\left[u_i(\mathbf{x}'_i)\right] \leq u_i\left(\mathbb{E}\left[\mathbf{x}'_i\right]\right)$ for all $i \in I$. Hence $u_i\left(\mathbb{E}\left[\mathbf{x}'_i\right]\right) > u_i(x^*_i)$ for every coalition member i. Thus, as in the proof of the first welfare theorem,

$$\sum_{i \in S} p^* \cdot \mathbb{E}\left[\mathbf{x}'_i\right] > \sum_{i \in S} p^* \cdot e_i + \sum_{i \in S} \sum_{j \in J} \theta_{ij} p^* \cdot y^*_j \geq \sum_{i \in S} p^* \cdot e_i + \sum_{i \in S} \sum_{j \in J} \theta_{ij} p^* \cdot \mathbb{E}\left[\mathbf{y}'_j \mid \mathbf{z}_j = 1\right],$$

where the last inequality follows from profit maximization and the fact that $\mathbb{E}\left[\mathbf{y}'_j \mid \mathbf{z}_j = 1\right]$ is a convex combination of elements of the convex set Y_j and hence belongs to Y_j. But then

$$p^* \cdot \sum_{i \in S} \mathbb{E}\left[\mathbf{x}'_i\right] > p^* \cdot \left[\sum_{i \in S} e_i + \sum_{j \in J} \sum_{i \in S} \theta_{ij} \mathbb{E}\left[\mathbf{y}'_j \mid \mathbf{z}_j = 1\right]\right].$$

This contradicts (20) because $\theta_{ij} = \hat{\theta}_{ij}$ by assumption. ■

Given whichever of the above interpretations, Theorem 2 stands as a contrast with the core equivalence result in the case of deterministic blocking. With either interpretation, the two theorems together convey the message that, once production is added, the scope of core equivalence, hence the cooperative foundation of the price-taking assumption, is sensitive to the allocation of corporate control that the notion of blocking captures.

5 Bibliography Note

Debreu and Scarf [5] considered a special case with production where the technology is constant returns to scale *and* is available to all coalitions. Other than the main differences in the treatment of production and the boundary aversion assumption, the assumption-differences between our Theorem 1 and theirs are minor. Debreu and Scarf assumed insatiability and continuity, we assume strong monotonicity but only lower semicontinuity. They assumed strict positivity of individual endowments, we assume nonvoid individual endowments and only strict positivity of the total endowment. Our Theorem 2 has the same minor differences, but it does not need boundary aversion, and it requires von Neuman Morgenstern rationality. Assuming strict concavity of production frontiers, we do not cover constant returns to scale in this paper, but we do in a companion paper, Xiong and Zheng [13].

The most general core convergence theorem for exchange economies is given by Anderson [3], who allowed nonconvex preferences. In [13], we prove an extension of Anderson's theorem to production economies based on stochastic blocking (Definition 4).

Other than Debreu and Scarf, there had been two approaches to include production to core equivalence, but none captured the interactions among shareholders. One group of authors such as Hildenbrand [11, Ch. 4] and Boehm [4] assumed that every possible coalition is endowed with a technology. Another group of authors, Allingham [2, pp52–53] and Aliprantis, Brown and Burkinshaw [1], assumed that technologies are divisible and a shareholder controls a fraction of the firm's production set.

Haller [10] is the first to consider the effect of corporate control on core equivalence. Our differences are highlighted in the Introduction and detailed in Appendix E.

Ellickson, Grodal, Scotchmer and Zame [7, 8, 9] prove core equivalence in their general equilibrium theory of clubs. Unlike in our model, a blocking coalition in their model is not interlocked with its outsiders in any firm, because a coalition forms clubs (firms) within itself and its members do not join any club (firm) containing outsiders.

A A Complete Second Welfare Theorem

The second welfare theorem proved here, Corollary 1, is used by Lemma 2.[4]

Lemma 7 *If $((x_i)_{i\in I}, (y_j)_{j\in J}; p)$ is a quasiequilibrium and $p \neq \mathbf{0}$, then $p \gg \mathbf{0}$ and the quasiequilibrium is a price equilibrium with possible transfers such that each person i's wealth is equal to $p \cdot x_i$.*

Proof Let w_i denote individual i's wealth at the quasiequilibrium $((x_i)_{i\in I}, (y_j)_{j\in J}; p)$. By definition of quasiequilibrium (e.g, [12, def. 16.D.1]), each firm's production plan is profit-maximizing given p and each consumer is quasi-optimizing. Consumers' quasi-optimization and strongly monotone preferences imply that the supporting price is nonnegative in each component, i.e., $p \geqq \mathbf{0}$. Then it follows from profit maximization and the assumptions "$\sum_{i\in I} e_i \gg \mathbf{0}$ and $\mathbf{0} \in \sum_{j\in J} Y_j$" that

$$\sum_{i\in I} w_i = \sum_{i\in I} p \cdot e_i + \sum_{j\in J} p \cdot y_j > 0.$$

Thus, there exists an individual i whose wealth w_i at this quasiequilibrium is positive. It follows that $p \gg \mathbf{0}$: Suppose not, say $p_k = 0$ for good k, then the new bundle x_i' obtained from adding one unit of good k to individual i's original bundle x_i would cost him at most w_i. As his preference is strongly monotone, $u_i(x_i') > u_i(x_i)$. Then the quasiequilibrium condition implies that $p \cdot x_i' = w_i$. By lower semicontinuity of his preference, we can scale down x_i' to $\lambda x_i'$ for some positive λ sufficiently close to one so that $p \cdot \lambda x_i' < w_i$ and $u_i(\lambda x_i') > u_i(x_i)$, contradicting the quasiequilibrium condition. Thus, $p \gg \mathbf{0}$.

It follows that x_i is optimum for each individual i with wealth $w_i^* := p \cdot x_i$: If $x_i \gneqq \mathbf{0}$ then $w_i^* > 0$ (since $p \gg \mathbf{0}$), i.e., the cheaper-consumption condition is met, hence x_i is optimum by lower semicontinuous preferences; else $x_i = \mathbf{0}$, then since $p \gg \mathbf{0}$, person i's

[4] In textbooks, the second welfare theorem is usually presented in two propositions, one says Pareto efficiency implies existence of a price that supports the allocation as a quasiequilibrium, and the other gives a sufficient condition, which however is about the endogenous supporting price, for a quasiequilibrium to be an equilibrium with transfers. Unlike such versions, Corollary 1 is based on purely primitive assumptions.

budget set is the singleton $\{x_i\} = \{\mathbf{0}\}$. Hence $(x, y; p)$ constitutes a price equilibrium with possible transfers such that each individual i's wealth is $w_i^* = p \cdot x_i$. ■

Corollary 1 (second-welfare theorem) *If (x, y) is a Pareo efficient allocation, then there exists a price vector $p \gg \mathbf{0}$ that supoorts (x, y) as a price equilibrium with transfers.*

Proof By the usual version of the second welfare theorem, there exists a $p \in \mathbb{R}^l_+ \setminus \{\mathbf{0}\}$ that supports the Pareto efficient allocation as a quasiequilibrium (e.g., [12, prop. 16.D.1; def. 16.D.1]). Then the corollary follows from Lemma 7. ■

B The Proof of Propositions 1 and 2

Proof of Proposition 1: The part $\mathbb{W} \subseteq \cap_{r=1}^{\infty} C^r_{\text{wish}}$ is easy, similar to the first welfare theorem. To prove the converse, $\cap_{r=1}^{\infty} C^r_{\text{wish}} \subseteq \mathbb{W}$, pick any allocation from $\cap_{r=1}^{\infty} C^r_{\text{wish}}$. Since wishful cores have the equal treatment property, denote this allocation by $((x_{i_1})_{i_1 \in I_1}, (y_{j_1})_{j_1 \in J_1})$. We shall prove $((x_{i_1})_{i_1 \in I_1}, (y_{j_1})_{j_1 \in J_1}) \in \mathbb{W}$ by mimicking Debreu and Scarf [5].

For each $i_1 \in I_1$, define

$$\Gamma_{i_1} := \left\{ z_{i_1} \in \mathbb{R}^l : z_{i_1} + e_{i_1} + \sum_{j_1 \in J_1} \theta_{i_1 j} y'_{j_1} \succ_{i_1} x_{i_1} \text{ for some } (y'_{j_1})_{j_1 \in J_1} \in (Y_{j_1})_{j \in J_1} \right\},$$

which is convex. Let Γ be the set of convex combinations of these sets, i.e.

$$\Gamma := \left\{ z = \sum_{i_1 \in I_1} \alpha_{i_1} z_{i_1} : \sum_{i_1 \in I_1} \alpha_{i_1} = 1; \forall i_1 \in I_1 \, [\alpha_{i_1} \geq 0; z_{i_1} \in \Gamma_{i_1}] \right\}.$$

Claim: $\mathbf{0} \notin \Gamma$. Prove by contradiction. Suppose $\mathbf{0} \in \Gamma$. Then $\sum_{i_1 \in I_1} \alpha_{i_1} z_{i_1} = \mathbf{0}$, with $\alpha_{i_1} \geq 0$, $\sum_{i_1 \in I_1} \alpha_{i_1} = 1$, and $z_{i_1} + e_{i_1} + \sum_{j_1 \in J_1} \theta_{i_1 j_1} y'_{j_1} \succ_{i_1} x_{i_1}$, for some $(y'_{j_1})_{j_1 \in J_1} \in (Y_{j_1})_{j_1 \in J_1}$ and for all $i_1 \in I_1$. Consider an arbitrary large integer k. Denote $\lceil k\alpha_{i_1} \rceil$ for the smallest integer greater or equal to $k\alpha_{i_1}$. Let S_1 be the set of $i_1 \in I_1$ for which $\alpha_{i_1} > 0$. For each $i_1 \in S_1$, let $z^k_{i_1} := (k\alpha_{i_1} / \lceil k\alpha_{i_1} \rceil) \, z_{i_1}$. Note that $z^k_{i_1}$ goes to z_{i_1} as k goes to infinity. By the lower semicontinuity of preference and the fact that $z_{i_1} + e_{i_1} + \sum_{j \in J} \theta_{i_1 j_1} y'_{j_1} \succ_{i_1} x_{i_1}$, there exists a large enough k such that $z^k_{i_1} + e_{i_1} + \sum_{j \in J} \theta_{i_1 j_1} y'_{j_1} \succ_{i_1} x_{i_1}$ and

$$\sum_{i_1 \in S_1} \lceil k\alpha_{i_1} \rceil z^k_{i_1} = k \sum_{i_1 \in S_1} \alpha_{i_1} z_{i_1} = \mathbf{0}.$$

In the $\sum_{i_1\in S_1}\lceil k\alpha_{i_1}\rceil$-replica economy, let the coalition consist of $\lceil k\alpha_{i_1}\rceil$ individuals of type i_1 for each type i_1, such that none of the coalition members live in the same unit. The blocking plan is: for every $i_1 \in S_1$, each coalition member of type i_1 consumes $z^k_{i_1}+e_{i_1}+\sum_{j_1\in J_1}\theta_{i_1 j}y'_{j_1}$, and each type-$j_1$ firm where this member is a shareholder produces y'_{j_1}. (Since no two coalition members share the same firm, they have no conflict on these production plans.) This coalition wishfully blocks $((x_{i_1})_{i_1\in I_1},(y_{j_1})_{j_1\in J_1})$, contradiction.

Hence $\Gamma\cap\{\mathbf{0}\}=\varnothing$. Also Γ is convex; by the separating hyperplane theorem, there exists a price $p\in\mathbb{R}^l\setminus\{\mathbf{0}\}$ such that $p\cdot z\geq 0$ for all $z\in\Gamma$.

To complete the proof, by Lemma 7, it suffices to show that (i) $((x_{i_1})_{i_1\in I_1},(y_{j_1})_{j_1\in J_1},p)$ is a quasiequilbrium and (ii) for all $i_1\in I_1$,

$$p\cdot x_{i_1}=p\cdot e_{i_1}+p\cdot\sum_{j_1\in J_1}\theta_{i_1 j_1}y_{j_1}, \tag{21}$$

because then Lemma 7 implies that it is a Walras equilibrium that requires zero transfer.

Let $\pi^*_{j_1}:=\sup\{p\cdot y'_{j_1}:y'_{j_1}\in Y_{j_1}\}$ for each firm-type j_1. Since

$$x'_{i_1}\succ_{i_1}x_{i_1}\Longrightarrow x'_{i_1}-e_{i_1}-\sum_{j_1\in J_1}\theta_{i_1 j_1}y'_{j_1}\in\Gamma_{i_1},\forall(y'_{j_1})_{j_1\in J_1}\in(Y_{j_1})_{j_1\in J_1},$$

the separating-hyperplane property of p implies

$$x'_{i_1}\succ_{i_1}x_{i_1}\Longrightarrow p\cdot x'_{i_1}\geq p\cdot e_{i_1}+\sum_{j_1\in J_1}\theta_{i_1 j_1}\pi^*_{j_1}. \tag{22}$$

By local non-satiation of preference, (22) implies, for all $i_1\in I_1$,

$$p\cdot x_{i_1}\geq p\cdot e_{i_1}+\sum_{j_1\in J_1}\theta_{i_1 j_1}\pi^*_{j_1}. \tag{23}$$

Since $((x_{i_1})_{i_1\in I_1},(y_{j_1})_{j_1\in J_1})$ is feasible, $\sum_{i_1\in I_1}x_{i_1}=\sum_{i_1\in I_1}e_{i_1}+\sum_{j_1\in J_1}y_{j_1}$. Then (23) implies

$$p\cdot\sum_{i_1\in I_1}e_{i_1}+p\cdot\sum_{j_1\in J_1}y_{j_1}=p\cdot\sum_{i_1\in I_1}x_{i_1}\geq p\cdot\sum_{i_1\in I_1}e_{i_1}+\sum_{j_1\in J_1}\pi^*_{j_1}\geq p\cdot\sum_{i_1\in I_1}e_{i_1}+p\cdot\sum_{j_1\in J_1}y_{j_1}, \tag{24}$$

where the last inequality follows from the definition of $\pi^*_{j_1}$. By (24), $\sum_{j_1}p\cdot y_{j_1}=\sum_{j_1}\pi^*_{j_1}$, hence every y_{j_1} maximizes profits for firms of type j_1 under the price p. From (24) we also have $p\cdot\sum_{i_1\in I_1}x_{i_1}=p\cdot\sum_{i_1\in I_1}e_{i_1}+\sum_{j_1\in J_1}\pi^*_{j_1}$; hence (23) implies

$$p\cdot x_{i_1}=p\cdot e_{i_1}+\sum_{j_1\in J_1}\theta_{i_1 j_1}\pi^*_{j_1}=p\cdot e_{i_1}+p\cdot\sum_{j_1\in J_1}\theta_{i_1 j_1}y_{j_1}$$

for all $i_1\in I_1$. This has two implications: First, Eq. (21) is true. Second, (22) implies $x'_{i_1}\succ_{i_1}x_{i_1}\Rightarrow p\cdot x'_{i_1}\geq p\cdot x_{i_1}$ for all $i_1\in I_1$. Hence $((x_{i_1})_{i_1\in I_1},(y_{j_1})_{j_1\in J_1},p)$ is a quasiequilbrium and (21) is true, as desired. ■

Proof of Proposition 2: To prove $\mathbb{W} \subseteq \cap_{r=1}^{\infty} C_{\text{incon}}^r$, just mimic the first welfare theorem.

To prove $\cap_{r=1}^{\infty} C_{\text{incon}}^r \subseteq \mathbb{W}$, we prove in the following that if an allocation is wishfully blocked in $\mathcal{E}^r$, then it is inconsiderately blocked in $\mathcal{E}^{2r}$. That means $\cap_{r=1}^{\infty} C_{\text{incon}}^r \subseteq \cap_{r=1}^{\infty} C_{\text{wish}}^r$. Then by Proposition 1 we have $\cap_{r=1}^{\infty} C_{\text{incon}}^r \subseteq \cap_{r=1}^{\infty} C_{\text{wish}}^r \subseteq \mathbb{W}$.

Suppose in an r-replica economy $\mathcal{E}^r$, a feasible allocation (x, y) is wishfully blocked by a coalition S with blocking plan $\big((x_i^*)_{i\in S}, (y_j^*)_{j\in J}\big)$. Hence

$$\sum_{i\in S} x_i^* = \sum_{i\in S} e_i + \sum_{j\in J}\sum_{i\in S} \theta_{ij} y_j^*. \tag{25}$$

With abuse of notation, let $\mathcal{E}^{r\prime}$ denote the other r-replica economy in the $2r$-replica economy $\mathcal{E}^{2r}$, i.e. $\mathcal{E}^{2r} = \mathcal{E}^r \cup \mathcal{E}^{r\prime}$ and $\mathcal{E}^r$ is disjoint with $\mathcal{E}^{r\prime}$. Let I, J be the sets of consumers and firms in $\mathcal{E}^r$, and we use $[']$ to denote the counterpart in $\mathcal{E}^{r\prime}$, so I', J' are the sets of consumers and firms in $\mathcal{E}^{r\prime}$, and S' is the $\mathcal{E}^{r\prime}$-counterpart of S.

In $\mathcal{E}^{2r}$ $(= \mathcal{E}^r \cup \mathcal{E}^{r\prime})$, consider the coalition, $S^* := S \cup I'$, with the blocking plan:

i. Consumption: $((x_i^*)_{i\in S}, (x_i)_{i\in I'})$, i.e. every agent in S consumes x_i^*, and every agent in $\mathcal{E}^{r\prime}$ takes the status quo consumption.

ii. Production: every firm in $\mathcal{E}^r$ takes the status quo prouduction; the firms in $\mathcal{E}^{r\prime}$ takes the following aggregate production:

$$\sum_{j\in J'}\sum_{i\in S'} \theta_{ij} y_j^* + \sum_{j\in J'}\sum_{i\in I'\setminus S'} \theta_{ij} y_j,$$

which is feasible since all Y_j are convex and all the firms in $\mathcal{E}^{r\prime}$ are controled by S^*.

To see that the above blocking plan is feasible, observe that pooling resources within this coalition amounts to a trade between S and I': I' gives S the production bundle $\sum_{j\in J'}\sum_{i\in S'} \theta_{ij} y_j^*$, and S gives I' the production bundle $\sum_{j\in J}\sum_{i\in S} \theta_{ij} y_j$. Since S' and J' are the $\mathcal{E}^{r\prime}$-counterparts of the sets S and J,

$$\begin{aligned}\sum_{j\in J'}\sum_{i\in S'} \theta_{ij} y_j^* &= \sum_{j\in J}\sum_{i\in S} \theta_{ij} y_j^*,\\ \sum_{j\in J}\sum_{i\in S} \theta_{ij} y_j &= \sum_{j\in J'}\sum_{i\in S'} \theta_{ij} y_j.\end{aligned}$$

Thus, it is feasible for members of S to consume $(x_i^*)_{i\in S}$ (recalling (25)), and it is feasible for members of I' to consume $(x_i)_{i\in I'}$, as $(x_i)_{i\in I'}$ is aggregate feasible in $\mathcal{E}^{r\prime}$.

Since this blocking plan makes each member of S better-off and none of I' worse-off, it can be slightly adjusted to make each member of I' also better-off by lower semicontinuous preferences. Thus, (x, y) is incosiderately blocked by the coalition S^* in $\mathcal{E}^{2r}$, as desired. ■

C An Inconsiderate Blocking Plan for Example 1

Here is a direct construction that inconsiderately blocks all non-equilibria in Example 1 without using Proposition 2.

We shall show that if an allocation is not Walras Equilibrium, then it can be inconsiderately blocked. Clearly there is no loss of generality to assume that a non-equilibrium allocation is Pareto optimal, then it is an element of the following set:

$$\{[x_1 = (0, a)\,;\ x_2 = (0, 2-a)\,;\ y = (-1, 1)] : a \in [0..2]\}.$$

There are two cases:

i. $a < 3/4$.

Consider the $(N+1)$-replica economy, and the coalition S that consists of all consumers except the $(N+1)$th type-2 consumer. The blocking plan is: Each firm in the first N-replica economy chooses the production plan $\left(-1 - \frac{1}{2N}, \sqrt{1 + \frac{1}{2N}}\right)$; the firm in the $(N+1)$th economy sticks to $(-1, 1)$; the consumers in first N-replica economy take the consumption $[x_1 = (0, a), x_2 = (0, 2-a)]$, and the $(N+1)$th type-1 consumer takes $\left(0, N\sqrt{1 + \frac{1}{2N}} - N + \frac{1}{2}\right)$. This blocking plan is feasible and inconsiderately blocks $[x_1 = (0, a), x_2 = (0, 2-a)]$ for sufficient large N, because

$$\begin{aligned}\lim_{N\to\infty} N\sqrt{1 + \frac{1}{2N}} - N + \frac{1}{2} &= \lim_{N\to\infty} \frac{\left(N\sqrt{1 + \frac{1}{2N}} - N + \frac{1}{2}\right)\left(N\sqrt{1 + \frac{1}{2N}} + N - \frac{1}{2}\right)}{N\sqrt{1 + \frac{1}{2N}} + N - \frac{1}{2}} \\ &= \lim_{N\to\infty} \frac{\frac{3}{2}N - \frac{1}{4}}{N\sqrt{1 + \frac{1}{2N}} + N - \frac{1}{2}} = \frac{3}{4}.\end{aligned}$$

ii. $a > 3/4$.

Hence type-2 person consumes $(0, 2-a)$ and $(2-a) < 5/4$.

Consider the $(N+1)$-replica economy, and the coalition S consisting of all consumers except the $(N+1)$th type-1 consumer. The blocking plan is: Each firm in the first

N-replica economy chooses the production plan $\left(-1+\frac{1}{2N}, \sqrt{1-\frac{1}{2N}}\right)$; the firm in the $(N+1)$th economy sticks to $(-1,1)$; the consumers in first N-replica economy take the consumption $[x_1=(0,a), x_2=(0,2-a)]$, and the $(N+1)$th type-1 consumer take $\left(0, N\sqrt{1-\frac{1}{2N}}-N+\frac{3}{2}\right)$. This blocking plan is feasible and inconsiderately blocks $[x_1=(0,a), x_2=(0,2-a)]$ for sufficient large N, because

$$\begin{aligned}\lim_{N\to\infty} N\sqrt{1-\frac{1}{2N}}-N+\frac{3}{2} &= \lim_{N\to\infty}\frac{\left(N\sqrt{1-\frac{1}{2N}}-N+\frac{3}{2}\right)\left(N\sqrt{1-\frac{1}{2N}}+N-\frac{3}{2}\right)}{N\sqrt{1-\frac{1}{2N}}+N-\frac{3}{2}}\\ &= \lim_{N\to\infty}\frac{\frac{5}{2}N-\frac{9}{4}}{N\sqrt{1-\frac{1}{2N}}+N-\frac{3}{2}}=\frac{5}{4}.\end{aligned}$$

D The Proof of Lemma 6

To prove Lemma 6, we first prove the following intermediate result.

Lemma 8 *For any feasible allocation (x'', y'') and for any $\epsilon > 0$, there exists a feasible allocation (x', y') such that x'_i is an interior point of $\mathbb{R}^l_+$ and $\|x'_i - x''_i\| < \epsilon$ for each $i \in I$.*

Proof By the feasibility of (x, y),

$$\mathbf{0} \leqq \sum_{i\in I} x_i = \sum_{i\in I} e_i + \sum_{j\in J} y_j. \tag{26}$$

Note that $(x_i)_k$, $(x'_i)_k$, $(e_i)_k$, and $(y_j)_k$ denote the kth coordinates of x_i, x'_i, e_i, and y_j.

Claim: there exist a $\lambda \in (0..1)$ sufficiently close to 1 and an $(x'_i)_{i\in I} \in \left(\mathbb{R}^l_+\right)^I$ such that

$$(\forall k = 1,\ldots,l)\ (1-\lambda)\left|\sum_{j\in J}(y_j)_k\right| < \frac{\epsilon}{2\sqrt{l}}; \tag{27}$$

$$\sum_{i\in I} x'_i \ = \ \sum_{i\in I} e_i + \sum_{j\in J}\lambda y_j \gg \mathbf{0}. \tag{28}$$

Proof of the claim: First, Ineq. (27) is true for any λ sufficiently close to 1. Second, we shall prove (28) for any $\lambda \in (0..1)$. For each good k there are only two possibilities:

i. Suppose $\sum_{j\in J}(y_j)_k \geq 0$. Then $\sum_{i\in I}(e_i)_k + \lambda\sum_{j\in J}(y_j)_k \geq \sum_{i\in I}(e_i)_k > 0$ for any $\lambda \in (0..1)$, hence the inequality in (28) is true for the kth coordinate. To get the equality in (28), for each $i \in I$, reduce $(x_i)_k$ by a nonnegative amount so that the new quantity $(x_i')_k$ is still nonnegative and the total reduction in good-k consumption across $i \in I$ is equal to the reduction in the aggregate output of good k, $(1-\lambda)\sum_j (y_j)_k$. (Thus, by the equality in (26), the equality in (28) is true for the kth coordinate.)

ii. Suppose $\sum_{j\in J}(y_j)_k < 0$. Then $\sum_{j\in J}(y_j)_k < 0$, so $\sum_{j\in J}(y_j)_k$ can only be less negative when $\sum_{j\in J} y_j$ is scaled down to $\lambda \sum_{j\in J} y_j$ for any $\lambda \in (0..1)$. (Hence the inequality in (28) is true for the kth coordinate.) This change in production frees up some individual endowments of good k, which totals $(1-\lambda)\sum_j (y_j)_k$. For each individual, add his freed up amount to his good k consumption. (Thus, by the equality in (26), the equality in (28) is true for the kth coordinate.)

Hence the claim is true.

Note that the new allocation (x', y') constructed above is feasible because $\lambda y_j \in Y_j$ by convexity of Y_j and $\mathbf{0} \in Y_j$. Also note that, for each good k,

$$|(x_i')_k - (x_i)_k| \le (1-\lambda)|\sum_j (y_j)_k| < \epsilon/(2\sqrt{l})$$

by (27), hence

$$\|x_i' - x_i\| \le (1-\lambda)\left\|\sum_j y_j\right\| < \epsilon/2.$$

Thus, we are done if x_i' is interior to $\mathbb{R}^l_+$ for all $i \in I$.

Suppose x_i' is not interior to $\mathbb{R}^l_+$ for some $i \in I$. Then conduct the following operation for any good k such that $(x_i')_k = 0$: By (28), $(x_{i_*}')_k > 0$ for some individual i_*; then reduce $(x_{i_*}')_k > 0$ by a sufficiently small amount less than $\min\{(x_{i_*}')_k, \epsilon\}/(2l|I|)$, and add this amount to $(x_i')_k$. Repeating this procedure for all boundary x_i', we are done. ■

Proof of Lemma 6: Let coalition S wishfully block the allocation (x, y) with a blocking plan (x', y') in economy $\mathcal{E}^m$. Suppose that x_i' is not interior to $\mathbb{R}^l_+$ for some $i \in S$. Consider the replica economy $\mathcal{E}^{m+1}$ and let the additional unit join the coalition, i.e., let

$$S^* := S \cup I_*, \quad \text{where} \quad I_* := I_1 \times \{m+1\}.$$

If S sticks to its initial blocking plan and I_* sticks to the status quo (x, y), then everyone in S is better-off and everyone in I_* is indifferent. We shall shuffle resources within the enlarged coaltion S^* so that everyone is better-off and everyone's blocking consumption is interior.

First, reduce the aggregate blocking consumption $\sum_S x'_i$ of S by a sufficiently small bundle $v \gneqq \mathbf{0}$ and distribute it evenly to every member of I_* so that every member of S remains better-off than the status quo allocation $(x_i)_{i\in S}$. This can be done because every member of S prefers x'_i to x_i and has lower semicontinuous preferences. Note that this transfer makes everyone in I_* better-off than the status quo, as his consumption becomes $x_i+\mathbf{v}/|I_*|$. Then, by lower semicontinuous preferences, there exists $\epsilon > 0$ such that everyone in I_* remains better-off than the status quo if his consumption is changed from $x_i+\mathbf{v}/|I_*|$ to $x'_i+\mathbf{v}/|I_*|$ for some $x'_i \in \mathbb{R}^l_+$ with $\|x'_i-x_i\| < \epsilon$. Since (x, y) is a feasible allocation within the unit I_*, Lemma 8 says that there exists another feasible allocation (x', y') within I_* such that for every $i \in I_*$, x'_i is interior to $\mathbb{R}^l_+$ and $\|x'_i - x_i\| < \epsilon$. Then every $i \in I_*$ has consumption $x'_i + \mathbf{v}/|I_*|$, which is interior to $\mathbb{R}^l_+$ and is still better than the status quo.

Finally, transfer a sufficiently small amount of every good from I_* to S to make the new allocation strictly interior while keeping everyone strictly better-off than the status quo allocation (again by lower semicontinuous preferences). ■

E Comparison with Haller [10]

The model of Haller [10] is 2-period (0 and 1) incomplete markets where the only way to reallocate the resources in period 1 is to trade the stocks of the firms in period 0. An allocation in his model specifies each consumer i's share $(\theta^*_{ij})_{j\in J}$ of the firms, besides i's consumption x^*_i and each firm j's production y^*_j. Part of his feasibility condition is that i's period-1 consumption should conform to this share (Haller[10, Eq. (I4), p830]):

$$x^*_{i1} = e_{i1} + \sum_{j\in J} \theta^*_{ij} y^*_{j1}.$$

Hence people cannot swap their period-1 resources; they can only trade their resources available in period 0. Thus, the only resources that a coalition can trade within itself are: period-0 endowments, period-0 productions, and the corporate shares its members are endowed with. Specifically, if none of the coalition members is endowed with any corporate share, every member's period-1 consumption has to be the same as the status quo.

Unlike his model, our model is an Arrow-Debreu economy where agents can trade any bundle in the commodity space.

Different from the main focus of our paper, Haller's notion of blocking does not require that a coalition's blocking plan should allow the outsiders to come up with a feasible consumption plan. Haller allows a blocking coalition to change the production plans of the firms that it controls, but he does not require any condition about the outsiders' consumptions given the new production plans. As explained in our Introduction, if it is infeasible for outsiders to have feasible consumptions, they have to change the actions of the firms that they can control and hence may upset the blocking plan of the coalition.

One of our intermediary notions of blocking, inconsiderate blocking, does not consider the feasibility condition for outsiders of a blocking coalition. Hence Haller's notion of blocking is a precedent of inconsiderate blocking. But the two notions are different because of Haller's restriction on the possible trades within a coalition, as explained at the first paragraph of this appendix. The next example illustrates this difference.

Example 2 (Haller's Example 2) *There are 2 goods, 3 consumer-types and 1 firm-type.*

$$\begin{array}{ll} \textit{firm:} & Y := \left\{(y_0, y_1) \in \mathbb{R}^2 : y_0 \leq 0, y_1 \leq 1, y_0 + y_1 \leq 0\right\} \\ \textit{type-1 consumer:} & e_1 = (0,0); u_1(x_0, x_1) = \frac{1}{2}x_0 + x_1, \theta_1 = 1 \\ \textit{type-2 consumer:} & e_2 = (1,0); u_1(x_0, x_1) = \frac{1}{2}x_0 + x_1, \theta_2 = 0 \\ \textit{type-3 consumer:} & e_3 = (1,1); u_1(x_0, x_1) = x_0 x_1, \theta_3 = 0 \end{array}$$

The following allocation with equal treatment property,

> $y^* = (-1, 1)$ *for the firm,* $[x_1^* = (0, 1/2), \theta_1^* = 1/2]$ *for a type-1 consumer,* $[x_2^* = (0, 1/2), \theta_2^* = 1/2]$ *for type 2, and* $[x_3^* = (1, 1), \theta_3^* = 0]$ *for type 3,*

cannot be blocked in Haller's model, but is inconsiderately blocked in our model.

The status quo allocation (x_i^*, y^*, θ^*) is inconsiderately blocked in our model even without replication: The blocking coalition consists of a type-2 and a type-3 individuals. The two coalition members are totally endowed with a consumption bundle $(2, 1)$ but zero corporate share. Hence the type-2 member can consume $(1/2, 1/4)$ and the type-3 member can consume $(3/2, 3/4)$. Then the type-2 member is not hurt and the type-3 member is better-off, so the former can be made also better-off by sharing a small bundle from the latter.

In Haller's model, by contrast, this blocking plan is not feasible. That is because none of the members of this coalition is endowed with any share of a firm, hence they cannot have period-1 consumptions different from the status quo (the first paragraph of this appendix).

Even if we set aside the difference between Haller's blocking notion and its counterpart in this paper, inconsiderate blocking, the core convergence results based on these two notions have different ranges. Haller's core convergence theorem says that if a non-equilibrium allocation satisfies an interiority condition and a liquidity condition, then the allocation is Haller-blocked.[5] By contrast, our Proposition 2, core convergence based on inconsiderate blocking, says that *any* non-equilibrium allocation is inconsiderately blocked.

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