Asset Valuation in Thin Markets*

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Abstract

In this paper we develop a model of asset pricing for thin financial markets – markets in which the number of institutional investors constantly monitoring the price and providing liquidity is small. Extensive empirical evidence shows that in such markets orders of institutional investors are large relative to the average volume of trade, and exert significant price impacts. Consequently, traditional competitive models of asset pricing, such as the Capital Asset Pricing Model (CAPM), cannot be applied. This paper departs from the price-taking assumption of CAPM to model investors who do have impact on prices and they take it into account when choosing their portfolios. We derive an asset pricing formula, explicitly modeling the price impacts of institutional traders. We find that an analog of the Security Market Line holds. The market portfolio is replaced by the average portfolio held by liquidity providers. We also show that narrow markets are associated with insufficient hedging of idiosyncratic risk among large institutional traders, and the inefficiency is proportional to the depth of the market. Finally, we address the well-known problem of how to valuate a large block of shares of a publicly traded company in a thin market. In thin markets blocks of shares are large relative to the average daily volume of trade on the market, and hence cannot be liquidated without depressing the price. Consequently, business valuation specialists discount their total value for blockages. We derive the formula for such discounts.

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1 Introduction

In this paper we develop an equilibrium asset pricing model for assets traded in thin markets – markets with a small number of large institutional investors, in which investors' orders may be large relative to the average volume of trade. Large institutional investors, such as mutual funds, hedge funds, pension funds, or dealers play a leading role in thin markets. Unlike individual traders, they have resources to continuously monitor asset fundamentals and prices, and therefore provide liquidity to the market. One of the key features of thin markets is that large institutional investors have a non-negligible market (price) impact, which they take into account when trading.

Thin trading becomes a more and more important phenomenon in contemporary financial markets. The empirical literature has documented that transactions of large institutional investors significantly affect market prices (see Keim and Madhavan [1996]; Holthausen, Leftwich, and Mayers [1987, 1990]; and Chan and Lakonishok [1995]). At the same time, leading equilibrium asset pricing models such as CAPM or Consumption CAPM rely on the assumption that markets are perfectly competitive. The assumption of price taking is often justified by the informal argument that individual traders are so small relative to the whole market that they perceive their impact on prices as at most negligible. By contrast, Chan and Lakonishok show that the trading orders of institutional traders constitute a significant fraction of the daily volume of trade. They estimate that the average orders of institutional investors account for 60% of the average daily volume of trade on the market, and therefore investors must break orders into smaller packages to mitigate, at least partially, the adverse effects of thin trading. On the macro level institutional trading becomes a dominating component of the total volume of trade. Schwartz and Shapiro [1990] estimated that in 1989 institutional trading accounted for 70% of the total volume of trade on the New York Stock Exchange [NYSE]. Gompers and Metrick [2001], cf. Friedman [1996], showed that large institutional traders nearly doubled their share in the U.S. stock markets between 1980 and 1996, exceeding 50% in 1996. Ownership became more concentrated within large institutions: 100 of the largest institutions increased market share from 19% to 37.1% during that period.

The empirical literature naturally leads one to wonder what the determinants of the market impacts of institutional traders are. We also would like to better understand how the presence of large institutional investors with market impact affects asset returns and optimal trading strategies. Therefore, we develop a static model of asset pricing for thin markets, which we call a *Thin Market-CAPM* (TM-CAPM). The major difference between the TM-CAPM and the standard CAPM or Consumption CAPM is that in the TM-CAPM the number of liquidity providers is small, and they have endogenously determined price impacts. We are particularly interested in the following questions: What are the determinants of the market impacts of institutional traders, and how do they affect asset prices? How should business valuation specialists account for a lack of perfect liquidity in thin markets when valuing an asset? Do the predictions of the thin market model match the empirical evidence on asset returns and trading strategies better than the traditional competitive models?

Among many other results we find that investors' ability to affect prices increases as market depth declines or the overall sensitivity of investors to risks goes up. In addition, the market impacts are stronger for assets with more risky payoffs. The overall price impact depends on the block size, and in the case of a CARA-Normal framework this relation is linear.

We also examine the effects of institutional trading on trading strategies and asset returns. The CAPM predicts that rational traders sell off their initial portfolios and put their wealth in a market portfolio and riskless asset (Two-Fund Separation). This way, they perfectly hedge their idiosyncratic risk. A large body of empirical evidence has shown that large investors do not diversify their risks perfectly.[1] In the TM-CAPM, traders sell only part of their initial portfolios, as aggressive trading worsens their terms of trade. Consequently post-trade risky portfolios are convex combinations of traders' initial portfolios and market portfolios (here we call it an average portfolio). The weight put on the market portfolio is positively correlated with the depth of the market (Three-Fund Separation). While equilibrium portfolios in the TM-CAPM differ from those predicted by the CAPM, in our baseline model the asset prices remain unchanged. Thus, one lesson from our model is that the assumption of price taking (or of perfectly elastic asset demands) *per se*, need not affect asset returns. We show, however, that the lack of non-competitive price biases in the TM-CAPM is due entirely to the fact that all traders have identical (and quadratic) utility functions.

The bulk of this paper is motivated by the pertinent question of how to assess the value of a large block of shares traded in a thin market. When appraising blocks of shares, business valuation specialists often apply a *blockage discount*, a deduction from the actively traded price of a stock, because the block of stock to be valued is so large relative to the volume of actual sales on the existing market that the block could not be liquidated within a reasonable time without depressing the market price. In practice, blockage discounts are applied not only to stocks, but also to real estate and personal property, such as collections of art, antiques, manuscripts, charitable gifts, etc. The discounts typically range between 0 and 15 percent. The concept of blockage has also been acknowledged by the IRS since 1937. According to the Federal Tax Regulations, the taxpayer is responsible for demonstrating that a blockage discount is justified. While practitioners have developed a range of methods to adjust the values of assets, there is no theoretical guidance on how to assess the discount. In this paper we derive a formula for blockage discounts. The blockage discount is quadratic in block size, and it is larger for assets with risky payoffs. It positively depends on the risk aversion of liquidity providers, and negatively on market depth. The discount can be decomposed into a fundamental and liquidity component, where the liquidity component is due to the market power of the liquidity providers.

We also contribute to the theoretical literature on modeling non-competitive trading. We will end up by demonstrating that the solution concept from this paper, a Perfect Conjectural Equilibrium (PCE) within the CARA-Normal model is equivalent to the Nash equilibrium in the Linear Supply Functions[2] (LSF), introduced to the financial literature by Kyle [1989], and to the industrial organization literature by Klemperer and Meyer [1989]. One of the benefits of the Equivalence Theorem from Section 7 is that the results characterizing the PCE in Weretka [2006a] under very general assumptions, apply to the LSF equilibrium. For example, in the literature to date the existence of an LSF equilibrium has been established only for models with one risky asset or, in the case of oligopolistic competition, only one good. Among many other results we show that the LSF equilibrium exists in the model

[1] Give the evidence on not perfect insurance

[2] Strictly speaking it is equivalent to a generalization of LSF equilibrium to a model with many risky assets, but without the asymmetry of information.

with many risky assets and arbitrarily heterogeneous traders. We also give results on Pareto inefficiency of equilibrium outcome and convergence to a competitive equilibrium in a deep finanancial market. To the best of our knowledge, the results are novel in the literature on LSF equilibrium. Finally, unlike LSF equilibrium, the PCE is defined and its properties are well understood also outside of the CARA-Normal framework. Therefore, using PCE gives the advantage that we can verify the robustness of the predictions for different utility functions. We demonstrate that some of the results of the TM-CAPM, especially the ones on asset returns, critically depend on the quadratic structure of the model.

The remainder of the paper proceeds as follows: In Section 2 we describe the model of thin markets and define a concept of equilibrium. In Section 3 we characterize an equilibrium. The two main results of the paper on asset returns and optimal trading strategies in thin markets are stated in Section 4. In Section 5 we derive a blockage discount formula, and in Section 6 we examine the robustness of the results from Section 4. Finally, Section 7 discusses this paper's the contribution to the literature on equilibrium in Linear Supply Functions.

2 The TM-CAPM

2.1 Market Microstructure

In thin markets the number of large institutional investors providing liquidity is small (I). Investors trade N risky assets, for example, shares of publicly traded companies and one riskless asset – a treasury bill. The trade is facilitated by a market maker: a specialist managing the stock in an exchange. The market maker is matching buyers and sellers and is assumed not to hold any inventory. The trades of institutional investors are large relative to the average daily volume of trade, and hence these investors have significant price impacts. We assume no asymmetry of information about the joint distribution of asset payoffs, and the initial portfolios of traders are in the asset span (which, as in the standard CAPM, can be shown to be equivalent to complete markets).

Large institutional investors, interchangeably called liquidity providers, or simply traders, are choosing portfolios to maximize expected CARA utility function, and asset payoffs are normally distributed. By the standard argument, under CARA-Normal assumption the traders maximize their expected payoffs less the variance.

$$u(x^i) = E\left(x^i\right) - \frac{\alpha}{2} Var\left(x^i\right). \tag{1}$$

Investor i is choosing a portfolio that consists of a riskless asset, θ_b^i, and N risky assets, $\theta^i = (\theta_1^i, ..., \theta_N^i)$. Without loss of generality, the treasury bill is in zero net supply and has a face value equal to one dollar. θ_b^i denotes the dollar value of bond holdings by trader i. R denotes the exogenously determined riskless rate of return. The payoffs of risky assets are random, and given by a commonly known, normally distributed random vector A. Assuming no default, a payoff of trader i is given by

$$x^i = R\theta_b^i + A \cdot \left(\theta^i + \theta_0^i\right), \tag{2}$$

where $\theta^i \in R^N$ is a risky portfolio demanded by investor i, and θ_0^i is her initial portfolio.

Let $\bar{A}$ be a vector of expected payoffs of risky assets normalized by the riskless return ($\bar{A} \equiv E(A)/R$) and $\bar{\mathcal{V}}$ is a variance-covariance matrix of the random vector A, also normalized by R ($\bar{\mathcal{V}} \equiv Var(A)/R$). Note that $\bar{\mathcal{V}}$ is symmetric and positive definite. Given (2), it is more convenient, as we shall see shortly, to rewrite (1) in terms of asset holdings:

$$U\left(\theta_b^i, \theta^i\right) = \theta_b^i + \bar{A}\left(\theta^i + \theta_0^i\right) - \frac{\alpha}{2}\left(\theta^i + \theta_0^i\right)\bar{\mathcal{V}}\left(\theta^i + \theta_0^i\right). \tag{3}$$

The indirect utility function is quasilinear in bond holdings and quadratic in risky assets.

Symbol γ denotes an index of a market depth, defined as

$$\gamma \equiv 1 - \frac{1}{I-1}. \tag{4}$$

When the number of traders I is large, γ is close to one, so markets are infinitely deep, and interactions are perfectly competitive. With only two traders γ is equal to zero. In this paper we restrict attention to markets with at least three liquidity providers and hence for which the market depth varies from one half to one. Finally, by $\theta^{Average}$ we denote a per capita initial portfolio for all institutional investors

$$\theta^{Average} \equiv \frac{1}{I}\sum_{i \in I} \theta_0^i. \tag{5}$$

We call it an *average portfolio.* The average portfolio, an analog of a market portfolio from CAPM, will play the role of a benchmark portfolio that will span the Security Market Line. The average portfolio from this paper, however, is not interpreted as a portfolio consisting of all assets available "in the world," but rather, it is a portfolio held by a small group of institutional traders who interact with each other on some specific thin markets, and it is expressed in per-capita terms.

2.2 Equilibrium

In the TM-CAPM, unlike in the CAPM, trades affect asset prices, and price impacts are reflected in choices of the investors. Therefore, instead of a competitive equilibrium, as a solution concept we use a Perfect Conjectural Equilibrium (PCE) introduced and studied in Weretka [2006a] in an abstract model of an exchange of goods.[3] In Section 7 we show that within the CARA-Normal framework the PCE is numerically equivalent to a Nash equilibrium in Linear Supply Functions (LSF) introduced to the literature on financial microstructure by Kyle [1989], and to the industrial organization literature by Klemperer and Meyer [1989]. Despite the equivalence, we prefer the PCE for the following reasons: The TM-CAPM belongs to a broader research program in which we study the properties of thin markets, with traders having different preferences, while the LSF equilibrium is well-defined only in the CARA-Normal framework. We also believe that the derivation of PCE offers new insights into the determinants of the traders' market impact. Finally, in Weretka [2006d] we show that the concept of equilibrium from this paper can be viewed as a refinement of a Subgame Perfect Nash equilibrium in the game defined by the Walrasian auction. We argue there that

[3] The concept of Perfect Conjectural Equilibirum is studied in detail in the context of strategic interactions in Rostek and Weretka [2006b]).

the equilibrium is likely to be observed in anonymous markets in which investors have no other information but their past trades and market prices, and therefore they discover market power through a statistical inference, for example by estimating their demands using the Least Squares method. Therefore PCE is endowed with the "learning story" of how anonymous markets converge to this equilibrium. We believe that the assumption of anonymity is approximately satisfied in financial markets, in which large institutional investors know that they have price impacts, but very often have no information about those with whom they are trading.

Let us now explain the idea behind the PCE. In the TM-CAPM traders have significant market impacts, and, therefore, apart from prices and traded quantities, the new components of an equilibrium are endogenously determined investors' price impacts. The price impact of trader i is quantified by a positive definite and symmetric, $N \times N$, matrix $\mathcal{M}^i$. A typical element of this matrix characterizes the impact on the price of one asset resulting from the increased demand for some other asset. We call $\mathcal{M}^i$ a *price impact matrix.* Given $\mathcal{M}^i \neq 0$, in equilibrium investors face not-perfectly-elastic linear demands for assets. The inverse demand for trader i is given by

$$p_{p,\bar{\theta}^i,\mathcal{M}^i}\left(\theta^i\right) = p + \mathcal{M}^i\left(\theta^i - \bar{\theta}^i\right). \tag{6}$$

The demand function is indexed by three parameters: price observed on the market, p, portfolio traded at this price, $\bar{\theta}^i$, and the slope, $\mathcal{M}^i$.

In equilibrium, the price impact matrices must be *consistent* in the following sense: Suppose trader i observes price p when her trade amounts to $\bar{\theta}^i$. Then selling extra shares above the equilibrium level $\bar{\theta}^i$ depresses the price to make other investors willing to purchase them: The price concession needed to clear the market is reflected in the price impacts $\mathcal{M} = \{\mathcal{M}^1, \mathcal{M}^2, \ldots, \mathcal{M}^I\}$

An equilibrium occurs when both in and off equilibrium (1) all investors optimize, given their perceived price impacts; (2) markets clear; (3) investors' perceived price impacts correspond to their actual price impacts. Formally the equilibrium is defined as:

Definition 1 *A vector $(\bar{p}, \bar{\theta}, \bar{\mathcal{M}})$ is an equilibrium if:*
1) markets clear, $\sum_i \bar{\theta}^i = 0$,
2) for any i trade $\bar{\theta}^i$ is optimal, given demand functions $p_{\bar{p},\bar{\theta}^i,\bar{\mathcal{M}}^i}(\cdot)$,
3) price impact matrices $\bar{\mathcal{M}}$ are mutually consistent.

To close the definition of equilibrium we must supply it with the definition of mutual consistency of $\bar{\mathcal{M}}$. Before doing so we have to endow the model with the story of what happens in financial markets when one of the traders, instead of trading optimally $\bar{\theta}^i$, offers extra shares $\theta^i > \bar{\theta}^i$. Unlike in a competitive model in which price is assumed to be constant, here price goes down sufficiently to encourage other traders to absorb additional shares to clear the market. Consequently, in our model markets clear and other investors respond optimally to prices,[4] *even* if one of the investors, i, is trading a suboptimal quantity of

[4]The rational response to market prices, off equilibrium, is not present in most of the existing non-competitive models. For example, in the Cournot model, after one of the duopolists deviates from the equilibrium production, the price goes down to clear the market but the other producer keeps its level of production on the original equilibrium level.

shares, θ^i. More formally, any deviation θ^i by trader i triggers a new subequilibrium in financial markets, defined as:

Definition 2 *Given $\bar{\mathcal{M}}^{-i}$, vector $\left(p, \theta^{-i}, \bar{\mathcal{M}}^{-i}\right)$ is a subequilibrium triggered by trade θ^i if:*
1) markets clear with the deviation, $\theta^i + \sum_{j \neq i} \theta^j = 0$,
2) for any $j \neq i$, trade θ^j is optimal, given demand functions $p_{p,\theta^j,\bar{\mathcal{M}}^j}(\cdot)$,

Consistent price impact reflects the price change needed to clear the market for any possible deviation θ^i, given that other traders rationally respond to market prices.

Definition 3 *$\bar{\mathcal{M}}^i$ is consistent with $\bar{\mathcal{M}}^{-i}$ if for any deviation θ^i of trader i*

$$p - \bar{p} \equiv \bar{\mathcal{M}}^i \left(\theta^i - \bar{\theta}^i\right), \tag{7}$$

where $\bar{p}$ is an equilibrium price and p is from a subequilibrium triggered by a deviation θ^i.

In the following section we show that the equilibrium has a closed form solution.

3 Derivation of an equilibrium

We now derive equilibrium prices, portfolios, and price impacts. Our strategy is first to take price impacts $\mathcal{M}^i$ and market price p as given and look at a portfolio $\bar{\theta}^i$ that is optimal given the demand $p_{p,\bar{\theta}^i,\mathcal{M}^i}(\cdot)$, expressed as functions of $\mathcal{M}^i$ and p. We call such a portfolio a *stable portfolio*.[5] A stable portfolio can be viewed as a generalization of competitive demand, for traders having different beliefs about their price impact (including a competitive one $\mathcal{M}^i$=0). Next, we use stable portfolio functions to determine consistent price impacts, $\bar{\mathcal{M}}$, for all traders, and finally we derive equilibrium prices $\bar{p}$ and trades $\bar{\theta}^i$.

3.1 Stable Portfolios

Investor i is choosing a portfolio from a budget set, a set of all financeable portfolios, defined by a budget constraint

$$p_{p,\bar{\theta}^i,\mathcal{M}^i}\left(\theta^i\right) \cdot \theta^i + \theta^i_b \leq 0. \tag{8}$$

Note that $p_{p,\bar{\theta}^i,\mathcal{M}^i}(\cdot)$ is linear in θ^i, and therefore the budget constraint is quadratic in θ^i. Consequently, in Figure 1 the budget set is represented by a paraboloid. The slope of the budget set is equal to a vector of marginal revenues from selling each asset (in a competitive framework marginal revenues are equal to prices). It is given by the derivative of the term $p_{p,\bar{\theta}^i,\mathcal{M}^i}\left(\theta^i\right) \cdot \theta^i$ with respect to θ^i. The derivative evaluated at $\bar{\theta}^i$ is given by $p + \mathcal{M}^i \bar{\theta}^i$.

[5] The demand function $p_{\bar{p},\bar{\mathcal{M}}^i,\bar{\theta}^i}(\cdot)$ is among others parameterized by $\bar{\theta}^i$ and hence the domain over which investors are optimizing depends on the optimal choice. Consequently $\bar{\theta}^i$ is not the best response to $p, \mathcal{M}^i$ but rather to a triple $p, \bar{\theta}^i, \mathcal{M}^i$ and hence is defined as a fixed point. For a detailed discussion, and formal derivation see Weretka [2006a].

Given p and $\mathcal{M}^i$, a rational investor i will choose the feasible portfolio $\bar{\theta}^i$, whose payoff maximizes his utility – the indifference curve is tangent to the budget set at $\bar{\theta}^i$. Therefore the traded portfolio $\bar{\theta}^i$ is determined by the equality of the marginal utility of the utility function (3) and marginal revenue. This is a necessary and sufficient condition for the stability[6] of $\bar{\theta}^i$.

$$\bar{A} - \alpha\bar{\mathcal{V}}\left(\bar{\theta}^i + \theta_0^i\right) = p + \mathcal{M}^i\bar{\theta}^i. \tag{9}$$

In Figure 1 condition (9) implies a tangency between the budget set and the indifference curve. With $\mathcal{M}^i$ positive definite, we can solve equality (6) to get function $\bar{\theta}^i\left(p, \mathcal{M}^i\right)$, that tells what the optimal portfolio is for any p and $\mathcal{M}^i$:

$$\bar{\theta}^i\left(p, \mathcal{M}^i\right) = \left(\mathcal{M}^i + \alpha\bar{\mathcal{V}}\right)^{-1}\left(\bar{A} - \alpha\bar{\mathcal{V}}\theta_0^i - p\right). \tag{10}$$

Function $\bar{\theta}^i\left(p, \mathcal{M}^i\right)$ has three interesting properties. First, the stable portfolio decreases in prices, with the coefficient $\left(\mathcal{M}^i + \alpha\bar{\mathcal{V}}\right)^{-1}$. For a competitive investor i, $\left(\mathcal{M}^i = 0\right)$, this coefficient is an inverse of the variance-covariance matrix times α, $\alpha\bar{\mathcal{V}}$. Interestingly, the slope can be further characterized as an inverse of the Hessian of the utility function (3). This is not coincidental, as the indirect utility function is quasilinear and as a result, the demand coincides with the marginal utility. Second, non-competitive investors are more reluctant to rebalance their portfolios to changing prices, as they take into account the adverse effects on prices. This is reflected in the value of the slope of $\bar{\theta}^i\left(p, \mathcal{M}^i\right)$, equal to $\left(\mathcal{M}^i + \alpha\bar{\mathcal{V}}\right)^{-1}$. Compared to a competitive trader, the slope is augmented by the additional term, $\mathcal{M}^i$. The belif about the price impact $\mathcal{M}^i$ makes the stable demand steeper than the competitive one, and its slope exceeds the slope of marginal utility $\left(\alpha\bar{\mathcal{V}}\right)^{-1}$.

We observe that the sum of stable portfolio functions across all investors evaluated at the equilibrium $\bar{p}$ and $\bar{\mathcal{M}}^i$ is equal to zero by the market clearing condition.

$$\sum_{i\in I}\bar{\theta}^i\left(\bar{p}, \bar{\mathcal{M}}^i\right) = 0. \tag{11}$$

3.2 Price impacts

We can now derive price impact, $\bar{\mathcal{M}}^i$, taking the price impacts of other traders $\bar{\mathcal{M}}^j$ as given. Suppose investor i, instead of his equilibrium trade $\bar{\theta}^i$, picks some other portfolio θ^i, hence offering an extra block of shares $\theta^i - \bar{\theta}^i$ above the equilibrium trade. For the markets to clear with this extra supply, prices must adjust so that the other investors are willing to purchase the additional shares. Market clearing and optimization by other traders implies that the required price change satisfies

$$\theta^i + \sum_{j\neq i}\bar{\theta}^j\left(p, \bar{\mathcal{M}}^j\right) = 0. \tag{12}$$

[6] This is a generalization of the well-known condition that the marginal rate of substitution is equal to a price ratio. In a more general, non-competitive framework the marginal rate of substitution should be equal to a ratio of marginal revenues. Since the marginal utility and the marginal revenue of a riskless asset (numeraire) is equal to one, such condition is equivalent to the equality of the marginal utility and the marginal revenue of risky (non-numeraire) assets (for details see Weretka [2006a].

Condition (12) determines the price level p that clears the market for any possible portfolio θ^i, and hence implicitly defines the inverse demand function faced by trader i. Substituting (10) into (12), solving for p, and using (11) gives

$$p_{\bar{p},\bar{\theta}^i,\bar{\mathcal{M}}^i}\left(\theta^i\right) = \bar{p} + \underbrace{\left(\sum_{j \neq i}\left(\bar{\mathcal{M}}^j + \alpha\bar{\mathcal{V}}\right)^{-1}\right)^{-1}}_{\bar{\mathcal{M}}^i}\left(\theta^i - \bar{\theta}^i\right). \tag{13}$$

By looking at (13) and (6) we know that the trader's i equilibrium price impact $\bar{\mathcal{M}}^i$ must be

$$\begin{aligned}\bar{\mathcal{M}}^i &= \left(\sum_{j \neq i}\left(\bar{\mathcal{M}}^j + \alpha\bar{\mathcal{V}}\right)^{-1}\right)^{-1} = \\ &= (1-\gamma)\,\mathcal{H}\left(\bar{\mathcal{M}}^j + \alpha\bar{\mathcal{V}} | j \neq i\right).\end{aligned} \tag{14}$$

Equation (14) offers an elegant characterization of the consistent price impact of trader i. The price impact is a harmonic average of the convexity of the utility functions for all other investors $\alpha\bar{\mathcal{V}}$ augmented by $\bar{\mathcal{M}}^j$, and discounted by factor $1-\gamma$. To better understand why (14) holds, recall that price impact corresponds to the price adjustment needed for the market to clear after a deviation of trader i from the equilibrium trade. The larger the variance of an asset payoff, the more reluctant the other investors are to absorb the deviation. They require a larger price discount to purchase an extra block. Similarly, the price impact of i is enhanced by the risk aversion of other traders α. On the other hand, the price effect is partially mitigated by the depth of the market γ, as in deep markets there are more potential buyers for the deviation. Furthermore, the price impacts of different traders reinforce each other: In equation (14) $\bar{\mathcal{M}}^i$ positively depends on $\bar{\mathcal{M}}^j$, and the other way round. Intuitively, the reinforcement relies on the following mechanism: the less competitive the other investors are, the less willing they are to absorb the deviations.

The system of consistent matrices $\bar{\mathcal{M}}$ is determined by I equations (14), each for one trader. Given the same risk aversion for all investors, the solution is symmetric across all investors $\bar{\mathcal{M}}^i = \bar{\mathcal{M}}^j$. The harmonic average in (14) can be written as $\mathcal{H}\left(\bar{\mathcal{M}}^j + \alpha\bar{\mathcal{V}} | j \neq i\right) = \bar{\mathcal{M}}^i + \alpha\bar{\mathcal{V}}$, and equation (14) simplifies to

$$\bar{\mathcal{M}}^i = (1-\gamma)\,\alpha\bar{\mathcal{V}} + (1-\gamma)\,\bar{\mathcal{M}}^i. \tag{15}$$

In the unique solution consistent price impacts are equal to

$$\bar{\mathcal{M}}^i = \frac{1-\gamma}{\gamma}\alpha\bar{\mathcal{V}}, \tag{16}$$

for all $i \in I$. The price impacts are non-zero only if investors are risk averse ($\alpha > 0$). Also the positive (negative) correlation between the asset payoffs of two assets implies that selling one asset depresses (increases) the price of the other asset.

The system (14) cannot be solved for $\gamma = 0$, a bilateral monopoly case. This is because with only two investors the price impacts mutually reinforce each other without any discounting, and the price impacts go off to infinity.

Given the equilibrium price impacts from (16), the non-competitive demand is proportional to the competitive one, for arbitrary price p

$$\bar{\theta}^i\left(p, \bar{\mathcal{M}}^i\right) = \gamma\left(\alpha\bar{\mathcal{V}}\right)^{-1}\left(\bar{A} - \alpha\bar{\mathcal{V}}\theta_0^i - p\right) = \gamma\bar{\theta}^i\left(p, 0\right). \tag{17}$$

The proportionality coefficient coincides with the depth of the market γ.

3.3 Asset Prices

Asset prices are determined by the market clearing condition. Combining (11) and (6), dividing both sides by the number of institutional investors I, we get

$$\begin{aligned} 0 &= \frac{1}{I}\sum_{i\in I}\bar{\theta}^i\left(\bar{p}, \bar{\mathcal{M}}^i\right) = \\ &= \frac{1}{I}\sum_{i\in I}\left(\bar{\mathcal{M}}^i + \alpha\bar{\mathcal{V}}\right)^{-1}\left(\bar{A} - \alpha\bar{\mathcal{V}}\theta_0^i - \bar{p}\right). \end{aligned} \tag{18}$$

Given identical price impacts for all traders, term $\left(\bar{\mathcal{M}}^i + \alpha\bar{\mathcal{V}}\right)^{-1}$ can be factored out. Multiplying both sides of (18) by $\bar{\mathcal{M}}^i + \alpha\bar{\mathcal{V}}$, using the definition of average portfolio (5) and solving for price vector $\bar{p}$ we get

$$\bar{p} = \bar{A} - \alpha\bar{\mathcal{V}}\theta^{Average}. \tag{19}$$

Surprisingly, equilibrium prices do not depend on the depth of the market γ. Intuitively, the symmetric price impacts reduce the demand and supply of each asset by the same factor γ, and therefore markets clear at the competitive price even when the trade is non-competitive. It follows that equilibrium prices from the TM-CAPM coincide with prices predicted by the CAPM (see Figure 2.)This result is stated formally in the following lemma.

Lemma 1 *Equilibrium asset prices are not affected by thin trading, that is* $\bar{p} = p^{Competitive}$

Proof. In (18) replace $\bar{\theta}^i\left(\bar{p}, \bar{\mathcal{M}}^i\right)$ with $\bar{\theta}^i\left(\bar{p}, 0\right)$ and solve for the competitive price, $p^{Competitive}$. ■

3.4 Equilibrium Portfolios

Plugging (5) in (6) and assuming $\bar{\mathcal{M}}^i = 0$ shows that in a competitive model demands are given by

$$\begin{aligned} \bar{\theta}^i\left(\bar{p}, 0\right) &= \left(\alpha\bar{\mathcal{V}}\right)^{-1}\left(\bar{A} - \alpha\bar{\mathcal{V}}\theta_0^i - \bar{p}\right) = \\ &= \theta^{Average} - \theta_0^i. \end{aligned} \tag{20}$$

The competitive trader sells off his initial portfolio and buys the average portfolio and riskless asset, perfectly hedging the idiosyncratic risk. This is a version of a well-known mutual fund theorem. Combining (20) and (17), we solve for a non-competitive trade

$$\bar{\theta}^i\left(\bar{p}, \bar{\mathcal{M}}^i\right) = \gamma\bar{\theta}^i\left(\bar{p}, 0\right) = \gamma\left(\theta^{Average} - \theta_0^i\right). \tag{21}$$

Equations (19), (21), and (16) fully characterize an equilibrium.

4 Main results

4.1 Security Market Line

Now we state two main results of the paper. The first result is a straightforward but remarkable consequence of the fact that prices are not affected by non-competitive trading. It asserts that the return of an asset traded in a thin market can be explained solely by the covariance of its return with the return of the average portfolio held by the liquidity providers in thin markets. By $\bar{R}^{Average}$ we denote the expected return of an average portfolio, $\beta_n = \mathcal{V}_{Average,n}/\mathcal{V}_{Average}$ is beta of asset n, and $\bar{R}_n$ is its expected return.

Theorem 1 *In thin markets with I liquidity providers the returns of individual assets are located on the Security Market Line*

$$\bar{R}_n - R = \beta_n \left(\bar{R}^{Average} - R\right), \tag{22}$$

Proof. The proof is immediate and follows from the two observations: 1) the prices, and hence the asset returns, are as in the competitive model, see Lemma 1 and 2) formula (22) holds in the competitive model. ■

The empirical testing of the relation between the average return and covariance predicted by (22) is not straightforward. In particular, one needs to properly identify thin markets. The average portfolio is a risky portfolio held by a certain group of liquidity providers trading in specific markets. Therefore, the standard approach to test CAPM predictions, based on a market portfolio defined, for example, as all assets traded on the NYSE (Fama and Frech) cannot be applied here.

4.2 Three-Fund Separation

The investment strategy of any liquidity provider in thin markets is quite simple. She should partially sell the risky assets she initially held, and invest her wealth in the average portfolio and bond. In contrast to the CAPM, in the TM-CAPM investors do not perfectly insure their individual risks

Theorem 2 *In thin markets with I liquidity providers the post-trade risky portfolio of investor i is given by*

$$\gamma\theta^{Average} + (1-\gamma)\,\theta_0^i, \tag{23}$$

and the remaining wealth is put in a riskless asset θ_b^i.

Proof. The result follows directly from equation (21). ■

We see that traders hold a convex combination of the average portfolio and their initial portfolio. The weight put on the market portfolio corresponds to market depth γ. In the TM-CAPM investors trade in order to hedge their idiosyncratic risk. In the competitive model they do it perfectly, by selling off their initial portfolios. In thin markets traders are reluctant to trade too aggressively, to minimize the adverse market impact effects on their terms of trade. The deeper the markets, the less important the second concern, and consequently the volume of trade is larger.

Risk aversion α does not affect the volume of trade. Such an observation need not be surprising if one notes that risk aversion affects traders in two ways. On the one hand, agents with higher α are very sensitive to risk, and will not so easily give up the opportunity to hedge their risk. On the other hand, higher risk aversion makes interactions less competitive, reducing the willingness to trade. With quadratic utility functions both effects offset each other, making the trading volume independent from α.

5 Market Value and Blockage Discount

Traditionally, the value of a block of shares, $\hat{\theta}$, is just the quantity of shares times the corresponding prices currently observed on the market

$$Value = \bar{p} \cdot \hat{\theta}. \tag{24}$$

In thin markets, however, selling a large block of shares is likely to have a significant, depressing effect on share prices as thin markets have a limited capacity to absorb large blocks of assets. In fact, (24) no longer corresponds to the amount of cash to which the block $\hat{\theta}$ can be converted by selling it on the market. This is because the liquidity providers will not buy $\hat{\theta}$ unless they are offered a satisfactory price reduction. Business valuation specialists have recognized the discrepancy between the market value and the liquidation value resulting from the blockage impact. For example, since 1937 the IRS has offers a *blockage discount* defined as "a deduction from the actively traded price of a stock, because the block of stock to be valued is so large relative to the volume of actual sales on the existing market that the block could not be liquidated within a reasonable time without depressing the market price" (Handbook of Advanced Business Valuation, p. 140). Blockage discounts are also applied to non-financial assets such as real estate, collections of art, antiques, manuscripts, charitable gifts etc, and typically they vary between 0 and 15 percent.

Practitioners have developed a number of heuristics to determine a blockage discount. However, in the theoretical literature to date there is no formal guidance that would provide a basis for the discount. We now derive a formula explicitly saying how much the value of a block of shares should be discounted due to thin trading. We also discuss the limitations of the model related to the static nature of the TM-CAPM from this paper.

The blockage discount can be interpreted as the difference between the value of a block evaluated at prices $\bar{p}$ observed when the block is not sold, and hypothetical prices $\hat{p}$, depressed if the block was offered on the market. The blockage discount is given by

$$BD \equiv \hat{\theta} \cdot (\bar{p} - \hat{p}). \tag{25}$$

Prices $\hat{p}$ can be found by adding to the market clearing condition (18), the extra block of shares and solving for $\hat{p}$. They are given by

$$\hat{p} = \bar{A} - \alpha \bar{\mathcal{V}} \theta^{Average} - \frac{\alpha}{\gamma} \frac{\bar{\mathcal{V}} \hat{\theta}}{I}, \tag{26}$$

and hence

$$BD = \frac{\alpha}{\gamma}\frac{\hat{\mathcal{V}}}{I} = \underbrace{\alpha\frac{\hat{\mathcal{V}}}{I}}_{funadamental\ c.} + \underbrace{\frac{\alpha}{(I-2)}\frac{\hat{\mathcal{V}}}{I}}_{liquidity\ c.}, \tag{27}$$

where $\hat{\mathcal{V}} \equiv Var\left(A \cdot \hat{\theta}\right)/R$ measures the risk of the block. The blockage discount is positively correlated with the riskiness of the block, and the risk aversion of the liquidity providers. This is because the higher the variance of an asset payoff, or the more cautious the liquidity providers, the harder it is to liquidate such an asset by selling it on the market. Market depth γ reduces the blockage discount. In the extreme, in perfectly competitive markets ($I = \infty$), the blockage discount is equal to zero. Finally, blockage discount depends linearly on the variance of the payoff, and therefore it is quadratic in the size of a block.

The blockage discount can be decomposed into two parts: the fundamental and liquidity component. The fundamental component results from the fact that the extra shares increase holdings of risky assets in the hands of liquidity providers, thus making them marginally less attractive. This effect would be present even if the liquidity providers were price takers. The second component arises because the liquidity providers are large, and hence they strategically reduce their demands for shares to exploit the monopolistic position to get a better price for extra shares.

In the derivation of formula (27) we made an implicit assumption that the block is being sold all at once (this is a one-period model). In practice, however, traders break up large blocks into smaller packages and sell them slowly, spreading the sales over time. This way they partially mitigate the adverse effects resulting from trading in thin markets. Therefore formula (27), intuitive as it is, formula is likely to overestimate the value of a blockage discount, and hence should be interpreted as an upper bound on the discount. In Rostek and Weretka [2006a] we derive a blockage discount formula in a dynamic model, in which we explicitly model the optimal selling strategy of a block.

6 Robustness of Model Predictions

6.1 Beyond homogeneity of risk aversion

In Section 4 we showed that in the TM-CAPM non-competitive trading does not affect asset returns, and hence the Security Market Line holds. We made a crucial assumption there, namely that all investors are equally risk averse. In this section we explain why, when traders differ in their risk aversion, our results begin to change.

Before proceeding with our example, we first state a lemma relating the relative strength of price impacts to a risk aversion of investors.

Lemma 2 *Suppose $\alpha^i > \alpha^j$. Then in equilibrium $\bar{\mathcal{M}}^i < \bar{\mathcal{M}}^j$*

Proof. In Appendix I. ■

Lemma 2 says that the more risk averse the trader, the smaller the market impact she has compared to other traders. Intuitively, the agent with the highest α^i is trading with partners that have flatter marginal utility. Such traders require only modest price concessions to absorb potential deviations.

We consider a simple example with one risky asset.[7] There are two types of traders buyers and sellers and their initial portfolios satisfy $\theta_0^s > \theta_0^b$.

The market clearing can be written as

$$\begin{aligned} \sum_{i \in I} \bar{\theta}^i \left(\bar{p}, \bar{\mathcal{M}}^i\right) &= \sum_{i \in I} \frac{\bar{A} - \alpha^i \bar{\mathcal{V}} \theta_0^i - \bar{p}}{\bar{\mathcal{M}}^i + \alpha^i \bar{\mathcal{V}}} = \\ &= \sum_{i \in I} \frac{\alpha^i \bar{\mathcal{V}}}{\bar{\mathcal{M}}^i + \alpha^i \bar{\mathcal{V}}} \frac{\bar{A} - \alpha^i \bar{\mathcal{M}}^i \theta_0^i - \bar{p}}{\alpha^i \bar{\mathcal{V}}} \\ &= \sum_{i \in I} \frac{\alpha^i \bar{\mathcal{V}}}{\bar{\mathcal{M}}^i + \alpha^i \bar{\mathcal{V}}} \bar{\theta}^i \left(\bar{p}, 0\right). \end{aligned} \tag{28}$$

If traders are equally risk averse, then their price impacts $\mathcal{M}^i$, and hence ratios, $\alpha^i \bar{\mathcal{V}} / \left(\bar{\mathcal{M}}^i + \alpha^i \bar{\mathcal{V}}\right)$ also coincide, and in the last line of (28) they can be factored out and ignored. Therefore, in Section 3 we got the conclusion that the price that clears the market in the TM-CAPM also clears the market when the interactions are competitive. Now, suppose that the buyers are more risk averse than the sellers $\alpha^b > \alpha^s$. By Lemma 2 the buyers have smaller market impacts than sellers $\mathcal{M}^b < \mathcal{M}$ and hence the ratios in (28) satisfy the following inequality

$$\frac{\alpha^b \bar{\mathcal{V}}}{\bar{\mathcal{M}}^b + \alpha^b \bar{\mathcal{V}}} > \frac{\alpha^s \bar{\mathcal{V}}}{\bar{\mathcal{M}}^s + \alpha^s \bar{\mathcal{V}}}, \tag{29}$$

Since in the last line of (28) sellers' competitive trades are negative and have smaller weights, replacing the ratios with weights equal to one for all traders makes the sum negative

$$\sum_{i \in I} \bar{\theta}^i \left(\bar{p}, 0\right) < 0. \tag{30}$$

It follows that at a (non-competitive) equilibrium price, $\bar{p}$, the competitive excess demand is strictly negative. Since the competitive aggregate demand is strictly decreasing in price, the price that clears the competitive market $\bar{p}^{Competitive}$ must be smaller than $\bar{p}$. Consequently, we observe a negative non-competitive bias in the asset return (for any possible realization of A)

$$R_n < R_n^{Competitive}. \tag{31}$$

Intuitively, in thin markets with $\alpha^b > \alpha^s$ the asset supply is affected more than the demand for any p, and hence price must go up to clear the market. By the symmetric argument, the sign of the price bias is reverted when the sellers are more risk averse than the buyers, and hence in terms of asset returns

$$R_n > R_n^{Competitive}. \tag{32}$$

The example demonstrates that in the IT-CAPM price can be greater, the same, or below the competitive one, depending on which side of the market is more risk averse. Consequently,

[7] This example was first given in Weretka [2005] for the abstract general equilibrium model, and served as an illustration of price biases in a more general model.

the expected asset returns may be located below or above the Security Market Line. Without extra assumptions about the heterogeneity of risk aversion, the model does not give systematic predictions about price biases generated by thin trading. In the next section we briefly discuss the systematic biases in the model with the CRRA utility functions.

6.2 Beyond CARA-Normal assumption

In Weretka [2006b] we examine the effects of thin trading on asset returns, without making any specific assumptions on the functional form of the utility function. We find that when investors have identical, but not necessarily quadratic, utility functions we typically observe price bias. The sign of the bias depends on the convexity of the marginal utility: A positive (negative) third derivative of the utility function is associated with a higher (lower) price than the competitive price. In this paper the indirect utility function is quadratic, and hence the third derivative is equal to zero. Therefore the TM-CAPM represents a knife-edge case for which price effects are non-existent, the predictions that is not robust.

In Weretka [2006b] we propose a model with more plausible utility functions characterized by constant relative risk aversion (CRRA). The third derivative of such function is positive, and hence we observe a positive price bias. In the financial markets this translates into spread between the risky and riskless return that is greater than that in the C-CAPM (the equity premium puzzle). In addition, the endogenously determined risk-free interest rate is lower than that predicted by the competitive model (the interest rate puzzle). Thus, thin trading is associated with the mechanism that has a potential to explain the empirically observed biases in asset returns. We should stress that so far we have not quantitatively assessed the relevance of this mechanism in explaining the two famous puzzles.

7 Relation of the PCE and the LSF equilibrium

7.1 Equivalence Theorem

This section is more theoretical. We examine the relation between the Perfect Conjectural Equilibrium (PCE) from this paper and the well-known concept of a Nash equilibrium in Linear Supply Functions (LSF). The LSF equilibrium was introduced to the literature on Financial Microstructure by Kyle [1989] and to Industrial Organization by Klemperer and Meyer [1989]. We consider a generalization of the LSF equilibrium to the model with many risky assets and traders heterogeneous in their risk aversion. We demonstrate that within the CARA-Normal framework the two concepts are equivalent in the sense that they predict identical outcomes. Consequently, they can be viewed as alternative representations of each other.

We consider a static game in which the set of players consists of all institutional investors from our model. The players simultaneously submit their demand functions for assets, contingent on prices $\tilde{\theta}^i(p)$. We restrict attention to the demand functions that are well behaved: the affinite linear functions, with the positive definite and symmetric Jacobians $D\tilde{\theta}^i$. Next the auctioneer sets the price $\tilde{p}$ that clears the market, and traders exchange assets at this price. It is well known that a Nash equilibrium in such a game is not determinate. Therefore, in order to refine the set of equilibria, we follow a standard procedure by adding a noise trade,

ε. When making decisions about $\tilde{\theta}^i(p)$, investors do not know the realization of ε, but only its normal distribution. The LSF equilibrium is defined as a specification of demand functions for all players that constitutes a Nash equilibrium, $\tilde{\theta}(\cdot) = \left\{\tilde{\theta}^1(\cdot), \tilde{\theta}^2(\cdot), \ldots, \tilde{\theta}^I(\cdot)\right\}$.

Before stating the equivalence theorem, we need to introduce some extra notation. For any LSF equilibrium, $\tilde{\theta}(\cdot)$ we define an *associated vector* of prices, trades, and price impacts $\left(\tilde{p}, \tilde{\theta}, \tilde{\mathcal{M}}\right)$ in the following way

$$\begin{aligned} \tilde{p} &\equiv -\left(\sum_{i\in I} D\tilde{\theta}^i\right)^{-1} \sum_{i\in I} \tilde{\theta}^i(0), \\ \tilde{\theta}^i &\equiv \tilde{\theta}^i(\tilde{p}) \text{ for any } i \in I, \\ \tilde{\mathcal{M}}^i &\equiv \left(\sum_{j\neq i} D\tilde{\theta}^j\right)^{-1} \text{ for any } i \in I. \end{aligned} \tag{33}$$

The associated vector gives equilibrium prices, traders, and price impacts when ε is equal to zero. By assumption $D\tilde{\theta}^i$ are positive definite and hence invertible, therefore vector $\left(\tilde{p}, \tilde{\theta}, \tilde{\mathcal{M}}\right)$ is well defined for any arbitrary LSF equilibrium.

The following theorem relates the two concepts of an equilibrium in the CARA-Normal framework.

Theorem 3 *For any LSF equilibrium $\tilde{\theta}(\cdot)$ its associated vector $\left(\tilde{p}, \tilde{\theta}, \tilde{\mathcal{M}}\right)$ constitutes a PCE. Conversely, for any PCE $\left(\bar{p}, \bar{\theta}, \bar{\mathcal{M}}\right)$, functions $\tilde{\theta}^i(\cdot) \equiv \bar{\theta}^i\left(\cdot, \bar{\mathcal{M}}\right)$ defined as in* (6) *are an LSF equilibrium.*

Proof. In Appendix II. ■

Theorem 3 allows us to better understand the relation between the two concepts of an equilibrium. The additional benefit of the equivalence result is that all the theorems that characterize one equilibrium automatically apply to the other one. In the next section we "arbitrage" our knowledge on the PCE to establish a number of properties of the LSF equilibrium.

7.2 Implications

In this section we combine implications of Theorem 3 and the results established in Weretka [2006a] for the PCE in a general framework with consumers and producers, to characterize the LSF equilibrium. Because of the special structure of the model implied by the CARA-Normal assumptions, some of the results are strengthened and the proofs are significantly simpler. We give them in Appendix II. The following corollaries with the exception of the last one hold in a CARA-Normal model, with *many* risky assets and many traders that are *heterogenous* in their risk aversion.

EXISTENCE. The existence of an LSF equilibrium has been established in the model with one risky asset (Kyle [1989], Vayanos [1999]), or in Industrial Organization literature with one good (Klemperer and Meyer [1999], Agkun [2001]). In addition, except in Agkun [2001], such models typically assume limited heterogeneity of utility or cost functions. We extend the existence results to a model with many risky assets and heterogenous traders.

Corollary 1 *The LSF equilibrium exists if, and only if, $I \geq 3$.*

Proof. In Appendix II. ■

The non-existence of an equilibrium in the model with two investors is caused by the mutual reinforcement effect between price impacts, discussed in Section 3. (In Weretka 2006a we show that one of the possible solutions to the problem is to introduce an outside option of trade for either of the two traders, for example inventories made by the market maker.)

Inefficiency. In an LSF equilibrium buyers and sellers have endogenously determined market power, and hence they strategically reduce their trade to improve the terms of trade. Consequently, the volume of trade is insufficient, unless *a priori* there are no gains to trade at all. The next result says that the allocation in an LSF equilibrium is Pareto inefficient.

Corollary 2 *In the LSF equilibrium post-trade portfolios are Pareto efficient if, and only if, investors a priori are not facing idiosyncratic risk (i.e. initial portfolios are Pareto efficient).*

Proof. In Appendix II. ■

Convergence. The negative message from Corollary 2 is partially mitigated by the following result, saying that the inefficiency associated with thin trading disappears as markets become deeper. We show it using a well-known tool, a $k-$replica[8] of financial markets.

Corollary 3 *For any ε there exists k, such that for any $k' \geq k$ in a $k'-$replica LSF equilibrium prices and allocations satisfy*

$$\left\| p^{k'} - p^{Competitive} \right\| \leq \varepsilon, \tag{34}$$

and

$$\left\| \tilde{\theta}^{i,k'} - \tilde{\theta}^{i,Competitive} \right\| \leq \varepsilon. \tag{35}$$

Proof. In Appendix II. ■

Global Uniqueness. In Appendix II we show that similarly to the PCE, the LSF equilibrium can be written as a solution to a system of $I \times N^2$ equations and the same number of unknowns. From an inspection of the conditions defining equilibrium, we believe that in a CARA-Normal framework, such a system has a unique solution. To date, we are able to formally show global uniqueness in a model with only one risky asset. This result can easily be extended to a model with many risky assets but in which asset payoffs are uncorrelated.

Corollary 4 *In the model with one risky asset the LSF equilibrium is (globally) unique.*

[8] By k replica of a market we mean the model in which there are k traders of each type from the original thin market.

Proof. In Appendix II. ■

It should be stressed that all the results from this section are easily extendable to the models of oligopolistic competition, with many goods produced with quadratic cost functions and with oligopolists facing linear demand functions.

In Weretka [2006d] we showed that the PCE from this paper can be viewed as a refinement of a Subgame Perfect Nash Equilibrium in a game defined by the Walrasian auction where the selection of an equilibrium follows from statistical inference about individual price impacts. We conclude this section with an observation that Theorem 3 implies that the learning "story" can also justify the LSF equilibrium as a solution concept in modeling non-competitive anonymous markets.

8 Conclusions

In this paper we have developed an asset pricing model for thin financial markets – markets with a small number of liquidity providers that have a significant price impact. We showed that the analog of the Security Market Line holds, but the market portfolio from the standard CAPM is replaced by the average portfolio of institutional investors. We have also demonstrated that narrow markets are associated with insufficient hedging of idiosyncratic risk, and that this distortion is proportional to the market depth. We have also derived the formula for discounting the value of assets that are traded in thin markets. The blockage discount is associated with the fact that in thin markets large blocks cannot be liquidated within a short time horizon without significantly depressing the market price. We have also contributed to theoretical literature on thin trading. We showed the equivalence of the PCE and equilibrium in Linear Supply functions within the CARA-Normal framework, and have given a number of results characterizing the LSF equilibrium in a model with many assets. These results include the existence, the uniqueness, and the inefficiency of an allocation, and the convergence towars a competitive outcome as the depth of the market increases.

This paper is part of a larger research program on thin markets. In Rostek and Weretka [2006a] we study a dynamic version of the model from this paper. We show that thin trading results in price overshooting due to a liquidity effect. The latter generates excess and clustering volatility of asset prices and their mean reversion – phenomena that are commonly observed in the data and that can hardly be reconciled with competitive asset-pricing models. We show that a probability that traders do not survive to the maturity of an asset results in a liquidity premium in asset returns, which has been well-documented. In Weretka [2006b] we discuss biases in asset returns due to thin trading with traders that have non-quadratic utility functions. In particular, we show that with CRRA functions, the model predicts asset returns that are consistent with the equity premium and the riskless rate puzzles. In Weretka [2006c] we demonstrate that a pricing kernel exists and the Modigiliani-Miller theorem holds in a model of thin trading.

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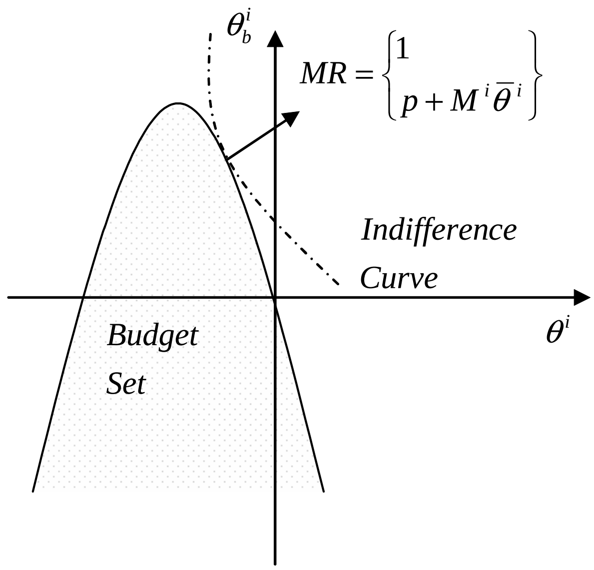

FIGURE 1. TANGENCY CONDITION WITH MARKET IMPACT.

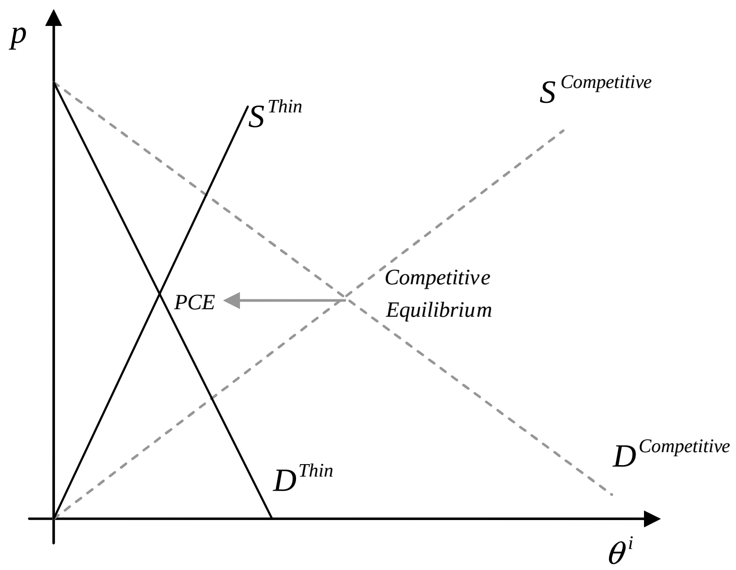

FIGURE 2. PRICES AND QUANTITIES IN PCE.

Appendix I

Proof. (Lemma 4: Price Impact Order)

Suppose $\alpha^i > \alpha^j$ and $\bar{\mathcal{M}}^i \geq \bar{\mathcal{M}}^j$. Since $\alpha^i > \alpha^i$ and $\bar{\mathcal{M}}^i \geq \bar{\mathcal{M}}^j$, therefore $\bar{\mathcal{M}}^i + \alpha^i \bar{\mathcal{V}} > \bar{\mathcal{M}}^j + \alpha^j \bar{\mathcal{V}}$. Consequently harmonic mean condition

$$\mathcal{H}\left(\bar{\mathcal{M}}^k + \alpha^k \bar{\mathcal{V}} | k \neq i\right) < \mathcal{H}\left(\bar{\mathcal{M}}^k + \alpha^k \bar{\mathcal{V}} | k \neq j\right), \tag{36}$$

as in a harmonic mean on the RHS the element $\bar{\mathcal{M}}^j + \alpha^j \bar{\mathcal{V}}$ is replaced with a larger element $\bar{\mathcal{M}}^i + \alpha^i \bar{\mathcal{V}}$. In equilibrium

$$\begin{aligned} \bar{\mathcal{M}}^i &= (1-\gamma)\,\mathcal{H}\left(\bar{\mathcal{M}}^k + \alpha^k \bar{\mathcal{V}} | k \neq i\right), \\ \bar{\mathcal{M}}^j &= (1-\gamma)\,\mathcal{H}\left(\bar{\mathcal{M}}^k + \alpha^k \bar{\mathcal{V}} | k \neq j\right), \end{aligned} \tag{37}$$

one concludes that $\bar{\mathcal{M}}^i < \bar{\mathcal{M}}^j$, a contradiction. ■

Appendix II

We first introduce some notation and state an auxiliary result useful in the following proofs. Let $\mathcal{M} = \left\{\mathcal{M}^1, \ldots, \mathcal{M}^I\right\}$, be a vector of I matrices where each $\mathcal{M}^i$ is a $N \times N$, positive semidefinite and symmetric matrix, and $\mathbf{M}$ is a set of all such $\mathcal{M}$. Note that $\mathbf{M} \subset \mathbb{R}^{I \times N \times N}$. We define function $\mathcal{H}^i : \mathbf{M} \to \mathbb{R}^{N \times N}$

$$\mathcal{H}^i(\mathcal{M}) = \mathcal{H}\left(\mathcal{M}^j + \alpha^j \bar{\mathcal{V}} | j \neq i\right), \tag{38}$$

as a harmonic mean of matrices $\mathcal{M}^j + \alpha^j \bar{\mathcal{V}}$ for all players but i and by $\mathcal{F} : \mathbf{M} \to \mathbf{M}$ define a function

$$\mathcal{F}(\mathcal{M}) = \frac{1}{(I-1)}\left(\mathcal{H}^1(\mathcal{M}), \ldots, \mathcal{H}^I(\mathcal{M})\right) \tag{39}$$

Now we give a lemma which offers a useful characterization of the LSF equilibrium $\tilde{\theta}(\cdot)$.

Lemma 3 *(*LSF Characterization*): $\tilde{\theta}(\cdot)$ is an LSF equilibrium if and only if for each individual player strategy $\theta^i(\cdot)$ is given by*

$$\tilde{\theta}^i(p) = \left(\hat{\mathcal{M}}^i + \alpha^i \bar{\mathcal{V}}\right)^{-1}\left(\bar{A} - \alpha \bar{\mathcal{V}} \theta_0^i - p\right), \tag{40}$$

and where $\hat{\mathcal{M}}$ is a fixed point of $\mathcal{F}$.

Observe that in the paper we have already shown that the consistency condition in the PCE is equivalent to $\bar{\mathcal{M}}$ being a fixed point of $\mathcal{F}$.

Proof. (Lemma 3: LSF Characterization):

Step 1. ε is perfectly revealed in the equilibrium prices p^ε.

Given equilibrium strategies of the players $\tilde{\theta}(\cdot)$ and realization of ε, the equilibrium price vector p^{ε}, is uniquely determined by the market clearing condition which defines a linear relation

$$\varepsilon = -\sum_{i\in I} \tilde{\theta}^{i}(p^{\varepsilon}). \tag{41}$$

By assumption Jacobians $D\tilde{\theta}^{i}$ are positive definite and hence invertible, this relation is one-to-one. Consequently the players will optimally respond to any realization of ε,even if they do not know it when making decisions, as they condition on p

Step 2. Given equilibrium strategies $\tilde{\theta}^{j}(\cdot)$ for players $j \neq i$, and the realization of ε we derive an optimal demand for player i denoted by θ^{i}_{ε}. Market clearing condition.

$$\theta^{i} + \sum_{j\neq i} \tilde{\theta}^{i}(p) + \varepsilon = 0, \tag{42}$$

defines a linear demand function faced by player i

$$p\left(\theta^{i}\right) = \tilde{\mathcal{M}}^{i}\left(\theta^{i} + \sum_{j\neq i} \tilde{\theta}^{i}(0) + \varepsilon\right), \tag{43}$$

where $\tilde{\mathcal{M}}^{i}$ is defined as in (33)

$$\tilde{\mathcal{M}}^{i} \equiv \left(\sum_{j\neq i} D\tilde{\theta}^{j}\right)^{-1}. \tag{44}$$

By assumption $D\tilde{\theta}^{j}$ are positive definite and symmetric therefore $\tilde{\mathcal{M}}^{i}$ is positive definite and symmetric too. Given demand function, marginal revenue evaluated at θ^{i}_{ε} can be found by differentiating $p\left(\theta^{i}\right)\theta^{i}$

$$\begin{aligned} MR\left(\theta^{i}_{\varepsilon}\right) &= D\left[p\left(\theta^{i}_{\varepsilon}\right)\theta^{i}_{\varepsilon}\right] = \\ &= Dp\left(\theta^{i}_{\varepsilon}\right)\theta^{i}_{\varepsilon} + p\left(\theta^{i}_{\varepsilon}\right) = \\ &= \tilde{\mathcal{M}}^{i}\theta^{i}_{\varepsilon} + p^{\varepsilon}, \end{aligned} \tag{45}$$

where in the last line we use the fact that the demand function $p()$ evaluated at θ^{i}_{ε} corresponds to an equilibrium price. The necessary and sufficient optimality condition requires that marginal utility $MU(\cdot)$ evaluated at optimal θ^{i}_{ε} coincides with the marginal revenue $MR(\cdot)$ also evaluated at θ^{i}_{ε}.

$$\bar{A} - \alpha^{i}\bar{\mathcal{V}}\left(\theta^{i}_{\varepsilon} + \theta^{i}_{0}\right) = \tilde{\mathcal{M}}^{i}\theta^{i}_{\varepsilon} + p^{\varepsilon}. \tag{46}$$

Condition (46) can be solved to give an optimal demand θ^{i}_{ε}

$$\theta^{i}_{\varepsilon} = \left(\tilde{\mathcal{M}}^{i} + \alpha^{i}\bar{\mathcal{V}}\right)^{-1}\left(\bar{A} - \alpha\bar{\mathcal{V}}\theta^{i}_{0} - p^{\varepsilon}\right). \tag{47}$$

Given the result from Step 1, in LSF the i will respond optimally to any possible realization of ε, hence best response of i to strategies of others is

$$\tilde{\theta}^{i}(p) = \left(\tilde{\mathcal{M}}^{i} + \alpha^{i}\bar{\mathcal{V}}\right)^{-1}\left(\bar{A} - \alpha\bar{\mathcal{V}}\theta^{i}_{0} - p\right). \tag{48}$$

Step 3. We observe that the functional form (40) is the same as the best response function (48). Therefore for "only if" part it suffices to show that $\tilde{\mathcal{M}}^i$ is a fixed point of $\mathcal{F}$. Using the definition of $\tilde{\mathcal{M}}^i$ and the fact that (48) is optimal for other players $j \neq i$, so $D\tilde{\theta}^j = \left(\tilde{\mathcal{M}}^j + \alpha^j \bar{\mathcal{V}}\right)^{-1}$, for any i we get

$$\begin{aligned} \tilde{\mathcal{M}}^i &\equiv \left(\sum_{j\neq i} D\tilde{\theta}^j\right)^{-1} = \\ &= \left(\sum_{j\neq i}\left(\tilde{\mathcal{M}}^j + \alpha^j \bar{\mathcal{V}}\right)^{-1}\right)^{-1} = \qquad (49) \\ &= \frac{1}{I-1}\mathcal{H}^i\left(\tilde{\mathcal{M}}\right), \end{aligned}$$

which with this implies that $\tilde{\mathcal{M}}$ is a fixed point of $\mathcal{F}$.

For "if" part we see that it is sufficient to verify that $\hat{\mathcal{M}}^i = \tilde{\mathcal{M}}^i$ for any i. $\hat{\mathcal{M}}$ is defined as a fixed point of $\mathcal{F}$ therefore its i^{th} component satisfies

$$\begin{aligned} \hat{\mathcal{M}}^i &= \frac{1}{I-1}\mathcal{H}^i\left(\hat{\mathcal{M}}\right) = \qquad (50) \\ &= \left(\sum_{j\neq i}\left(\hat{\mathcal{M}}^j + \alpha^j \bar{\mathcal{V}}\right)^{-1}\right)^{-1}. \end{aligned}$$

Given strategies of other investors given by (40), their derivatives are $D\tilde{\theta}^j = \left(\hat{\mathcal{M}}^j + \alpha^j \bar{\mathcal{V}}\right)^{-1}$. Plugging it in (50) gives

$$\hat{\mathcal{M}}^i = \left(\sum_{j\neq i} D\tilde{\theta}^j\right)^{-1} \equiv \tilde{\mathcal{M}}^i, \qquad (51)$$

and hence (40) corresponds to (48) ∎

Proof. (Theorem 3: Equivalence):

Step 1. In this step we show that for any LSF equilibrium $\tilde{\theta}(\cdot)$ its associated vector $\left(\tilde{p}, \tilde{\theta}, \tilde{\mathcal{M}}\right)$ constitutes a PCE. We need to verify three conditions: Market clearing, optimization by all traders and consistency of price impacts

For market clearing observe that

$$\begin{aligned} \sum_{i\in I}\tilde{\theta}^i &= \sum_{i\in I}\tilde{\theta}^i(\tilde{p}) = \sum_{i\in I}\left(\tilde{\theta}^i(0) + D\tilde{\theta}^i\cdot(\tilde{p}-0)\right) = \qquad (52) \\ &= \sum_{i\in I}\tilde{\theta}^i(0) - \left(\sum_{i\in I} D\tilde{\theta}^i\right)\left(\sum_{i\in I} D\tilde{\theta}^i\right)^{-1}\sum_{i\in I}\tilde{\theta}^i(0) = 0, \end{aligned}$$

where the first equality holds because $\tilde{\theta}^i(\cdot)$ are linear and hence coincide with their first order Taylor expansions and equality in the second line results from the definition of $\tilde{p}$.

The necessary and sufficient condition for $\tilde{\theta}^i$ to be optimal given $p_{\tilde{p},\tilde{\theta}^i,\bar{\mathcal{M}}^i}(\cdot)$ is equality of marginal revenue and marginal utility at $\tilde{\theta}^i$

$$\bar{A} - \alpha\bar{\mathcal{V}}\left(\tilde{\theta}^i + \theta_0^i\right) = \tilde{\mathcal{M}}^i\tilde{\theta}^i + \tilde{p}. \tag{53}$$

But this condition corresponds to a necessary condition for optimally of i for a particular realization of $\varepsilon = 0$. Finally from Lemma 3 it follows that $\tilde{\mathcal{M}}$ is a fixed point of $\mathcal{F}$ which as shown in equation 14 is also sufficient for consistency of $\tilde{\mathcal{M}}$.

Step 2. The supply functions $\tilde{\theta}^i(\cdot) \equiv \bar{\theta}^i\left(\cdot, \bar{\mathcal{M}}\right)$ have a functional form as in Lemma 3 and by equation 14 $\bar{\mathcal{M}}$ is a fixed point of $\mathcal{F}$. Hence Lemma 3 applies and $\tilde{\theta}^i(\cdot)$ is an LSF equilibrium. ■

Proof. (Corollary 1: Existence of LSF):

Given Lemma 3, the existence of LSF is equivalent to the existence of a fixed point $\hat{\mathcal{M}}$ of function $\mathcal{F}$. Therefore first we establish existence of fixed points $\hat{\mathcal{M}}$ for $I \geq 3$ (Steps 1-2) and then we show that $\hat{\mathcal{M}}$ does not exit for $I = 2$ (Step 3)

Step 1: In this step we construct a non-empty convex and compact set $\mathbf{M}^\lambda$

For any non-negative scalar $0 \leq \lambda < \infty$, we define a set $\mathbf{M}^\lambda \subset \mathbf{M}$

$$\mathbf{M}^\lambda = \left\{\mathcal{M} \in \mathbf{M} \middle| \left\|\mathcal{M}^i\right\| \leq \lambda \text{ for any } i\right\}. \tag{54}$$

where $\left\|\mathcal{M}^i\right\|$ is a sup norm on a linear operator. We argue that $\mathbf{M}^\lambda$ is a non-empty, convex and compact subset in $\mathbb{R}^{I\times N\times N}$.

For non-emptiness, observe that $0 \in \mathbf{M}^\lambda$. For convexity, consider two matrices $\mathcal{M}_1$, $\mathcal{M}_2$ $\in \mathbf{M}^\lambda$. Then $\alpha\mathcal{M}_1 + (1-\alpha)\mathcal{M}_2$ is a vector of I matrices for which each i^{th} component $\alpha\mathcal{M}_1^i + (1-\alpha)\mathcal{M}_2^i$ is a positive semi-definite and symmetric matrix. In addition, by the triangle inequality and the linearity of a norm for any i

$$\left\|\alpha\mathcal{M}_1^i + (1-\alpha)\mathcal{M}_2^i\right\| \leq \alpha\left\|\mathcal{M}_1^i\right\| + (1-\alpha)\left\|\mathcal{M}_2^i\right\| \leq \bar{\lambda}, \tag{55}$$

This implies that $\alpha\mathcal{M}_1 + (1-\alpha)\mathcal{M}_2 \in \mathbf{M}^\lambda$ so it is convex. To show that $\mathbf{M}^\lambda$ is compact in $\mathbb{R}^{\ N\times L\times L}$ it suffices to show that $\mathbf{M}^\lambda$ is closed and bounded in $R^{I\times N\times N}$. For closedness, consider any sequence $\{\mathcal{M}_k\}_{k=0,1..\infty}$ in $\mathbf{M}^\lambda$, converging to some $\mathcal{M} \in \mathbb{R}^{I\times N\times N}$. We now show that $\mathcal{M} \in \mathbf{M}^\lambda$ $\mathcal{M}$ is a positive definite an symmetric matrix as such properties are preserved in the limit. In addition since $\left\|\mathcal{M}_k^i\right\| \leq \lambda$ for any k and norm $\|\cdot\|$ is continuous, therefore for any i

$$\lambda \geq \lim_{k\to\infty}\left\|\mathcal{M}_k^i\right\| = \left\|\lim_{k\to\infty}\mathcal{M}_k^i\right\| = \left\|\mathcal{M}^i\right\|, \tag{56}$$

Hence $\mathcal{M} \in \mathbf{M}^\lambda$ so $\mathbf{M}^\lambda$ is closed. Boundedness of $\mathbf{M}^\lambda$ follows from the fact that for the positive definite and symmetric matrices diagonal elements are positive and bounded be λ and absolute values of the off-diagonal elements are bounded in turn by the diagonal ones. We conclude that $\mathbf{M}^\lambda$ is non-empty convex and compact.

Step 2: In this step we show that there exists $\bar{\lambda}$ for which $\mathcal{F}$ maps continuously set $\mathbf{M}^\lambda$ into itself and hence by Brouwer fixed point theorem the fixed point exists.

Define $\bar{\lambda}$ as

$$\bar{\lambda} \equiv \frac{\bar{\alpha}}{I-2} \left\| \bar{\mathcal{V}} \right\|, \tag{57}$$

where $\bar{\alpha}$ is a maximal risk aversion coefficient among all the traders, $\alpha^i \equiv \max_i \left\{ \alpha^i \right\}$. Since $I > 2$, we have $0 < \bar{\lambda} < \infty$. For any i we define auxiliary function $\mathcal{A}^i(\cdot) : \mathbf{M} \to \mathbb{R}^{N \times N}$ as

$$\mathcal{A}^i(\mathcal{M}) = \frac{1}{I-1} \sum_{j \neq i} \left(\mathcal{M}^j + \alpha^j \bar{\mathcal{V}} \right). \tag{58}$$

$\mathcal{A}^i(\cdot)$ is an arithmetic mean of matrices $\left(\mathcal{M}^j + \alpha^j \bar{\mathcal{V}} \right)$ for all $j \neq i$. For any $\mathcal{M} \in \mathbf{M}$, $\mathcal{A}^i(\mathcal{M})$ defines a positive definite and symmetric matrix as properties are preserved under addition. By the triangle inequality and the linearity of norm $\|\cdot\|$, for each i

$$\begin{aligned} \left\| \mathcal{A}^i(\mathcal{M}) \right\| &= \frac{1}{I-1} \left\| \sum_{j \neq i} \left(\mathcal{M}^j + \alpha^j \bar{\mathcal{V}} \right) \right\| \leq \\ &\leq \frac{1}{I-1} \sum_{j \neq i} \left(\left\| \mathcal{M}^j \right\| + \bar{\alpha} \left\| \bar{\mathcal{V}} \right\| \right). \end{aligned} \tag{59}$$

For any $\mathcal{M} \in \mathbf{M}^{\bar{\lambda}}$, $\left\| \mathcal{M}^j \right\| \leq \bar{\lambda}$ so by inequality (59), and the definition of $\bar{\lambda}$ from (57) we get

$$\begin{aligned} \left\| \mathcal{A}^i(\mathcal{M}) \right\| &\leq \frac{1}{I-1} \sum_{j \neq i} \left(\underbrace{\left\| \mathcal{M}^j \right\|}_{\leq \bar{\lambda}} + \underbrace{\bar{\alpha} \left\| \bar{\mathcal{V}} \right\|}_{\leq (I-2)\bar{\lambda}} \right) \\ &\leq \bar{\lambda} + (I-2)\bar{\lambda} = (I-1)\bar{\lambda}. \end{aligned} \tag{60}$$

By a well-known harmonic-arithmetic inequality for positive definite matrices, for any i

$$\left\| \mathcal{H}^i(\mathcal{M}) \right\| \leq \left\| \mathcal{A}^i(\mathcal{M}) \right\| \leq (I-1)\bar{\lambda}. \tag{61}$$

This implies that for any $\mathcal{M} \in \mathbf{M}^{\bar{\lambda}}$, and for any i

$$\left\| \mathcal{F}^i(\mathcal{M}) \right\| = \frac{\left\| \mathcal{H}^i(\mathcal{M}) \right\|}{I-1} \leq \bar{\lambda}, \tag{62}$$

and hence $\mathcal{F}(\mathcal{M}) \in \mathbf{M}^{\bar{\lambda}}$. We conclude that $\mathcal{F} : \mathbf{M}^{\bar{\lambda}} \to \mathbf{M}^{\bar{\lambda}}$. Because $\alpha\bar{\mathcal{V}}$ is strictly positive definite, harmonic mean $\mathcal{H}(\cdot)$ is well defined and it is a continuous function on $\mathbf{M}^{\bar{\lambda}}$. Recall that $\mathbf{M}^{\bar{\lambda}}$ is non-empty convex and compact in $\mathbb{R}^{I \times N \times N}$. We apply the Brouwer fixed point theorem to establish the existence of $\hat{\mathcal{M}} \in \mathbf{M}^{\bar{\lambda}}$ satisfying $\mathcal{F}\left(\hat{\mathcal{M}} \right) = \hat{\mathcal{M}}$.

Step 3. Fixed point does not exist when $I = 2$.)

In case of $I = 2$, the $\mathcal{F}$ function becomes

$$\mathcal{F}^i(\mathcal{M}) = \mathcal{M}^j - \alpha^j \bar{\mathcal{V}}, \tag{63}$$

for $i, j = 1, 2$. Suppose there exists fixed point $\tilde{\mathcal{M}}$. Its two components $\left\{\tilde{\mathcal{M}}^1, \tilde{\mathcal{M}}^2\right\}$ must satisfy

$$\begin{aligned} \hat{\mathcal{M}}^1 &= \hat{\mathcal{M}}^2 - \alpha^2 \bar{\mathcal{V}}, \\ .\hat{\mathcal{M}}^2 &= \hat{\mathcal{M}}^1 - \alpha^1 \bar{\mathcal{V}}. \end{aligned} \tag{64}$$

Equations (63) and (64) imply

$$\alpha^2 = -\alpha^1. \tag{65}$$

which is impossible since by assumption all traders are risk averse, $\alpha^1, \alpha^2 > 0$. ■

Proof. (Corollary 2: Inefficiency):

Step 1: In this step we show that for any LSF price impacts matrices $\tilde{\mathcal{M}}^i$ are strictly positive definite. For any i equilibrium price impact is given by

$$\tilde{\mathcal{M}}^i = \left(\sum_{j \neq i} \left(\tilde{\mathcal{M}}^j + \alpha^j \bar{\mathcal{V}} \right)^{-1} \right)^{-1}, \tag{66}$$

where by Lemma 3we know that $\tilde{\mathcal{M}}^j$ are positive semidefinite and symmetric. But then each matrix $\tilde{\mathcal{M}}^j + \alpha^j \bar{\mathcal{V}}$ is strictly positive definite and since inversion and summation preserves strict positive definiteness, $\tilde{\mathcal{M}}^i$ must also be strictly positive definite and symmetric.

Step 2: (only if part) Suppose the equilibrium trade is Pareto efficient. Because the marginal utility of riskless asset is equal to one for all investors, the necessary condition for Pareto efficiency is the equality of marginal utilities for all other assets, across all the players. That is for any i, j

$$\bar{A} - \alpha^i \bar{\mathcal{V}} \left(\tilde{\theta}^i + \theta_0^i \right) = \bar{A} - \alpha^j \bar{\mathcal{V}} \left(\tilde{\theta}^j + \theta_0^j \right). \tag{67}$$

In equilibrium marginal utilities are equal to marginal revenues

$$\bar{A} - \alpha^i \bar{\mathcal{V}} \left(\tilde{\theta}^i + \theta_0^i \right) = \tilde{p} + \tilde{\mathcal{M}}^i \tilde{\theta}^i. \tag{68}$$

The two equations imply that $\tilde{\mathcal{M}}^i \tilde{\theta}^i$ coincide for all traders, and are equal to some vector x. For any player its trader can be written as

$$\left(\tilde{\mathcal{M}}^i \right)^{-1} x = \tilde{\theta}^i. \tag{69}$$

Summing over all i, factoring out x, and applying market clearing condition gives

$$x = \left(\sum_i \left(\tilde{\mathcal{M}}^i \right)^{-1} \right)^{-1} 0 = 0, \tag{70}$$

which in turn by (69) implies no trade

$$\tilde{\theta}^i = 0, \tag{71}$$

hence with (68) we get

$$\bar{A} - \alpha^i \bar{\mathcal{V}} \theta_0^i = \tilde{p} = \bar{A} - \alpha^j \bar{\mathcal{V}} \theta_0^j. \tag{72}$$

Equation (72) is sufficient for the initial portfolios to be Pareto efficient.

Step 3: (if part) Suppose now that endowments are Pareto efficient and hence marginal utilities evaluated at zero trade are equal for all traders. That is for any i, j

$$\bar{A} - \alpha^i \bar{\mathcal{V}} \theta_0^i = \bar{A} - \alpha^j \bar{\mathcal{V}} \theta_0^j. \tag{73}$$

With (73) equilibrium condition (68) can be rewritten as

$$\bar{A} - \alpha^i \bar{\mathcal{V}} \theta_0^i - \tilde{p} \equiv y = \left(\alpha^i \bar{\mathcal{V}} + \tilde{\mathcal{M}}^i \right) \tilde{\theta}^i, \tag{74}$$

where vector y has the same value for all traders. Because $\alpha^i \bar{\mathcal{V}} + \tilde{\mathcal{M}}^i$ is (strictly) positive definite for all traders it is invertible so

$$\tilde{\theta}^i = \left(\alpha^i \bar{\mathcal{V}} + \tilde{\mathcal{M}}^i \right)^{-1} y. \tag{75}$$

Summing it over all i, applying market clearing condition implies that $y = \tilde{\theta}^i = 0$. Consequently traders consume only their initial endowments. ■

Proof. (Corollary 3: Convergence):

Step 1: For $k-$replica of a thin market prices are determined by a market clearing condition

$$0 = k \times \sum_{i \in I} \left(\hat{\mathcal{M}}^i + \alpha^i \bar{\mathcal{V}} \right)^{-1} \left(\bar{A} - \alpha \bar{\mathcal{V}} \theta_0^i - p^k \right), \tag{76}$$

where $\hat{\mathcal{M}}$ is a fixed point of $\mathcal{F}^k$, where $\mathcal{F}^k$ is an function $\mathcal{F}$ modified for k replica. Dividing it by k and solving for p^k gives

$$p^k = \left(\sum_{i \in I} \left(\hat{\mathcal{M}}^{i,k} + \alpha^i \bar{\mathcal{V}} \right)^{-1} \right)^{-1} \sum_{i \in I} \left(\hat{\mathcal{M}}^{i,k} + \alpha^i \bar{\mathcal{V}} \right)^{-1} \left(\bar{A} - \alpha \bar{\mathcal{V}} \theta_0^i \right), \tag{77}$$

and the traded portfolios are given by

$$\tilde{\theta}^{i,k} \left(p^k \right) = \left(\hat{\mathcal{M}}^{i,k} + \alpha^i \bar{\mathcal{V}} \right)^{-1} \left(\bar{A} - \alpha \bar{\mathcal{V}} \theta_0^i - p^k \right). \tag{78}$$

It can be seen that p^k and hence $\tilde{\theta}^{i,k} \equiv \tilde{\theta}^{i,k} \left(p^k \right)$ do not depend on k except through $\hat{\mathcal{M}}^k$ and are continuous in $\hat{\mathcal{M}}^k$. Therefore for any ε there exists ε' such that $\left\| p^k - p^{Competitive} \right\|^E \leq \varepsilon$ and $\left\| \tilde{\theta}^{i,k} - \tilde{\theta}^{i,Competitive} \right\|^E \leq \varepsilon$ for any $\left\| \hat{\mathcal{M}}^k \right\|^E \leq \varepsilon'$, where $\left\| \cdot \right\|^E$ is an Euclidian norm.

Step 2: We show by contradiction that in $k-$replica there cannot exist a fixed point of $\mathcal{F}^k$, with $\left\| \mathcal{M}^i \right\| > \bar{\lambda}(k)$ for one or more traders, where

$$\bar{\lambda}(k) \equiv \frac{\bar{\alpha}}{k \times I - 2} \left\| \bar{\mathcal{V}} \right\|. \tag{79}$$

Fix k and consider any $\mathcal{M} \in \mathbf{M}$ for which at least for one i $\left\|\mathcal{M}^i\right\| > \bar{\lambda}(k)$ and without loss of generality assume that, $\left\|\mathcal{M}^i\right\| \geq \left\|\mathcal{M}^j\right\|$ for all j. Then

$$\begin{aligned}\left\|\mathcal{F}^{i,k}(\mathcal{M})\right\| &= \frac{\left\|\mathcal{H}^i(\mathcal{M})\right\|}{k \times I - 1} \leq \frac{\left\|\mathcal{A}^i(\mathcal{M})\right\|}{k \times I - 1} = \\ &\leq \frac{1}{k \times I - 1}\left(\left\|\mathcal{M}^i\right\| + \alpha^j \left\|\bar{\mathcal{V}}\right\|\right) \leq \\ &\leq \frac{1}{k \times I - 1}\left\|\mathcal{M}^i\right\| + \frac{k \times I - 2}{k \times I - 1}\bar{\lambda}(k) < \left\|\mathcal{M}^i\right\|.\end{aligned} \tag{80}$$

where the first equality holds by the definition of $\mathcal{F}^k$, the first inequality by the harmonic algebraic mean inequality, the equality in the second line by the definition of $\mathcal{A}^i(\cdot)$ and the fact that $\left\|\mathcal{M}^i\right\|$ is maximal among all traders, the first inequality in the third line by the definition of $\bar{\lambda}$ and strict inequality by the fact that a convex combination of two different elements, with both weights positive is always smaller than the larger of the two elements. But this inequality implies that

$$\mathcal{F}^{i,k}(\mathcal{M}) \neq \mathcal{M}^{i,k}, \tag{81}$$

and hence $\mathcal{M}$ is not a fixed point of $\mathcal{F}^k$. This implies that any fixed point $\left\|\hat{\mathcal{M}}\right\| \leq \bar{\lambda}(k)$. By monotonicity of $\bar{\lambda}(k)$ it follows that for any $k' \geq k$ any fixed point of $\mathcal{F}^{k'}$ must satisfy $\left\|\hat{\mathcal{M}}\right\| \leq \bar{\lambda}(k') \leq \bar{\lambda}(k)$.

Step 3: $\mathcal{M}^i$ are positive definite and symmetric, therefore $\left\|\mathcal{M}^i\right\| \leq \bar{\lambda}(k)$ implies that $\bar{\lambda}(k)$ is an upper bound for the absolute value of any on and off diagonal element of $\mathcal{M}^i$. Hence their Euclidian norm are bounded $\left\|\tilde{\mathcal{M}}\right\|^E \leq \varepsilon' \equiv N^2\bar{\lambda}(k)$. Since

$$\lim_{k\to\infty} \varepsilon' = \lim_{k\to\infty} \frac{N^2\bar{\alpha}}{k \times I - 2}\left\|\bar{\mathcal{V}}\right\| = 0, \tag{82}$$

we conclude that for any ε' there exists sufficiently large k, for which any fixed points of function $\mathcal{F}^{k'}$ for any $k' \geq k$ satisfies $\left\|\hat{\mathcal{M}}\right\|^E \leq \varepsilon'$. Consequently, by Step 1 and Lemma 3 for any ε there exists sufficiently large k, such that for any $k' \geq k$ in any $k'-$replica equilibrium prices and allocations satisfy

$$\left\|p^{k'} - p^{Competitive}\right\|^E \leq \varepsilon, \tag{83}$$

and

$$\left\|\tilde{\theta}^{i,k'} - \tilde{\theta}^{i,Competitive}\right\|^E \leq \varepsilon \tag{84}$$

. ■

Proof. (Corollary 4: Global Uniqueness):

By Lemma 3 to establish global uniqueness of an LSF equilibrium it is sufficient to show that function $\mathcal{F}$ has only one fixed point $\hat{\mathcal{M}}$. Observe that with one risky asset each individual player price impact matrix becomes a scalar, hence $\hat{\mathcal{M}}$ is a vector. Let $\hat{\mathcal{M}}_1$ and $\hat{\mathcal{M}}_2$ be fixed points of $\mathcal{F}$. We have

$$\hat{\mathcal{M}}_1 - \hat{\mathcal{M}}_2 = \mathcal{F}\left(\hat{\mathcal{M}}_1\right) - \mathcal{F}\left(\hat{\mathcal{M}}_2\right) = D\mathcal{F}\left[\hat{\mathcal{M}}_1 - \hat{\mathcal{M}}_2\right], \tag{85}$$

where the first equality comes from the fact that $\tilde{\mathcal{M}}_1 - \tilde{\mathcal{M}}_2$ are fixed points and second from the mean value theorem, where $D\mathcal{F}$ is a Jacobian of $\mathcal{F}$ evaluated at some intermediate point $\mathcal{M}$ between $\hat{\mathcal{M}}_1$ and $\hat{\mathcal{M}}_2$. With one risky asset, the Jacobian becomes

$$D\mathcal{F} = \begin{pmatrix} 0 & \phi_{12} & \dots & \phi_{1I} \\ \phi_{21} & 0 & \dots & \phi_{2I} \\ \dots & \dots & \dots & \dots \\ \phi_{I1} & \phi_{I2} & \dots & 0 \end{pmatrix}, \tag{86}$$

where the typical element $(i \neq j)$ is given by

$$\phi_{i,j} = \frac{\left(\left(\mathcal{M}^j + \alpha^j \bar{\mathcal{V}}\right)^{-1}\right)^2}{\left(\sum_{k \neq i} \left(\mathcal{M}^k + \alpha^k \bar{\mathcal{V}}\right)^{-1}\right)^2}. \tag{87}$$

Note that because for trader i the $\phi_{i,i}$ is 0 and sum of all elements in the row is

$$\bar{\phi} \equiv \sum_j \phi_{i,j} = \frac{\sum_{k \neq i} \left(\left(\mathcal{M}^j + \alpha^j \bar{\mathcal{V}}\right)^{-1}\right)^2}{\left(\sum_{k \neq i} \left(\mathcal{M}^k + \alpha^k \bar{\mathcal{V}}\right)^{-1}\right)^2} < 1. \tag{88}$$

Without loss of generality we assume that the difference between the price impacts of two fixed points is the largest for trader i and it is non-negative, that is $\hat{\mathcal{M}}_1^i - \hat{\mathcal{M}}_2^i \geq \hat{\mathcal{M}}_1^j - \hat{\mathcal{M}}_2^j$ for any j and $\hat{\mathcal{M}}_1^i - \hat{\mathcal{M}}_2^i \geq 0$. But then from (85) we get

$$\begin{aligned} \hat{\mathcal{M}}_1^i - \hat{\mathcal{M}}_2^i &= \sum_j \alpha_{i,j} \left[\hat{\mathcal{M}}_1^j - \hat{\mathcal{M}}_2^j\right] \leq \\ &\leq \sum_j \alpha_{i,j} \left[\hat{\mathcal{M}}_1^i - \hat{\mathcal{M}}_2^i\right] = \\ &= \phi \left[\hat{\mathcal{M}}_1^i - \hat{\mathcal{M}}_2^i\right], \end{aligned} \tag{89}$$

which implies that

$$\hat{\mathcal{M}}_1^i - \hat{\mathcal{M}}_2^i = \frac{1}{1-\phi} 0 = 0. \tag{90}$$

This implies that the maximal positive difference between any two components of $\hat{\mathcal{M}}_1$ and $\hat{\mathcal{M}}_2$ is zero. Replacing the roles of $\hat{\mathcal{M}}_1$ and $\hat{\mathcal{M}}_2$ implies that maximal absolute difference between the two vectors is zero. Consequently $\hat{\mathcal{M}}_1 = \hat{\mathcal{M}}_2$, so they are the same fixed point.
■