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Equilibrium and Media Exchange in a Convex Trading Post Economy with  
Transaction Costs

BY

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# Equilibrium and Media of Exchange in a Convex Trading Post Economy with Transaction Costs

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PRELIMINARY: NOT FOR QUOTATION

”[An] important and difficult question...[is] not answered by the approach taken here: the integration of money in the theory of value...”

— Gerard Debreu, *Theory of Value* (1959)

## Abstract

General equilibrium is investigated with  $N$  commodities traded at  $N(N - 1)/2$  commodity-pairwise trading posts. Trade is a resource-using activity undertaken by firms recovering transaction costs through the spread between bid (wholesale) and ask (retail) prices (quoted as commodity rates of exchange). Budget constraints are enforced at each trading post separately so that there is demand for a carrier of value between trading posts, commodity money. Existence of general equilibrium is established under conventional convexity and continuity conditions and technical assumptions assuring boundedness of price ratios. Trade in media of exchange (commodity money) is the difference between household gross and net trades.

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# 1 Introduction

It is well-known that the Arrow-Debreu model of Walrasian general equilibrium cannot account for money. Professor Hahn (1982) writes

”The most serious challenge that the existence of money poses to the theorist is this: the best developed model of the economy cannot find room for it. The best developed model is, of course, the Arrow-Debreu version of a Walrasian general equilibrium. A first, and...difficult...task is to find an alternative construction without...sacrificing the clarity and logical coherence ... of Arrow-Debreu.”

This paper pursues development of foundations for a theory of money based on elaborating the detail structure of an Arrow-Debreu model. The elementary first step is to create a general equilibrium where there is a well defined demand for a medium of exchange — a carrier of value between transactions. This is arranged by replacing the single budget constraint of the Arrow-Debreu model with the requirement that the typical household or firm pays for its purchases at each of many separate transactions. Transactions take place at commodity-pairwise trading posts. Then a well-defined demand for media of exchange (not necessarily unique) arises endogenously as an outcome of the market equilibrium. Media of exchange (commodity monies) are characterized as the carrier of value between transactions (not fulfilling final demands or input requirements themselves), the difference between gross and net trades.

## 1.1 Structure of the Model

Trade takes place at commodity pairwise trading posts (Cournot (1838), Shapley and Shubik (1977), Walras (1874)) with budget constraints (you pay for what you get in commodity terms) enforced at each post. Prices — bid (wholesale) and ask (retail) — are quoted as commodity rates of exchange. Trade across trading posts is arranged by firms, typically buying at bid prices and selling at ask prices, incurring transaction costs (resources used up in the transaction process) and recouping them through the bid/ask spread. Market equilibrium occurs when bid and ask prices at each trading post have adjusted so that all trading posts clear.

## 1.2 Structure of the Proof

The structure of the proof of existence of general equilibrium follows the approach of Arrow and Debreu (1954), Debreu (1959), and Starr (1997). The usual assumptions of continuity, convexity (traditional but by no means innocuous in this context), and no free lunch/irreversibility are used. There is one additional (objectionable) family of assumptions, strong substitutability between goods and the existence of a linear

backup transactions technology for all goods so that the equilibrium price ratios are necessarily bounded, allowing characterization of the price space as a compact convex set (a cube interior to the positive quadrant in  $R^{N(N-1)}$ ). The attainable set of trading post transactions is compact. The model considers transaction plans of firms and households artificially bounded in a compact set including the attainable set as a proper subset. Price adjustment to a fixed point with market clearing leads to equilibrium of the artificially bounded economy. But the artificial bounds are not a binding constraint in equilibrium. The equilibrium of the artificially bounded economy is as well an equilibrium of the original economy.

### 1.3 Conclusion: The medium(a) of exchange

The general equilibrium specifies each household and firm's trading plan. At the conclusion of trade, each has achieved a net trade. Gross trades include trading activity that goes to paying for acquisitions and accepting payment for sales rather than directly implementing desired net trades. It's easy to calculate gross trades and net trades at equilibrium. For households, the difference — gross trades minus net trades — represents trading activity in carriers of value between trades, media of exchange. Since firms perform a market-making function, identification of media of exchange used by firms is not so straightforward. In some examples (see Starr (2003A, 2003B)) the medium of exchange may be a single specialized commodity (the common medium of exchange). The approach of this model is intended to provide a Walrasian general equilibrium alternative to the overlapping generations, Wallace (1980), and search theory, Kiyotaki and Wright (1989), models of the foundations of monetary theory. The present model is sufficiently general to include both a single common medium of exchange and many goods simultaneously acting as media of exchange. There is nothing in the present model designed to encourage concentration of trade on a *single* common medium of exchange.

## 2 Trading Posts

There are  $N$  tradeable goods denoted 1, 2, ...,  $N$ . They are traded for one another pairwise at trading posts.  $\{i, j\}$  (or equivalently  $\{j, i\}$ ) denotes the trading post where goods  $i$  and  $j$  are traded for one another. There are  $N(N - 1)/2$  distinct trading posts.

### 3 Prices

Goods are traded directly for one another without distinguishing any single good as 'money'. Prices are then quoted as rates of exchange between goods. We distinguish between bid (selling or wholesale) prices and ask (buying or retail) prices. Thus the ask price of a hamburger might be 5.0 chocolate bars and the bid price 3.0 chocolate bars. Note that the ask price of a chocolate bar then is the inverse of bid price of a hamburger. That is, the ask price of a chocolate bar is 0.333 hamburger and the bid price of a chocolate bar is 0.2 hamburger.

More formally, denote the bid price of  $i$  at trading post  $\{i,j\}$  as  $q_i^{\{i,j\}}$  expressed in units of  $j$ . Then the ask price of  $j$  at  $\{i,j\}$  expressed in units of  $i$  is  $[q_i^{\{i,j\}}]^{-1}$ . Thus with  $N$  commodities, there are  $N(N - 1)$  distinct prices, the bid price of each of  $N$  goods versus  $(N-1)$  counterparts. The array of prices is then an  $N(N - 1)$ -dimensional vector,  $q$  in  $R_+^{N(N-1)}$ .

Once  $q$  is specified, showing all bid prices  $q_i^{\{i,j\}}$  for all  $1 \leq i, j \leq N, i \neq j$ , that implies as well all the resulting ask prices. In principle, any nonnegative value of  $q_i^{\{i,j\}}$  is possible, though a value of nil implies an undefined value of  $[q_i^{\{i,j\}}]^{-1}$ . We'll find below that bounding the price space above and away from the boundary makes it more manageable.

### 4 Budget Constraints and Trading Opportunities

The budget constraint is simply that at each pairwise trading post, at prevailing prices, in each transaction, payment is given for goods received. That is, at trading post  $\{i, j\}$ , a bid price for  $i$  is quoted  $q_i^{\{i,j\}}$  in terms of  $j$  and a bid price of  $j$  is quoted,  $q_j^{\{i,j\}}$  in terms of  $i$ . Suppose a typical firm or household is considering a trading plan  $(y, x) \in R^{2N(N-1)}$ . That plan specifies the following transactions at trading post  $\{i,j\}$ :  $y_i^{\{i,j\}}$  (at ask prices — retail) in  $i$ ,  $y_j^{\{i,j\}}$  (at ask prices — retail) in  $j$ ,  $x_i^{\{i,j\}}$  (at bid prices — wholesale) in  $i$ ,  $x_j^{\{i,j\}}$  (at bid prices — wholesale) in  $j$ . Positive values of these transactions are purchases. Negative values are sales. At each trading post (of two goods) there are four quantities to specify in a trading plan. Then value delivered must equal value received. That is

$$0 = y_j^{\{i,j\}} + q_i^{\{i,j\}} x_i^{\{i,j\}}, \text{ or equivalently, } 0 = [q_j^{\{i,j\}}]^{-1} y_i^{\{i,j\}} + x_j^{\{i,j\}} \quad (\mathbf{B})$$

(B) says — evaluating all trades in terms of good  $j$  (without loss of generality) — that trades in  $i$  and  $j$  at retail and wholesale at the  $\{i,j\}$  trading post must sum to a zero value of  $j$ . Each retail purchase is paid for by a wholesale delivery.

Given a price vector  $q \in R_+^{N(N-1)}$  the array of trades fulfilling **(B)** is the set of trades fulfilling the  $N(N-1)/2$  local budget constraints at the trading posts. Denote this set

$$\mathbf{M}(q) \equiv \{(y, x) \in R^{2N(N-1)} | (y, x) \text{ fulfills } \mathbf{(B)} \text{ at } q \text{ for all } i, j = 1, \dots, N, i \neq j\}$$

## 5 Firms

The heavy lifting in this model is done by firms. They perform the market-making function, incurring transaction costs. The population of firms is a finite set denoted  $F$ , with typical element  $f \in F$ . Thus, firm  $f$ 's technology set may specify that  $f$ 's purchase of labor (retail) in exchange for  $i$  on the  $\{i, \text{labor}\}$  market and purchase of  $i$  and  $j$  wholesale on the  $\{i, j\}$  market allows  $f$  to sell  $i$  and  $j$  (retail) on the  $\{i, j\}$  market. That's how  $f$  can become a market maker. If there is a sufficient difference between bid and ask prices so that  $f$  can cover the cost of its  $i, j$  inputs with a surplus left over, that surplus becomes  $f$ 's profits, to be rebated to  $f$ 's shareholders.

### 5.1 Technology Set

Firm  $f$ 's technology set is  $Y^f$ . We assume

$$\mathbf{P.0} \quad Y^f \subset R^{2N(N-1)}$$

The typical element of  $Y^f$  is  $(y^f, x^f)$ , a pair of  $N(N-1)$ -dimensional vectors. The  $N(N-1)$ -dimensional vector  $y^f$  represents  $f$ 's transactions at ask (retail) prices; the  $N(N-1)$ -dimensional vector  $x^f$  represents  $f$ 's transactions at bid (wholesale) prices. The 2-dimensional vector  $y^{f\{i,j\}}$  represents  $f$ 's transactions at ask (retail) prices at trading post  $\{i, j\}$ ; the 2-dimensional vector  $x^{f\{i,j\}}$  represents  $f$ 's transactions at bid (wholesale) prices at trading post  $\{i, j\}$ . The typical co-ordinates  $y_i^{f\{i,j\}}, x_i^{f\{i,j\}}$  are  $f$ 's action with respect to good  $i$  at the  $\{i, j\}$  trading post. Since  $f$  may act as a wholesaler/retailer/market maker, entries anywhere in  $(y^{f\{i,j\}}, x^{f\{i,j\}})$  may be positive or negative — subject of course to constraints of technology  $Y^f$  and prices  $\mathbf{M}(q)$ . This distinguishes the firm from the typical household. The typical household can only sell at bid prices and buy at ask prices.

The entry  $y_i^{f\{i,j\}}$ , represents  $f$ 's actions at ask prices with regard to good  $i$  at trading post  $\{i, j\}$ .  $y_i^{f\{i,j\}} > 0$  represents a purchase of  $i$  at the  $\{i, j\}$  trading post (at the ask price).  $y_i^{f\{i,j\}} < 0$  represents a sale of  $i$  at the ask price.

The entry  $x_i^{f\{i,j\}}$ , represents  $f$ 's actions at bid prices with regard to good  $i$  at trading post  $\{i, j\}$ .  $x_i^{f\{i,j\}} > 0$  represents a purchase of  $i$  at the trading post (at the bid price).  $x_i^{f\{i,j\}} < 0$  represents a sale of  $i$  at the bid price.

A firm that is an active market-maker at  $\{i,j\}$  will typically buy at the bid price and sell at the ask price. A firm that is not a market-maker may have to pay retail — like the rest of us — selling at the bid price and buying at the ask price.

In addition to indicating the transaction possibilities,  $Y^f$  includes the usual production possibilities. The usual assumptions on production technology apply. For each  $f \in F$ , assume

**P.I**  $Y^f$  is convex.

**P.II**  $0 \in Y^f$ , where 0 indicates the zero vector in  $R^{2N(N-1)}$ .

**P.III**  $Y^f$  is closed.

The aggregate technology set is the sum of individual firm technology sets.  $Y \equiv \sum_{f \in F} Y^f$ . It fulfills the familiar *no free lunch* and *irreversibility* conditions.

**P.IV** [(a)] if  $(y, x) \in Y$  and  $(y, x) \neq 0$ , then  $y_i^{\{i,j\}} + x_i^{\{i,j\}} > 0$  for some  $i, j$ .

[(b)] if  $(y, x) \in Y$  and  $(y, x) \neq 0$ , then  $-(y, x) \notin Y$ .

Denote the initial resource endowment of the economy as  $r \in R_+^N$ . Then we define the attainable production plans of the economy as

$$\hat{Y} \equiv \{(y, x) \in Y \mid r_i \geq \sum_j (y_i^{\{i,j\}} + x_i^{\{i,j\}})\}$$

Lemma 5.1: Assume P.0 - P.IV. Then  $\hat{Y}$  is closed, convex, and bounded.

Attainable production plans for firm  $f$  can then be described as

$$\hat{Y}^f \equiv \{(y^f, x^f) \in Y^f \mid \text{there is } (y^k, x^k) \in Y^k \text{ for each } k \in F, k \neq f, \text{ so that } [\sum_{k \in F, k \neq f} (y^k, x^k) + (y^f, x^f)] \in \hat{Y}\}$$

## 5.2 Firm Maximand and Transactions Function

The firm formulates a production plan and a trading plan. The firm's opportunity set for net yields after transactions fulfilling budget is  $E^f(q) \equiv [\mathbf{M}(q) - Y^f] \cap R_+^{2N(N-1)}$ . That is, consider the firm's production, purchase, and sale possibilities, net after paying for them, and what's left is the net yield. Using the sign conventions we've adopted — purchases are positive co-ordinates, sales are negative co-ordinates — the net yield is then the negative co-ordinates (supplies) in a trading plan that are not absorbed by payments due. The supplies are subtracted out, so the surpluses enter  $E^f(q)$  as positive co-ordinates.

A typical element of these surplus supplies is denoted  $(y', x') \in E^f(q)$ . Note that in the notation  $(y', x')$ ,  $y'$  and  $x'$  are dummies, not actual marketed supplies and demands.

Now consider  $(y', x') \in E^f(q)$ . In each good  $i$ , the net surplus available in good  $i$  is  $w_i^f \equiv \sum_{j=1}^N (y_i^{\{i,j\}} + x_i^{\{i,j\}})$  and firm  $f$ 's surplus is the vector  $w^f$  of these co-ordinates.

In the absence of a single family of well-defined prices, it is difficult to characterize optimizing behavior for the firm. *Fautes de mieux* we'll give the firm a scalar maximand with argument  $q, y', x'$ . Firm  $f$  is assumed to have a real-valued, continuous maximand  $v^f(q; y', x')$ . We take  $v^f$  to be strictly concave in  $w^f$ , defined above.

The firm's (unconstrained) market behavior then is described by

$$S^f(q) \equiv \{(y, x) | (y, x) - (y^o, x^o) = (y', x'), \text{ where } (y', x') \text{ maximizes } v^f(q; y', x') \\ \text{subject to } (y', x') \in E^f(q) \text{ and } (y^o, x^o) \in Y^f \text{ and } (y, x) \in \mathbf{M}(q)\}$$

The logic of this definition is that  $(y', x') \geq 0$  is the surplus left over after the firm  $f$  has performed according to its technology and subject to prevailing prices.

It is possible that  $S^f(q)$  is not well defined, since the opportunity set may be unbounded. In the light of Lemma 5.1, there is a constant  $c > 0$  sufficiently large so that for all  $f \in F$ ,  $\hat{Y}^f$  is strictly contained in a closed ball, denoted  $B_c$  of radius  $c$  centered at the origin of  $R^{2N(N-1)}$ . Following the technique of Arrow and Debreu (1954), constrained market behavior for the firm will consist of limiting its production choices to  $Y^f \cap B_c$ . This leads to the constrained surplus

$$\tilde{E}^f(q) \equiv [\mathbf{M}(q) - [Y^f \cap B_c]] \cap R_+^{2N(N-1)}$$

Lemma 5.2: Assume P.0 - P.IV. Then  $\tilde{E}^f(q)$  is nonempty, upper and lower hemicontinuous.

Proof: Upper hemicontinuity and convexity follow from closedness and convexity of the underlying sets.  $0 \in \tilde{E}^f(q)$  always, so nonemptiness is fulfilled. Lower hemicontinuity requires some work.

Let  $q^\nu \rightarrow q^o$ ,  $(y^o, x^o) \in \tilde{E}^f(q^o)$ . We seek  $(y^\nu, x^\nu) \in \tilde{E}^f(q^\nu)$  so that  $(y^\nu, x^\nu) \rightarrow (y^o, x^o)$ . If  $(y^o, x^o) = 0$ , lower hemicontinuity is trivially satisfied. Suppose instead  $(y^o, x^o) \geq 0$  (the inequality applies co-ordinatewise). Then in an  $\epsilon$ -neighborhood of  $(y^o, x^o)$ , for  $\nu$  sufficiently large, there is  $(y^\nu, x^\nu) \in \tilde{E}^f(q^\nu)$ .  $(y^\nu, x^\nu)$  is the required sequence.

The firm's constrained market behavior then is defined as

$$\tilde{S}^f(q) \equiv \{(y, x) | (y, x) - (y^o, x^o) = (y', x'), \text{ where } (y', x') \text{ maximizes } v^f(q; y', x') \\ \text{subject to } (y', x') \in \tilde{E}^f(q) \text{ and } (y^o, x^o) \in Y^f \cap B_c \text{ and } (y, x) \in \mathbf{M}(q)\}$$

Lemma 5.3: Assume P.0 - P.IV. Then  $\tilde{S}^f(q)$  is well defined, non-empty, upper hemi-continuous, and convex-valued for all  $q \in R_+^{N(N-1)}$ .

Proof: Theorem of the Maximum.

### 5.3 Profits

When (constrained) firm  $f$  supplies  $\tilde{S}^f(q)$  to the market, it retains as surplus  $\tilde{\pi}^f(q) \equiv \arg \max v^f(q; y', x') \in \tilde{E}^f(q)$ . Under strict concavity of  $v^f$ ,  $\tilde{\pi}^f(q)$  is point-valued and well-defined. When (unconstrained) firm  $f$  supplies  $S^f(q)$  to the market, it retains as surplus  $\pi^f(q) \equiv \arg \max v^f(q; y', x') \in E^f(q)$ .

Note that  $\tilde{S}^f(q) + \tilde{\pi}^f(q)$  is the set of gross production activity planned (subject to length constraint) by firm  $f$  at prices  $q$ . When it is well defined  $S^f(q) + \pi^f(q)$  is the corresponding gross activity plan without length constraint.

Lemma 5.4: Assume P.0 - P.IV. Then  $\tilde{\pi}^f(q)$  is point-valued and continuous for all  $q \in R_+^{2N(N-1)}$ .

Proof: Continuity and strict concavity of  $v^f$ , Theorem of the Maximum.

## 5.4 Inclusion of constrained supply in unconstrained supply

Lemma 5.5: Let  $q \in R_+^{N(N-1)}$  such that  $[\tilde{S}^f(q) + \tilde{\pi}^f(q)] \subseteq \hat{Y}^f$ . Then  $\pi^f(q)$  and  $S^f(q)$  are well defined and nonempty. Further  $\pi^f(q) = \tilde{\pi}^f(q)$  and  $S^f(q) = \tilde{S}^f(q)$ .

Proof: Recall that  $B_c$  strictly includes  $\hat{Y}^f$ . Then the result follows from convexity of  $Y^f$  and  $\hat{Y}^f$  and concavity of  $v^f(q; y', x')$ .

## 6 Bounding the Price Space

Though it is logically possible for any  $q \in R_+^{N(N-1)}$  to be the array of bid prices, this leads to conceptual and mathematical difficulties. A price space as large as  $R_+^{N(N-1)}$  is unbounded, not compact, and hence lacks the fixed point property. In order to avoid these difficulties (which are far from the focus of this study) we will introduce sufficient conditions so that the space of equilibrium prices is necessarily bounded. Then, without loss of generality, we can confine the price space to a bounded set.

This calls for a special assumption.

**P.V** (Backstop Technologies) Let  $i, j$  be integers,  $i \neq j$ ,  $N \geq i, j \geq 1$ .

(a) Let  $\Upsilon^{ijk} \subset R^{2N(N-1)}$ ,  $\Upsilon^{ijk} \equiv \{(y, x) | y_n^{\{n,m\}} = 0, \text{ for } n, m \neq i; x_n^{\{n,m\}} = 0, \text{ for } n, m \neq j; y_i^{\{i,j\}} < 0, x_j^{\{i,j\}} > 0; k|x_j^{\{i,j\}}| = |y_i^{\{i,j\}}|\}$ . For all  $i, j = 1, 2, \dots, N$ ,  $i \neq j$ , there is  $k > 0$  and  $f \in F$  so that  $\Upsilon^{ijk} \subset Y^f$  and  $v^f(q; y', x') = y_i^{\{i,j\}} + x_j^{\{i,j\}}$ .

(b) Let  $\Psi^{ijk} \subset R^{2N(N-1)}$ ,  $\Psi^{ijk} \equiv \{(y, x) | y_n^{\{n,m\}} = 0, \text{ for } n, m \neq i; x_n^{\{n,m\}} = 0, \text{ for } n, m \neq j; y_i^{\{i,j\}} > 0, x_j^{\{i,j\}} < 0, k|y_i^{\{i,j\}}| = |x_j^{\{i,j\}}|\}$ . For all  $i, j = 1, 2, \dots, N$ ,  $i \neq j$ , there is  $k > 0$  and  $f \in F$  so that  $\Psi^{ijk} \subset Y^f$  and  $v^f(q; y', x') = y_j^{\{i,j\}} + x_i^{\{i,j\}}$ .

P.V is simpler than it looks. It asserts the inclusion among firm technologies of cones (rays from the origin) transforming good  $j$  (wholesale) into  $i$  (retail) and vice versa, with firm maximands so that the technology will be used when it is profitable. P.V says that there is a backstop technology for transforming every good purchased wholesale into every good sold retail. Similarly there is a backstop technology for

transforming any good acquired retail into any good delivered wholesale. The back-stop technology may be terribly inefficient — using a conversion ratio of  $k$ , where  $k$  may be very large (or very small) and positive. Nevertheless, under P.V, it follows from simple arbitrage that equilibrium prices  $q_j^{\{i,j\}}$  are bounded above and below by  $k$  and  $(1/k)$ . Then take the maximum,  $K$ , of the values  $k$  and  $1/k$  in P.V(a) and P.V(b). The price space can then be confined to the rectangular prism

$$\mathbf{Q} \subset R^{N(N-1)}, \mathbf{Q} \equiv \{q \in R^{N(N-1)} | K \geq q_i^{\{i,j\}} \geq (1/K)\}$$

$\mathbf{Q}$  is a compact convex subset of  $R_+^{N(N-1)}$ . This is the price space where we can confine the search for an equilibrium price vector, under assumption P.V.

An assumption on the household side, C.II, corresponding to P.V, is introduced below, to assure that  $\mathbf{Q}$  is the largest price space we need.

## 7 Households

There is a finite set of households,  $H$  with typical element  $h$ .

### 7.1 Endowment and Consumption Set

$h \in H$  has a possible consumption set, taken for simplicity to be the nonnegative quadrant of  $R^N$ ,  $R_+^N$ .  $h \in H$  is endowed with  $r^h \gg 0$  assumed to be strictly positive to avoid boundary problems.  $h \in H$  has a share  $\alpha^{hf} \geq 0$  of firm  $f$ , so that  $\sum_{h \in H} \alpha^{hf} = 1$ .

### 7.2 Trades and Payment Constraint

$h \in H$  chooses  $(y^h, x^h) \in R^{2N(N-1)}$  subject to the following restrictions. A household always balances its budget, sells wholesale and buys retail:

- (i)  $0 \geq x_i^{h\{i,j\}}$  for all  $i, j$ .
- (ii)  $y_i^{h\{i,j\}} \geq 0$  for all  $i, j$ .
- (iii)  $(y^h, x^h) \in \mathbf{M}(q)$

### 7.3 Maximand and Demand

Household  $h$ 's share of profits from firm  $f$ ,  $\alpha^{hf} \pi^f(q)$ , is part of  $h$ 's endowment and enters directly into consumption. When the profits of all firms  $\pi^f(q)$  are well defined  $h$ 's consumption of good  $i$  is

$$(iv) c_i^h \equiv r_i^h + [\sum_{f \in F} \alpha^{hf} \pi^f(q)]_i + \sum_{j=1}^N x_i^{h\{i,j\}} + \sum_{j=1}^N y_i^{h\{i,j\}}$$

However, prices  $q$  may be such that  $\pi^f(q)$  is not well defined for some  $f$ . Then we may wish to discuss the constrained version of (iv),

$$(iv') c_i^h \equiv r_i^h + [\sum_{f \in F} \alpha^{hf} \tilde{\pi}^f(q)]_i + \sum_{j=1}^N x_i^{h\{i,j\}} + \sum_{j=1}^N y_i^{h\{i,j\}}$$

In addition,  $h$ 's consumption must be nonnegative.

(v)  $c^h \geq 0$ . The inequality applies co-ordinatewise.

**C.I** For all  $h \in H$ ,  $h$ 's maximand is the continuous, concave, real-valued, strictly monotone, utility function  $u^h(c^h)$ .

**C.II** For all  $h \in H$ , whenever  $c_i^h$  and  $c_j^h$  are  $> 0$ ,  $MRS_{ij}$  is well defined (either as a point value or a range in  $R_+$ ) and  $K > MRS_{ij} > (1/K)$ .

Assumption C.II says that indifference curves do not become very steep or very flat. Hence extremely high price ratios result in zero household net purchase transactions for the high-price good. Since the bounding parameter,  $K$ , is the same one that characterizes bounds on the technology side, the same bounded price space,  $\mathbf{Q}$ , will fully encompass relevant prices.

$h$ 's demand/supply function is defined as  $D^h: \mathbf{Q} \rightarrow R^{2N(N-1)}$ .

$D^h(q) \equiv \{(y^h, x^h) \in R^{2N(N-1)} | (y^h, x^h) \text{ maximizes } u^h(c^h), \text{ subject to (i), (ii), (iii), (iv) and (v)}\}$ . However,  $D^h(q)$  may not be well defined for  $q$  such that  $\pi^f(q)$  is not well defined for some  $f$ . To treat this issue, we define  $\tilde{D}^h: \mathbf{Q} \rightarrow R^{2N(N-1)}$ .

$\tilde{D}^h(q) \equiv \{(y^h, x^h) \in R^{2N(N-1)} | (y^h, x^h) \text{ maximizes } u^h(c^h), \text{ subject to (i), (ii), (iii), (iv')} \text{ and (v)}\}$ .

Lemma 7.1: Assume P.0 - P.IV, C.I, C.II. Then  $\tilde{D}^h(q)$  is nonempty, upper hemi-continuous and convex-valued, for all  $q \in \mathbf{Q}$ . The range of  $\tilde{D}^h(q)$  is compact. For  $q$  such that  $\tilde{\pi}^f(q) = \pi^f(q)$  for all  $f \in F$ ,  $\tilde{D}^h(q) = D^h(q)$ .

## 8 Excess Demand

Let  $q \in \mathbf{Q}$ . Constrained excess demand at  $q$  is defined as

$$\tilde{Z}(q) \equiv \sum_{f \in F} \{(y, x) | (y, x) \in \tilde{S}^f(q)\} + \sum_{h \in H} \{(y, x) | (y, x) \in \tilde{D}^h(q)\}.$$

Lemma 8.1: Assume P.0 - P.IV, C.I and C.II.  $\tilde{Z}: \mathbf{Q} \rightarrow R^{2N(N-1)}$ . The range of  $\tilde{Z}$  is bounded.  $\tilde{Z}$  is upper hemi-continuous and convex-valued for all  $q \in \mathbf{Q}$ .

Let  $\Xi$  denote a closed convex subset of  $R^{2N(N-1)}$  including the range of  $\tilde{Z}$ .

Lemma 8.2 (Walras' Law): Let  $q \in \mathbf{Q}$ . Let  $(y, x) \in \tilde{Z}(q)$ . Then for each  $i, j = 1, \dots, N, i \neq j$ , we have

$$0 = y_j^{\{i,j\}} + q_i^{\{i,j\}} x_i^{\{i,j\}}, \text{ or equivalently, } 0 = [q_j^{\{i,j\}}]^{-1} y_i^{\{i,j\}} + x_j^{\{i,j\}}$$

Proof of Lemma 8.2:  $\tilde{Z}(q)$  is the sum of elements  $(y^f, x^f)$  of  $S^f(q)$  and  $(y^h, x^h)$  of  $D^h(q)$  each of which is subject to **(B)**:  $0 = y_j^{\{i,j\}} + q_i^{\{i,j\}} x_i^{\{i,j\}}$ , or equivalently,  $0 = [q_j^{\{i,j\}}]^{-1} y_i^{\{i,j\}} + x_j^{\{i,j\}}$ . QED

## 9 Equilibrium

Define  $\rho : \Xi \times \mathbf{Q} \rightarrow \mathbf{Q}$

$$\rho(z, q) \equiv \{q^o \in \mathbf{Q} \mid z = (y, x) \in R^{2N(N-1)}. q_i^{o\{i,j\}} = \text{med}[K, q_i^{\{i,j\}} + x_i^{\{i,j\}}, 1/K]\}$$

where "med" denotes median .

Lemma 9.1:  $\rho$  is upper hemi-continuous and convex-valued for all  $(z, q) \in \Xi \times \mathbf{Q}$ .

Define  $\Gamma : \mathbf{Q} \times \Xi \rightarrow \mathbf{Q} \times \Xi$ .  $\Gamma(q, z) \equiv \rho(z, q) \times \tilde{Z}(q)$  .

Lemma 9.2: Assume P.0 - P.IV, C.I and C.II. Then  $\Gamma$  is upper hemi-continuous and convex-valued on  $\mathbf{Q} \times \Xi$  .  $\Gamma$  has a fixed point  $(q^*, z^*)$  and  $0 = z^*$  .

Proof: Upper hemicontinuity and convexity are established in lemmas 8.1 and 9.1. Existence of the fixed point  $(q^*, z^*)$  then follows from the Kakutani fixed point theorem. To demonstrate that  $z^* = 0$  , let  $z^* = (y^*, x^*)$ . We claim  $x^* = 0$ . Use a proof by contradiction. Suppose not. Then (case 1)  $x_i^{*\{i,j\}} > 0$  for some i, j. Then  $q_i^{*\{i,j\}} = K$ . By P.V and C.II it follows that  $x_i^{*\{i,j\}} < 0$ , a contradiction. Alternatively, (case 2)  $x_i^{*\{i,j\}} < 0$  for some i, j. Then  $q_i^{*\{i,j\}} = 1/K$ . By P.V and C.II it follows that  $x_i^{*\{i,j\}} > 0$ , a contradiction. Thus,  $x^* = 0$ . By lemma 8.2 it follows that  $y^* = 0$ . QED

Definition:  $q^* \in \mathbf{Q}$  is said to be an equilibrium if

$$0 \in \sum_{f \in F} S^f(q^*) + \sum_{h \in H} D^h(q^*).$$

**Theorem 9.1:** Assume P.0 - P.V, C.I and C.II. Then there is an equilibrium  $q^* \in \mathbf{Q}$ .

Proof: Lemmas 5.5, 7.1, 9.2.

## 10 Media of Exchange

Firms perform a market-making function, both buying and selling the same good in  $x$  and  $y$  co-ordinates. Hence distinguishing between firms' medium of exchange transactions and directly productive transactions is problematic. However the situation is simpler for households. Let  $(y^h, x^h) \in D^h(q)$  be household  $h$ 's  $2N(N-1)$ -dimensional transaction vector. The  $x$  co-ordinates are typically sales (negative sign) at bid prices; the  $y$  co-ordinates are typically purchases (positive sign) at ask prices. Then we can characterize  $h$ 's gross transactions in good  $i$  as

$$\sum_j y_i^{h\{i,j\}} - \sum_j x_i^{h\{i,j\}} \equiv \gamma_i^h .$$

Further, the absolute value of h's net transactions in good i, is

$$|\sum_j y_i^{h\{i,j\}} + \sum_j x_i^{h\{i,j\}}| \equiv \nu_i^h .$$

The  $N$ -dimensional vector  $\gamma^h$  with typical element  $\gamma_i^h$  is h's gross trade. The  $N$ -dimensional vector  $\nu^h$  with typical element  $\nu_i^h$  is h's net trade vector (in absolute value).  $\mu^h \equiv \gamma^h - \nu^h$  is h's flow of goods as media of exchange. The total flow of media of exchange among households is then  $\sum_{h \in H} \mu^h$

Thus the trading post equilibrium establishes a well-defined household demand for media of exchange as an outcome of the market equilibrium. Media of exchange (commodity monies) are characterized as goods flows acting as the carrier of value between transactions (not fulfilling final demands or input requirements themselves), the difference between gross and net trades.

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