When Worlds Collide: Different Comparative Static Predictions of Continuous and Discrete Agent Models with Land[1]

by

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Abstract

Since the number of agents in the real world is finite, models with an infinite number of agents make no sense unless they are close, in terms of equilibria and comparative statics, to models with a finite number of agents. The scattered literature on such models with land to this point has generally tried, with little success, to take a standard model with a continuum of consumers and associate with it a model with a finite number of consumers so that the equilibria of the two models are similar. Here we take a different approach, and examine a comparative static. We employ closed city models (with a fixed number of consumers and an exogenous central business district) but with an endogenous city boundary. The comparative static that looks at the effect of an increase in (marginal) commuting cost on the average land rent at various distances from the central business district is studied. The canonical result for the model with a continuum of agents can be found in a number of sources, including Fujita's text. It states that the per unit cost of land increases close to the central business district, is constant at some intermediate distance, and decreases for all consumers farther from the central business district. Below we show that for the discrete model with a finite population, the per unit cost of land must increase for all consumers. This is important for both urban economic theory and empirical work. On the theoretical front, it provides evidence that models with a finite number and a continuum of consumers are qualitatively different from each other, so it will be impossible to say that their equilibria are similar in general. On the empirical front, it provides a potentially testable hypothesis to distinguish the two types of theories.

1 Introduction

Models with a continuum of consumers are often employed for reasons of mathematical convenience or simplicity. Moreover, they can make precise the notion of perfect competition. Since the number of agents in the real world is finite, models with an infinite number of agents make no sense unless they are close, in terms of equilibria and comparative statics, to reasonable models with a finite number of agents. The scattered literature on such models with land to this point has generally tried, with little success, to take a standard model with a continuum of consumers and associate with it a model with a finite number of consumers so that the equilibria of the two models are similar. See Berliant (1985, 1991), Asami *et al* (1991), Kamecke (1993), Papageorgiou and Pines (1990) and Berliant and ten Raa (1991). The intuition is that any partition of a σ-finite measure space, such as a Euclidean space, can have only countably many elements of positive measure. So except for a negligible set of consumers out of a continuum, all must consume or even be endowed with a set of measure zero. A corollary is that economies with a finite number of consumers approximating these continuum economies must have land consumption tending to zero almost surely.

Here we take a different approach, and examine a comparative static. We employ closed city models (with a fixed number of consumers and an exogenous central business district) but with an endogenous city boundary. The comparative static that looks at the effect of an increase in (marginal) commuting cost on the average land rent at various distances from the central business district is studied. The canonical result for the model with a continuum of agents can be found in a number of sources, including Fujita's text. Fujita (1989, p. 81, Proposition 3.14 part (iii)) states that with an increase in the (marginal) cost of commuting, the per unit cost of land increases close to the central business district, is constant at some intermediate distance, and decreases for all consumers farther from the central business district. Below we show that for the discrete model with a finite population, the per unit cost of land must increase for all consumers.

This is important for both urban economic theory and empirical work. On the theoretical front, it provides evidence that models with a finite number and a continuum of consumers are qualitatively different from each other, so it will be impossible to say that their equilibria are similar in general. On the empirical front, it provides a potentially testable hypothesis to distinguish the two types of theories.

Recent literature on city formation, for example Lucas and Rossi-Hansberg (2002), or the new economic geography, for example Fujita and Thisse (2002), generally employ a continuum of consumers and ordinarily have land as a commodity at least implicitly. We have postulated in our work an exogenously given CBD. In most models of city formation, the CBD or location of firms is endogenous, and there is an agglomeration externality used to determine these locations. However, these models all have embedded in them a model of consumer location and commuting, making our analysis relevant. For example, conditional on the spatial distribution of firms, one might want to consider the consumer location problem.

In the next section we introduce the notation and present the comparative static in the case of quasi-linear utility for the model with a finite number of consumers. This is essentially the model of Berliant and Fujita (1991) but with an endogenous city boundary that is determined using an exogenous agricultural rent. The last section presents our conclusions.

2 Model and Comparative Statics

Land is modeled in a long, narrow city with land density 1 available at all non-negative distances from the Central Business District (CBD or 0). Let m be Lebesgue measure on the reals. Consider the quasi-linear model with endogenous city size.

Suppose there are $N \geq 2$ agents, and two goods – land, and a freely mobile composite good. A bundle $(x_n, s_n, z_n) \in \mathbb{R}^3_+$ specifies the consumption of land and composite commodity of agent n, where x_n is distance from the CBD to the front of the parcel of agent n, s_n is size of parcel, and therefore, land consumption is represented by the interval

$[x_n, x_n+s_n)$, while composite commodity consumption is z_n. Each consumer's utility is given by $U(s,z) = v(s) + z$, with $v' > 0$ and $v'' < 0$. Moreover, the marginal rate of substitution $MRS = v'(s) \equiv MU(s)$. Each consumer is endowed with the same $\omega \in \mathbb{R}_{++}$ units of the composite commodity, but no land.

Agricultural rent is given by $\xi > 0$. Cost of commuting is $t > 0$ per unit distance to the CBD. As usual, composite good is taken to be numéraire. The price of land is an integrable price density $p : [0, \sum_{n=1}^{N} s_n) \to \mathbb{R}_+$.

An absentee landlord is endowed with all land but has utility equal to composite good consumption less rent paid for agricultural land, which we call $z_{N+1} \in \mathbb{R}_+$.

The definition of equilibrium is the standard one; see, for example Berliant and Fujita (1992). An *allocation* is a vector of intervals and of consumption bundles $\{([x_n, x_n+s_n), z_n - x_n t)_{n=1}^{N}, z_{N+1}\}$, where for all $n = 1, ..., N$, $z_n \geq x_n t$, and $z_{N+1} \geq 0$. An allocation $\{([x_n, x_n + s_n), z_n - x_n t)_{n=1}^{N}, z_{N+1}\}$ is called *feasible* if $[x_n, x_n + s_n)_{n=1}^{N}$ partition $[0, \sum_{n=1}^{N} s_n)$ (formally, $\cup_{n=1}^{N}[x_n, x_n+s_n) = [0, \sum_{n=1}^{N} s_n)$ and for all $i \neq j$, $1 \leq i, j \leq N$, $[x_i, x_i+s_i) \cap [x_j, x_j+s_j) = \varnothing$), $v'(s_N) = \xi$, and $\sum_{n=1}^{N+1} z_i = N\omega$. A feasible allocation $\{([x_n, x_n + s_n), z_n - x_n t)_{n=1}^{N}, z_{N+1}\}$ and an integrable price density $p : [0, \sum_{n=1}^{N} s_n) \to \mathbb{R}_+$ constitute an *equilibrium* if for each consumer $n = 1, ..., N$, $\int_{x_n}^{x_n+s_n} p(x)dm(x) + z_n \leq \omega_n$, (that is, belongs to the budget set,) $U(s'_n, z'_n - x'_n t) > U(s_n, z_n - x_n t) \Rightarrow \int_{x'_n}^{x'_n+s'_n} p(x)dm(x) + z'_n > \omega_i$, (that is, it is optimal,) and for the landlord, $z_{N+1} = \int_0^{\sum_{n=1}^{N} s_n} p(x)dm(x) - \sum_{n=1}^{N} s_n \cdot \xi$, by definition of the landlord's consumption.

2.1 Equilibrium Parcels and Their Comparative Statics

As usual, first order conditions imply that in equilibrium, for $n = 1, \ldots, N-1$, $MRS_n = MRS_{n+1} + t$, and for $n = N$, $MRS_N = \xi$, where agent N is farthest from the CBD. Therefore,

$$\begin{aligned} MU(s_n) &= MU(s_{n+1}) + t, && \text{for } n = 1, \ldots, N-1, \quad \text{and} \\ MU(s_N) &= \xi && \text{for } n = N. \end{aligned}$$

In particular, $MU(s_N) = \xi$ implies that $MU(s_{N-1}) = \xi + t$, and proceeding inductively, $MU(s_{N-k}) = \xi + kt$, for $k = 0, \ldots, N-1$. Changing index yields

$$MU(s_n) = \xi + (N-n)t \quad \text{for} \quad n = 1, \ldots, N.$$

This further implies that

$$s_n = MU^{-1}(\xi + (N-n)t) \quad \text{for} \quad n = 1, \ldots, N,$$

and correspondingly,

$$\frac{\partial s_n}{\partial t} = \frac{N-n}{v''(\xi + (N-n)t)} \quad \begin{cases} < 0 & n = 1, \ldots, N-1, \\ = 0 & n = N. \end{cases}$$

2.2 Equilibrium Prices

Consistent with Berliant and Fujita (1992), the equilibrium price density is as follows.

$$p(s) = \begin{cases} MU(s_1) & \text{on } [0, s_1] \\ MU(s - \sum_{k=1}^{n-1} s_k) & \text{on } \left[\sum_{k=1}^{n} s_k, \sum_{k=1}^{n-1} s_k + s_{n+1}\right] \quad n = 1, \ldots, N-1, \\ MU(s_{n+1}) & \text{on } \left[\sum_{k=1}^{n-1} s_k + s_{n+1}, \sum_{k=1}^{n+1} s_k\right] \quad n = 1, \ldots, N-1. \\ & \text{Note. For } n = 1, \sum_{k=1}^{n-1} s_k \equiv 0. \end{cases}$$

Notice that $\left[\sum_{k=1}^{n} s_k, \sum_{k=1}^{n+1} s_k\right]$ is the parcel for agent $n+1$, and it can be decomposed into $\left[\sum_{k=1}^{n} s_k, \sum_{k=1}^{n-1} s_k + s_{n+1}\right] \cup \left[\sum_{k=1}^{n-1} s_k + s_{n+1}, \sum_{k=1}^{n+1} s_k\right]$. Thus, at the front end of the parcel of agent $n+1$, price decreases with the MRS of agent n, and at the back end, price is constant at the MRS of agent $n+1$. For example, for agent 1, price is constant at $MU(s_1)$, for agent 2, price is $MU(s)$ on $[s_1, s_2]$, and $MU(s_2)$ on $[s_2, s_1 + s_2]$, for agent 3, price is $MU(s - s_1)$

on $[s_1+s_2, s_1+s_3]$, and $MU(s_3)$ on $[s_1+s_3, s_1+s_2+s_3]$, and so on.

2.3 Comparative Statics of Rent per Unit Parcel

Notice that for agent 1, $rent_1 = MU(s_1)s_1$, and for $n = 1, \ldots, N-1$,

$$\begin{aligned}
\text{rent}_{n+1} &= \int_{\sum_{k=1}^{n} s_k}^{\sum_{k=1}^{n-1} s_k + s_{n+1}} MU(s - \textstyle\sum_{k=1}^{n-1} s_k) dm(s) + \int_{\sum_{k=1}^{n-1} s_k + s_{n+1}}^{\sum_{k=1}^{n+1} s_k} MU(s_{n+1}) dm(s) \\
&= v(s_{n+1}) - v(s_n) + MU(s_{n+1})s_n \\
&= v(s_{n+1}) - v(s_n) + (\xi + (N-n-1)t)s_n.
\end{aligned}$$

Consequently, for $n = 1, \ldots, N-1$,

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\text{rent}_{n+1}}{s_{n+1}}\right) &= \frac{1}{s_{n+1}^2} s_{n+1}\left[MU(s_{n+1})\frac{\partial s_{n+1}}{\partial t} - MU(s_n)\frac{\partial s_n}{\partial t} + (\xi + (N-n-1)t)\frac{\partial s_n}{\partial t}\right] \\
&\quad + \frac{1}{s_{n+1}^2} s_{n+1}(N-n-1)s_n \\
&\quad - \frac{1}{s_{n+1}^2}\left[v(s_{n+1}) - v(s_n) + MU(s_{n+1})s_n\right]\frac{\partial s_{n+1}}{\partial t} \\
&= \frac{1}{s_{n+1}^2} s_{n+1}\left[MU(s_{n+1})\frac{\partial s_{n+1}}{\partial t} - (\xi + (N-n)t)\frac{\partial s_n}{\partial t} + (\xi + (N-n-1)t)\frac{\partial s_n}{\partial t}\right] \\
&\quad + \frac{1}{s_{n+1}^2} s_{n+1}(N-n-1)s_n \\
&\quad - \frac{1}{s_{n+1}^2}\left[v(s_{n+1}) - v(s_n) + MU(s_{n+1})s_n\right]\frac{\partial s_{n+1}}{\partial t} \\
&= \frac{1}{s_{n+1}^2}\left[(N-n-1)s_n s_{n+1} - t s_{n+1}\frac{\partial s_n}{\partial t}\right] \\
&\quad - \frac{1}{s_{n+1}^2}\left[v(s_{n+1}) - v(s_n) + MU(s_{n+1})(s_n - s_{n+1})\right]\frac{\partial s_{n+1}}{\partial t} \\
&> 0.
\end{aligned}$$

For the last inequality, notice that $t s_{n+1}\frac{\partial s_n}{\partial t} < 0$, by the comparative statics for parcel size, $v(s_{n+1}) - v(s_n) + MU(s_{n+1})(s_n - s_{n+1}) > 0$, by the concavity of v, and combined with comparative statics for parcel size, $(v(s_{n+1}) - v(s_n) + MU(s_{n+1})(s_n - s_{n+1}))\frac{\partial s_{n+1}}{\partial t} < 0$ for $n = 1, \ldots, N-2$, and this expression equals 0 for $n = N-1$. This shows that for

$n = 1, \ldots, N-1$, $\frac{\partial}{\partial t}\left(\frac{\text{rent}_{n+1}}{s_{n+1}}\right) > 0$. Moreover,

$$\frac{\partial}{\partial t}\left(\frac{\text{rent}_1}{s_1}\right) = v''(s_1)\frac{\partial s_1}{\partial t} > 0.$$

Therefore, rent per unit parcel increases with t for all agents, in contrast with the result for the model with a continuum of agents.

3 Conclusions

We have examined a comparative static in closed city models with an endogenous city boundary both with a continuum and a finite number of consumers, and we have found a difference. For researchers in urban economic theory, the implication is that there are qualitative differences between the models. For empiricists, the possibility of testing the models against one another is real.

Although the quasi-linear utility case is sufficient to make our point that the comparative statics in the model with a finite number of consumers and the model with a continuum of consumers can differ,[2] we are working on proving the general comparative static for the finite model (along the same lines as the quasi-linear case). If we can prove the general result, it must be backed up by a theorem on existence of equilibrium for the finite model with an endogenous city boundary.[3] We conjecture that we can do this using the existence theorem in Berliant and Fujita (1991). Assuming all the assumptions of that paper, we have an existence theorem for any fixed length of city. We also have the exogenous agricultural land rent. Then it's just a question of adjusting the MRS of the last consumer to equal the agricultural land rent by varying the length of the city. So we just have to show that equilibrium MRS of the last consumer is continuous in the length of the city and assume some boundary conditions, and the intermediate value theorem should yield the theorem on

[2]It is easy to verify that the result in Fujita (1989) applies to quasi-linear utility.

[3]Notice that existence of equilibrium in the case of quasi-linear utility is not an issue, since we find the equilibrium explicitly, and thus we have proved that it exists.

existence of equilibrium.

Which model, finite or continuum of agents, will be verified empirically? Probably this depends on the context. One obvious way to test the models is to look at the per unit cost of land parcels in a city, say Chicago, before and after a change in commuting cost, say the introduction of a new "el" line. Dan McMillen's work in this area should be useful, but one must account for firm relocation.

It is unclear if the difference in the comparative static presented extends to other comparative statics as well.

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