

A note on the convexity and compactness of the integral of a Banach space valued correspondence*

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Abstract

We characterize the class of finite measure spaces (T, \mathcal{T}, μ) which guarantee that for a correspondence ϕ from (T, \mathcal{T}, μ) to a general Banach space the Bochner integral of ϕ is convex. In addition, it is shown that if ϕ has weakly compact values and is integrably bounded, then, for this class of measure spaces, the Bochner integral of ϕ is weakly compact, too. Analogous results are provided with regard to the Gelfand integral of correspondences taking values in the dual of a separable Banach space, with “weakly compact” replaced by “weak*-compact.” The crucial condition on the measure space (T, \mathcal{T}, μ) concerns its measure algebra and is consistent with having $T = [0, 1]$ and μ to be an extension of Lebesgue measure.

1 Introduction

Recall that a correspondence from a set A to a set B is a set-valued mapping, i.e. a mapping from A to 2^B . Classical results due to Richter (1963) and Aumann (1965) state that if ϕ is a correspondence from an atomless measure space to \mathbb{R}^ℓ , then the integral of ϕ (defined as the set of integrals of all integrable selections) is convex and, if ϕ is integrably bounded¹ and has closed values, also compact. These results are of substantial importance in economic theory and have found numerous applications (see e.g. Aumann (1966) or Hildenbrand (1974)).

There are question in economic theory for which an infinite dimensional version of the results of Richter (1963) and Aumann (1965) would be appropriate; e.g., question regarding the existence of competitive equilibria in models with an atomless measure space of agents and an infinite dimensional commodity space, or questions regarding the existence of pure strategy Nash equilibria in games with an atomless measure space of players and an infinite action set (see e.g. Rustichini and Yannelis, 1991).

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¹See Section 2.3 for the definition.

Unfortunately, the results of Richter (1963) and Aumann (1965) about the integral of an \mathbb{R}^ℓ -valued correspondence over an atomless measure space do not generalize to correspondences with infinite dimensional range spaces. These results are based on Liapounoff's theorem on the convexity of the range of an \mathbb{R}^ℓ -valued atomless vector measure, and this theorem does not carry over to infinite dimensional spaces. In fact, there is an example in Yannelis (1991) of a correspondence from a closed interval in \mathbb{R} endowed with Lebesgue measure to the Banach space ℓ_2 such that the Bochner integral of this correspondence (i.e. the set of Bochner integrals of all Bochner integrable selections) is neither convex nor weakly compact, not even norm-closed, even though the correspondence is integrably bounded and has norm-compact values.

This example shows, in particular, that non-atomicity is not a very strong property of a measure space in the context of integration of correspondences with infinite dimensional range spaces.

In this note we present a strengthening of non-atomicity—which we call “super-atomless,” see Section 3 for the actual definition²—that will be shown to be both a necessary and sufficient condition on a finite measure space in order that it is guaranteed that the Bochner integral of a correspondence to a general Banach space is convex and, if the correspondence has weakly compact values and is integrably bounded, also weakly compact. The class of super-atomless finite measure spaces will be shown to also be the class of finite measure spaces for which analogous facts—with “weakly compact” replaced by “weak*-compact”—are guaranteed to hold for the Gelfand integral of a correspondence to the dual of a separable Banach space.

Previous results in this context have been established by Rustichini and Yannelis (1991) and Sun (1997) (see (c) and (e) of the remark following the statement of Theorem 2 below). The results of our note provide, in particular, a simultaneous generalization of the corresponding ones of Rustichini and Yannelis (1991) and Sun (1997). Also, our results will show that those of Sun (1997), where the domain of a correspondence is assumed to be an atomless Loeb measure space, do not depend on any specific properties of a Loeb measure space but only on the properties of its measure algebra.

We want to remark here that if a measure space (T, \mathcal{T}, μ) satisfies our condition of being super-atomless, then this does not mean that this measure space must be extraordinarily large in terms of the cardinality of $L^\infty(\mu)$ or that of T . In fact, we shall show, using a construction due to Fremlin (2005), that Lebesgue measure on $[0, 1]$ can be extended to a super-atomless measure on $[0, 1]$. Thus, in regard to economic applications, if a measure space of agents in an economic model is assumed to be super-atomless, then this is consistent with having, as in Aumann (1964, 1966), a set of agents not larger than the continuum. (See also (g) of the remarks following the statement of Theorem 2 below.)

²The name “super-atomless” was suggested to me by Erik Balder.

The rest of the paper is organized as follows. In Section 2 some notation and terminology are introduced. In Section 3 the definition of “super-atomless” is given and some equivalent properties are stated. Section 4 contains the results and proofs concerning the Bochner integrable setting, and Section 5 those concerning the Gelfand integrable setting. In Section 6 it is shown that Lebesgue measure on $[0, 1]$ can be extended to a super-atomless measure.

2 Notation and terminology

2.1 Measure spaces (a) The term “measure space” always means that the measure in question is non-negative. A measure space (T, \mathcal{T}, μ) is called *finite* if $\mu(T) < \infty$; it is called *non-trivial* if $\mu(T) \neq 0$.

(b) Let (T, \mathcal{T}, μ) be a measure space.

- For a subset $A \subset T$,
 - 1_A denotes the characteristic function of A ;
 - \mathcal{T}_A denotes the trace σ -algebra on A , i.e. $\mathcal{T}_A \equiv \{A \cap E : E \in \mathcal{T}\}$;
 - μ_A denotes the subspace measure on A , i.e. the measure whose domain is \mathcal{T}_A and whose value at $F \in \mathcal{T}_A$ is given by $\mu_A(F) = \mu^*(F)$ where μ^* is the outer measure induced by μ . Of course, if $A \in \mathcal{T}$, then $\mu_A(F) = \mu(F)$ for every $F \in \mathcal{T}_A$.
- $L^1(\mu)$ denotes the Banach space of all (equivalence classes of) μ -integrable functions from T into \mathbb{R} , endowed with its usual norm $\|\cdot\|_1$.
- $L^\infty(\mu)$ denotes the Banach space of all (equivalence classes of) μ -essentially bounded functions from T into \mathbb{R} , endowed with its usual norm $\|\cdot\|_\infty$.
- For $E \in \mathcal{T}$,
 - $L_E^1(\mu)$ denotes the subspace of $L^1(\mu)$ consisting of the elements of $L^1(\mu)$ vanishing off E ;
 - $L_E^\infty(\mu)$ denotes the subspace of $L^\infty(\mu)$ consisting of the elements of $L^\infty(\mu)$ vanishing off E .

2.2 The measure algebra of a measure space (a) Let (T, \mathcal{T}, μ) be a measure space and let $\mathcal{N}(\mu)$ denote the σ -ideal of null sets in T . The *measure algebra* of (T, \mathcal{T}, μ) (or, for short, of μ) is the pair $(\mathfrak{A}, \hat{\mu})$ given as follows:

- \mathfrak{A} is the quotient Boolean algebra $\mathcal{T}/(\mathcal{N}(\mu) \cap \mathcal{T})$. That is, denoting by \sim the equivalence relation on \mathcal{T} given by $E \sim F$ if $E \Delta F \in \mathcal{N}(\mu)$, \mathfrak{A} is the set of equivalence classes in \mathcal{T} for \sim , considered with operations \cap^\bullet , \cup^\bullet , \setminus^\bullet , and Δ^\bullet , and a partial ordering \subset^\bullet that are given in the following way: If $E^\bullet, F^\bullet \in \mathfrak{A}$ and E, F are any elements of \mathcal{T} determining E^\bullet and F^\bullet , respectively, then $E^\bullet \subset^\bullet F^\bullet$ if and only if $E \setminus F \in \mathcal{N}(\mu)$, $E^\bullet \cap^\bullet F^\bullet = (E \cap F)^\bullet$, and analogously for $\cup^\bullet, \setminus^\bullet$, and Δ^\bullet .
- $\hat{\mu}: \mathfrak{A} \rightarrow [0, \infty]$ is the functional given by $\hat{\mu}(E^\bullet) = \mu(E)$ where E is any element of \mathcal{T} determining E^\bullet .

- (b) Let (T, \mathcal{T}, μ) be a measure space and $(\mathfrak{A}, \hat{\mu})$ its measure algebra.
- A *subalgebra* of \mathfrak{A} is a non-empty subset of \mathfrak{A} that is closed under \cup^\bullet and \setminus^\bullet (and thus also under \cap^\bullet and Δ^\bullet).
 - A subalgebra \mathfrak{B} of \mathfrak{A} is *order-closed* if, with respect to \subset^\bullet , any non-empty upwards directed subset of \mathfrak{B} has its supremum in \mathfrak{B} in case the supremum is defined in \mathfrak{A} .
 - A subset $A \subset \mathfrak{A}$ is said to *completely generate* \mathfrak{A} if the smallest order closed subalgebra in \mathfrak{A} containing A is \mathfrak{A} itself.
 - The *Maharam type* of μ (or of (T, \mathcal{T}, μ)) is the least cardinal number of any subset $A \subset \mathfrak{A}$ which completely generates \mathfrak{A} .
 - μ is said to be *Maharam homogeneous* if for each $E \in \mathcal{T}$ with $\mu(E) > 0$ the Maharam type of the subspace measure μ_E is equal to the Maharam type of μ .

(c) Note that a measure space (T, \mathcal{T}, μ) is atomless if and only if for each $E \in \mathcal{T}$ with $\mu(E) > 0$ the Maharam type of the subspace measure μ_E is infinite. Indeed, for $E \in \mathcal{T}$, write $(\mathfrak{A}_E, \hat{\mu}_E)$ for the measure algebra of μ_E , and note that if μ is atomless then \mathfrak{A}_E must be an infinite set for each $E \in \mathcal{T}$ with $\mu(E) > 0$. Given $E \in \mathcal{T}$, note that if B is a finite subset of \mathfrak{A}_E , then the subalgebra in \mathfrak{A}_E generated by B is a finite set, and that a finite subalgebra of \mathfrak{A}_E is automatically order closed in \mathfrak{A}_E . Thus if the Maharam type of μ_E is finite, so is \mathfrak{A}_E . Hence if $\mu(E) > 0$ and the Maharam type of μ_E is finite, then the measure μ cannot be atomless. For the other direction, note that if E is an atom of μ then \mathfrak{A}_E contains no proper subalgebra, so the Maharam type of μ_E is zero.

2.3 Miscellany (a) When we speak of a correspondence, say ϕ from a set A to a set B , i.e. a mapping from A to 2^B , it is always understood that $\phi(a)$ is non-empty for every $a \in A$.

(b) Let (T, \mathcal{T}, μ) be a measure space, X a Banach space, and $\phi: T \rightarrow 2^X$ a correspondence.

- ϕ is said to be *integrably bounded* if there is an integrable function $\rho: T \rightarrow \mathbb{R}_+$ such that, denoting by B the unit ball of X , $\phi(t) \subset \rho(t)B$ for almost all $t \in T$.
- G_ϕ denotes the graph of a ϕ , i.e. the set $\{(t, x) \in T \times X: x \in \phi(t)\}$.

(c) If X is a Banach space, then X^* denotes the dual space of X . For a subset A of X or X^* , $\overline{\text{co}} A$ denotes the closed convex hull of A , and for subset A of X^* , $\overline{\text{co}}^* A$ denotes the weak*-closed convex hull of A .

(d) $\mathcal{B}(Z)$ denotes the Borel σ -algebra of a topological space Z .

3 A strengthening of the non-atomicity condition

As noted in 2(c) of the previous section, a measure space (T, \mathcal{T}, μ) is atomless if and only if for each $E \in \mathcal{T}$ with $\mu(E) > 0$ the Maharam type of the subspace measure μ_E is infinite. Thus a natural way to strengthen the condition that a measure space (T, \mathcal{T}, μ) be atomless is to require that for each $E \in \mathcal{T}$ with $\mu(E) > 0$ the Maharam type of the subspace measure μ_E be uncountable.

We will call a measure with this latter property super-atomless.

Definition. Let (T, \mathcal{T}, μ) be a measure space. The measure μ (or the measure space (T, \mathcal{T}, μ)) is said to be *super-atomless* if for each $E \in \mathcal{T}$ with $\mu(E) > 0$ the Maharam type of the subspace measure μ_E is uncountable.

See the remark following the statement of Theorem 2 for examples.

For sake of illustration and later reference we will collect some properties of a finite measure space that are equivalent to being super-atomless. To this end, recall that it is a consequence of Maharam's theorem that if (T, \mathcal{T}, μ) is a non-trivial atomless finite measure space, then there is a finite or countable infinite set of distinct infinite cardinal numbers $\{\kappa_i: i \in I\}$ and a corresponding partition of T into measurable subsets $\{T_i: i \in I\}$ such that for each $i \in I$ the subspace measure μ_{T_i} is Maharam homogeneous with Maharam type κ_i ; moreover, the measure algebra of μ_{T_i} is, up to renormalization of the measure, isomorphic to the measure algebra of the usual measure on $\{0, 1\}^{\kappa_i}$, i.e. the product measure on $\{0, 1\}^{\kappa_i}$ when $\{0, 1\}$ is endowed with the coin flipping measure. Following Jin and Keisler (2000), we will call the set $\{\kappa_i: i \in I\}$ the Maharam spectrum of μ . For a trivial measure space (T, \mathcal{T}, μ) , i.e. when μ is the zero measure, we define the Maharam spectrum of μ to be the singleton $\{0\}$.

The following fact holds.

Fact. Let (T, \mathcal{T}, μ) be a finite measure space. Then the following are equivalent.

- (i) *The measure μ is super-atomless.*
- (ii) *μ is atomless and the Maharam spectrum of μ does not contain \aleph_0 .*
- (iii) *For every $E \in \mathcal{T}$ with $\mu(E) > 0$, the cardinality of any set $\mathcal{G} \subset \mathcal{T}_E$ such that for any $\epsilon > 0$ and any $F \in \mathcal{T}_E$ there is a $G \in \mathcal{G}$ with $\mu(F \triangle G) < \epsilon$ is uncountable.*
- (iv) *For every $E \in \mathcal{T}$ with $\mu(E) > 0$, $L_E^1(\mu)$ is non-separable.*

Proof. (i) \Rightarrow (ii): Evidently not(ii) \Rightarrow not(i) holds.

(ii) \Rightarrow (i): Note first that if $A, B \in \mathcal{T}$ and $A \subset B$, then the Maharam type of B is at least as large as the Maharam type of A (see Fremlin, 2002, 331H(c)). Now pick any $E \in \mathcal{T}$ and suppose $\mu(E) > 0$. Let $\{T_i: i \in I\}$ be a partition of T , chosen according to the definition of the Maharam spectrum. Then $\mu(E \cap T_i) > 0$ for some i , and since the subspace measure μ_{T_i} is Maharam homogeneous, (ii) \Rightarrow (i) follows by what has been noted above.

(i) \Leftrightarrow (iii): See Fremlin (2002, 323A(d)) together with Fremlin (2002, 331Y(e)) or together with Fremlin (2005, 524D), and apply the facts stated there to the measure algebras of the subspaces $(E, \mathcal{T}_E, \mu_E)$ for $E \in \mathcal{T}$.

(iii) \Rightarrow (iv): This follows because for each $E \in \mathcal{T}$ the subspace measure μ_E is a finite measure by hypothesis, so the set of (equivalence classes of) characteristic functions of measurable subsets of E belongs to $L_E^1(\mu)$.

(iv) \Rightarrow (iii): This follows because for each $E \in \mathcal{T}$ the set of elements of $L_E^1(\mu)$ that are (equivalence classes of) characteristic functions of measurable subsets of E has a dense linear span in $L_E^1(\mu)$. \square

4 The Bochner integrable setting

Notation. In this section, if (T, \mathcal{T}, μ) is a measure space, X a Banach space, and $f: T \rightarrow X$ a Bochner integrable function, then for any $E \in \mathcal{T}$, $\int_E f d\mu$ denotes the Bochner integral of f over E , and if $\phi: T \rightarrow 2^X$ is a correspondence, $\int_T \phi d\mu$ denotes the Bochner integral of ϕ , i.e. the set

$$\left\{ \int_T f d\mu : f \text{ is a Bochner integrable (almost everywhere) selection of } \phi \right\}.$$

Our results for the Bochner integrable setting are as follows.

Theorem 1. *Let (T, \mathcal{T}, μ) be a finite measure space and X an infinite dimensional Banach space (which need not be separable). Then the following are equivalent.*

- (i) $\int_T \phi d\mu$ is convex for every correspondence $\phi: T \rightarrow 2^X$.
- (ii) The measure μ is super-atomless.

Theorem 2. *Let (T, \mathcal{T}, μ) be a finite measure space and X an infinite dimensional Banach space (which need not be separable). Then the following are equivalent.*

- (i) $\int_T \phi d\mu$ is convex and weakly compact for every correspondence $\phi: T \rightarrow 2^X$ which has weakly compact values and is integrably bounded.
- (ii) The measure μ is super-atomless.

Some remarks are in order before the proofs will be presented.

Remarks. (a) The condition in Theorems 1 and 2 that the Banach space X be infinite dimensional is in order for (i) \Rightarrow (ii) to be true. If X is finite dimensional then, as is well known, (i) holds if and only if (T, \mathcal{T}, μ) is atomless.

(b) Concerning (ii) \Rightarrow (i) of Theorems 1 and 2, note that it is not assumed that a correspondence has a measurable graph. On the other hand, in the counterexample that will establish not(ii) \Rightarrow not(i), the correspondence will have this property. Actually, when X is separable and ϕ is integrably bounded and has a measurable graph (for the μ -completion of (T, \mathcal{T}, μ)) then, by a standard fact, $\int_T \phi d\mu$ is non-empty, regardless of whether (ii) holds or not.

(c) Theorems 1 and 2 generalize results of Sun (1997) which state that (i) of these theorems holds if (T, \mathcal{T}, μ) is an atomless finite Loeb measure space (see also (d), (f), and (g) below). In particular, Theorems 1 and 2 imply (by the equivalence of (i) and (ii)) that an atomless finite Loeb measure space is super-atomless.

These theorems also show, by the definition of “super-atomless,” that the corresponding results of Sun (1997) depend not on any special properties of a Loeb measure space but only on the properties of its measure algebra. Of course, the fact that an atomless finite Loeb measure space is super-atomless can be deduced without appealing to our theorems. E.g. in Jin and Keisler (2000) it is shown that the Maharam spectrum of an atomless finite Loeb measure space does not contain \aleph_0 , which, by the Fact stated in Section 3, means that such a measure space is super-atomless in our terminology.

(d) Denote by $\text{card} S$ the cardinality of a set S , and by \mathfrak{c} the cardinality of the continuum. There are finite measure spaces (T, \mathcal{T}, μ) such that μ is super-atomless but such that $\text{card} L^\infty(\mu) = \mathfrak{c}$ (and hence also $\text{card} L^1(\mu) = \mathfrak{c}$ because $L^\infty(\mu)$ is norm-dense in $L^1(\mu)$.) One example is provided by the fact that an atomless Loeb probability space (T, \mathcal{T}, μ) can be constructed so that $\text{card} L^\infty(\mu) = \mathfrak{c}$. For another example, let μ be the usual measure on $\{0, 1\}^\mathfrak{c}$, i.e. the product measure on $\{0, 1\}^\mathfrak{c}$ each of whose factors is the coin flipping measure. Then it is easily seen that $L_E^1(\mu)$ is non-separable for each μ -measurable subset E of $\{0, 1\}^\mathfrak{c}$ with $\mu(E) > 0$. That is, μ is super-atomless by the Fact stated in Section 3; on the other hand, $\text{card} L^\infty(\mu) = \mathfrak{c}$. (To see this latter property, recall that if $((T_i, \mathcal{T}_i, \mu_i))_{i \in I}$ is a family of probability spaces and μ the product measure on $\prod_{i \in I} T_i$, then each element of $L^\infty(\mu)$ has a version depending on only countably many coordinates.) A similar example is provided by letting μ be the product measure on $[0, 1]^\mathfrak{c}$ when $[0, 1]$ is endowed with Lebesgue measure. C.f. Rosenthal (1970, Remark following the proof of Theorem 3.5), where it is shown that if κ is any infinite cardinal and μ the product measure on $[0, 1]^\kappa$ corresponding to Lebesgue measure on $[0, 1]$, then $\text{card} L^\infty(\mu) = \kappa^{\aleph_0}$.

(e) In Rustichini and Yannelis (1991), convexity of the Bochner integral of a correspondence from a finite measure space (T, \mathcal{T}, μ) to an infinite dimensional Banach space X has been shown under the condition that for each $E \in \mathcal{T}$ with $\mu(E) > 0$, the algebraic dimension of $L_E^\infty(\mu)$ (i.e. cardinality of a Hamel basis) be larger than the algebraic dimension of X . Under the continuum hypothesis, this condition requires (because an infinite dimensional Banach space cannot have a countable Hamel basis) that for any $E \in \mathcal{T}$ with $\mu(E) > 0$, $\text{card} L_E^\infty(\mu) > \mathfrak{c}$. Theorem 1 together with the previous remark shows that such a requirement is stronger than necessary. Also, Theorem 1 shows that a dimensional condition on the range space X is not necessary.³

(f) There are non-trivial finite measure spaces (T, \mathcal{T}, μ) such that μ is super-atomless but $\text{card} T = \mathfrak{c}$. In fact, based on arguments due to Fremlin (2002, 2005), we show in Section 6 that Lebesgue measure on the unit interval $[0, 1]$ can

³That no dimensional, or cardinality, or density condition on X appears in Theorems 1 and 2 (except for the assumption on X to be infinite dimensional, which is needed only for (i) \Rightarrow (ii)) reflects the fact that Bochner integrable functions are essentially separably valued. In our result below for the Gelfand integrable setting, we require the range space of a correspondence to be the dual of a separable Banach space.

be “enriched” so that the resulting measure is super-atomless. More precisely, we show there is a σ -algebra \mathcal{T} on $[0, 1]$ and a super-atomless measure μ with domain \mathcal{T} such that, writing Λ for the domain of Lebesgue measure on $[0, 1]$, \mathcal{T} includes Λ and μ agrees on Λ with Lebesgue measure. In addition, one can have $\text{card } L^\infty(\mu) = \mathfrak{c}$ for such an extension of Lebesgue measure.

(g) For a non-trivial atomless Loeb measure space (T, \mathcal{T}, μ) it is also possible to have $\text{card } T = \mathfrak{c}$, so that T can be identified with $[0, 1]$. However, under any such identification, the domain of an atomless Loeb measure cannot include the Borel σ -algebra of $[0, 1]$ (see Keisler and Sun, 2002). In fact, as shown in Keisler and Sun (2002), if (T, \mathcal{T}, μ) is any atomless Loeb measure space, Z any Polish space, $f: T \rightarrow Z$ any \mathcal{T} - $\mathcal{B}(Z)$ -measurable mapping, and ν denotes the image measure of μ under f , then for ν -almost every $z \in Z$ the inverse image $f^{-1}(\{z\})$ has cardinality $\geq \mathfrak{c}$. In an economic model where (T, \mathcal{T}, μ) is the space of agents, Z is a space of agents’ types, and f assigns to each agent her type, the specification of (T, \mathcal{T}, μ) to be an atomless Loeb measure space therefore implies that there are “continuum many” agents of almost every type. Of course, such an implication need not hold for a measure space that is an extension of Lebesgue measure. Thus an atomless Loeb measure space is maybe not the most flexible choice of an atomless measure space for specifying a large number of agents in economic models.

The proofs of Theorems 1 and 2 are split into a series of lemmata.

Lemma 1. *Let (T, \mathcal{T}, μ) be a finite measure space, let X be a separable Banach space, and let $R: L^\infty(\mu) \rightarrow X$ be a linear operator which is continuous for the weak*-topology of $L^\infty(\mu)$ and the weak topology of X . Let $E \in \mathcal{T}$ and suppose $L_E^1(\mu)$ is non-separable. Then R is not one-to-one on $L_E^\infty(\mu)$.*

Proof. We will show the contrapositive. Thus suppose R is one-to-one on $L_E^\infty(\mu)$. We have to show that $L_E^1(\mu)$ is separable. Note that $L^\infty(\mu) \subset L^1(\mu)$ by finiteness of the measure μ . Let B denote the closed unit ball in $L^\infty(\mu)$. Thus B is a weak*-compact subset of $L^\infty(\mu)$. Set $B_E = B \cap L_E^\infty(\mu)$ and note that $L_E^\infty(\mu)$ is a weak*-closed subspace of $L^\infty(\mu)$. Thus B_E is weak*-compact, and it follows that, on B_E , the weak*-topology of $L^\infty(\mu)$ coincides with the weak topology of $L^1(\mu)$ (because $L^\infty(\mu) \subset L^1(\mu)$ and hence, on B_E , the weak topology of $L^1(\mu)$ is formally weaker than the weak*-topology of $L^\infty(\mu)$). By the supposed properties of the operator R , then, the restriction of R to B_E is a homeomorphism between B_E and $R(B_E)$ for the (restricted) weak topologies of $L^1(\mu)$ and X . By the hypothesis on X , $R(B_E)$ is norm-separable and hence weakly separable. Consequently, B_E is a weakly separable subset of $L^1(\mu)$. Since B_E is convex, it follows that B_E is actually a norm-separable subset of $L^1(\mu)$ (see below). But $\bigcup_{n=1}^\infty nB_E$ is norm-dense in $L_E^1(\mu)$, and we may conclude that $L_E^1(\mu)$ is separable.

(To see that a weakly separable and convex subset C of a Banach space is norm-separable, let D be a countable weakly dense subset of such a set C . Let

D' be the set of all (finite) convex combinations of elements of D such that the coefficients are rational. Then D' is countable, $D' \subset C$ since C is convex, and D' is dense in $\text{co} D'$, the convex hull of D' , for any linear topology. By the Hahn-Banach theorem, the weak closure of $\text{co} D'$ agrees with the norm closure of $\text{co} D'$, and it follows that D' is norm dense in C . \square

Lemma 2. *Let X be a Banach space, let (T, \mathcal{T}, μ) be a finite measure space, and let $f: T \rightarrow X$ be a Bochner integrable function. Suppose μ is super-atomless. Then the set $\{\int_E f d\mu: E \in \mathcal{T}\}$ is a convex subset of X .*

*Proof.*⁴ We may assume that X is separable. Indeed, since f is Bochner integrable, f is essentially separably valued. By modifying f on a null set if necessary, we may assume that f is separably valued. Note that the set $\{\int_E f d\mu: E \in \mathcal{T}\}$ belongs to the closed linear span of the set $\{f(t): t \in T\}$. Thus we may as well assume that X is separable.

Let

$$A = \{g \in L^\infty(\mu): 0 \leq g \leq 1_T\} \quad \text{and} \quad B = \left\{ \int_T g f d\mu: g \in A \right\}.$$

Then A is convex and hence so is B . Also $\{\int_E f d\mu: E \in \mathcal{T}\} \subset B$. Thus we have to show that the reverse inclusion holds, i.e. that given any $g \in A$ there is an $E \in \mathcal{T}$ such that $\int_T g f d\mu = \int_E f d\mu$.

To this end, let $R: L^\infty(\mu) \rightarrow X$ be the linear operator defined by

$$R(g) = \int_T g f d\mu, \quad g \in L^\infty(\mu),$$

and note that R is continuous for the weak*-topology of $L^\infty(\mu)$ and the weak topology of X . (Indeed, if (g_α) is a net in $L^\infty(\mu)$, weak*-converging to some $g \in L^\infty(\mu)$, then for each $x^* \in X^*$,

$$x^* \int_T g_\alpha f d\mu = \int_T g_\alpha x^* f d\mu \rightarrow \int_T g x^* f d\mu = x^* \int_T g f d\mu$$

because $x^* f \in L^1(\mu)$.) In terms of the operator R , we have to show that given any $g \in A$ we have $R(g) = R(1_E)$ for some $E \in \mathcal{T}$. Thus let $g \in A$ be given and let

$$H = \{h \in A: Rg = Rh\}.$$

Then, by the mentioned continuity properties of R , since A is a convex and weak*-compact subset of $L^\infty(\mu)$, so is H . Hence the set H has an extreme point, say \bar{h} . We claim there is an $E \in \mathcal{T}$ such that $\bar{h} = 1_E$. Suppose, if possible, otherwise. Then we can find an $\epsilon > 0$ and an $E \in \mathcal{T}$ with $\mu(E) > 0$ such that $\epsilon 1_E \leq 1_E \bar{h} \leq (1 - \epsilon) 1_E$. Now by the Fact stated in Section 3, the hypothesis about the measure space (T, \mathcal{T}, μ) implies that $L_E^1(\mu)$ is non-separable. Hence

⁴The proof uses arguments from the proofs of Lemma IX.1.3 and Theorem IX.1.4 in Diestel and Uhl (1977, pp. 263), specialized to the setting considered here.

by Lemma 1, since X is separable, we can find an $h_1 \in L_E^\infty(\mu)$ such that $Rh_1 = 0$ but $\|h_1\|_\infty > 0$. We may assume that $\|h_1\|_\infty < \epsilon$. Then $\bar{h} + h_1 \in A$ as well as $\bar{h} - h_1 \in A$ (note: h_1 vanishes off E according to the definition of $L_E^\infty(\mu)$) and we have both $R(\bar{h} + h_1) = Rg$ and $R(\bar{h} - h_1) = Rg$. That is, $\bar{h} + h_1 \in H$ as well as $\bar{h} - h_1 \in H$. But since $\|h_1\|_\infty > 0$, this contradicts the property of \bar{h} being an extreme point of H . Thus for some $E \in \mathcal{T}$, $\bar{h} = 1_E$ whence $R1_E = R\bar{h} = Rg$. \square

Lemma 3. *Let X be a Banach space and let (T, \mathcal{T}, μ) be a finite measure space. Suppose that μ is super-atomless. Then for any two Bochner integrable functions $f_1: T \rightarrow X$ and $f_2: T \rightarrow X$, and any real number $0 < \alpha < 1$, there is a set $E \in \mathcal{T}$ such that*

$$\int_E f_1 d\mu + \int_{T \setminus E} f_2 d\mu = \alpha \int_T f_1 d\mu + (1 - \alpha) \int_T f_2 d\mu .$$

Proof. Given any such functions f_1 and f_2 , set $f = f_1 - f_2$. Fix any real number $0 < \alpha < 1$. By Lemma 2, there is a set $E \in \mathcal{T}$ such that $\int_E f d\mu = \alpha \int_T f d\mu$. (Note that $0 = \int_\emptyset f d\mu \in \{\int_E f d\mu: E \in \mathcal{T}\}$.) Then

$$\begin{aligned} \int_E f_1 d\mu + \int_{T \setminus E} f_2 d\mu &= \int_E f d\mu + \int_T f_2 d\mu \\ &= \alpha \int_T f_1 d\mu - \alpha \int_T f_2 d\mu + \int_T f_2 d\mu \\ &= \alpha \int_T f_1 d\mu + (1 - \alpha) \int_T f_2 d\mu . \end{aligned} \quad \square$$

Lemma 4. *Let (T, \mathcal{T}, μ) be a finite measure space and X an infinite dimensional Banach space. If for some $E \in \mathcal{T}$, $\mu(E) > 0$ and $L_E^1(\mu)$ is separable, then there exists a Bochner integrable function $f: T \rightarrow X$ such that the set $\{\int_F f d\mu: F \in \mathcal{T}\}$ is non-convex.*

Proof. Our construction follows that in the proof of Corollary IX.1.6 in Diestel and Uhl (1977, p. 267). Fix an $E \in \mathcal{T}$. Suppose that $\mu(E) > 0$ and that $L_E^1(\mu)$ is separable. Then we can select a countable family $(h_i)_{i=1,2,\dots}$ in $L_E^1(\mu)$ which separates the points of $L_E^1(\mu)$ and such that $\|h_i\|_1 = 1$ for each i . Also, since X is infinite dimensional, we can select a sequence $(x_i, x_i^*)_{i=1,2,\dots}$ in $X \times X^*$ such that $x_i^*(x_i) = 1$ for each i and $x_i^*(x_j) = 0$ whenever $i \neq j$. Let a function $f: T \rightarrow X$ be defined by

$$f(t) = \begin{cases} \sum_{i=1}^{\infty} h_i(t) \frac{1}{\|x_i\|} \frac{1}{2^i} x_i & \text{if } t \in E \\ 0 & \text{if } t \in T \setminus E. \end{cases}$$

It is readily seen that f is Bochner integrable with $\int_T f d\mu = \int_E f d\mu \neq 0$. Fix any real number $0 < \alpha < 1$. We claim that $\alpha \int_T f d\mu \neq \int_F f d\mu$ for all $F \in \mathcal{T}$. Arguing by contradiction, suppose that for some $F \in \mathcal{T}$ we have $\alpha \int_T f d\mu = \int_F f d\mu$, or, in other words, $\int_T (\alpha 1_E - 1_F) f d\mu = 0$. Then for each $i = 1, 2, \dots$,

$$0 = x_i^* \int_T (\alpha 1_E - 1_F) f d\mu = \frac{1}{\|x_i\|} \frac{1}{2^i} \int_E (\alpha 1_E - 1_F) h_i d\mu .$$

By choice of the functions h_1, h_2, \dots , it follows that $\alpha 1_E - 1_{F \cap E} = 0$, which is impossible since $0 < \alpha < 1$ and $\mu(E) > 0$. Thus $\alpha \int_T f d\mu \neq \int_F f d\mu$ for all $F \in \mathcal{T}$, showing that the set $\{\int_F f d\mu: F \in \mathcal{T}\}$ is non-convex. \square

The proof of Theorem 1 is now a straightforward matter.

Proof of Theorem 1. (ii) \Rightarrow (i) is immediate from Lemma 3.

not(ii) \Rightarrow not(i): First note that if $h: T \rightarrow X$ is any Bochner integrable function then, according to a standard fact, h is measurable for the Borel σ -algebra of X and the μ -completion \mathcal{T}_μ of \mathcal{T} ; in particular, the set $\{t \in T: h(t) = 0\}$ is an element of \mathcal{T}_μ .

Now suppose that the measure μ is not super-atomless. Then by the Fact stated in Section 3, for some $E \in \mathcal{T}$ with $\mu(E) > 0$, $L_E^1(\mu)$ is separable. Let $f: T \rightarrow X$ be a Bochner integrable function chosen according to Lemma 4. Thus the set $\{\int_F f d\mu: F \in \mathcal{T}\}$ is non-convex. Let $\phi: T \rightarrow 2^X$ be the correspondence given by $\phi(t) = \{f(t)\} \cup \{0\}$, $t \in T$. Then, from what has been noted above, a function $h: T \rightarrow X$ is a Bochner integrable selection of ϕ if and only if for some $F \in \mathcal{T}$, $h(t) = 1_F(t)f(t)$ for almost all $t \in T$. Thus $\int_T \phi d\mu = \{\int_F f d\mu: F \in \mathcal{T}\}$ and we conclude that $\int_T \phi d\mu$ is non-convex. \square

Remark. Since a Bochner integrable function is essentially separably valued, it can be assumed for the function f in this latter proof that for some closed separable subspace Y of X , $f(t) \in Y$ for all $t \in T$ (by modifying f on a null set if necessary). Then the correspondence ϕ constructed in this proof has actually a measurable graph for the μ -completion of \mathcal{T} and the Borel σ -algebra of X . Indeed, separability of Y implies that we can select a countable subset of Y^* separating the points of Y . Extending each element of such a subset of Y^* to an element of X^* , we get a countable subset $\{d_1, d_2, \dots\}$ of X^* separating the points of Y . Then since $f(t) \in Y$ for all $t \in T$,

$$G_\phi = (T \times \{0\}) \cup \left((T \times Y) \cap \bigcap_{i=1}^{\infty} \{(t, x) \in T \times X: d_i x = d_i f(t)\} \right)$$

whence $G_\phi \in \mathcal{T}_\mu \otimes \mathcal{B}(X)$.

For the proof of Theorem 2 it is convenient to first provide two additional lemmata.

Lemma 5. *Let (T, \mathcal{T}, μ) be a finite measure space, let X be a Banach space, and let $h: T \rightarrow X$ be a Bochner integrable function. Suppose μ is super-atomless, and that $\int_T \|h(t)\| d\mu(t) > 0$, where $\|\cdot\|$ denotes the norm of X . Then there is a $g \in L^\infty(\mu)$ such that $\int_T gh d\mu = 0$ but $\int_T \|g(t)h(t)\| d\mu(t) > 0$.*

Proof. By Lemma 2 and the hypothesis on the measure μ , there is an $F \in \mathcal{T}$ such that $\int_F h d\mu = \frac{1}{2} \int_T h d\mu$. Set $g = \frac{1}{2} 1_T - 1_F$. Then

$$\int_T gh d\mu = \frac{1}{2} \int_T h d\mu - \int_F h d\mu = 0,$$

and since $|g(t)| = \frac{1}{2}$ for all $t \in T$ and $\int_T \|h(t)\| d\mu(t) > 0$, we must have $\int_T \|g(t)h(t)\| d\mu(t) > 0$. \square

Lemma 6. *Let (T, \mathcal{T}, μ) be a finite measure space, Z a locally convex Suslin space, and $\xi: T \rightarrow 2^Z$ a correspondence such that $\xi(t)$ is convex for each $t \in T$ and such that $G_{\xi} \in \mathcal{T} \otimes \mathcal{B}(Z)$. Let \bar{h} be a \mathcal{T} - $\mathcal{B}(Z)$ -measurable selection of ξ and suppose that the set $\{t \in T: \bar{h}(t) \text{ is not an extreme point of } \xi(t)\}$ is not a null set in T . Then there are \mathcal{T} - $\mathcal{B}(Z)$ -measurable selections h_1 and h_2 of ξ such that for some $C \in \mathcal{T}$ with $\mu(C) > 0$, $h_1(t) \neq h_2(t)$ for all $t \in C$, but $\frac{1}{2}h_1 + \frac{1}{2}h_2 = \bar{h}$.*

Proof. See Castaing and Valadier (1977, Theorem IV.14, p. 108). \square

Proof of Theorem 2. (ii) \Rightarrow (i): Let $L^1(\mu, X)$ be the Banach space of (equivalence classes) of Bochner integrable functions from (T, \mathcal{T}, μ) into X , endowed with its usual norm $\|f\|_1 = \int_T \|f(t)\| d\mu(t)$, where $\|\cdot\|$ is the norm of X . Let $\phi: T \rightarrow 2^X$ be an integrably bounded correspondence with weakly compact values, and let

$$S_{\phi}^1 = \{f \in L^1(\mu, X): f \text{ is a selection of } \phi\}.$$
⁵

Since ϕ has weakly compact values and is integrably bounded, it follows from Diestel, Ruess, and Schachermayer (1993, Corollary 2.6) that S_{ϕ}^1 is weakly relatively compact in $L^1(\mu, X)$. Hence $\int_T \phi d\mu$ is weakly relatively compact in X (since the operator $f \mapsto \int_T f d\mu$ from $L^1(\mu, X)$ to X is continuous for the weak topologies of these spaces). By Theorem 1, and the hypothesis about (T, \mathcal{T}, μ) , $\int_T \phi d\mu$ is convex, and it thus remains to show that $\int_T \phi d\mu$ is a weakly closed subset of X .

Since $\int_T \phi d\mu$ is weakly relatively compact, and because a Banach space is angelic in its weak topology, it is actually enough to show that $\int_T \phi d\mu$ is weakly sequentially closed. Thus let (x_n) be a sequence in $\int_T \phi d\mu$, weakly converging to some $\bar{x} \in X$.⁶ We have to show that $\bar{x} \in \int_T \phi d\mu$. (Actually, since $\int_T \phi d\mu$ is convex, it would be sufficient to consider a norm-convergent sequence.)

Select a sequence (f_n) in S_{ϕ}^1 such that $x_n = \int_T f_n d\mu$ for each n . Recall that weak relative compactness and weak relative sequential compactness are the same in a Banach space. Thus we may assume that the sequence f_n converges weakly in $L^1(\mu, X)$ to some \bar{f} . In particular, then, $\bar{x} = \int_T \bar{f} d\mu$.

Identify each f_n with one of its versions. Arguing similarly as in the first paragraph of the proof of Lemma 2, we may assume that there is a closed separable subspace Y of X such that $f_n(t) \in Y$ for all n and all $t \in T$. Let a correspondence $\psi: T \rightarrow 2^Y$ be defined by

$$\psi(t) = \text{weak-cl}\{f_n(t): n = 1, 2, \dots\}, t \in T,$$

(where “cl” means “closure”). Then by choice of the functions f_n , we have $\psi(t) \subset \phi(t)$ for almost all $t \in T$. In particular, for almost all $t \in T$, $\psi(t)$ is

⁵More precisely, a μ -equivalence class of a selection.

⁶We can assume here that $\int_T \phi d\mu$ is non-empty, for otherwise there is nothing to be proved.

a weakly compact subset of X , therefore also a weakly compact subset of Y . Modifying the functions f_n on a null set if necessary, we can assume that $\psi(t)$ is a weakly compact subset of Y for all $t \in T$.

Consider the correspondence $\overline{\text{co}}\psi: T \rightarrow 2^Y$ (defined as usual pointwise by $(\overline{\text{co}}\psi)(t) = \overline{\text{co}}\psi(t)$). Observe the following facts about $\overline{\text{co}}\psi$. First, by the Krein-Smulian theorem, $\overline{\text{co}}\psi$ has weakly compact values since ψ does. Second, since $\psi(t) \subset \phi(t)$ for almost all $t \in T$ and ϕ is integrably bounded, so is ψ , and hence so is $\overline{\text{co}}\psi$.

We may assume in the sequel that the measure space (T, \mathcal{T}, μ) is complete. Then $G_{\overline{\text{co}}\psi}$, the graph of $\overline{\text{co}}\psi$, belongs to the product σ -algebra $\mathcal{T} \otimes \mathcal{B}(Y)$. Indeed, it is plain that for each $\gamma^* \in Y^*$ the function $\gamma \mapsto \sup \gamma^* \overline{\text{co}}\psi(t)$ is the pointwise supremum of the measurable functions $\gamma^* f_n$, $n = 1, 2, \dots$. Thus the function $t \mapsto \sup \gamma^* \overline{\text{co}}\psi(t)$ is measurable for each $\gamma^* \in Y^*$. According to Castaing and Valadier (1977, p.84, Theorem III.37 and the remark at the end of p. 73), this implies that $G_{\overline{\text{co}}\psi} \in \mathcal{T} \otimes \mathcal{B}(Y)$, since the Banach space Y is separable, since $\overline{\text{co}}\psi$ has weakly compact values, and since (T, \mathcal{T}, μ) is complete.

Now let $L^1(\mu, Y)$ be the Banach space of Bochner integrable functions from (T, \mathcal{T}, μ) into Y and let

$$S_{\overline{\text{co}}\psi}^1 = \{f \in L^1(\mu, Y) : f \text{ is an (almost everywhere) selection of } \overline{\text{co}}\psi\}.$$

Since $\overline{\text{co}}\psi$ is integrably bounded and has weakly compact values, using Diestel et al. (1993, Corollary 2.6) again, it follows that $S_{\overline{\text{co}}\psi}^1$ is weakly relatively compact in $L^1(\mu, Y)$. Evidently $S_{\overline{\text{co}}\psi}^1$ is a norm-closed subset of $L^1(\mu, Y)$. (To see this, use the fact that if a sequence in $L^1(\mu, Y)$ norm-converges to some element h of this space, then some subsequence converges to h pointwise almost everywhere in the norm of Y .) It is also evident that $S_{\overline{\text{co}}\psi}^1$ is convex. Consequently $S_{\overline{\text{co}}\psi}^1$ is a weakly closed subset of $L^1(\mu, Y)$, and it follows that $S_{\overline{\text{co}}\psi}^1$ is in fact weakly compact in $L^1(\mu, Y)$.

Now $L^1(\mu, Y)$ is naturally identifiable as a closed subspace of $L^1(\mu, X)$, and since $f_n \in L^1(\mu, Y)$ for each n by construction and $f_n \rightarrow \bar{f}$ weakly in $L^1(\mu, X)$, it follows that $\bar{f} \in L^1(\mu, Y)$. In particular, $f_n \rightarrow \bar{f}$ weakly in $L^1(\mu, Y)$. Also by construction, $f_n \in S_{\overline{\text{co}}\psi}^1$ for each n , and since $S_{\overline{\text{co}}\psi}^1$ is weakly closed in $L^1(\mu, Y)$ it follows that $\bar{f} \in S_{\overline{\text{co}}\psi}^1$. Let

$$H = \left\{ h \in S_{\overline{\text{co}}\psi}^1 : \int_T \bar{f} d\mu = \int_T h d\mu \right\}.$$

Evidently H is convex. In addition, H is weakly compact in $L^1(\mu, Y)$ since $S_{\overline{\text{co}}\psi}^1$ is and since the operator $h \mapsto \int_T h d\mu$ from $L^1(\mu, Y)$ to Y is continuous for the weak topologies of these spaces. Consequently H has an extreme point, say \bar{h} . We claim that \bar{h} is an extreme point of $S_{\overline{\text{co}}\psi}^1$, too. For suppose otherwise. Then there is an $h_1 \in L^1(\mu, Y)$ with $h_1 \neq 0$ such that $\bar{h} + h_1 \in S_{\overline{\text{co}}\psi}^1$ as well as $\bar{h} - h_1 \in S_{\overline{\text{co}}\psi}^1$. By Lemma 5, there is a $g \in L^\infty(\mu)$ such that $\int_T g h_1 d\mu = 0$ but

$\|gh_1\|_1 > 0$. In particular, then, $\int_T \bar{h} + gh_1 d\mu = \int_T \bar{h} - gh_1 d\mu = \int_T \bar{h} d\mu$. Clearly we may assume that $\|g\|_\infty \leq 1$. Then $\bar{h} + gh_1 \in S_{\bar{c}\bar{o}\psi}^1$ as well as $\bar{h} - gh_1 \in S_{\bar{c}\bar{o}\psi}^1$.⁷ It follows that both $\bar{h} + gh_1 \in H$ and $\bar{h} - gh_1 \in H$. But this is impossible because \bar{h} was chosen to be an extreme point of H and $gh_1 \neq 0$. Thus \bar{h} must be an extreme point of $S_{\bar{c}\bar{o}\psi}^1$ as claimed.

Recall that Y is a separable Banach space and that we have assumed (T, \mathcal{T}, μ) to be complete. Thus a function $h: T \rightarrow Y$ is strongly measurable if and only if it is \mathcal{T} - $\mathcal{B}(Y)$ -measurable. In particular, since the correspondence $\bar{c}\bar{o}\psi: T \rightarrow 2^Y$ is integrably bounded, every \mathcal{T} - $\mathcal{B}(Y)$ -measurable selection of $\bar{c}\bar{o}\psi$ is Bochner integrable. Hence, in view of Lemma 6, since $G_{\bar{c}\bar{o}\psi} \in \mathcal{T} \otimes \mathcal{B}(Y)$, the fact that \bar{h} is an extreme point of $S_{\bar{c}\bar{o}\psi}^1$ implies that for almost all $t \in T$, $\bar{h}(t)$ is an extreme point of $\bar{c}\bar{o}\psi(t)$. Now as noted above, $\psi(t)$ is a weakly compact subset of Y , so all extreme points of $\bar{c}\bar{o}\psi(t)$ are elements of $\psi(t)$. Consequently $\bar{h}(t) \in \psi(t)$ for almost all $t \in T$. By construction, $\psi(t) \subset \phi(t)$ for almost all $t \in T$, and it follows that $\bar{h} \in S_\phi^1$. Also by construction, $\int_T \bar{h} d\mu = \int_T \bar{f} d\mu = \bar{x}$, and thus $\bar{x} \in \int_T \phi d\mu$ as was to be shown. This completes the proof of (ii) \Rightarrow (i).

not(ii) \Rightarrow not(i): See the proof of Theorem 1 not(ii) \Rightarrow not(i). The correspondence constructed there evidently has weakly compact values (in fact norm-compact values) and is integrably bounded, this latter property following from the fact that if $f: T \rightarrow X$ is Bochner integrable, then $\int_T \|f(t)\| d\mu(t) < \infty$. (Actually, the correspondence in the proof of Theorem 1 not(ii) \Rightarrow not(i) can be constructed so as to have a measurable graph for the completion of (T, \mathcal{T}, μ) ; see the remark following that proof.) \square

5 The Gelfand integrable setting

In this section we will treat the Gelfand integral of correspondences taking values in a dual Banach space. Accordingly, from now on some change of notation will be in force.

Notation. If X is a Banach space and f a Gelfand integrable mapping from a finite measure space (T, \mathcal{T}, μ) to X^* , then $\int_E f d\mu$ means the Gelfand integral of f over $E \in \mathcal{T}$. Similarly, if $\phi: T \rightarrow 2^{X^*}$ is a correspondence, $\int_T \phi d\mu$ means the Gelfand integral of ϕ , i.e. the set

$$\left\{ \int_T f d\mu : f \text{ is a Gelfand integrable (almost everywhere) selection of } \phi \right\}.$$

Given a finite measure space (T, \mathcal{T}, μ) and a Banach space X , recall that a mapping $f: T \rightarrow X^*$ is Gelfand integrable if and only if $xf \in L^1(\mu)$ for all $x \in X$. Note also that if $f: T \rightarrow X^*$ is Gelfand integrable then so is the function gf for

⁷Note that if $\bar{h}(t) \in \bar{c}\bar{o}\psi(t)$ and both $\bar{h}(t) + h_1(t) \in \bar{c}\bar{o}\psi(t)$ and $\bar{h}(t) - h_1(t) \in \bar{c}\bar{o}\psi(t)$, then for any number α with $0 \leq |\alpha| \leq 1$ (and not just for α with $0 \leq \alpha \leq 1$), $\bar{h}(t) + \alpha h_1(t) \in \bar{c}\bar{o}\psi(t)$ as well as $\bar{h}(t) - \alpha h_1(t) \in \bar{c}\bar{o}\psi(t)$.

each $g \in L^\infty(\mu)$; in particular $g \mapsto \int_T g f d\mu$ is a well defined linear operator from $L^\infty(\mu)$ to X^* .

In the sequel, (X^*, weak^*) means X^* in its weak^* -topology. Here are the results for the Gelfand integrable setting.

Theorem 3. *Let (T, \mathcal{T}, μ) be a finite measure space and X a separable infinite dimensional Banach space. Then the following are equivalent.*

- (i) $\int_T \phi d\mu$ is convex for every correspondence $\phi: T \rightarrow 2^{X^*}$.
- (ii) The measure μ is super-atomless.

Theorem 4. *Let (T, \mathcal{T}, μ) be a finite measure space and X a separable infinite dimensional Banach space. Then the following are equivalent.*

- (i) $\int_T \phi d\mu$ is convex and weak^* -compact for every correspondence $\phi: T \rightarrow 2^{X^*}$ which has weak^* -compact values, is integrably bounded, and has the property that $G_\phi \in \mathcal{T}_\mu \otimes \mathcal{B}(X^*, \text{weak}^*)$, where \mathcal{T}_μ denotes the μ -completion of \mathcal{T} .
- (ii) The measure μ is super-atomless.

The proofs will appear after the following remark.

Remark. (a) The conditions required for ϕ in (i) of Theorem 4 actually guarantee that $\int_T \phi d\mu$ is non-empty. In fact, since X is separable, (X^*, weak^*) is a Suslin space (see the proof of that theorem below). Therefore, if the graph of ϕ belongs to $\mathcal{T}_\mu \otimes \mathcal{B}(X^*, \text{weak}^*)$, then ϕ has a \mathcal{T}_μ - $\mathcal{B}(X^*, \text{weak}^*)$ -measurable selection (see Castaing and Valadier, 1977, Theorem III.22, p. 74). Evidently such a selection is Gelfand integrable if ϕ is integrably bounded.

(b) Theorems 3 and 4 generalize corresponding results of Sun (1997); cf. the remark following the statement of Theorem 2 above.

Preparing the proofs of Theorems 3 and 4, we state two lemmata.

Lemma 7. *Let (T, \mathcal{T}, μ) be a finite measure space, let X be a separable Banach space, and let $R: L^\infty(\mu) \rightarrow X^*$ be a linear operator which is continuous for the weak^* -topology of $L^\infty(\mu)$ and the weak^* -topology of X^* . Let $E \in \mathcal{T}$ and suppose $L_E^1(\mu)$ is non-separable. Then R is not one-to-one on $L_E^\infty(\mu)$.*

Proof. Recall that a weak^* -compact subset of the dual of a separable Banach space is weak^* -metrizable, and consequently weak^* -separable. With this fact, the proof goes as that of Lemma 1, but now working with the weak^* topology of X^* (instead with the weak topology of X). \square

Lemma 8. *Let X be a separable Banach space, (T, \mathcal{T}, μ) a finite measure space, and $f: T \rightarrow X^*$ a Gelfand integrable function. Suppose μ is super-atomless. Then the set $\{\int_E f d\mu: E \in \mathcal{T}\}$ is a convex subset of X^* .*

Proof. The proof of Lemma 2 applies, modulo the following modifications.

- Skip the first paragraph of that proof.
- The integral that matters is now understood as the Gelfand integral.
- The operator R given by $Rg = \int_T gf \, d\mu$ is now an operator taking values in X^* ; substitute X by X^* and x by x^* in its discussion in the proof of Lemma 2 to see that it is continuous for the weak*-topology of $L^\infty(\mu)$ and the weak*-topology of X^* .
- Replace the appeal to Lemma 1 by an appeal to Lemma 7. □

Proof of Theorem 3. (ii) \Rightarrow (i): If (ii) holds, then by Lemma 8 and an argument analogous to that of the proof of Lemma 3, given any two Gelfand integrable functions f_1 and f_2 from T to X^* , and any real number $0 < \alpha < 1$, there is an $E \in \mathcal{T}$ such that $\alpha \int_T f_1 \, d\mu + (1 - \alpha) \int_T f_2 \, d\mu = \int_E f_1 \, d\mu + \int_{T \setminus E} f_2 \, d\mu$. From this, (i) follows.

not(ii) \Rightarrow not(i): First note the following two facts. (a) If a function $h: T \rightarrow X^*$ is Bochner integrable, then it is also Gelfand integrable and for any $E \in \mathcal{T}$, the Gelfand integral of h over E coincides with the Bochner integral of h over E . (b) If a function $h: T \rightarrow X^*$ is Gelfand integrable then, since X is separable, i.e. contains a countable set separating the points of X^* , the set $\{t \in T: h(t) = 0\}$ belongs to the μ -completion of \mathcal{T} .

Now suppose that the measure space (T, \mathcal{T}, μ) is not super-atomless, or, equivalently by the Fact stated in Section 3, that there is an $E \in \mathcal{T}$ with $\mu(E) > 0$ such that $L_E^1(\mu)$ is separable. Following the proof of Theorem 1 not(ii) \Rightarrow not(i), let $f: T \rightarrow X^*$ be a Bochner integrable function chosen according to Lemma 4, and let $\phi: T \rightarrow 2^{X^*}$ be the correspondence given by $\phi(t) = \{f(t)\} \cup \{0\}$, $t \in T$. Then by the proof of Theorem 1 not(ii) \Rightarrow not(i), the Bochner integral of ϕ is non-convex. By (a) of the previous paragraph, the Bochner integral of ϕ is included in the Gelfand integral of ϕ . Using (b), we see that if h is any Gelfand integrable selection of ϕ , then for some $F \in \mathcal{T}$, $h(t) = 1_F(t)f(t)$ for almost all $t \in T$, and thus h is actually Bochner integrable; in particular, again from (a), the Gelfand integral of h coincides with the Bochner integral of h . Hence the Gelfand integral of ϕ is included in the Bochner integral of ϕ . Thus both these integrals of ϕ coincide, and we conclude that the Gelfand integral of ϕ is non-convex. □

The next lemma prepares the proof of Theorem 4. It is the analogue of Lemma 5 for the Gelfand integrable setting now under discussion.

Lemma 9. *Let (T, \mathcal{T}, μ) be a finite measure, let X be a separable Banach space, and let $h: T \rightarrow X^*$ be a Gelfand integrable function. Suppose μ is super-atomless, and that h is not equal to zero almost everywhere, i.e. the set $\{t \in T: h(t) \neq 0\}$ is not a null set. Then there exists a $g \in L^\infty(\mu)$ such that $\int_T gh \, d\mu = 0$ but gh is not equal to zero almost everywhere.*

Proof. By Lemma 8 and the hypothesis on the measure space (T, \mathcal{T}, μ) , there is

an $F \in \mathcal{T}$ such that $\int_F h d\mu = \frac{1}{2} \int_T h d\mu$. Set $g = \frac{1}{2}1_T - 1_F$. Then

$$\int_T gh d\mu = \frac{1}{2} \int_T h d\mu - \int_F h d\mu = 0,$$

and since $|g(t)| = \frac{1}{2}$ for all $t \in T$ and, by hypothesis, the set $\{t \in T: h(t) \neq 0\}$ is not a null-set, the set $\{t \in T: g(t)h(t) \neq 0\}$ is not a null-set either. \square

Proof of Theorem 4. (ii) \Rightarrow (i): Note first that since X is separable by hypothesis, (X^*, weak^*) is a Suslin space. Indeed, since X is separable, the closed unit ball of X^* is weak*-metrizable in addition to being weak*-compact. Thus (X^*, weak^*) is the countable union of subsets that are Suslin in their subspace topology. According to Schwartz (1973, p. 96, Theorem 3), this implies that (X^*, weak^*) , being a Hausdorff space, is Suslin.

Now let $\phi: T \rightarrow 2^{X^*}$ by a correspondence satisfying the properties listed in (i) of the theorem. We may assume in the following that the measure space (T, \mathcal{T}, μ) is complete. Then, in particular, $G_\phi \in \mathcal{T} \otimes \mathcal{B}(X^*, \text{weak}^*)$. Consider the correspondence $\overline{\text{co}}^*\phi: T \rightarrow 2^{X^*}$ (defined in the usual way pointwise by $(\overline{\text{co}}^*\phi)(t) = \overline{\text{co}}^*\phi(t)$). To collect some facts about this latter correspondence that are needed in the sequel, observe first that $\overline{\text{co}}^*\phi$ has weak*-compact values and is integrably bounded since ϕ has these properties. Now for each $x \in X$, define functions $\overline{\pi}_x: T \rightarrow \mathbb{R}$ and $\underline{\pi}_x: T \rightarrow \mathbb{R}$ by

$$\overline{\pi}_x(t) = \sup x \overline{\text{co}}^*\phi(t) \quad \text{and} \quad \underline{\pi}_x(t) = \inf x \overline{\text{co}}^*\phi(t), \quad t \in T.$$

Evidently we have $\overline{\pi}_x(t) = \sup x \phi(t)$ and $\underline{\pi}_x(t) = \inf x \phi(t)$ for each $x \in X$ and each $t \in T$. Hence, because $G_\phi \in \mathcal{T} \otimes \mathcal{B}(X^*, \text{weak}^*)$ and (X^*, weak^*) is a Suslin space, the functions $\overline{\pi}_x$ and $\underline{\pi}_x$ are \mathcal{T} -measurable for each $x \in X$ (see Castaing and Valadier, 1977, Lemma III.39, p.86, and recall that (T, \mathcal{T}, μ) is assumed to be complete). In particular, then, since the correspondence $\overline{\text{co}}^*\phi$ is integrably bounded, the functions $\overline{\pi}_x$ and $\underline{\pi}_x$, $x \in X$, are integrable as may readily be seen. Thus these functions determine elements of $L^1(\mu)$. Since the correspondence $\overline{\text{co}}^*\phi$ has weak*-compact values and again since (X^*, weak^*) is a Suslin space, the fact that for each $x \in X$ the function $\overline{\pi}_x$ is \mathcal{T} -measurable also implies that $G_{\overline{\text{co}}^*\phi} \in \mathcal{T} \otimes \mathcal{B}(X^*, \text{weak}^*)$ (see Castaing and Valadier, 1977, Theorem III.37, p. 84).

Now let $L^{1,*}(\mu, X^*)$ denote the vector space of all μ -equivalence classes of Gelfand integrable functions $f: T \rightarrow X^*$. Further, with $L^\infty(\mu) \otimes X$ denoting the algebraic tensor product of $L^\infty(\mu)$ and X , let $L^\infty(\mu) \hat{\otimes} X$ be the Banach space that results by completing $L^\infty(\mu) \otimes X$ when $L^\infty(\mu) \otimes X$ is equipped with the greatest reasonable crossnorm.⁸ As we will now show, $L^{1,*}(\mu, X^*)$ can be naturally identified with a subspace of $(L^\infty(\mu) \hat{\otimes} X)^*$.⁹

⁸See Diestel and Uhl (1977, Chapter VIII) for the definition of “greatest reasonable crossnorm” and for general material about tensor products of Banach spaces.

⁹“Subspace” does not necessarily mean “closed subspace.”

To see this, let $B(L^\infty(\mu), X)$ denote the Banach space of bounded bilinear real-valued functions on $L^\infty(\mu) \times X$, the norm of $u \in B(L^\infty(\mu), X)$ being given by $\|u\|_B = \sup \{|u(g, x)| : g \in L^\infty(\mu), x \in X, \|g\|_\infty \leq 1, \|x\| \leq 1\}$. (“Bilinear” for a function u on $L^\infty(\mu) \times X$ means $u(\cdot, x)$ is linear on $L^\infty(\mu)$ for each fixed $x \in X$ and $u(g, \cdot)$ is linear on X for each fixed $g \in L^\infty(\mu)$; “bounded” means $\|u\|_B < \infty$ for the just noted norm $\|u\|_B$.) According to Diestel and Uhl (1977, Theorem VIII.2.1, p. 230), $(L^\infty(\mu) \hat{\otimes} X)^*$ with the dual norm can be identified with $B(L^\infty(\mu), X)$ by a linear isometry such that, in a natural way, $u \in (L^\infty(\mu) \hat{\otimes} X)^*$ corresponds to $\tilde{u} \in B(L^\infty(\mu), X)$ if and only if $u(g \otimes x) = \tilde{u}(g, x)$ for each $(g, x) \in L^\infty(\mu) \times X$. From now on, this identification will be in force. In particular, we may speak of a $u \in (L^\infty(\mu) \hat{\otimes} X)^*$ as a bilinear function on $L^\infty(\mu) \times X$ and denote its value at $g \otimes x$ simply by $u(g, x)$.

Now let $f: T \rightarrow X^*$ be a Gelfand integrable function. Evidently the mapping $(g, x) \mapsto \int_T gxf \, d\mu$, $g \in L^\infty(\mu)$, $x \in X$, is a bilinear function on $L^\infty(\mu) \times X$. We claim that this function is bounded. To see this, recall from the proof of Lemma 8 that the operator $g \mapsto \int_T gf \, d\mu$ from $L^\infty(\mu)$ to X^* is continuous for the weak*-topologies of $L^\infty(\mu)$ and X^* . Therefore, since the closed unit ball in $L^\infty(\mu)$ is weak*-compact, the set $\{\int_T gf \, d\mu : g \in L^\infty(\mu), \|g\|_\infty \leq 1\}$ is a weak*-compact subset of X^* and hence norm bounded. Consequently, the set $\{|\int_T gxf \, d\mu| : g \in L^\infty(\mu), x \in X, \|g\|_\infty \leq 1, \|x\| \leq 1\}$ is bounded, or, in other words, $\sup\{|\int_T gxf \, d\mu| : g \in L^\infty(\mu), x \in X, \|g\|_\infty \leq 1, \|x\| \leq 1\} < \infty$.

Clearly if f and f' are any two Gelfand integrable functions from T into X^* such that $f(t) = f'(t)$ for almost all $t \in T$, then f and f' define the same bilinear function on $L^\infty(\mu) \times X$. On the other hand, since X is separable by hypothesis, if such functions f and f' do not agree on the complement of some null set, then they define different bilinear functions on $L^\infty(\mu) \times X$. Indeed, if $\int_T 1_E xf \, d\mu = \int_T 1_E xf' \, d\mu$ for all $E \in \mathcal{T}$ and all $x \in X$, then for each $x \in X$, $xf(t) = xf'(t)$ for almost all $t \in T$, which, since X is separable, implies that $f(t) = f'(t)$ for almost all $t \in T$.

Finally, it is clear that if f and f' are any two Gelfand integrable functions from T into X^* , and α and β any two real numbers, then for all $g \in L^\infty(\mu)$ and all $x \in X$, $\int_T gx(\alpha f + \beta f') \, d\mu = \alpha \int_T gxf \, d\mu + \beta \int_T gxf' \, d\mu$.

Thus, by the facts just noted, identifying each Gelfand integrable function $f: T \rightarrow X^*$ with the bilinear function $(g, x) \mapsto \int_T gxf \, d\mu$ on $L^\infty(\mu) \times X$, we have a one-to-one linear mapping from $L^{1,*}(\mu, X)$ to $(L^\infty(\mu) \hat{\otimes} X)^*$. Henceforth, $L^{1,*}(\mu, X)$ will be identified with its image in $(L^\infty(\mu) \hat{\otimes} X)^*$ under this mapping. In particular, then, for any $g \in L^\infty(\mu)$ and $x \in X$, the value of $f \in L^{1,*}(\mu, X)$ taken at $g \otimes x$ is given by $\int_T gxf \, d\mu$, and will be denoted in this way.

Note for the sequel that if a net (u_α) in $(L^\infty(\mu) \hat{\otimes} X)^*$ is weak*-convergent to some $u \in (L^\infty(\mu) \hat{\otimes} X)^*$ then, under the natural identification of $(L^\infty(\mu) \hat{\otimes} X)^*$ with $B(L^\infty(\mu), X)$, we have $u_\alpha(g, x) \rightarrow u(g, x)$ for each $(g, x) \in L^\infty(\mu) \times X$. In particular, we may see that the linear operator $f \mapsto \int_T f \, d\mu$ from $L^{1,*}(\mu, X^*)$

to X^* is continuous for the weak*-topology of X^* and the restriction of the weak*-topology of $(L^\infty(\mu) \hat{\otimes} X)^*$ to $L^{1,*}(\mu, X^*)$. For if (f_α) is a net in $L^{1,*}(\mu, X^*)$, weak*-converging in $(L^\infty(\mu) \hat{\otimes} X)^*$ to some $f \in L^{1,*}(\mu, X^*)$, then the net (f_α) must converge to f in particular at each point of $L^\infty(\mu) \times X$ that has the form $(1_T, x)$. Thus

$$x \int_T f_\alpha d\mu = \int_T x f_\alpha d\mu = \int_T 1_T x f_\alpha d\mu \rightarrow \int_T 1_T x f d\mu = x \int_T f d\mu$$

for each $x \in X$ whence $\int_T f_\alpha d\mu \rightarrow \int_T f d\mu$ weak* in X^* .

Now let

$$S_{\overline{\text{co}}^*\phi}^{1,*} = \{f \in L^{1,*}(\mu, X^*) : f \text{ is a } (\mu\text{-equivalence class of a selection of } \overline{\text{co}}^*\phi)\}.$$

Obviously $S_{\overline{\text{co}}^*\phi}^{1,*}$ is a convex subset of $L^{1,*}(\mu, X)$, and hence of $(L^\infty(\mu) \hat{\otimes} X)^*$. Also, $S_{\overline{\text{co}}^*\phi}^{1,*}$ is a norm-bounded subset of $(L^\infty(\mu) \hat{\otimes} X)^*$. Indeed, as noted above, the correspondence $\overline{\text{co}}^*\phi$ is integrably bounded. That is, we can select an integrable function $\rho: T \rightarrow \mathbb{R}_+$ such that $\sup\{\|x^*\| : x^* \in \overline{\text{co}}^*\phi(t)\} \leq \rho(t)$ for almost all $t \in T$. Then for any $f \in S_{\overline{\text{co}}^*\phi}^{1,*}$, $g \in L^\infty(\mu)$, and $x \in X$, with $\|g\|_\infty \leq 1$ and $\|x\| \leq 1$,

$$\left| \int_T g x f d\mu \right| \leq \int_T |g(t) x f(t)| d\mu(t) \leq \|g\|_\infty \|x\| \int_T \|f(t)\| d\mu(t) \leq \int_T \rho d\mu,$$

showing that $\|f\|_B \leq \int_T \rho d\mu < \infty$ for each $f \in S_{\overline{\text{co}}^*\phi}^{1,*}$. Thus, as claimed, $S_{\overline{\text{co}}^*\phi}^{1,*}$ is norm-bounded in $(L^\infty(\mu) \hat{\otimes} X)^*$, since $(L^\infty(\mu) \hat{\otimes} X)^*$ is isometrically isomorphic to $B(L^\infty(\mu), X)$.

We are going to show that $S_{\overline{\text{co}}^*\phi}^{1,*}$ is also a weak*-closed subset of $(L^\infty(\mu) \hat{\otimes} X)^*$. To this end, suppose (f_α) is a net in $S_{\overline{\text{co}}^*\phi}^{1,*}$, weak*-converging in $(L^\infty(\mu) \hat{\otimes} X)^*$ to some $u \in (L^\infty(\mu) \hat{\otimes} X)^*$. We must show that $u = f$ for some $f \in S_{\overline{\text{co}}^*\phi}^{1,*}$. Note for the sequel that u being a bounded bilinear function on $L^\infty(\mu) \times X$ implies that for any $g \in L^\infty(\mu)$, $u(g, \cdot) \in X^*$ and for any $x \in X$, $u(\cdot, x) \in L^\infty(\mu)^*$. (To see this, note simply that $|u(g, x)| \leq \|u\|_B \|g\|_\infty \|x\|$ for any $g \in L^\infty(\mu)$ and $x \in X$.)

We claim that, in the present context, for each $x \in X$, $u(\cdot, x)$ must actually belong to $L^1(\mu)$. To see this, fix any $x \in X$. Since $f_\alpha \rightarrow u$ weak* in $(L^\infty(\mu) \hat{\otimes} X)^*$, for each $g \in L^\infty(\mu)$ we have $\int_T g x f_\alpha d\mu \rightarrow u(g, x)$. That is, $x f_\alpha \rightarrow u(\cdot, x)$ weak* in $L^\infty(\mu)^*$. Now by the construction in the second paragraph of this proof, for each α , $x f_\alpha$ is an element the order interval $[\underline{\pi}_x, \overline{\pi}_x]$ in $L^1(\mu)$. Observing that $[\underline{\pi}_x, \overline{\pi}_x]$, when viewed as a subset of $L^\infty(\mu)^*$, is a weak*-closed subset of $L^\infty(\mu)^*$, it follows that $u(\cdot, x) \in [\underline{\pi}_x, \overline{\pi}_x]$ and thus $u(\cdot, x)$ can indeed be represented by an element of $L^1(\mu)$.

We write h_x for the element of $L^1(\mu)$ representing $u(\cdot, x)$. Note that, by bilinearity of u , the mapping from X to $L^1(\mu)$ that takes x to h_x is linear and that for each $x \in X$, $\underline{\pi}_x(t) \leq h_x(t) \leq \overline{\pi}_x(t)$ for almost all $t \in T$.

Denote by $\mathcal{L}^1(\mu)$ the vector space of all integrable functions on (T, \mathcal{T}, μ) (different functions agreeing almost everywhere *not* being identified) and let

$\rho: L^1(\mu) \rightarrow \mathcal{L}^1(\mu)$ be a linear lifting of $L^1(\mu)$; that is, ρ is a linear mapping and for each $h \in L^1(\mu)$, $\rho(h)$ is a version of h , i.e. the μ -equivalence class of $\rho(h)$ is h . (Such a mapping ρ can be constructed by taking an arbitrary algebraic basis of $L^1(\mu)$ and assigning to each element of this basis one of its versions. We do of course not claim that ρ is positive or multiplicative.)

Define a mapping $\lambda: T \times X \rightarrow \mathbb{R}$ by setting

$$\lambda(t, x) = \rho(h_x)(t).$$

Then for each $x \in X$, $\lambda(\cdot, x)$ is a version of h_x . Also, since the mapping $x \mapsto h_x$ is linear, so is the mapping $x \rightarrow \rho(h_x)$, and hence $\lambda(t, \cdot)$ is a linear functional on X for each $t \in T$.

By the hypothesis that X is separable, select a countable dense subset D of X and let D^r be the set of all (finite) linear combinations of elements of D such that the coefficients are rational. Thus D^r is countable, and it follows that we can fix a null set N in T such that for each $t \in T \setminus N$, $\underline{\pi}_d(t) \leq \lambda(t, d) \leq \overline{\pi}_d(t)$ for all $d \in D^r$. Let Z be the linear span of D^r . Then D^r is dense in Z , and since $\overline{\text{co}}^*\phi(t)$ is a weak*-compact subset of X^* for each $t \in T$, it follows that for each $t \in T \setminus N$, $\underline{\pi}_z(t) \leq \lambda(t, z) \leq \overline{\pi}_z(t)$ for all $z \in Z$. But this implies that for each $t \in T \setminus N$, the linear functional $\lambda(t, \cdot)$ is bounded on $B \cap Z$, where B denotes the closed unit ball of X . (Indeed, otherwise there would be sequence (z_n) in $B \cap Z$ with $\|z_n\| \rightarrow 0$ but such that $\lambda(t, z_n) = 1$ for all n . But this is impossible if $\underline{\pi}_{z_n}(t) \leq \lambda(t, z_n) \leq \overline{\pi}_{z_n}(t)$, because by weak*-compactness of $\overline{\text{co}}^*\phi(t)$, $\|z_n\| \rightarrow 0$ entails $\overline{\pi}_{z_n}(t) \rightarrow 0$ as well as $\underline{\pi}_{z_n}(t) \rightarrow 0$.)

It follows that for each $t \in T \setminus N$ there is a uniquely determined element $x^*(t) \in X^*$ such that $x^*(t)z = \lambda(t, z)$ for all $z \in Z$. Let $\overline{f}: T \rightarrow X^*$ be the function defined by setting $\overline{f}(t) = x^*(t)$ for $t \in T \setminus N$ and $\overline{f}(t) = 0$ for $t \in N$. Then for almost all $t \in T$, $\underline{\pi}_z(t) \leq \overline{f}(t)z \leq \overline{\pi}_z(t)$ for all $z \in Z$. Because Z is dense in X and $\overline{\text{co}}^*\phi(t)$ is a weak*-compact subset of X^* , we must have, in fact, $\underline{\pi}_x(t) \leq \overline{f}(t)x \leq \overline{\pi}_x(t)$ for almost all $t \in T$ and all $x \in X$. Consequently, $x\overline{f} \in L^1(\mu)$ for all $x \in X$ and, by the Hahn-Banach theorem, $\overline{f}(t) \in \overline{\text{co}}^*\phi(t)$ for almost all $t \in T$. Thus \overline{f} is a Gelfand integrable selection of $\overline{\text{co}}^*\phi$, i.e. determines an element of $S_{\overline{\text{co}}^*\phi}^{1,*}$.

Now by construction, for each $z \in Z$, $\overline{f}(t)z = \lambda(t, z)$ for almost all $t \in T$ and $\lambda(\cdot, z)$ is a version of h_z . It follows that $\int_T g z \overline{f} d\mu = \int_T g h_z d\mu = u(g, z)$ for each $g \in L^\infty(\mu)$ and each $z \in Z$. In particular, for each fixed $g \in L^\infty(\mu)$, the two elements $\int_T g \overline{f} d\mu$ and $u(g, \cdot)$ of X^* agree on the dense subspace Z of X . Thus they must agree on the entire space X . Consequently $u = \overline{f}$ (as elements of $B(L^\infty(\mu), X) \equiv (L^\infty(\mu) \hat{\otimes} X)^*$), and thus we have shown that $S_{\overline{\text{co}}^*\phi}^{1,*}$ is a weak*-closed subset of $(L^\infty(\mu) \hat{\otimes} X)^*$. As remarked above, $S_{\overline{\text{co}}^*\phi}^{1,*}$ is convex and norm-bounded in $(L^\infty(\mu) \hat{\otimes} X)^*$, and we conclude that $S_{\overline{\text{co}}^*\phi}^{1,*}$ is, in fact, a convex weak*-compact subset of $(L^\infty(\mu) \hat{\otimes} X)^*$.

Now suppose $\bar{f} \in S_{\overline{co}^*\phi}^{1,*}$. Let

$$H = \left\{ h \in S_{\overline{co}^*\phi}^{1,*} : \int_T \bar{f} d\mu = \int_T h d\mu \right\}.$$

As seen above, the linear operator $f \mapsto \int_T f d\mu$ from $L^{1,*}(\mu, X^*)$ to X^* is continuous for the (restricted) weak*-topology of $(L^\infty(\mu) \hat{\otimes} X)^*$ and the weak*-topology of X^* . Hence, since the subset $S_{\overline{co}^*\phi}^{1,*}$ of $L^{1,*}(\mu, X^*)$ is a convex weak*-compact subset of $(L^\infty(\mu) \hat{\otimes} X)^*$, so is H . Thus H has an extreme point, say \bar{h} . Arguing as in the corresponding passage of the proof of Theorem 2, modulo now invoking Lemma 9 instead of Lemma 5, we see that \bar{h} must be an extreme point of $S_{\overline{co}^*\phi}^{1,*}$, too. Now because (X^*, weak^*) is a Suslin space, and since the measure space (T, \mathcal{T}, μ) is complete, a function $h: T \rightarrow X^*$ is Gelfand measurable if and only if it is \mathcal{T} - $\mathcal{B}(X^*, \text{weak}^*)$ -measurable (see e.g. Thomas, 1975, Theorem 1). In particular, since the correspondence $\overline{co}^*\phi$ is integrably bounded, any \mathcal{T} - $\mathcal{B}(X^*, \text{weak}^*)$ -measurable selection of $\overline{co}^*\phi$ is Gelfand integrable. Therefore by Lemma 6, since $G_\phi \in \mathcal{T} \otimes \mathcal{B}(X^*, \text{weak}^*)$, the fact that \bar{h} is an extreme point of $S_{\overline{co}^*\phi}^{1,*}$ means that $\bar{h}(t)$ is an extreme point of $\overline{co}^*\phi(t)$ for almost all $t \in T$. Since $\phi(t)$ is a weak*-compact subset of X^* , it follows from this that $\bar{h}(t) \in \phi(t)$ for almost all $t \in T$. Thus we have shown that given $f \in S_{\overline{co}^*\phi}^{1,*}$ there is a Gelfand integrable selection h of ϕ such that $\int_T f d\mu = \int_T h d\mu$. In other words,

$$\left\{ \int_T f d\mu : f \in S_{\overline{co}^*\phi}^{1,*} \right\} \subset \int_T \phi d\mu.$$

Obviously, the reverse inclusion holds too, and by the fact that $S_{\overline{co}^*\phi}^{1,*}$ is convex and weak*-compact in $(L^\infty(\mu) \hat{\otimes} X)^*$, in conjunction with the above noted continuity properties of the linear operator $f \mapsto \int_T f d\mu$, we conclude that $\int_T \phi d\mu$ is a convex and weak*-compact subset of X^* . Thus (ii) \Rightarrow (i) of the theorem is proved.

not(ii) \Rightarrow not(i): See the proof of Theorem 3 not(ii) \Rightarrow not(i) and observe that given a Bochner integrable function $f: T \rightarrow X^*$, the correspondence $\phi: T \rightarrow 2^{X^*}$ defined by $\phi(t) = \{f(t)\} \cup \{0\}$, $t \in T$, trivially has weak*-compact values (in fact norm-compact values) and is integrably bounded (since Bochner integrability of f implies $\int_T \|f(t)\| d\mu(t) < \infty$). Moreover, choosing any countable subset $\{d_1, d_2, \dots\}$ of X separating the points of X^* (which is possible by hypothesis), we have $G_\phi = (T \times \{0\}) \cup \bigcap_{i=1}^\infty \{(t, x^*) \in T \times X^* : x^* d_i = f(t) d_i\}$, showing that $G_\phi \in \mathcal{T}_\mu \otimes \mathcal{B}(X^*, \text{weak}^*)$ (because Bochner integrability of $f: T \rightarrow X^*$ implies that $x^{**} f$ is \mathcal{T}_μ -measurable for each $x^{**} \in X^{**}$ and therefore, since $X \subset X^{**}$, that $x f$ is \mathcal{T}_μ -measurable for each $x \in X$). \square

6 An extension of Lebesgue measure to a super-atomless measure

Let $[0, 1]$ denote the closed unit interval in \mathbb{R} , let λ denote Lebesgue measure on $[0, 1]$, and let Λ denote the σ -algebra of Lebesgue measurable subsets of $[0, 1]$. Fix any cardinal κ with $\aleph_1 \leq \kappa \leq 2^{\mathfrak{c}}$ and let ν_κ be the usual measure on the product $\{0, 1\}^\kappa$. Let γ be the product measure on $\{0, 1\}^\kappa \times [0, 1]$ corresponding to ν_κ and λ . According to Fremlin (2002, p. 161, Exercise 334X(g)), the measure algebra of γ is Maharam homogeneous with Maharam type κ . Now by the proof of Proposition 521P(b) in Fremlin (2005)¹⁰ there is a subset C of $\{0, 1\}^\kappa \times [0, 1]$ such that the following hold: First, C has full outer measure for γ ; in particular, the measure algebra of γ can be identified with the measure algebra of γ_C , where γ_C denotes the subspace measure on C induced by γ . Thus the measure algebra of γ_C is Maharam homogeneous with Maharam type κ . Second, denoting by π the projection of $\{0, 1\}^\kappa \times [0, 1]$ onto $[0, 1]$ and by π_C its restriction to C , π_C is an injective mapping from C into $[0, 1]$. Now let Γ_C be the σ -algebra of γ_C -measurable subsets of C , let \mathcal{T} be the σ -algebra on $[0, 1]$ defined by

$$\mathcal{T} = \{A \subset [0, 1] : \pi_C^{-1}(A) \in \Gamma_C\},$$

and let μ be the measure on $([0, 1], \mathcal{T})$ given by $\mu(A) = \gamma_C(\pi_C^{-1}(A))$ for $A \in \mathcal{T}$. Then since π_C is injective, π_C induces an isomorphism between the measure algebras of μ and γ_C . Thus the measure algebra of μ is Maharam homogeneous with Maharam type κ . In particular, the measure space $([0, 1], \mathcal{T}, \mu)$ is super-atomless. Now pick any $B \in \Lambda$. Evidently $\pi_C^{-1}(B) = (\{0, 1\}^\kappa \times B) \cap C$. Hence $\pi_C^{-1}(B) \in \Gamma_C$. In particular, $B \in \mathcal{T}$ and $\mu(B) = \gamma_C(\pi_C^{-1}(B))$. On the other hand, since C has full outer measure for γ , we have

$$\gamma_C((\{0, 1\}^\kappa \times B) \cap C) = \gamma(\{0, 1\}^\kappa \times B) = \lambda(B).$$

Consequently $\mu(B) = \lambda(B)$. Thus the σ -algebra \mathcal{T} includes the σ -algebra Λ and the measure μ is an extension of λ to \mathcal{T} . Note also that if $\kappa \leq \mathfrak{c}$, then $\text{card } L^\infty(\mu) = \mathfrak{c}$. Indeed, being Maharam homogeneous with Maharam type κ , the measure algebra of μ is isomorphic to the measure algebra of ν_κ . Consequently the spaces $L^\infty(\mu)$ and $L^\infty(\nu_\kappa)$ are isomorphic. But $\text{card } L^\infty(\nu_\kappa) = \mathfrak{c}$ for $\kappa \leq \mathfrak{c}$, because each element of $L^\infty(\nu_\kappa)$ has a version that depends on only countably many coordinates.

¹⁰In regard to the applicability of the arguments in that proof to the context here, note that by the Hewitt-Marczewski-Pondiczerny theorem (see e.g. Hodel, 1984, Theorem 11.2, p. 42), $\{0, 1\}^\kappa$ with the product topology corresponding to the discrete topology of $\{0, 1\}$ has a dense subset of cardinality \mathfrak{c} for any cardinal κ with $\aleph_0 \leq \kappa \leq 2^{\mathfrak{c}}$.

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