Nash Implementation and Opportunity Equilibrium

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Abstract

An economic institution in equilibrium analysis is formulated by a system "opportunities" the institution offers to its members. Extending this formulation, an *opportunity system* is a non-empty set of profiles of individual *opportunity sets* (subsets of alternatives). An alternative is an *opportunity equilibrium* under a system if there is an opportunity profile in the system such that the alternative maximizes each agent's well-being over his opportunity set in the profile. Examples are Walrasian equilibrium, Lindahl equilibrium, valuation equilibrium by Mas-Colell (1980), *equal* opportunity equilibrium by Thomson (1994), etc. The main results show that this formulation of economic institutions by opportunity systems is closely related with the alternative formulation by game forms in Implementation Theory and that in some well-known environments, they are equivalent. A useful by-product is a decomposition of implementation procedure into two steps: the first step is to identify an opportunity system supporting a rule as the opportunity equilibrium correspondence, and the second step is to use this system to design a game form implementing the rule. Thus, informational efficiency in the opportunity system, if any, can be embedded in the game form implementing the rule.

Keywords: opportunity system; opportunity equilibrium; Nash implementation; monotonicity

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1 Introduction

A standard formulation of economic institutions in equilibrium analysis is by way of a system "opportunities" an institution offers to its members. Each member is assumed to choose rationally an element from his opportunity set. An equilibrium is reached when such decentralized decisions are mutually compatible. A best-known example in the private goods economy is the unit-pricing system which offers to consumers the so-called Walrasian budget sets as opportunity sets. The corresponding notion of equilibrium is known as Walrasian equilibrium. Another example in the public goods economy is the individualized unit-pricing system offering consumers the so-called Lindahl budget sets with the corresponding notion of equilibrium known as Lindahl equilibrium. A generalized notion of opportunity system with much greater freedom in the choice of pricing schemes, called "valuation system", is studied by Mas-Colell (1980) in a model of pure public goods. Thomson (1994) introduces a general notion of equal opportunity system and opportunity equilibrium that are applicable in a variety of economic environments with private goods or public goods.

The main objective of this paper is to study how this formulation of economic institutions based on oppoutunity sysyems and opportunity equilibrium is related with the alternative formulation in the literature of mechanism design that makes use of game forms and game theoretic equilibrium concepts such as Nash equilibrium. The main results show that the two formulations are very closely related and in some well-known economic environments, they are equivalent.

A useful by-product of the main results is a decomposition of implementation procedure into two steps: in order to implement a social choice rule, in the first step, we identify an opportunity system supporting the rule, and in the second step, we use this system to design a game form implementing the rule. In this game form, a player's strategy consists of an opportunity profile, his demand over his opportunity set, and an integer. Thus, informational efficiency in the opportunity system, if any, can be embedded in the game form implementing the rule.[1] For example, Walrasian rule in exchange economies is supported by the opportunity system characterized by the set of price vectors. Thus it is implemented by a simple market-like game form. Other examples of rules implemented by informationally efficient game forms are Lindahl rule in public goods economies, uniform rule in the single-peaked preferences economies studied by Sprumont (1991), no-envy rule (Crawford 1977; Thomson 2005), etc.

For a formal explanation of our results, let us define some basic terms. An

[1] Williams (1986) establishes another way of embeding informational efficiency in game forms.

opportunity set is a non-empty subset of social alternatives (or allocations). An *opportunity system* is a non-empty family of profiles of opportunity sets indexed by agents. An alternative is an *opportunity equilibrium under a system* if there is an opportunity profile in the system such that the alternative maximizes each agent's well-being over his opportunity set in the profile. In economic environments with individualistic (or selfish) preferences, this can be translated into standard notions of market equilibrium, such as Walrasian or Lindahl equilibrium, defined by optimal choices over individual budget sets and market clearing conditions. A (social choice) rule is called an *opportunity rule* if it is supported by an opportunity system, that is, there is an opportunity system such that at any social choice problem (or any economy) in the domain, each alternative chosen by the rule is an opportunity equilibrium under the system and vice versa.[2]

Under a "relative denseness" condition of the domain together with some additional domain conditions or fairly weak axioms for rules, we show that any Nash implementable rule is supported by an opportunity system. Conversely, within rules satisfying the so called no-veto-power and under a very weak domain condition called "weak separated variability", we show that any opportunity rule is Nash implementable. The proof makes use of the "opportunity game form" in which each player's strategy space is given by the product of the opportunity system, the alternative space, and the set of integers. This construction enables us to implement any rule supported by a simple opportunity system by a game form with an equally simple strategy space, thus offering a novel approach to the study of strategy space reduction in Maskin's game form by Williams (1986), Saijo (1988), McKelvey (1989), Moore and Repullo (1990), etc.

Thomson (1994) focuses on opportunity systems which are composed of profiles of *equal* opportunity sets across agents. Also his opportunity sets are subsets of *feasible* alternatives. We do not impose either the equality condition or feasibility.

Our opportunity game form when applied in economic environments does not have some nice features that numerous authors in the market-game literature have tried to embed in their game forms implementing Walrasian or Lindahl rules: see Hurwicz (1979), Schmeidler (1980), Dubey (1982), Simon (1984), Bennassy (1986), Bevia, Corchon, and Wilkie (1998), etc. In particular, our opportunity game form crucially relies on the special handling of integer announcements as in Maskin's game form. This is inevitable because like Maskin (1999) our result provides a canonical relationship between implementable rules and game forms.

[2] Greenberg (1990) and Miyagawa (2001) consider a general notion of mechanisms, called "effectivity forms". These mechanisms encompass both opportunity systems and game forms.

On the other hand, our opportunity game form gives us a simple and unified game form that allows us to implement constrained or unconstrained Walrasian rule with or without taxation in exchange economies and constrained or unconstrained Lindahl rule in public goods economies, all at once.

The strategy spaces in Saijo (1988) and Moore and Repullo (1990) contain the set of preference profiles and so they are quite large, particularly when the alternative space is infinite. Moreover, in their frameworks, it is not allowed to tailor the structure of strategy space to each rule. The strategy space in McKelvey (1989) has a similarity to ours because instead of preference profiles, he uses the opportunity profiles consisting of lower counter sets of preferences in the definition of strategy space. Relying on richness of the domain of preferences, he offers a way of reducing the strategy space relative to each rule. In particular, he shows that for Walrasian rule the strategy space reduction can be substantial so that price announcements instead of preferences are sufficient for the implementation. The strategy space reduction in our result is stronger because our reduction does not rely on what preferences are admissible in the domain.

The rest of the paper is organized as follows. Section 2 gives basic definitions. Section 3 gives general results in the abstract model. Section 4 gives further results and applications in economic environments.

2 The Model and Basic Concepts

Let $N \equiv \{1, \cdots, n\}$ be the set of agents with $n \geq 3$. Let A be the set of *alternatives*. Individuals have complete and transitive binary relations over A, namely *preferences*. Let $\mathcal{R}$ be the set of all such preferences and $\mathcal{R}^N$ the set of n-tuples of preferences. Generic notation for preferences of agent i is R_i and the corresponding strict and indifference relations are denoted by P_i and I_i, respectively.

Let $\mathcal{D}$ be a subset of $\mathcal{R}^N$. In order to deal with numerous applications where alternatives in A are not always feasible, let $Z \subseteq A$ be the set of all *feasible* alternatives. A *social choice rule*, or simply, a *rule*, on $\mathcal{D}$ is a correspondence $\varphi\colon \mathcal{D} \to Z$ associating with each preference profile in the domain a *nonempty* set of *feasible* alternatives.

Example 1 (Exchange economy). In the exchange economy with l goods and social endowment $\Omega \in \mathbb{R}^l_+$, $A = \mathbb{R}^{l\cdot n}_+$ and $Z = \{z \in \mathbb{R}^{l\cdot n}_+ \colon \sum_i z_i \leq \Omega\}$. For each $i \in N$, let $z_i \in \mathbb{R}^l_+$ be i's bundle at z. It is standard to assume that each agent $i \in N$ has a preference R_i over $\mathbb{R}^{l\cdot n}_+$, which is *individualistic*, that is, for each pair $z, z' \in \mathbb{R}_+^{l \cdot n}$,

$$\text{if } z_i = z_i', \text{ then } z \; I_i \; z'.$$

Other standard assumptions on preferences in the exchange economy are continuity, monotonicity, and convexity to be defined in Section 4.

Game Forms and Nash Implementation

For each $i \in N$, let S_i be a set of strategies for i. Let $S \equiv \times_i S_i$. Let $g\colon S \to A$ be an *outcome function.* A *game form* is defined by the pair (S, g).

A rule φ is *Nash implementable* if there is a game form $G \equiv (S, g)$ such that for each $R \in \mathcal{D}$, $g(NE(G, R)) = \varphi(R)$. It is *feasibly Nash implementable* if, in addition, $g(S) \subseteq Z$. Our results are obtained for a more general notion, Nash implementation under an arbitrary range restriction. Let $Y \subseteq A$ be a set of alternatives to which any possible outcome of a game form is desired to belong. A rule φ is *Nash implementable under the range-restriction of* Y if φ is Nash implementable by a game form $G \equiv (S, g)$ and the range of g is in Y, that is, $g(S) \subseteq Y$. Thus when $Y = A$ or Z, this notion coincides with the above mentioned Nash implementability (without any constraint) and feasible Nash implementability, respectively.

Here are some examples of range-restrictions in exchange economies.

Example 2 (Continuation: exchange economy). (i) An allocation is *efficient* at $R \in \mathcal{D}$ if there is no other feasible allocation making an agent better off without making someone else worse off. One may want to consider outcome functions which do not take a value that is never effcient on $\mathcal{D}$. The range-restriction associated with this objective is given by the set of potentially efficient allocations, that is, $Y_{\text{Eff.}} \equiv \cup_{R \in \mathcal{D}} \{a \in A : a \text{ is efficient at } R\}$.
(ii) An allocation z is *envy free* at $R \in \mathcal{D}$ if for each pair $i, j \in N$, $z_i \; R_i \; z_j$. Let $Y_{\text{NVF}} \equiv \{z \in \mathbb{R}_+^{l \cdot n} : \text{for some } R \in \mathcal{D}, z \text{ satisfies no-envy at } R\}$ be the set of potentially envy free allocations on $\mathcal{D}$.
(iii) An allocation z satisfies *no-domination* at $R \in \mathcal{D}$ if for each pair $i, j \in N$, $z_i \ngeq z_j$. Let $Y_{\text{ND}} \equiv \{z \in \mathbb{R}_+^{l \cdot n} : \text{for all } i, j \in N, z_i \ngeq z_j\}$.
(iv) An allocation z satisfies *average-no-domination* at $R \in \mathcal{D}$ if for all $i \in N$, $\sum_{j \neq i} z_j/(n-1) \ngeq z_i$. Let $Y_{\text{AND}} \equiv \{z \in \mathbb{R}_+^{l \cdot n} : \text{for all } i \in N, \sum_{j \neq i} z_i/(n-1) \ngeq z_j\}$.
(v) An allocation z satisfies *egalitarian equivalence* at $R \in \mathcal{D}$ if there exists $z_0 \in \mathbb{R}_+^l$ such that for all $i \in N$, $z \; I_i \; (z_0, \cdots, z_0)$. Let $Y_{\text{EE}} \equiv \{z \in \mathbb{R}_+^{l \cdot n} : \text{for some } R \in \mathcal{D}, z \text{ satisfies egalitarian equivalence at } R\}$.
(vi) An allocation z satisfies the *equal division lower bound property* at $R \in \mathcal{D}$ if

for all $i \in N$, $z \; R_i \; \Omega/n$. Let $Y_{\text{EDB}} \equiv \{z \in \mathbb{R}_+^{l \cdot n}$: for some $R \in \mathcal{D}$, z satisfies the equal division lower bound property at $R\}$.

Important Axioms for Rules

We now define generalization of the important conditions for Nash implementation introduced by Maskin (1977, 1999). We transform his conditions into conditions relative to an arbitrary range-restriction Y. In what follows, fix a range-restriction $Y \subseteq A$.

Monotonicity on Y. For each $R \in \mathcal{D}$, each $a \in \varphi(R)$ and each $R' \in \mathcal{D}$, if for each $i \in N$, $LC(R_i, a; Y) \subseteq LC(R'_i, a; Y)$, then $a \in \varphi(R')$.

When $Y = Z$, monotonicity on Z coincides with the so-called *Maskin monotonicity* ("monotonicity" in Maskin, 1977, 1999). When $Y = A$, monotonicity on A coincides with the so-called *weak monotonicity* (Gevers, 1986).

No-Veto-Power on Y. For each $R \in \mathcal{D}$ and each $i \in N$, if $a \in \cap_{j \neq i} \max[R_j, Y] \cap Z$, then $a \in \varphi(R)$.

When $Y = A$, no-veto-power on A can be substantially mild. In some domains, it could be met by any rule. For example, when for each feasible alternative a, individuals always have alternatives in A (feasible or not) preferred to a, no-veto-power in A is met trivially by any rule.

In a companion paper, Ju (2005), we extend Maskin's results for feasible Nash implementation (with $Y = Z$). We show that if a rule is Nash implementable under the range-restriction of Y, then it is a subcorrespondence of Y and satisfies monotonicity on Y and that when there are more than two agents, if a rule is a subcorrespondence of Y and satisfies monotonicity and no-veto-power on Y, then it is Nash implementable under the range-restriction of Y.

The following axiom is a standard unanimity condition. No-veto-power implies this axiom.

Unanimity on Y. For each $R \in \mathcal{D}$, if $a \in \cap_{i \in N} \max[R_i, Y] \cap Z$, then $a \in \varphi(R)$.

The next axiom says that no one should be worst off at any alternative chosen by a rule.

No Punishment on Y. For each $R \in \mathcal{D}$, if $a \in Y$ is such that for some $i \in N$, $a \in \min[R_i, Y]$, $a \notin \varphi(R)$.

Note that this axiom may not be compatible with *no-veto-power*.

Opportunity Equilibrium

An *opportunity set in* $Y \subseteq A$ is a subset of Y. An *opportunity profile in* Y is a list of opportunity sets in Y, indexed by agents. Notation for opportunity profiles are O, O', O'', etc. The i-th component of O is the opportunity set for agent i and is denoted by O_i. An *opportunity system in* Y, denoted by $\mathcal{O}$, is a family of opportunity profiles in Y. Notation for opportunity systems are $\mathcal{O}, \mathcal{O}', \mathcal{O}''$, etc. A *feasible opportunity system* is an opportunity system in Z.

Given an opportunity system $\mathcal{O}$ in Y, $a \in A$ is an *opportunity equilibrium at* $R \in \mathcal{D}$ if $a \in Z$ and there is an opportunity profile $O \in \mathcal{O}$ such that for each $i \in N$, $a \in \max[R_i, O_i]$. Let $\varphi^{\mathcal{O}}$ be the correspondence associating with each $R \in \mathcal{D}$ the set of opportunity equilibria at R. When the correspondence $\varphi^{\mathcal{O}}$ is non-empty valued, we call it the *opportunity rule associated with* $\mathcal{O}$. A rule φ is *supported by an opportunity system in* Y if there is an opportunity system $\mathcal{O}$ in Y such that $\varphi = \varphi^{\mathcal{O}}$: that is, for each $R \in \mathcal{D}$ and each $a \in A$, $a \in \varphi(R)$ if and only if $z \in Z$ and there is $O \equiv (O_i)_{i \in N}$ such that for each $i \in N$, $a \in \max[R_i, O_i]$.

Lemma 1. *A rule φ is supported by an opportunity system in Y if and only if for each $R \in \mathcal{D}$ and each $a \in \varphi(R)$, there is an opportunity profile O in Y such that* (i) $a \in \cap_{i \in N} \max[R_i, O_i]$ *and* (ii) *for each $R' \in \mathcal{D}$ and each $a' \in Z$, if $a' \in \cap_{i \in N} \max[R'_i, O_i]$, then $a' \in \varphi(R')$.*

Proof. Let φ be supported by an opportunity system $\mathcal{O}$ in Y, that is, $\varphi(\cdot) = \varphi^{\mathcal{O}}(\cdot)$. Let $R \in \mathcal{D}$ and $a \in \varphi(R) = \varphi^{\mathcal{O}}(R)$. Then since $a \in \varphi^{\mathcal{O}}(R)$, there is $O \in \mathcal{O}$ such that (i) holds. Let $R' \in \mathcal{D}$ and $a' \in Z$ be such that $a' \in \cap_{i \in N} \max[R'_i, O_i]$. Then $a' \in \varphi^{\mathcal{O}}(R')$ and so $a' \in \varphi(R')$.

In order to prove the converse, for each $R \in \mathcal{D}$ and each $a \in \varphi(R)$, let $O(R, a) \equiv (O_i(R, a))_{i \in N}$ be the opportunity profile in Y satisfying (i) and (ii). Let $\mathcal{O} \equiv \{O(R, a) : R \in \mathcal{D}, a \in \varphi(R)\}$. If $a \in \varphi(R)$, then $a \in Z$ and by (i), $a \in \cap_{i \in N} \max[R_i, O_i(R, a)]$. Thus $a \in \varphi^{\mathcal{O}}(R)$. If $a \in \varphi^{\mathcal{O}}(R)$, then $a \in Z$ and there is $R' \in \mathcal{D}$ and $a' \in \varphi(R')$ such that $a \in \cap_{i \in N} \max[R_i, O_i(R', a')]$ (note that since $a' \in \varphi(R')$, $O(R', a') \in \mathcal{O}$). Then by (ii), $a \in \varphi(R)$. Therefore, $\varphi(R) = \varphi^{\mathcal{O}}(R)$. ∎

Domain Properties

We define some important properties of domains for our results.

The first one is a separability-type property saying that for each alternative $a \in Y$ and each agent $i \in N$, it is always possible to fix i's welfare at the level of a and to make a certain variation of a.

Weak Separated Variability on Y, briefly **WSV on Y.** For each $a \in Y$ and each $i \in N$, there is an alterative in Y, denoted by $e_i(a)$, such that for each $R \in \mathcal{D}$, (i) $e_i(a)\ I_i\ a$ and (ii) either $e_i(a) \in Z$ or for some $j \neq i$ and some $a' \in Y$, $a'\ P_j\ e_i(a)$.

When $Y = A$, we call this property *weak separated variability*, or *WSV*. Note that when $A = Z$ as in Maskin (1977, 1999) and Moore and Repullo (1990), for every Y, WSV on Y is satisfied trivially by letting $e_i(a) \equiv a$ for all $a \in Y$ and all $i \in N$. In domains where any alternative is never a best one over Y, WSV on Y is also met trivially by letting $e_i(a) \equiv a$, for all $a \in A$ and all $i \in N$.

The second property is a denseness-type property. It roughly says that it is always possible to find alternatives between any pair of alternatives.

Relative Denseness on Y, briefly **RD on Y.** For each pair $R, R' \in \mathcal{D}$, each $i \in N$, and each pair $a, a' \in Y$, if $a\ R_i\ a'$ and $a'\ P'_i\ a$, then
(i) when $a \notin \min[R_i, Y]$, there is $\bar{a} \in Y$ such that $a\ P_i\ \bar{a}$ and $\bar{a}\ P'_i\ a$;[3]
(ii) when $a \notin \max[R_i, Y]$, there is $\bar{a}' \in Y$ such that $\bar{a}'\ P_i\ a$, and $a'\ P'_i\ \bar{a}'$.

The next property says that the agreement among all agents except one is never possible.

No-Agreement on Y. For each $R \in \mathcal{D}$ and each $i \in N$, $\cap_{j \neq i} \max[R_j, Y] = \emptyset$.

We show later that when $Y = A$ or Z, these properties are very mild in several economic environments including exchange economies and public goods economies. Note that since the agreement among all agents except one is never possible, NA on Y implies WSV on Y (simply, let $e_i(a) = a$ for each $i \in N$ and each $a \in Y$).

Single-Minimum on Y. For each $R \in \mathcal{D}$ and each $i \in N$, if $\min[R_i, Y] \neq \emptyset$, then $\min[R_i, Y]$ is a singleton.

Thus if there is a R_i-minimal alternative over Y, it is the only R_i-minimal element in Y. Note that this property is different from the so-called single-dippedness. Clearly, single-dipped preferences are examples but there a variety other examples that are not single-dipped.

[3] When $a\ P_i\ a'$, this part is met trivially be letting $\bar{a} \equiv a'$.

3 General Results

We use the following notation. Let R_i be a preference relation of agent $i \in N$ and a an alternative. For each $Y \subseteq A$, let $LC(R_i, a, Y) \equiv \{a' \in Y : a \ R_i \ a'\}$ and $SLC(R_i, a, Y) \equiv \{a' \in Y : a \ P_i \ a'\}$ be the lower contour set and the strict lower contour set in Y of R_i at a. Given a strategy space $S \equiv \Pi_{i=1}^n S_i$, for each $s \in S$ and each $i \in N$, let $s_{-i} \equiv (s_j)_{j \in N \setminus i}$ and $Atn_i(s) \equiv \{g(s'_i, s_{-i}) : s'_i \in S_i\}$ (i's attainable set at s).

Theorem 1. *Assume that $\mathcal{D}$ satisfies RD on Y. If a rule on $\mathcal{D}$ is Nash implementable under the range-restriction of Y and satisfies unanimity and no-punishment on Y, then it is supported by an opportunity system in Y.*

Proof. Assume that $\mathcal{D}$ satisfies RD on Y. Let φ be a rule on $\mathcal{D}$ and $G \equiv (S, g)$ a game form such that $g(S) \subseteq Y$ and G implements φ. We constructs an opportunity system in Y as follows. For each $R \in \mathcal{D}$, each $a \in \varphi(R)$, and each $i \in N$, let $O_i(R, a) \equiv SLC(R_i, a, Y) \cup \{a\}$. Let $O(R, a) \equiv (O_i(R, a))_{i \in N}$. Let $\mathcal{O} \equiv \{O(R, a) : R \in \mathcal{D} \text{ and } a \in \varphi(R)\}$.

By Lemma 1, we only have to show that for each $R \in \mathcal{D}$ and each $a \in \varphi(R)$, $O(R, a)$ satisfies the two properties (i) and (ii) in the lemma. Let $R \in \mathcal{D}$ and $a \in \varphi(R)$. Since for each $i \in N$, by definition, $O_i(R, a) = SLC(R_i, a, Y) \cup \{a\}$, then $a \in \max[R_i, O_i(R, a)]$. Thus property (i) in Lemma 1 holds. Let $R' \in \mathcal{D}$ and $a' \in Z$ be such that $a' \in \cap_{i \in N} \max[R'_i, O_i(R, a)]$. We prove property (ii) in Lemma 1 in two steps.

Step 1. If $a' \notin \cap_{i \in N} \max[R'_i, Y]$, then $a' = a$.

Assume $a' \notin \cap_{i \in N} \max[R'_i, Y]$. Suppose by contradiction $a' \neq a$. Let $i \in N$ be such that $a' \notin \max[R'_i, Y]$. Then $a' \in SLC(R_i, a, Y)$ and so $a \ P_i \ a'$. Hence for each $i \in N$, $a' \ R'_i \ a$ and $a \ P_i \ a'$. By RD on Y, since $a' \notin \max[R'_i, Y]$, then there is $\bar{a} \in Y$ such that $a \ P_i \ \bar{a}$ and $\bar{a} \ P'_i \ a'$. Hence $\bar{a} \in SLC(R_i, a, Y) \subseteq O_i(R, a)$ and $\bar{a} \ P'_i \ a'$. This contradicts $a' \in \max[R'_i, O_i(R, a)]$.

Step 2. $a' \in \varphi(R')$.

By Step 1, $a' \in \cap_{i \in N} \max[R'_i, Y]$ or $a' = a$. In the former case, $a' \in \varphi(R')$ by unanimity on Y. In what follows, we consider the latter case $a' = a$.

Since $a \in \varphi(R)$ and φ is Nash implementable by $G \equiv (S, g)$, then there is $s \in NE(G, R)$ such that $g(s) = a$. Because s is a Nash equilibrium, for each $i \in N$, $a \in \max[R_i, Atn_i(s)]$, that is, $Atn_i(s) \subseteq LC(R_i, a)$.

In order to show $Atn_i(s) \subseteq LC(R'_i, a)$, suppose that there is $a'' \in Atn_i(s) \setminus \{a\}$ such that $a'' \ P'_i \ a$. Then $a \ R_i \ a''$ and $a'' \ P'_i \ a$. By no punishment on Y, there is no $i \in N$ such that $a \in \min[R_i, Y]$. Hence by part (i) of RD on Y applied to the quadruple a, a'', R_i, and R'_i, there is $\bar{a} \in Y$ such that $a \; P_i \; \bar{a}$ and $\bar{a} \; P'_i \; a$. Hence $\bar{a} \in SLC(R_i, a, Y)$ and $\bar{a} \; P'_i \; a'$ (note $a' = a$), contradicting $a' \in \max[R'_i, O_i(R, a)]$. Therefore for each $i \in N$, $Atn_i(s) \subseteq LC(R'_i, a)$. This shows $s \in NE(G, R')$. Since φ is Nash implementable by (S, g), then $g(s) = a \in \varphi(R')$. ∎

Adding other domain properties, we obtain:

Corollary 1. *Consider a domain $\mathcal{D}$ that satisfies RD on Y.*
(i) *Assume that $\mathcal{D}$ also satisfies no-agreement over Y. Then if a rule on $\mathcal{D}$ is Nash implementable under the range-restriction of Y and satisfies no-punishment on Y, then it is supported by an opportunity system in Y.*
(ii) *Assume that $\mathcal{D}$ also satisfies single-minimum over Y. Then if a rule on $\mathcal{D}$ is Nash implementable under the range-restriction of Y and satisfies unanimity on Y, then it is supported by an opportunity system in Y.*
(iii) *Assume that $\mathcal{D}$ satisfies both no-agreement and single-minimum over Y. Then if a rule on $\mathcal{D}$ is Nash implementable under the range-restriction of Y, it is supported by an opportunity system in Y.*

Proof. (i) If the domain satisfies no-agreement over Y, then unanimity on Y is trivially satisfied by any rule.

(ii) To prove this part, we modify the proof of the theorem in the following way. To show $Atn_i(s) \subseteq LC(R'_i, a)$ in Step 2, we divide two cases. First is when $a \in \min[R_i, Y]$. Then by the single-minimum property, $LC(R_i, a) = \{a\}$ and so $Atn_i(s) = \{a\} \subseteq LC(R'_i, a)$. Second is when $a \notin \min[R_i, Y]$. In this case, we apply part (i) of RD as in the above proof.

(iii) Thus on the domain satisfying both no-agreement and single-minimum, both unanimity and no-punishment in the theorem can be dropped. ∎

We show by an example that without the domain property, RD on Y, Theorem 1 does not hold.

Proposition 1. *If the domain $\mathcal{D}$ does not satisfy RD on Y, then a Nash implementable rule on $\mathcal{D}$ may not be supported by an opportunity system in Y.*

Proof. Let $A = Z = Y \equiv \{a, b, c, d\}$ and $N \equiv \{1, 2, 3\}$. We write, for example, $abcd$ to denote the preference relation that ranks a the first, b the second, c the third, and d the fourth. Consider the domain $\mathcal{D}$ consisting of the following five profiles of preference relations, $R^0 \equiv (abcd, bacd, cbad)$, $R^1 \equiv (dabc, bacd, cbad)$, $R^2 \equiv (abcd, dbac, cbad)$, $R^3 \equiv (abcd, bacd, cdba)$, and $R^4 \equiv (adbc, dabc, cdab)$. Let φ be defined as follows: $\varphi(R^0) \equiv \{b, c\}$ and $\varphi(R^1) = \cdots = \varphi(R^4) \equiv \{c\}$. Then in order to check monotonicity, we only need to confirm there is no $R \in \mathcal{D} \backslash \{R^0\}$ such that for each $i \in N$, $LC(R_i^0, b) \subseteq LC(R_i, b)$ and $b \notin \varphi(R)$. This is true because for each $R \in \mathcal{D} \backslash \{R^0\}$, there is $i \in N$ such that $LC(R_i^0, b) \nsubseteq LC(R_i, b)$. Note that the domain $\mathcal{D}$ satisifes no agreement. Thus φ satisfies no-veto-power trivially.

Note that $b\ P_1^0\ d$ and $d\ P_1^4\ b$. However two sets $\{x \in A : x\ P_1^0\ b\} = \{a\}$ and $\{x \in A : d\ P_1^4\ x\} = \{b, c\}$ are disjoint. Therefore $\mathcal{D}$ violates RD on Y.

We now show that φ is not supported by any opportunity system. Suppose by contradiction that there is a opportunity system $\mathcal{O}$ that supports φ. Then since $b \in \varphi(R^0)$, there exists $O \in \mathcal{O}$ such that $b \in \cap_{i \in N} \max[R_i^0, O_i]$. Therefore, for each $i \in N$, $O_i \subseteq LC(R_i^0, b)$, that is, $O_1 \subseteq \{b, c, d\}$, $O_2 \subseteq \{a, b, c, d\}$, and $O_3 \subseteq \{a, b, d\}$. We derive a contradiction for each of the following cases.

Case 1: $d \notin \cap_{i \in N} O_i$.

Subcase 1.1: $d \notin O_1$. Then $O_1 \subseteq \{b, c\}$. Since $b \in O_1 \subseteq \{b, c\} = LC(R_1^1, b)$, then $b \in \max[R_1^1, O_1]$. Note $b \in \max[R_2^1, O_2] \cap \max[R_3^1, O_3]$. Therefore $b \in \varphi(R^1)$, contradicting $\varphi(R^1) = \{c\}$.

Subcase 1.2: $d \notin O_2$. Then $O_2 \subseteq \{a, b, c\}$. Since $b \in O_2 \subseteq \{a, b, c\} = LC(R_2^2, b)$, then $b \in \max[R_2^2, O_2]$. Note $b \in \max[R_1^2, O_1] \cap \max[R_3^2, O_3]$. Therefore $b \in \varphi(R^2)$, contradicting $\varphi(R^2) = \{c\}$.

Subcase 1.3: $d \notin O_3$. Then $O_3 \subseteq \{a, b\}$. Since $b \in O_3 \subseteq \{a, b\} = LC(R_3^3, b)$, then $b \in \max[R_3^3, O_3]$. Note $b \in \max[R_1^3, O_1] \cap \max[R_2^3, O_2]$. Therefore $b \in \varphi(R^3)$, contradicting $\varphi(R^3) = \{c\}$.

Case 2: $d \in \cap_{i \in N} O_i$.

Then $d \in \cap_{i \in N} \max[R_i^4, O_i]$. Therefore $d \in \varphi(R^4)$, contradicting $\varphi(R^4) = \{c\}$. ∎

We next consider the converse of Theorem 1. The next example shows that the converse does not hold.

Example 3. This is an example by Maskin (1999) showing that Maskin monotonicity is not sufficient for feasible Nash implementability. We use this example to show that supporting a rule by an opportunity system is not sufficient for Nash implementability. Let $A = Z = Y \equiv \{a, b, c\}$. As in the proof of Proposition 1, write, for example, abc to denote the preference relation that ranks a in the top, b in the second, and c in the third. Let $\mathcal{D}$ be the domain consisting of the following three profiles of preference relations:

$$R \equiv (bca, cab, cab)\,; R' \equiv (abc, cba, cab)\,; R'' \equiv (bac, abc, abc)\,.$$

Let φ be the rule on $\mathcal{D}$ such that $\varphi(R) \equiv \{b, c\}$, $\varphi(R') \equiv \{a\}$, and $\varphi(R'') \equiv \{b\}$. Then it can be shown that φ is supported by the opportunity system $\mathcal{O}$ given by

$$\mathcal{O} \equiv \{(\{a,b,c\}, \{b\}, \{b\}), (\{a,c\}, \{b,c\}, \{a,b,c\}), (\{a,b,c\}, \{a\}, \{a,b\})\}.$$

As is shown by Maskin (1999), φ is not Nash implementable.

In our next result, we focus on rules satisfying no-veto-power and establish the converse of Theorem 1. For the implementation of a opportunity rule, we use the following special game form that can be tailored for the opportunity system supporting the rule.

Consider a domain satisfying WSV on Y. Let $\mathcal{O}$ be an opportunity system in Y.

Definition 1 (Opportunity Game Form $G^{\mathcal{O}}$). For each $i \in N$, let $S_i \equiv \mathcal{O} \times Y \times \mathbb{Z}$ be i's strategy set with the generic element $s^i \equiv (O^i, a^i, t^i)$. Let $g\colon S \to A$ be the outcome function defined by the following three states.
State I: There is $(O, a, t) \in \mathcal{O} \times Y \times \mathbb{Z}$ such that for each $i \in N$, $(O^i, a^i, t^i) = (O, a, t)$ and $a \in \cap_{i \in N} O_i \cap Z$. Then let

$$g(s) \equiv a.$$

State II: There are $i \in N$ and $(O, a, t) \in \mathcal{O} \times Y \times \mathbb{Z}$ such that for each $j \neq i$, $(O^j, a^j, t^j) = (O, a, t)$, $a \in \cap_{j \in N} O_j \cap Z$, and $(O^i, a^i, t^i) \neq (O, a, t)$. Then let

$$g(s) \equiv \begin{cases} e_i\left(a^i\right), & \text{if } a^i \in O_i, \\ a, & \text{otherwise.} \end{cases}$$

State III: In all other cases, let

$$g(s) \equiv e_h\left(a^h\right)$$

where $h \equiv \min\{i \in N : t^i \in \max\{t^1, \cdots, t^n\}\}$.

Thus each player announces a profile of opportunity sets for all players, his demand, and an integer. When all players reach an agreement, the agreed outcome prevails. If a partial agreement is reached among all players except one, then the agreed outcome prevails unless the violator demands an outcome that is allowed by others ($a^i \in O_i$), in which case the violater gets his demand. When there is no full or partial agreement, the player with the largest voice t^i can get what he wants. Note that in this game, players' preferences are not necessarily a part of players' strategies.[4] We allow strategies to depend on a opportunity system supporting the opportunity rule. Thus, opportunity rules with "simple" opportunity systems can have equally simple strategy spaces.

Note that if the opportunity system $\mathcal{O}$ is in Y, then the opportunity game form $G^{\mathcal{O}}$ has the range (of the outcome function) in Y.

Theorem 2. *Let $\mathcal{D}$ be a domain satisfying WSV on Y and $|N| \geq 3$. If a rule on $\mathcal{D}$ is supported by an opportunity system $\mathcal{O}$ in Y and satisfies no-veto-power on Y, then it is Nash implementable in Y by the opportunity game form $G^{\mathcal{O}}$.*

Proof. Let φ be a rule supported by an opportunity system $\mathcal{O}$ in Y. Under the stated assumptions, we show that φ is Nash implemented by the opportunity game form $G^{\mathcal{O}}$. In what follows, we fix $R^* \in \mathcal{D}$ and show that $\varphi(R^*) = g(NE(G^{\mathcal{O}}, R^*))$ in two steps.

Step 1: $\varphi(R^*) \subseteq g\left(NE\left(G^{\mathcal{O}}, R^*\right)\right).$

Let $a^* \in \varphi(R^*)$. Then $a^* \in Z$ and there exists $O \in \mathcal{O}$ such that for each $i \in N$, $a^* \in \max[R_i^*, O_i]$. Let $s_i = (O, a^*, 1)$ for each $i \in N$. Then State I applies at s.

Let $i \in N$. Then for each $a' \in O_i \backslash \{a^*\}$, i can attain $e_i(a')$ by announcing a'. Thus $Atn_i(s_{-i}) = \{e_i(a') : a' \in O_i\} \cup \{a^*\}$. Since $a^* \in \max[R_i^*, O_i]$ and for each $a' \in O_i$, $a' \; I_i^* \; e_i(a')$, then $g(s_i, s_{-i}) = a^* \in \max[R_i^*, Atn_i(s_{-i})]$. Therefore $s \in NE(G^{\mathcal{O}}, R^*)$ and $a^* \in g(NE(G^{\mathcal{O}}, R^*))$.

Step 2: $g\left(NE\left(G^{\mathcal{O}}, R^*\right)\right) \subseteq \varphi(R^*).$

Let $a^* \in g(NE(G^{\mathcal{O}}, R^*))$ and $s^* \in NE(G^{\mathcal{O}}, R^*)$ be such that $g(s^*) = a^*$. We consider each of the three states one by one.

State I: *There is $(O, a, t) \in \mathcal{O} \times Y \times \mathbb{Z}$ such that for each $i \in N$, $s_i^* = (O, a, t)$ and $a \in \cap_N O_i \cap Z$.*

Clearly, then $a = a^*$ and as shown above, for each $i \in N$, $Atn_i\left(s_{-i}^*\right) = \{e_i(a') : a' \in O_i\} \cup \{a^*\}$. Since s^* is a Nash equilibrium and for each $a' \in O_i$, $e_i(a') \; I_i^* \; a'$, then for each $i \in N$, $a^* \in \max[R_i^*, O_i]$. Therefore $a \in \varphi(R^*)$.

State II: *There exist $i \in N$ and $(O, a, t) \in \mathcal{O} \times Y \times \mathbb{Z}$ such that for each $j \neq i$, $s_j^* = (O, a, t)$, $a \in \cap_{j \in N} O_j \cap Z$,* and $s_i^* = (O', a', t') \neq (O, a, t)$.

[4] Preferences announcement is essential in the game forms used by Maskin (1977, 1999), Saijo (1988), and Moore and Repullo (1990).

We first show that the Nash equilibrium outcome $g(s^*) = a^*$ is feasible. Note $g(s^*) = a$ or $e_i(a')$. Since $a \in Z$, then if $g(s^*) = a^* \notin Z$, $g(s^*) = e_i(a')$ and by WSV on Y, there are $j \neq i$ and $a'' \in Y$ such that $a'' \; P_j^* \; a^*$. Since $e_j(a'') \in Atn_j(s^*_{-j})$, this contradicts s^* being a Nash equilibrium. Therefore, $g(s^*) = a^* \in Z$.

For each $j \neq i$ and each $a'' \in Y$, j can attain $e_j(a'')$ by announcing a sufficiently large integer. Thus $\{e_j(a'') : a'' \in Y\} \subseteq Atn_j\left(s^*_{-j}\right)$. Since $a^* \in Z$ and s^* is a Nash equilibrium, then $a^* \in \cap_{j \in N \setminus i} \max[R_j^*, Y] \cap Z$. Therefore by *no-veto-power on* Y, $a^* \in \varphi(R^*)$.

State III. We can show $g(s^*) = a^* \in Z$ using the same argument as in State II. On the other hand, for each $i \in N$, $\{e_i(a') : a' \in Y\} \subseteq Atn_i(s^*_{-i})$. Therefore, since s^* is a Nash equilibrium, $a^* \in \cap_{i \in N} \max[R_i^*, Y] \cap Z$. By *no-veto-power on* Y, $a^* \in \varphi(R^*)$. ∎

Remark 1. By Theorem 2, implementation of a rule can be decomposed into two processes: (i) the first is to identify an opportunity system supporting the rule and (ii) the second is to use this opportunity system in the definition of the game form implementing the rule.

The assumption $|N| \geq 3$ plays a crucial role in Theorem 2. When there are two agents, even if an opportunity rule satisfies no-veto-power, it may not be Nash implementable. This is shown by the next example.

Example 4. This example is due to Moore and Repullo (1990), where they use this example to show that in the two-agent case, Maskin monotonicity and no-veto-power do not imply feasible Nash implementability. Let $A \equiv Z \equiv \{a, b\}$. Using the same notational convention used before, let $R_0 \equiv ab$ and $R_0' \equiv ba$. Let $\mathcal{R} \equiv \{R_0, R_0'\}$ and $\mathcal{D} \equiv \mathcal{R}^2$. Let φ be defined as follows: $\varphi(R_0, R_0') \equiv \{a, b\} \equiv \varphi(R_0', R_0)$, $\varphi(R_0, R_0) \equiv \{a\}$, and $\varphi(R_0', R_0') \equiv \{b\}$. Then φ is supported by the following opportunity system $\mathcal{O}$:

$$\mathcal{O} \equiv \{(\{a\}, \{a, b\}), (\{a, b\}, \{a\}), (\{b\}, \{a, b\}), (\{a, b\}, \{b\}), (\{a, b\}, \{a, b\})\},$$

Clearly, φ satisfies *no-veto-power.* Since $A = Z$, the domain satisfies WSV. However, φ is not *Nash implementable.* This can be shown as follows. Suppose to the contrary that there exists a game form $G \equiv (S, g)$ such that $\varphi = g(NE(\cdot))$. Then given (R_0, R_0'), there exists a Nash equilibrium strategy $\bar{s}$ such that $g(\bar{s}) = a$. Since $b \; P_0' \; a$, then for all $s_2 \in S_2$, $g(\bar{s}_1, s_2) = a$. Also, given (R_0', R_0), there exists a Nash equilibrium strategy $\bar{s}'$ such that $g(\bar{s}') = a$. Then similarly, we can show that for all $s'_1 \in S_1$, $g(s'_1, \bar{s}'_2) = a$. Therefore, for $(\bar{s}_1, \bar{s}'_2)$, both individuals have attainable set $\{a\}$. Therefore, $a \in g(NE(R'_0, R'_0))$. This contradicts $\varphi(R'_0, R'_0) = \{b\}$.

We next establish a direct logical relation between *opportunity supportability in* Y and *monotonicity on* Y.

Theorem 3. *If a rule is supported by an opportunity system in Y, then it is a subcorrespondence of Y and satisfies monotonicity on Y. Conversely, when the domain satisfies RD on Y, if a rule is a subcorrespondence of Y and satisfies monotonicity and unanimity on Y, then it is supported by an opportunity system in Y.*

Proof. We omit the trivial proof of the first statement.

Let $\mathcal{D}$ satisfy RD on Y. Let φ be a subcorrespondence of Y satisfying *monotonicity on* Y. We show that the opportunity system constructed in Proof of Theorem 1 supports φ. For each $R \in \mathcal{D}$, each $a \in \varphi(R)$, and each $i \in N$, let $O_i(R, a) \equiv SLC(R_i, a; Y) \cup \{a\}$. Let $\mathcal{O}^{\varphi} \equiv \{O(R, a) : R \in \mathcal{D} \text{ and } a \in \varphi(R)\}$. Since φ is a subcorrespondence of Y, $\mathcal{O}^{\varphi}$ is a opportunity system in Y.

Let $R \in \mathcal{D}$ and $a \in \varphi(R)$ be given. We only have to show that $O(R, a)$ satisfies (i) and (ii) in Lemma 1. Clearly, by construction, for each $i \in N$, $a \in \max[R_i, O_i(R, a)]$. Hence part (i) holds. Let $R' \in \mathcal{D}$ and $a' \in Z$ be such that $a' \in \cap_{i \in N} \max[R'_i, O_i(R, a)]$. We prove part (ii) of Lemma 1 in two steps.

Step 1: If $a' \notin \cap_{i \in N} \max[R'_i, Y]$, then $a' = a$.

Assume $a' \notin \cap_{i \in N} \max[R'_i, Y]$. Suppose by contradiction $a' \neq a$. Let $i \in N$ be such that $a' \notin \max[R'_i, Y]$. Then $a' \; R'_i \; a$ and $a \; P_i \; a'$. By RD on Y, since $a' \notin \max[R'_i, Y]$, then there is $\bar{a} \in Y$ such that $a \; P_i \; \bar{a}$ and $\bar{a} \; P'_i \; a'$. Hence $\bar{a} \in SLC(R_i, a; Y) \subseteq O_i(R, a)$ and $\bar{a} \; P'_i \; a'$. This contradicts $a' \in \max[R'_i, O_i(R, a)]$.

Step 2: $a' \in \varphi(R')$.

By Step 1, $a' \in \cap_{i \in N} \max[R'_i, Y]$ or $a' = a$. In the former case, $a' \in \varphi(R')$ by unanimity on Y. Now consider the latter case $a' = a$.

Since $a \in \cap_{i \in N} \max[R'_i, O_i(R, a)]$, then for each $i \in N$, $SLC(R_i, a; Y) \subseteq LC(R'_i, a; Y)$.

If there exist $i \in N$ and $\hat{a} \in LC(R_i, a; Y) \backslash LC(R'_i, a; Y)$, then, $\hat{a} \; I_i \; a$ and $\hat{a} \; P'_i \; a$. And by RD on Y, there is $\bar{a} \in Y$ such that $a \; P_i \; \bar{a}$ and $\bar{a} \; P'_i \; a$, contradicting $a \in \max[R'_i, O_i(R, a)]$. Therefore, for each $i \in N$, $LC(R_i, a; Y) \subseteq LC(R'_i, a; Y)$. Since $a' = a$, $a \in \varphi(R)$ and φ satisfies monotonicity on Y, then $a' \in \varphi(R')$. ∎

4 Applications and Further Results in Economic Environments

In this section, we assume that A is a subset of a topological vector space. For simplicity, we consider the case when A is a subset of a Euclidean space $\mathbb{R}^l$ with $l \in \mathbb{N} \cup \{\infty\}$.

Additional Properties of Preferences

Associated with the added structure of the alternative space A, there are some widely considered properties of preferences. A preference R_0 is *continuous* if for each $z \in A$, both $SUC(R_0, z)$ and $SLC(R_0, z)$ are open. It is *convex* if for each pair $z, z' \in A$ and each $\lambda \in [0, 1]$ with $z \; R_0 \; z'$, $\lambda z + (1 - \lambda)z' \; R_0 \; z'$. It is *strictly convex* if for each pair $z, z' \in A$ and each $\lambda \in (0, 1)$ with $z \; R_0 \; z'$ and $z \neq z'$, $\lambda z + (1 - \lambda)z' \; P_0 \; z'$. It is *locally non-satiated* if for each $z \in A$ and each open neighborhood $\mathcal{U} \subseteq A$ of z, there is $z' \in \mathcal{U}$ such that $z' \; P_0 \; z$.

Let Y be a subset of A that does not have any isolated point. A preference R_0 has *no non-global-satiation over* Y if for each $z \in Y$ with $z \notin \max[R_0, Y]$[5] and each open neighborhood $\mathcal{U} \subseteq A$ of z, there is $z' \in \mathcal{U} \cap Y$ such that $z' \; P_0 \; z$. Thus R_0 may have a "global satiation point" over Y, $z \in \max[R_0, Y]$, but it cannot have any local satiation point that is not a global satiation point. Examples are monotonic preferences, single-peaked preferences, multiple-peaked preferences where all peak points are indifferent, etc. When $Y = A$, local non-satiation implies no non-global-satiation over A but the converse does not hold (for example, consider single-peaked preferences). A preference R_0 is *conditionally non-inverse-satiated over* Y if for each pair $z, z' \in Y$ with $z \; R_0 \; z'$ and $z \neq z'$ and each open neighborhood $\mathcal{U} \subseteq A$ of z', there is $\bar{z} \in \mathcal{U} \cap Y$ such that $z \; P_0 \; \bar{z}$ (when $z \; P_0 \; z'$, this requirement is trivial because we may let $\bar{z} = z'$). Thus any alternative with a non-singleton lower contour set (or indifference set, resp.) has an inferior alternative in any of open neghborhoods of each element in the lower contour set (or indifference set, resp.). Note that if a preference satisfies condition non-inverse-satiation over Y, then it cannot have two minimal elements in Y. Thus the domain satisfies single-minimum over Y. For example, in exchange economies with two private goods, convex and conditionally non-inverse-satiated preferences cannot have two indifferent alternatives with zero consumption of a good.

[5]Without this condition, the property is too strong. This is because when Y is closed and R_0 is continuous, z can be R_0-maximal in Y and in this case, the condition to follow will not hold.

4.1 Semi-Individualistic Environments

Alternatives in standard economic environments have both public and private components. To capture this aspect and also to differentiate various examples, the following domain property is useful. A domain $\mathcal{D}$ is *semi-individualistic* if there exist a nontrivial partition $\Pi \equiv \{\pi_1, \cdots, \pi_p\}$ of N (nontriviality means $p \geq 2$) and $(p+1)$-component sets denoted by $X_0, X_1, \cdots, X_p$ such that $A \equiv X_0 \times X_1 \times \cdots \times X_p$ and for each $R \in \mathcal{D}$, each $q \in \{1, \cdots, p\}$, and each pair $z \equiv (x_0, x_1, \cdots, x_p)$, $z' \equiv (x'_0, x'_1, \cdots, x'_p) \in A$, whenever $(x_0, x_q) = (x'_0, x'_q)$, all agents in π_q are indifferent between z and z'. When each group of the partition Π is a singleton, we say $\mathcal{D}$ is *individualistic.* An element $w_q \in X_q$ is a *never-a-best bundle for π_q over Y* if for each $R \in \mathcal{D}$, each $x_0 \in X_0$, and each $x_{-0,q} \in \Pi_{q' \in \{1,\ldots,p\} \setminus \{q\}} X_{q'}$, there exist $i \in \pi_q$ and $z \in Y$ such that $z\ P_i\ (x_0, x_q, x_{-0,q})$.[6] Note that each never-a-best bundle over Y can never be a part of a best alternative over Y, whatever other components are combined with it.

When a domain is semi-individualistic with respect to both Π and Π', the domain is also semi-individualistic with respect to the coarsest common refinement of the two partition. Therefore, for each semi-individualistic domain, there is the unique finest partition with respect to which it is semi-individualistic.

Proposition 2. *If a domain is composed of profiles of preferences that satisfy continuity, no non–global-satiation and conditional non-inverse-satiation over Y, then the domain satisfies RD and single-minimum over Y.*

Proof. Let $\mathcal{D}$ be a domain composed of profiles of preferences that satisfy continuity, no non-global-satiation over Y and conditional non-inverse-satiation over Y. Let $R, R' \in \mathcal{D}$, $i \in N$, and $z, z' \in Y$ be such that $z\ R_i\ z'$ and $z'\ P'_i\ z$. By continuity of R'_i, there is an open neighborhood $\mathcal{U}$ of z' such that $\mathcal{U} \cap Y \subseteq SUC(R'_i, z; Y)$. Since R_i is conditionally non-inverse-satiated over Y, there is $\bar{z} \in \mathcal{U} \cap Y$ such that $z\ P_i\ \bar{z}$. Therefore, $z\ P_i\ \bar{z}$ and $\bar{z}\ P'_i\ z$. Assume $z \notin \max[R_i, Y]$. By continuity of R'_i, there is an open neighborhood $\mathcal{U}'$ of z such that $\mathcal{U}' \cap Y \subseteq SLC(R'_i, z'; Y)$. Since R_i has no non-global-satiation over Y and $z \notin \max[R_i, Y]$, there is $\bar{z}' \in \mathcal{U}' \cap Y$ such that $\bar{z}'\ P_i\ z$. Therefore $\bar{z}'\ P_i\ z$ and $z'\ P'_i\ \bar{z}'$.

Single-minimum over Y follows from conditional non-inverse-satiation over Y. ∎

Proposition 3. *If a domain is semi-individualistic with respect to a partition Π and each group in Π has a never-a-best bundle over Y, then the domain satisfies WSV on Y.*

[6] Note the difference between never-a-best *bundles* and never-a-best *alternatives*.

Proof. Let $\mathcal{D}$ be semi-individualistic with respect to a partition $\Pi \equiv \{\pi_1, \cdots, \pi_p\}$ of N and $X_0, X_1, \cdots, X_p$. Thus $A \equiv X_0 \times X_1 \times \cdots \times X_p$. For each $q \in \{1, \cdots, p\}$, let $w_q \in X_q$ be never-a-best for π_q over Y. For each $z \equiv (x_0, x_1, \cdots, x_p) \in Y$, each $q \in \{1, \cdots, p\}$, and each $i \in \pi_q$, let $e_i(z) \equiv (x_0, w_1, \ldots, w_{q-1}, x_q, w_{q+1}, \ldots, w_p)$. Then by semi-individualisticity, for each $R \in \mathcal{D}$, $z \ I_i \ e_i(z)$. For each $q' \neq q$, since $w_{q'}$ is never-a-best for $\pi_{q'}$ over Y, then there exist $j \in \pi_{q'}$ and $z' \in Y$ such that $z' \ P_j \ e_i(z)$. ∎

Next are some well-known examples of economic environments where the two propositions and our general results in Section 3 apply.

Example 5 (Classical Private Goods Economy). There are $m \in \mathbb{N}$ private goods. For each $i \in N$, let $X_i \equiv \mathbb{R}^m_+$. Let $\omega_i \in X_i$ be the initial endowment of i. Let $X_0 \equiv \varnothing$ and $A \equiv \mathbb{R}^{|N| \times m}_+$. Let $T \subseteq \mathbb{R}^m$ be the production possibility set and $Z \equiv \{(x_i)_{i \in N} \in \mathbb{R}^{|N| \times m}_+ : \sum_i (x_i - \omega_i) \in T\}$ the set of feasible allocations. Let $\mathcal{D}$ be the class of all profiles of preferences over A that satisfy the following property: for each $i \in N$ and each pair $z \equiv (x_j)_{j \in N}, z' \equiv (x'_j)_{j \in N} \in A$, if $x_i = x'_i$, then i is indifferent between z and z'. Then $\mathcal{D}$ is individualistic with respect to the finest partition. Assume further that each preference in the domain satisfies continuity, local non-satiation, and conditional non-inverse-satiation over A. Then each preference in the domain has no non-global-satiation over A. Therefore, by Proposition 2, $\mathcal{D}$ satisfies RD and single-minimum over A. Since preferences are locally non-satiated, then for each $i \in N$, any bundle for agent i is never-a-best bundle for i over A. Thus by Proposition 3, $\mathcal{D}$ satisfies WSV on A.
If all preferences in $\mathcal{D}$ satisfy, in addition, monotonicity and convexity, then $\mathcal{D}$ has no non-global-satiation "over Z" and so by Proposition 2, $\mathcal{D}$ satisfies RD and single-minimum over Z.[7] Since for each $i \in N$, there is a bundle, e.g. $0 \in \mathbb{R}^m_+$, which is a never-a-best bundle for i over Z, by Proposition 3, $\mathcal{D}$ satisfies WSV on Z.

Example 6 (Classical Public Goods Economy). There are $m \in \mathbb{N}$ private goods and $l \in \mathbb{N}$ public goods. For each $i \in N$, let $X_i \equiv \mathbb{R}^m_+$ be i's private goods consumption space and $\omega_i \in \mathbb{R}^{m \cdot l}_+$ i's endowment. Let $Y \equiv \mathbb{R}^l_+$ be the public goods consumption space. Let $T \subseteq \mathbb{R}^{m \cdot l}$ be the production possibility set and $Z \equiv \{(x_1, \cdots, x_n, y) \in A : ((\sum_i x_i, y) - \sum_i \omega_i) \in T\}$ the set of feasible allocations. Let $\mathcal{D}$ be the class of all profiles of preferences over A that satisfy the following property: for each $i \in N$ and each pair $z \equiv ((x_j)_{j \in N}, y), z' \equiv ((x'_j)_{j \in N}, y') \in A$,

[7] Without either monotonicity or convexity, we can find preferences that have non-global satiation points over Z.

and each $R \in \mathcal{D}$, if $(x_i, y) = (x'_i, y')$, i is indifferent between z and z', that is, $z \; I_i \; z'$. Then $\mathcal{D}$ is individualistic with respect to the finest partition. Assume further that each preference satisfies continuity, local non-satiation, and conditional non-inverse-satiation over A. Then as in the previous example, each preference in the domain has no non-global-satiation over A. Therefore, by Proposition 2, $\mathcal{D}$ satisfies RD and single-minimum over A. Since preferences are locally non-satiated, then for each $i \in N$, any private bundle for agent i is never-a-best bundle for i over A. Thus by Proposition 3, $\mathcal{D}$ satisfies WSV on A.
Assume that all preferences in $\mathcal{D}$ satisfy, in addition, monotonicity and convexity and that zero private consumption $0 \in \mathbb{R}^m_+$ is a never-a-best bundle for each agent over Z. Then $\mathcal{D}$ has no non-global-satiation "over Z" and so by Proposition 2, $\mathcal{D}$ satisfies RD and single-minimum over Z. Since for each $i \in N$, there is a bundle, e.g. $0 \in \mathbb{R}^m_+$, which is a never-a-best bundle for i over Z, by Proposition 3, $\mathcal{D}$ satisfies WSV on Z.

Example 7 (Single-Peaked Preferences). Let $A = Z \equiv \mathbb{R}$ (we can also consider the general case of multi-dimensional alternative space). Assume that each agent has a single-peaked preference over A, R_0, that is, there is a peak location $x_0 \in [a, b]$ such that i's welfare strictly increases by moving toward x_0 in either direction. Given such a domain, each preference has the unique global satiation point and no other local satiation point. Thus it has no non-global-satiation point. Since moving away from the peak always decreases the welfare and $\mathbb{R}$ is open, the preference has no minimal point and so it satisfies conditional non-inverse satiation. This domain is not semi-individualistic. Thus we cannot apply Proposition 3 to show WSV. However, because $A = Z$, WSV on Z is trivially satisfied. Note that openness of $A = Z$ plays an important role for showing RD.

4.2 Closed or Convex Opportunity Systems and Asymptotically Feasible Nash Implementation

The opportunity sets used for proving Theorem 1 are not closed in economic environments. This was inevitable to prevent unwanted alternatives from becoming opportunity equilibria while constructing a generally applicable opportunity system. A *closed opportunity system* is an opportunity system with closed opportunity sets. We next investigate what rules are represented by a closed opportunity system.

The next axiom is crucial. It says that if an alternative is considered as being desirable, then any other alternative that is indifferent to this alternative for all agents should also be considered as being desirable.

Pareto Indifference. For each $R \in \mathcal{D}$ and each pair $z, z' \in Z$, if $z \in \varphi(R)$ and for each $i \in N$, $z'\ I_i\ z$, then $z' \in \varphi(R)$.

Although this axiom sounds very reasonable, some well known rules violate it. No-envy rule is an example.

We now strengthen Theorem 1 by focusing on Pareto indifferent rules and considering the case $Y = A$. Unlike in Theorem 1, we do not need unanimity on A for the next result.

Theorem 4. *Assume that $\mathcal{D}$ satisfies RD on A and preferences in $\mathcal{D}$ are continuous and locally non-satiated. If a rule on $\mathcal{D}$ is Nash implementable (without any range-restriction) and satisfies Pareto indifference and no punishment on A, then it is supported by a closed opportunity system.*

Proof. Let $\mathcal{D}$ satisfy RD on A. Let φ be a rule on $\mathcal{D}$ that is Nash implementable (under no range-restriction) and satisfies Pareto indifference. Let $G \equiv (S, g)$ be the game form implementing φ. For each $R \in \mathcal{D}$, each $z \in \varphi(R)$, and each $i \in N$, let $\bar{O}_i(R, z) \equiv LC(R_i, z)$. Let $\bar{O}(R, z) \equiv (\bar{O}_i(R, z))_{i \in N}$. Let $\overline{\mathcal{O}}^{\varphi} \equiv \{\bar{O}(R, z) : R \in \mathcal{D}, z \in \varphi(R)\}$. We only have to show that for each $R \in \mathcal{D}$ and each $z \in \varphi(R)$, $\bar{O}(R, z)$ satisfies (i) and (ii) of Lemma 1.

Let $R \in \mathcal{D}$ and $z \in \varphi(R)$. Then for each $i \in N$, by definition ($\bar{O}_i = LC(R_i, z, Y)$), $z \in \max\left[R_i, \bar{O}_i(R, z)\right]$ and so (i) holds. Let $R' \in \mathcal{D}$ and $z' \in Z$ be such that $z' \in \cap_{i \in N} \max[R'_i, \bar{O}_i(R, z)]$.

Step 1. For each $i \in N$, $z'\ I_i\ z$.

Let $i \in N$. Note that $z'\ R'_i\ z$ and $z\ R_i\ z'$. Suppose $z\ P_i\ z'$. Then $z'\ R'_i\ z$ and $z\ P_i\ z'$. By continuity of R_i, there is an open neighborhood $\mathcal{U}$ of z' such that $\mathcal{U} \subseteq SLC(R_i, z)$. By local non-satiation of R'_i, there is $\bar{z}' \in \mathcal{U}$ such that $\bar{z}'\ P'_i\ z'$. Then $z\ P_i\ \bar{z}'$ and $\bar{z}'\ P'_i\ z'$, contradicting $z' \in \max[R'_i, \bar{O}_i]$ (recall $\bar{O}_i = LC(R_i, z)$).

Step 2. $z' \in \varphi(R')$.

By Step 1 and Pareto indifference, $z' \in \varphi(R)$. Note that for each $i \in N$, $LC(R_i, z') = LC(R_i, z)$. Since φ is implemented by $G \equiv (S, g)$ and $z' \in \varphi(R)$, there is a Nash equilibrium strategy profile $s \in S$ such that $g(s) = z'$. Suppose that there exist $i \in N$ and $s'_i \in S_i$ such that $g(s'_i, s_{-i})\ P'_i\ z'$. Then $z'\ R_i\ g(s'_i, s_{-i})$ and $g(s'_i, s_{-i})\ P'_i\ z'$. By no punishment on A, $z' \notin \min[R_i, A]$. Hence applying part (i) of RD on A for the quodruple z', $g(s'_i, s_{-i})$, R_i, and R'_i, there is $\bar{z} \in A$ such that $z' \; P_i \; \bar{z}$ and $\bar{z} \; P'_i \; z'$, contradicting $z' \in \max[R'_i, \bar{O}_i]$ (recall $\bar{O}_i = LC(R_i, z)$). Therefore, s is a Nash equilibrium strategy profile for R' too and the result follows from Nash implementability. ∎

The converse of Theorem 4 does not hold. We later show that in exchange economies, no-envy rule is supported by a closed opportunity system but it does not satisfy Pareto indifference.

When $Y \not\subset Z$, the outcome functions we used for the implementation of opportunity rules associated with an opportunity system in Y may take infeasible values. We next show that this infeasibility can be avoided approximately, when the opportunity system has the convexity property. Formally, a system is called a *convex opportunity system* if each opportunity set in a profile in the system is convex. Examples are Walrasian rule in exchange economies, Lindahl rule in public goods economies, etc.

A rule φ is *asymptotically feasibly Nash implementable with the range-restriction of* Y if there is a sequence of game forms $(G^m \equiv (S^m, g^m) : m \in \mathbb{N})$ such that for each $a \in Y \backslash Z$, there is $\bar{m}$ such that when $m \geq \bar{m}$, $a \notin g^m(S^m)$. When Z is compact and all ranges of g^m are compact, the convergence of $(g^m(S^m) : m \in \mathbb{N})$ to a subset of Z in Hausdorff topology implies asymptotically feasible Nash implementability with the range restriction of Y.

The next lemma is a well-known fact saying that for continuous and convex preferences, a local maximum over a convex opportunity set is a global maximum. Although the proof is standard, we add it for completeness.

Lemma 2. *Let $B \subseteq A$. Consider a continuous and convex preference relation R_0. If $O_0 \subseteq A$ is convex, z is an interior point of B and $z \in \max[R_0, O_0 \cap B]$, then $z \in \max[R_0, O_0]$.*

Proof. Let B, O_0, z, and R_0 be given as stated above. Assume $z \in \max[R_0, O_0 \cap B]$. Suppose to the contrary that there exists $z' \in O_0$ such that $z' \; P_0 \; z$. Then by convexity of O_0, for all $\lambda \in (0,1)$, $\lambda z' + (1-\lambda)z \in O_0$. Since $z' \; P_0 \; z$, then by convexity and continuity of R_0, for all $\lambda \in (0,1)$, $[\lambda z' + (1-\lambda)z] \; P_0 \; z$. Since z is an interior point of B, there is $\lambda \in (0,1)$ such that $\lambda z' + (1-\lambda)z \in O_0 \cap B$. This contradicts $z \in \max[R_0, O_0 \cap B]$. ∎

We now show that under some additional but mild assumptions on the domain, opportunity rules associated with a convex opportunity system are asymptotically feasibly Nash implementable.

Theorem 5. *Assume that* $|N| \geq 3$ *and* Y *is convex. Assume that there is a decreasing sequence of subsets of* Y, $(Y^k\colon k \in \mathbb{N}, Y^k \subseteq Y)$, *such that* $\cap_{k\in\mathbb{N}}[Y^k \cap Z] = Y \cap Z$, *for each* $k \in \mathbb{N}$, *all alternatives of* $Y \cap Z$ *are in the interior of* Y^k, *and domain* $\mathcal{D}$ *satisfies WSV on* Y^k.
Then if a rule on $\mathcal{D}$ *is supported by a convex opportunity system in* Y *and satisfies no-veto-power on* Y, *then it is asymptotically feasibly Nash implementable under the range-restriction of* Y.

Proof. Let Y, $(Y^k\colon k \in \mathbb{N}, Y^k \subseteq Y)$ and $\mathcal{D}$ be given as stated in the theorem. Let φ be a rule on $\mathcal{D}$ that is supported by a convex opportunity system $\mathcal{O}$ in Y and satisfies no-veto-power on Y. In order to show that φ is asymptotically feasibly Nash implementable, we make use of the following variant of the opportunity game form in Definition 1 that is used in the proof of Theorem 2.

For each $k \in \mathbb{N}$, let $Y^k \equiv Y \cap Z^k$ and let $G^{k,\mathcal{O}}$ be the game form defined by the following strategy sets and outcome function. For each $i \in N$, let $S_i = \mathcal{O} \times Y^k \times \mathbb{N}$ be i's strategy set with generic element $s^i = (O^i, z^i, t^i)$. Let $g^k\colon S \to A$ be the same outcome function as in Definition 1 except for the above difference in its domain: that is, g^k is defined by the following states.

State I: There is $(O, a, t) \in \mathcal{O} \times Y^k \times \mathbb{Z}$ such that $a \in \cap_{i\in N} O_i \cap Z$ and for each $i \in N$, $(O^i, a^i, t^i) = (O, a, t)$. Then let

$$g^k(s) \equiv a.$$

State II: There exist $(O, a, t) \in \mathcal{O} \times Y^k \times \mathbb{Z}$ and $i \in N$ such that $a \in \cap_{j\in N} O_j \cap Z$, for each $j \neq i$, $(O^j, a^j, t^j) = (O, a, t)$, and $(O^i, a^i, t^i) \neq (O, a, t)$. Then let

$$g^k(s) \equiv \begin{cases} e_i\left(a^i\right), \text{ if } a^i \in O_i, \\ a, \text{ otherwise,} \end{cases}$$

where $e_i\left(a^i\right) \in Y^k$ is such that for each $R \in \mathcal{D}$, (i) $e_i(a^i)\ I_i\ a^i$ and (ii) either $e_i(a^i) \in Z$ or for some $j \neq i$ and some $b \in Y^k$, $b\ P_j\ e_i(a^i)$ (note that such $e_i(a^i)$ exists because of WSV on Y^k).

State III: In all other cases, let

$$g^k(s) \equiv e_h\left(a^h\right),$$

where $h \equiv \min\left\{i \in N : t^i \in \max\left\{t^1, \cdots, t^n\right\}\right\}.$

In what follows, we fix $R^* \in \mathcal{D}$ and show $\varphi(R^*) = g^k(NE(G^{k,\mathcal{O}}, R^*))$ in two steps.

Step 1: $\varphi(R^*) \subseteq g^k\left(NE\left(G^{k,\mathcal{O}}, R^*\right)\right)$.

Let $a^* \in \varphi(R^*)$. Then $a^* \in Z$ and there exists $O \in \mathcal{O}$ such that for each $i \in N$, $a^* \in \max[R_i^*, O_i]$. Since $\mathcal{O}$ is an opportunity system in Y, $a^* \in Y \cap Z$. Thus $a^* \in Y^k \cap Z$. For each $i \in N$, let $s_i \equiv (O, a^*, 1)$. Then State I applies at s.

For each $i \in N$, $Atn_i(s) = \left\{e_i(a') : a' \in O_i \cap Y^k\right\} \cup \{a^*\}$. Since $a^* \in \max[R_i^*, O_i]$ and for each $a' \in O_i \cap Y^k$, $a'\ I_i^*\ e_i(a')$, then $g^k(s_i, s_{-i}) = a^* \in \max[R_i^*, Atn_i(s)]$. Therefore $s \in NE(G^{k,\mathcal{O}}, R^*)$ and $a^* \in g^k(NE(G^{k,\mathcal{O}}, R^*))$.

Step 2: $g^k\left(NE\left(G^{k,\mathcal{O}}, R^*\right)\right) \subseteq \varphi(R^*)$.

Let $a^* \in g^k(NE(G^{k,\mathcal{O}}, R^*))$ and $s^* \in NE(G^{k,\mathcal{O}}, R^*)$ be such that $g^k(s^*) = a^*$. We consider each of the three possible states one by one.

State I: There is $(O, a, t) \in \mathcal{O} \times Y^k \times \mathbb{Z}$ such that $a \in \cap_{i \in N} O_i \cap Z$ and for each $i \in N$, $s_i^* = (O, a, t)$.

Clearly, then $a = a^*$ and for each $i \in N$, $Atn_i(s^*) = \left\{e_i(a') : a' \in O_i \cap Y^k\right\} \cup \{a^*\}$. Since s^* is a Nash equilibrium and for each $a' \in O_i \cap Y^k$, $e_i(a')\ I_i^*\ a'$, then for each $i \in N$, $a^* \in \max[R_i^*, O_i \cap Y^k]$. Since $a^* \in Y \cap Z$, a^* is an interior point of Y^k. Thus by convexity of O_i, applying Lemma 2, $a \in \max[R_i^*, O_i]$. Since φ is supported by $\mathcal{O}$, $a = a^* \in \varphi(R^*)$.

State II: There exist $(O, a, t) \in \mathcal{O} \times Y^k \times \mathbb{Z}$ and $i \in N$ such that $a \in \cap_{j \in N} O_j \cap Z$, for each $j \neq i$, $s_j^* = (O, a, t)$, and $s_i^* = (O', a', t') \neq (O, a, t)$.

We first show that the Nash equilibrium outcome $g^k(s^*) = a^*$ is feasible. Note that $g^k(s^*) = a$ or $e_i(a')$. Since $a \in Z$, then if $g^k(s^*) = a^* \notin Z$, $g^k(s^*) = e_i(a')$ and by WSV on Y^k, there exist $j \neq i$ and $b \in Y^k$ such that $b\ P_j^*\ a^*$. Since j can attain $e_j(b)$, this contradicts s^* being a Nash equilibrium.

For each $j \neq i$, $\{e_j(b) : b \in Y^k\} \subseteq Atn_i(s^*)$. Since $a^* \in Z$ and s^* is a Nash equilibrium, $a^* \in \cap_{j \in N \setminus \{i\}} \max[R_j^*, Y^k] \cap Z$. Since $a^* \in Y \cap Z$, then a^* is an interior point of Y^k. Thus, since Y is convex and $Y^k \cap Y = Y^k$, then by Lemma 2, $a^* \in \cap_{j \in N \setminus \{i\}} \max[R_j^*, Y]$. Therefore, by no-veto-power on Y, $a^* \in \varphi(R^*)$.

State III. We can show $a^* = g^k(s^*) \in Z$ using the same argument as in State II. On the other hand, for each $i \in N$, $Atn_i(s) = \{e_i(a') : a' \in Y^k\}$. Then since s^* is a Nash equilibrium, $a^* \in \cap_{i \in N} \max[R_i^*, Y^k] \cap Z$. Using Lemma 2 and the same argument as in State II, we can show $a^* \in \cap_{j \in N} \max[R_j^*, Y] \cap Z$. Hence by no-veto-power on Y, $a^* \in \varphi(R^*)$. ∎

In classical private goods economies, Walrasian rule is supported by the convex opportunity system consisting of Walrasian budget sets and so is asymptotically feasibly Nash implementable. We will provide some other examples in Section 4.3. In classical public goods economies, Lindahl rule is asymptotically feasibly Nash implementable.

4.3 Opportunity Rules in Exchange Economies

In this section, we focus on exchange economies with m goods and provide some examples of opportunity rules. Let $\Omega \in \mathbb{R}^m_+$ be the social endowment. Let $A \equiv \mathbb{R}^{m \cdot n}_+$, $Z \equiv \{z \in \mathbb{R}^{m \cdot n}_+ : \sum z_i \leq \Omega\}$ and $Z_0 \equiv \{x \in \mathbb{R}^m_+ : 0 \leq x \leq \Omega\}$.

Assume that agents have individualistic preferences: for each $i \in N$ and each pair $z \equiv (x_j)_{j \in N}, z' \equiv (x'_j)_{j \in N} \in A$, if $x_i = x'_i$, then i is indifferent between z and z'. Then each preference relation R_0 over A can also be considered as a preference relation over $\mathbb{R}^m_+$ without any confusion. Let $\mathcal{D}$ be the domain of profiles of individualistic preferences that satisfy continuity, monotonicity and convexity.

Walrasian, $\mathbf{W}(\cdot)$. For each $R \in \mathcal{D}$, let $W(R) \equiv \{z \in Z :$ there is $p \in \Delta^{m-1}$ such that for each $i \in N$, $p \cdot z_i \leq p \cdot \Omega/n$ and for each $z'_i \in \mathbb{R}^m_+$ with $p \cdot z'_i \leq p \cdot \Omega/n$, $z_i \; R_i \; z'_i\}$.

Constrained Walrasian, $\mathbf{W^c}(\cdot)$. For each $R \in \mathcal{D}$, let $W^c(R) \equiv \{z \in Z :$ there is $p \in \Delta^{m-1}$ such that for each $i \in N$, $p \cdot z_i \leq p \cdot \Omega/n$ and for each $z'_i \in Z_0$ with $p \cdot z'_i \leq p \cdot \Omega/n$, $z_i \; R_i \; z'_i\}$.

Pareto, $\mathbf{P}(\cdot)$. For each $R \in \mathcal{D}$, let $P(R) \equiv \{z \in Z :$ there is no $z' \in Z$ such that for each $i \in N$, $z'_i \; R_i \; z_i$ and for some $j \in N$, $z'_j \; P_j \; z_j\}$.

Strong Pareto, $\mathbf{P}^*(\cdot)$. For each $R \in \mathcal{D}$, let $P^*(R) \equiv \{z \in P(R) :$ for each $i \in N$, $z_i \; P_i \; 0\}$.

No-Envy, $\mathbf{F}(\cdot)$. For each $R \in \mathcal{D}$, let $F(R) \equiv \{z \in Z :$ for each pair $i, j \in N$, $z_i \; R_i \; z_j\}$.

Super No-Envy, $\mathbf{K}(\cdot)$. For each $R \in \mathcal{D}$, let $K(R) \equiv \{z \in Z :$ for each $i \in N$ and each $x \in co\{z_1, \ldots, z_n\}$, $z_i \; R_i \; x\}$, where $co\{z_1, \ldots, z_n\}$ is the set of convex combinations of $z_1, \ldots, z_n$.

Equal division lower bound rule, $\mathbf{B_{ed}}(\cdot)$. For each $R \in \mathcal{D}$, let $B_{ed}(R) \equiv \{z \in Z :$ for each $i \in N$, $z_i \; R_i \; \Omega/n\}$.

We denote the intersection of two rules $\varphi(\cdot)$ and $\varphi'(\cdot)$ by $\varphi\varphi'(\cdot)$ whenever it is well-defined: for example, $FP(\cdot)$, $FP^*(\cdot)$, $B_{ed}P(\cdot)$, etc. We give examples of opportunity systems that support the above rules.

Example 8 (Walrasian and Constrained Walrasian Rules). For each price vector $p \in \Delta^{m-1}$, let $B(p) = \{x \in \mathbb{R}^m_+ : p \cdot x \le p \cdot \Omega/n\}$ be the budget set with price p. Let $B^c(p) = \{x \in Z_0 : p \cdot x \le p \cdot \Omega/n\}$ be the intersection of $B(p)$ and Z_0, called the constrained budget set with price p. We extend these two sets as sets of alternatives (allocations) which give the same opportunities. For each $i \in N$, let $\mathbf{B}_i(p) \equiv \{z \in A : z_i \in B(p)\}$ and $\mathbf{B}^c_i(p) \equiv \{z \in Z : z_i \in B^c(p)\}$. Then $\mathbf{B}_i(p)$ gives the same opportunity for agent i as $B(p)$ does and $\mathbf{B}^c_i(p)$ gives the same opportunity as $B^c(p)$ does. Let $\mathcal{O}_W \equiv \{(\mathbf{B}_i(p))_{i\in N} : p \in \Delta^{m-1}\}$ and $\mathcal{O}_{W^c} \equiv \{(\mathbf{B}^c_i(p))_{i\in N} : p \in \Delta^{m-1}\}$. Then it can be easily shown that Walrasian rule is supported by $\mathcal{O}_W$ and constrained Walrasian rule is supported by $\mathcal{O}_{W^c}$.

Example 9 (No-Envy and Super No-Envy Rules). Let Π be the set of all permutations on N. For each $z \in Z$ and each $i \in N$, let $O^F_i(z) \equiv \{z_\pi$: for all $\pi \in \Pi\}$, where for each $j \in N$, the j^{th} component of z_π is given by $z_{\pi(j)}$. And let $O^K_i(z) \equiv \{z' \in Z : z'_i \in co\{z_1, \cdots, z_n\}\}$. Let $\mathcal{O}_F \equiv \{(O^F_i(z))_N : z \in Z\}$ and $\mathcal{O}_K \equiv \{(O^K_i(z))_N : z \in Z\}$. Then it is easy to show that no-envy rule F is supported by $\mathcal{O}_F$ and super no-envy rule K is supported by $\mathcal{O}_K$.

Example 10 (Equal Division Lower Bound Rule). Let $z^{ed} \equiv (\Omega/n, \ldots, \Omega/n)$ be the equal division allocation. For each $z \in Z$ and each $i \in N$, let $O^{B_{ed}}_i(z) \equiv \{z, z^{ed}\}$. Let $\mathcal{O}_{B_{ed}} \equiv \{(O^{B_{ed}}_1(z), \cdots, O^{B_{ed}}_n(z)) : z \in Z\}$. It is easy to show that equal division lower bound rule is supported by $\mathcal{O}_{B_{ed}}$.

Example 11 (Pareto and Strong Pareto Rules). Let $\mathcal{O}_P \equiv \{(O_1(p,z), \cdots, O_n(p,z)) :$ there exists $z \in Z$ and $p \in \Delta^{m-1}$ such that for all $i \in N$, $O_i(p,z) = \{z' \in Z : z'_i \in B^t(p, z_i)\}$. Let $\mathcal{O}_{P^*} \equiv \{(O_1(p,z), \cdots, O_n(p,z))$: there exists $z \in Z$ and $p \in \Delta^{m-1}$ such that for all $i \in N$, $O_i(p,z) = \{z' \in Z : p \cdot z'_i \le p \cdot z_i \neq 0\}$. We show that Pareto rule P and strong Pareto rule P^* are supported by $\mathcal{O}_P$ and $\mathcal{O}_{P^*}$ respectively. Since the proofs for the two rules are similar, we only show this for P^*.

Suppose $z \in P^*(R)$. Then by definition of P^*, there exists a price vector $p \in \Delta^{m-1}$ such that for all $i \in N$, $p \cdot z_i > 0$ and $z_i \in \max[R_i, B^t(p, z_i)]$. Then for all $i \in N$, $z \in \max[R_i, O_i(p,z)]$. Hence $z \in \varphi^{\mathcal{O}_{P^*}}(R)$. Suppose $z \in \varphi^{\mathcal{O}_{P^*}}(R)$. Then there exists $z' \in Z$ and $p \in \Delta^{m-1}$ such that for all $i \in N$, $p \cdot z'_i > 0$ and $z_i \in \max[R_i, O(p, z')]$. Then by strict monotonicity of preference, if there exists $z'' \in Z$ such that for all $i \in N$, $z''_i \ R_i \ z_i$ and for some $j \in N$, $z''_j \ P_j \ z_j$, then for all $i \in N$, $p \cdot z_i'' \geq p \cdot z_i'$ and for some $j \in N$, $p \cdot z_j'' > p \cdot z_j'$. This implies $p \cdot \sum z_j'' = p \cdot \Omega > p \cdot \Omega = p \cdot \sum z_j'$. This is a contradiction. Therefore $z \in P^*(R)$.

Example 12 (The Intersection of No-Envy and Pareto Rules). Let $B_0^t(p, z_i) = \{z_i' \in Z_0 : p \cdot z_i' < p \cdot z_i\}$. For each $z \in Z$ and each $p \in \Delta^{m-1}$, let $O_i(p, z) \equiv \{z' \in Z : z_i' \in B_0^t(p, z_i) \cup \{z_1, \cdots, z_n\}\}$ for all $i \in N$. Let $\mathcal{O}_{FP} \equiv \{(O_i(p, z))_N : z \in Z, p \in \Delta^{m-1}\}$. It is clear that $\mathcal{O}_{FP}$ is a feasible and simple opportunity system. In the following argument we prove that $\varphi^{\mathcal{O}_{FP}} = FP$.

If $z \in FP(R)$, then it is easy to show that there exists $p \in \Delta^{m-1}$ such that for all $i \in N$, $z_i \in \max[B^t(p, z_i) \cup \{z_1, \cdots, z_n\}; R_i]$ hence $z_i \in \max[B_0^t(p, z_i) \cup \{z_1, \cdots, z_n\}; R_i]$. Therefore $z \in \varphi^{\mathcal{O}_{FP}}(R)$. To prove the converse inclusion, let $R = (R_i)_N \in \mathcal{R}^N$ and $z \in \varphi^{\mathcal{O}_{FP}}(R)$. Then $z \in Z$ and there exists $z' \in Z$ and $p \in \Delta^{m-1}$ such that for all $i \in N$, $z_i \in \max[B_0^t(p, z_i') \cup \{z_1', \cdots, z_n'\}; R_i]$. Then since $\sum_N z_i = \sum_N z_i' = \Omega$, whenever there exists $i \in N$ such that $p \cdot z_i > p \cdot z_i'$, there exists $j \in N$ such that $p \cdot z_j < p \cdot z_j'$ and so by the strict monotonicity of preference, $z_j \notin \max[B_0^t(p, z_j'); R_j]$. This contradicts $z_i \in \max[B_0^t(p, z_i') \cup \{z_1', \cdots, z_n'\}; R_i]$. Hence for all $i \in N$, $p \cdot z_i \leq p \cdot z_i'$. Therefore for all $i \in N$, since $z_i \in M[B_0^t(p, z_i') \cup \{z_1', \cdots, z_n'\}; R_i]$ and preferences are strict monotonic, $p \cdot z_i = p \cdot z_i'$. This implies that for all $i \in N$, $z_i \in \{z_1', \cdots, z_n'\}$: i.e. $\{z_1, \cdots, z_n\} \subseteq \{z_1', \cdots, z_n'\}$. Therefore for all $i \in N$, $B^t(p, z_i) = B^t(p, z_i')$ and $z_i \in \max[B_0^t(p, z_i') \cup \{z_1', \cdots, z_n'\}; R_i]$ implies that $z_i \in \max[B^t(p, z_i); R_i]$ and $z_i \in \max[\{z_1, \cdots, z_n\}; R_i]$. This implies $z \in FP(R)$.

Example 13 (The Intersection of Equal Division Lower Bound and Pareto Rules). For each $z \in Z$ and each $p \in \Delta^{m-1}$, let $O_i(p, z) \equiv \{z' \in Z : z_i' \in B^t(p, z_i) \cup \{\Omega/n\}\}$, for all $i \in N$. Let $\mathcal{O}_{B_{ed}P} = \{(O_i(p, z))_{i \in N} : p \in \Delta^{m-1}, z \in Z\}$. It is clear that $\mathcal{O}_{B_{ed}P}$ is a feasible, closed, and simple opportunity system. We have only to prove that $\varphi^{\mathcal{O}_{B_{ed}P}} = B_{ed}P$.

Let $z \in B_{ed}P(R)$. Then since $z \in P(R)$, there exists a normal vector $p \in \Delta^{m-1}$ such that for all $i \in N$, $z_i \in \max[B^t(p, z_i); R_i]$. Then it is clear by definition of $B_{ed}P$ that for all $i \in N$, $z_i \in \max(R_i; B^t(p, z_i) \cup \{\Omega/n\})$. Therefore $z \in \varphi^{\mathcal{O}_{B_{ed}P}}(R)$. In order to prove the converse inclusion, let $R = (R_i)_N \in \mathcal{R}^N$ and $z \in \varphi^{\mathcal{O}_{B_{ed}P}}(R)$. Then $z \in Z$ and there exists $z' \in Z$ and $p \in \Delta^{m-1}$ such that for all $i \in N$, $z_i \in \max[B^t(p, z_i') \cup \{\Omega/n\}; R_i]$. Since $z_i \in \max[B^t(p, z_i') \cup \{\Omega/n\}; R_i]$ and preferences are monotonic, for all $i \in N$, $p \cdot z_i \geq p \cdot z_i'$. Since $\sum z_i = \sum z_i' = \Omega$, for all $i \in N$, $p \cdot z_i = p \cdot z_i'$. Hence for all $i \in N$, $z_i \in \max[B^t(p, z_i) \cup \{\Omega/n\}; R_i]$. This implies that $z \in P(R)$ and for all $i \in N$, $z_i \; R_i \; \Omega/n$. Therefore $z \in B_{ed}P(R)$.

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