Existence of Equilibrium for Continuum Economies with Bads*

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Abstract: We prove existence of equilibrium in a continuum economy with bads. A fundamental condition is that no group of consumers, however small, has too little distaste for the bads.

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1. Introduction

The study of the existence of an equilibrium in a competitive economy has proceeded for more than a century. Walras (1877) was the first[1] to formulate the state of a competitive economy as the solution of a system of simultaneous equations representing the demand for goods by consumers, the supply of goods by producers, and the equilibrium condition of supply equalizing demand in every market. He counted equations and incognitas and was happy with this. This was not a satisfactory basis for establishing the existence of equilibrium and Wald (1936)[2] gave a rigorous proof. This line of research lead to the classic paper of Arrow and Debreu (1954) which does not require monotonic preferences but allows costless disposal of goods. In an economy with bads, i.e. goods which reduce utility if consumption increases, free disposal of the bads makes the existence of equilibrium a trivial problem (the bads will have a zero price and any excess will be disposed off). However free disposal of bads is an unrealistic assumption.Usually it is costly to dispose of bads. Thus we require a model in which no goods/bads can be costlessly disposed off. In a finite economy (i.e. finite number of goods, producers and consumers) Debreu (1962) proves quasi-equilibrium existence under quite general conditions. His result includes equilibrium existence for an economy with bads as long as preferences are weakly convex non-satiated.

The economic hypothesis that consumers take prices as given is strong if there are just a few consumers. In contrast–intuitively–the larger the num-

[1]But see http:\ \cepa.newschool.edu for a brief history on this subject.

[2]Translated to English in Wald (1951).

ber of consumers, none of them having a considerable fraction of total economic resources, the hypothesis that consumers take prices as given is more satisfactory. Aumann takes this reasoning to the limit and consider a continuum of agents. In his words:

> ...a mathematical model appropriate to the intuitive notion of perfect competition must contain infinitely many participants.

He proves core equivalence (Aumann (1964)) and equilibrium existence (Aumann (1966)). Subsequent work in continuum economies have considered: incomplete preferences (Schmeidler (1969)); production economies with free disposal (Hildenbrand (1970)); a reduced free disposal cone (Cornet, Topuzu, and Yildiz (2003)) and infinitely many goods (see among others Ostroy and Zame (1994), Podczeck (1997) and Araujo, Martins-da-Rocha, and Monteiro (2004)). The problem we consider– existence of equilibrium for continuum economies with bads–has not been considered in the literature. One might think that such a generalization would be routine. However the paper Hara (2005) shows that there are deep difficulties to overcome. He shows that even for quite simple examples[3] there is no equilibrium. Why is the combination of continuum of agents and bads so problematic for existence? To begin, a bad has a negative price. Thus a consumer may increase his income by consuming a bad and therefore buy more of the goods. If there is a group of consumers that has little distaste for the bad, they will consume a lot of the bad and this can make the optimal consumption of bads non-integrable. Thus for the existence of equilibrium the dislike of the bads must be sufficiently

[3]Like $u(a, x_g, x_b) = x_g - ax_b^2$, for each $0 < a < 1$.

intense. This is the key to our proof.

2. The model

We consider a pure exchange economy with a finite set G of goods and a finite set B of bads. The commodity space is $\mathbb{R}^G_+ \times \mathbb{R}^B_+$ and a consumption bundle is a vector $z = (z_G, z_B) \in \mathbb{R}^G_+ \times \mathbb{R}^B_+$ where $z_G \in \mathbb{R}^G_+$ and $z_B \in \mathbb{R}^B_+$.

The space of agents is a complete positive measure space $(A, \mathscr{A}, \mu)$ with $m(A) = 1$. The set A represents the names of agents, the σ-algebra $\mathscr{A}$ the admissible coalitions, and the number $\mu(E)$ the fraction of agents belonging to the coalition $E \in \mathscr{A}$. The consumption set $X(a)$ of agent $a \in A$ is a subset of $X := \mathbb{R}^G_+ \times \mathbb{R}^B_+$. Each agent $a \in A$ is characterized by an initial endowment vector $e(a) = (e_G(a), e_B(a)) \in X(a)$ and a preference relation defined by a utility function $u(a,.) : X(a) \to \mathbb{R}$.

We will maintain in this paper the following assumptions on the economy.

Definition 2.1. An economy $\mathscr{E} = \{X(a), u(a,.), e(a)\}$ is *standard* if it satisfies the following list of assumptions:

(S.1) for almost every a,

- **(S.1.a)** the consumption set $X(a)$ coincides with $X = \mathbb{R}^G_+ \times \mathbb{R}^B_+$,
- **(S.1.b)** the initial endowment $e(a)$ is a non-zero vector in X,
- **(S.1.c)** the function $z \longmapsto u(a,z)$ is continuous on X,
- **(S.1.d)** the function $z_G \longmapsto u(a, z_G, z_B)$ is strictly increasing on $\mathbb{R}^G_+$,
- **(S.1.e)** the function $z_B \longmapsto u(a, z_G, z_B)$ is strictly decreasing on $\mathbb{R}^B_+$;

(S.2) for every atom $E \in \mathscr{A}$, for almost every $a \in E$, the function $z \longmapsto u(a,z)$ is quasi-concave on X;

(S.3) for every $z \in X$, the function $a \longmapsto u(a,z)$ is measurable;

(S.4) the function $e : a \longmapsto e(a)$ is integrable and satisfies

$$\omega_G := \int_A e_G(a)\mu(da) \in \mathbb{R}^G_{++} \qquad \text{and} \qquad \omega_B := \int_A e_B(a)\mu(da) \in \mathbb{R}^B_{++}.$$

A price system π is a vector $\pi = (\pi_G, \pi_B) \in \mathbb{R}^G \times \mathbb{R}^B$ where $\pi_G(g)$ represents the unit price of good $g \in G$ and $\pi_B(b)$ the unit price of bad $b \in B$. We denote by Π the compact subset of $\mathbb{R}^G_+ \times \mathbb{R}^B_-$ defined by

$$\Pi = \left\{\pi = (\pi_G, \pi_B) \in \mathbb{R}^G_+ \times \mathbb{R}^B_- \ : \ \pi_G \cdot \mathbf{1}_G - \pi_B \cdot \mathbf{1}_B = 1\right\}.$$[4]

For every price system $\pi = (\pi_G, \pi_B)$, we denote by $B(a,\pi)$ the budget set of agent a of all consumption bundles $z \in X$ such that $\pi \cdot z \leqslant \pi \cdot e(a)$. In other words,

$$B(a,\pi) = \{z = (z_G, z_B) \in X \ : \ \pi_G \cdot z_G + \pi_B \cdot z_B \leqslant \pi_G \cdot e_G(a) + \pi_B \cdot e_B(a)\}.$$

An integrable function x from A to X is called an allocation; it is feasible (or attainable) if

$$\int_A x(a)\mu(da) = \int_A e(a)\mu(da).$$

The aggregate initial endowment $\int_A e d\mu$ is denoted by ω.

Definition 2.2. A pair (π, x) consisting of a non-zero price system $\pi \in \Pi$ and a feasible allocation x is said to be a *competitive equilibrium* if for almost

[4]For every finite set K, we denote by $\mathbf{1}_K$ the vector in $\mathbb{R}^K$ defined by $\mathbf{1}_K(k) = 1$ for every $k \in K$.

every agent $a \in A$ we have

$$x(a) \in \text{Argmax}\{u(a,z) \,:\, z \in B(a,\pi)\}.$$

3. Examples for non-existence

Before introducing our main hypothesis we discuss a few examples. First note that since no consumer wants to consume the bads, theirs price must be negative. Now if a group of consumers has just a small dislike of the bads they will consume large amounts so that they can buy more of the goods. Without any restriction, this precludes equilibrium. The following example shows this point clearly.

In this section, we consider a standard economy with one good $G = \{g\}$ and one bad $B = \{b\}$, and where the space of agents $(A, \mathscr{A}, \mu)$ is the continuum $[0,1]$ endowed with the Lebesgue measure.

Example 3.1. For every $a \in [0,1]$ the utility function is $u(a, x_g, x_b) = x_g - ax_b$ and the initial endowment $(e_g(a), e_b(a))$ is such that $e_g(a) > 0$ and $e_b(a) > 0$. If (π_g, π_b) is an equilibrium price then $\pi_g > 0$ and $\pi_b < 0$. The consumer problem is to maximize $x_g - ax_b$ subject to

$$\pi_g x_g + \pi_b x_b = \pi_g e_g(a) + \pi_b e_b(a).$$

Denoting $\pi_g e_g(a) + \pi_b e_b(a)$ by $w(a)$ and substituting $\pi_g x_g$ by $w(a) - \pi_b x_b$, the consumer maximizes

$$\frac{w(a)}{\pi_g} - \frac{\pi_b}{\pi_g} x_b - ax_b = \frac{w(a)}{\pi_g} - x_b \left(a + \frac{\pi_b}{\pi_g} \right),$$

under the constraint $x_b \geqslant 0$. Thus $a + (\pi_b/\pi_g) \geqslant 0$ for almost every $a \in [0, 1]$. It then follows that $\pi_b \geqslant 0$: contradiction.

Thus what drives the non-existence in this example is that the demand is only defined if relative price $-\pi_b/\pi_g$ is smaller than the disutility a of the bad. The next example is taken from Hara (2005).

Example 3.2 (Hara (2005))**.** For every $a \in [0, 1]$ the utility is $u(a, x) = x_g - a x_b^2$ and the initial endowment is $e(a) = (2, 1)$. Now the consumer problem is to maximize

$$\frac{2\pi_g + \pi_b}{\pi_g} - \frac{\pi_b}{\pi_g} x_b - a x_b^2,$$

under the constraint $x_b \geqslant 0$. This gives $x_b(a) = -\pi_b/(2a\pi_g)$ for every $a > 0$. But this function x_b is not integrable.

This example shows that even if the marginal disutility from consuming a bad increases with its consumption still an equilibrium may not exist. The main objective of this paper is to precise the relationship between the rate at which the disutility from consuming x_b has to increase compared to the utility from consuming x_b for equilibrium to exist.

4. Conditions for existence

Before presenting our main condition for existence, we introduce some notations. For every $\varepsilon > 0$, we denote by $\Pi(\varepsilon)$ the subset of Π defined by

$$\Pi(\varepsilon) := \{\pi = (\pi_G, \pi_B) \in \Pi : \pi_G \geqslant \varepsilon \mathbf{1}_G\}.$$

For every $a \in A$ and every $z \in X$, we denote by $P(a,z)$ the set of consumption bundles $z' \in X$ which are strictly preferred to z, i.e. $P(a,z) = \{z' \in X : u(a,z') > u(a,z)\}$.

Assumption (M). *For every $\varepsilon > 0$, there exists $r \in L^1(\mu, \mathbb{R})$ such that for every price system $\pi \in \Pi(\varepsilon)$, for every feasible allocation $x : A \to X$, for almost every a, if*

$$x(a) \in B(a,\pi) \text{ and } \|x_B(a)\| > r(a)$$

then

$$\exists y \in B(a,\pi) \cap P(a, x(a)), \quad \|y_B\| \leqslant \|x_B(a)\|.$$

Remark 4.1. If the measure space $(A, \mathscr{A}, \mu)$ has finitely many atoms, then Assumption M is automatically satisfied. Indeed, it follows from feasibility that there exists $r \in L^1(\mu, \mathbb{R})$ such that for every feasible allocation $x : A \to X$, we have $\|x_B(a)\| \leqslant r(a)$ for almost every $a \in A$.

Remark 4.2. Observe that if there are only goods in the economy, i.e. $B = \emptyset$, then Assumption M is automatically satisfied. Indeed, if $\pi \in \Pi(\varepsilon)$ for some $\varepsilon > 0$ then the budget set correspondence $a \mapsto B(a,\pi)$ is integrably bounded by the function $r \in L^1(\mu, \mathbb{R})$ defined by

$$\forall a \in A, \quad r(a) := \frac{1}{\varepsilon} \max_{g \in G} \|e_G(a)\|.$$

Remark 4.3. If π is a price system in Π, we denote by $d_B(a,\pi)$ the demand for bads of agent a, i.e.

$$d_B(a,\pi) = \left\{ z_B \in \mathbb{R}^B_+ : \exists z_G \in \mathbb{R}^G_+, \quad (z_G, z_B) \in d(a,\pi) \right\}$$

where $d(a,\pi)$ is the demand of agent a under the price system π, i.e.

$$d(a,\pi) = \text{Argmax}\{u(a,z) \, : \, z \in B(a,\pi)\}.$$

If an economy satisfies Assumption M then the demand correspondence for bads, $a \mapsto d_B(a,\pi)$, is uniformly integrably bounded on every set

$$\Pi(\varepsilon) := \{\pi = (\pi_G, \pi_B) \in \Pi \colon \pi_G \geqslant \varepsilon \mathbf{1}_G\}$$

with $\varepsilon > 0$, i.e.

$$\forall \varepsilon > 0, \quad \exists r \in L^1(\mu, \mathbb{R}), \quad \sup_{\pi \in \Pi(\varepsilon)} \|d_B(a,\pi)\| \leqslant r(a),^5 \quad \mu\text{-a.e.}$$

We can now state our main result. The proof is postponed to Appendix A.

Theorem 4.1. *If a standard economy satisfies Assumption M then there exists a competitive equilibrium.*

Following Remarks 4.1 and 4.2, this theorem provides a generalization of the existence results in the literature with bads and finitely many agents (e.g. McKenzie (1959)[6], Bergstrom (1976)[7], Hart and Kuhn (1975) and Polemarchakis and Siconolfi (1993)) together with the existence results in the literature with a continuum of agents and goods (e.g. Aumann (1966), Schmeidler (1969) and Hildenbrand (1974)).

[5] If K is a finite set and Z is a subset of $\mathbb{R}^K$, then we denote by $\|Z\|$ the extended real number $\sup\{\|z\| : z \in Z\}$.

[6] See also McKenzie (1961) and McKenzie (1981).

[7] See also Gay (1979).

5. Examples for existence

We propose a list of explicit conditions on the primitives of an economy which ensures the validity of the main Assumption M. In this section, we consider a standard economy with one good and one bad, and where the space of agents $(A, \mathscr{A}, \mu)$ is the continuum $[0,1]$ endowed with the Lebesgue measure. Moreover we assume that for every $a \in A$, the utility function $u(a,.)$ has the following form

$$\forall x = (x_g, x_b) \in \mathbb{R}^2_+, \quad u(a,x) = v_g(a,x_g) - v_b(a,x_b)$$

where $v_g(a,.)$ and $v_b(a,.)$ are continuous and strictly increasing functions from $\mathbb{R}_+$ to $\mathbb{R}$ and $v_g(.,x_g)$ and $v_b(.,x_b)$ are measurable functions from A to $\mathbb{R}$.

Proposition 5.1. *Assume that for almost every $a \in A$,*

1. *the function $v_g(a,.)$ is differentiable and concave,*
2. *the function $v_b(a,.)$ is differentiable and convex,*
3. *for every $\varepsilon > 0$, there exists an integrable function $\rho \in L^1(\mu, \mathbb{R})$ such that for almost every $a \in A$,*

$$v_b'(a, \rho(a)) > \frac{1}{\varepsilon} v_g'(a, e_g(a)) \tag{1}$$

then Assumption M is satisfied.

By convexity, the greater ρ is, the greater $v_b'(a,\rho)$ is. The bite of the assumption is to do this in an integrable way. The proof of Proposition 5.1 is postponed to Appendix B.

Example 5.1. For every $a \in [0,1]$, consider the utility function defined by

$$u(a, x_g, x_b) = v_g(a, x_g) - a x_b^{\phi}$$

where $\phi > 2$. If the initial endowment $e : A \to \mathbb{R}^2$ is an integrable and strictly positive function such that the function $a \mapsto v_g'(a, e_g(a))$ is bounded, then we can apply Proposition 5.1 to get the existence of a competitive equilibrium.

Remark 5.1. Following the previous example, in order to get existence of equilibrium, we can choose utility functions as follows

$$\forall a \in A, \quad u(a, x_g, x_b) = x_g - a x_b^{\phi}$$

where $\phi > 2$. It appears that the counterexample provided by Hara (2005) corresponds to the limit case: $\phi = 2$.

Proposition 5.2. *Assume that there exist $(\theta, \phi) \in \mathbb{R}^2_{++}$ with $\phi > \theta + 1$ and two measurable functions γ and β from A to $(0, \infty)$ such that*

1. *uniformly on $a \in A$ we have*

$$\lim_{x_g \to \infty} \frac{v_g(a, x_g)}{\gamma(a) x_g^{\theta}} = 1 \quad \text{and} \quad \lim_{x_b \to \infty} \frac{v_b(a, x_b)}{\beta(a) x_b^{\phi}} = 1,$$

2. *the following function*

$$a \longmapsto \left(\frac{\gamma(a)}{\beta(a)} \right)^{\frac{1}{\phi - \theta}}$$

   is integrable,

3. *the endowment function e is bounded and the function $a \mapsto u(a, e(a))$ is a.e. strictly positive.*

Then Assumption M is satisfied.

The proof of Proposition 5.2 is postponed to Appendix C.

Example 5.2. For every $a \in [0,1]$, consider the utility function defined by

$$u(a, x_g, x_b) = x_g^\theta - a x_b^\phi$$

where $\theta > 0$ and $\phi > \theta + 1$. If for each a, the initial endowment is $e(a) = (1,1)$, then we can apply Proposition 5.2 to get the existence of a competitive equilibrium.

Appendix A: Proof of Theorem 4.1

Let $\mathscr{E} = \{X(a), u(a,.), e(a)\}$ be a standard economy satisfying Assumption M. For each $n \in \mathbb{N}$, there exists $r_n \in L^1(\mu, \mathbb{R})$ such that for every price $\pi \in \Pi$ satisfying $\pi_G \geqslant 1/(n+1)\mathbf{1}_G$, for every feasible allocation $x : A \to X$, for almost every a, if

$$x(a) \in B(a,\pi) \quad \text{and} \quad \|x_B(a)\| > r_n(a)$$

then

$$\exists z(a) \in B(a,\pi) \cap P(a, x(a)), \quad \|z_B(a)\| \leqslant \|x_B(a)\|.$$

Fix $n \in \mathbb{N}$ and let $\mathscr{E}^n = \{X^n(a), u(a,.), e(a)\}$ be the economy defined by $X^n(a) := \mathbb{R}^G_+ \times X^n_B(a)$ where for every a,

$$X^n_B(a) := \{z \in \mathbb{R}^B_+ : \|z\| \leqslant \xi^n(a) := \max(r_n(a), \|e_B(a)\| + n)\}.$$

The economy $\mathscr{E}^n$ satisfies the assumptions of Theorem 3.1 in Cornet, Topuzu, and Yildiz (2003).[8] Therefore there exist a price $\pi^n \in \mathbb{R}^G \times \mathbb{R}^B$ with $\|\pi^n\| = 1$

[8]The boundedness assumption $\mathbf{B}_c$ in Cornet, Topuzu, and Yildiz (2003) is satisfied for the cone $C = \mathbb{R}^G_+ \times \{0\}$.

and an allocation $x^n : A \to \mathbb{R}^G_+ \times \mathbb{R}^B_+$ such that for almost every a,

$$\pi^n \cdot x^n(a) \leqslant \pi^n \cdot e(a) \tag{2}$$

$$\forall z \in X^n(a), \quad u(a,z) > u(a,x^n(a)) \Longrightarrow \pi^n \cdot z \geqslant \pi^n \cdot e(a) \tag{3}$$

and

$$\int_A x^n_B d\mu = \int_A e_B d\mu \quad \text{and} \quad \int_A x^n_G d\mu \leqslant \int_A e_G d\mu. \tag{4}$$

Claim A.1. *The price π^n satisfies*

$$\pi^n_G \in \mathbb{R}^G_{++}, \tag{5}$$

for almost every a, one has

$$\forall z \in X^n(a), \quad u(a,z) > u(a,x^n(a)) \Longrightarrow \pi^n \cdot z > \pi^n \cdot e(a), \tag{6}$$

and markets clear

$$\int_A x^n d\mu = \int_A e d\mu. \tag{7}$$

Proof of Claim A.1. It follows from Assumption S.1.d and relation (3) that prices of goods are non-negative, i.e. $\pi^n_G \in \mathbb{R}^G_+$. From Assumption S.4 we have that $\omega_B \in \mathbb{R}^B_{++}$. Then there exists $\alpha > 0$ such that $\omega_B > \alpha \mathbf{1}_B$. In particular it follows from (4) there exists a measurable set $E \in \mathscr{A}$ with $\mu(E) > 0$ such that

$$\forall a \in E, \quad x^n_B(a) \geqslant \alpha \mathbf{1}_B.$$

We let $F \in \mathscr{A}$ be a subset of E with $\mu(F) > 0$ such that relation (3) is satisfied for every $a \in F$. Fix any $a \in F$, from Assumption S.1.e, we have that

$$\forall b \in B, \quad u(a, x^n(a) - (\alpha/2)(0, \mathbf{1}_{\{b\}}) > u(a, x^n(a)).$$

It then follows from (3) that $\pi^n_B \in \mathbb{R}^B_-$. We have thus proved that $\pi^n \in \mathbb{R}^G_+ \times \mathbb{R}^B_-$. Since $\pi^n \neq 0$ it follows that either $\pi^n_G > 0$ or $\pi^n_B < 0$.

If $\pi^n_G > 0$ then $\pi^n_G \cdot \omega_G > 0$. In particular, there exists a measurable set $H \in \mathscr{A}$ with $\mu(H) > 0$ such that for every $a \in H$, we have $\pi^n_G \cdot e_G(a) > 0$, in particular

$$\forall a \in G, \quad \pi^n \cdot e(a) > \pi^n \cdot (0, e_B(a)) \geqslant \inf \pi \cdot X^n(a).$$

Following standard arguments, we can prove that for almost every $a \in H$, we have

$$\forall z \in X^n(a), \quad u(a,z) > u(a,x^n(a)) \Longrightarrow \pi^n \cdot z > \pi^n \cdot e(a).$$

Applying Assumption S.1.d, we deduce that $\pi^n_G \in \mathbb{R}^G_{++}$.

Assume now that $\pi^n_B > 0$. Since $\xi^n(a) > \|e_B(a)\|$, one has

$$\pi^n \cdot e(a) \geqslant \pi^n \cdot (0, e_B(a)) > \inf \pi^n \cdot X^n(a).$$

Following standard arguments, it follows from relation (3) that for almost every $a \in A$, we have

$$\forall z \in X^n(a), \quad u(a,z) > u(a,x^n(a)) \Longrightarrow \pi^n \cdot z > \pi^n \cdot e(a).$$

Applying Assumption S.1.d, we deduce that $\pi^n_G \in \mathbb{R}^G_{++}$.

We have thus proved that $\pi^n_G \in \mathbb{R}^G_{++}$. From Assumption S.1.b, it follows that for almost every $a \in A$,

$$\pi^n \cdot e(a) > \pi^n \cdot (0, e_G(a)) \geqslant \inf \pi^n \cdot X^n(a).$$

Following standard arguments, we can deduce from (3) the required property (6).

We denote by z^n_G the aggregate excess demand for goods, and z^n_B the excess demand for bads, i.e.

$$z^n_G := \int_A (x^n_G - e_G) d\mu \quad \text{and} \quad z^n_B := \int_A (x^n_B - e_B) d\mu.$$

From (4) we have $z^n_G \in -\mathbb{R}^G_+$ and $z^n_B = 0$. But from (6) we know that for almost every $a \in A$, the bundle $x^n(a)$ is optimal in the budget set, i.e.

$$x^n(a) \in \operatorname{Argmax}\{u(a,z) \, : \, z \in X^n(a) \cap B(a, \pi^n)\}.$$

Using Assumption S.1.d, we get that budget constraints are binding, i.e. for almost every a, we have $\pi^n \cdot x^n(a) = \pi^n \cdot e(a)$. In particular $\pi^n \cdot z^n = 0$ and then $\pi^n_G \cdot z^n_G = 0$. But since π^n_G is strictly positive we conclude that $z^n_G = 0$ and get the desired (exact) market clearing condition (7). □

Without any loss of generality we can assume that for each $n \in \mathbb{N}$, the price π^n belongs to Π. Passing to a subsequence if necessary, we can assume that the sequence $\{\pi^n\}$ converges to a vector $\pi \in \Pi$.

Claim A.2. *The price of every good is strictly positive, i.e.* $\pi_G \in \mathbb{R}^G_{++}$.

Proof of Claim A.2. We already know that $\pi_G \in \mathbb{R}^G_+$. Assume by way of contradiction that there exists $g \in G$ such that $\pi_G(g) = 0$. We let $E \in \mathscr{A}$ be the set defined by

$$E := \{a \in A \, : \, \pi \cdot e(a) > \inf \pi \cdot X\}.$$

Since $\pi \in \Pi$, applying Assumption S.4 we have $\mu(E) > 0$.[9] We let F be a measurable subset of E with $\mu(F) = \mu(E)$ and such that relations (2) and (6) are satisfied for every $a \in F$. We claim that

$$\forall a \in F, \quad \lim_{n\to\infty} \|x^n(a)\| = \infty. \tag{8}$$

Fix $a \in F$ and assume by way of contradiction that the sequence $\{\|x^n(a)\|\}$ is bounded. Passing to a subsequence if necessary, we can assume that the sequence $\{x^n(a)\}$ converges to a bundle $y \in X$. Fix $\nu \in \mathbb{N}$ such that $\|y_B\| < \xi^\nu(a) + 1$. It follows from (6) and Assumption S.1.c that

$$\forall z \in X, \quad u(a,z) > u(a,y) \Longrightarrow \pi \cdot z \geqslant \pi \cdot y.$$

Since $\pi \cdot e(a) > \inf \pi \cdot X$, it is standard to deduce that we actually have

$$\forall z \in X, \quad u(a,z) > u(a,y) \Longrightarrow \pi \cdot z > \pi \cdot y. \tag{9}$$

Now if we pose $z := y + (0, \mathbf{1}_{\{g\}})$, then, by Assumption S.1.d we have $u(a,z) > u(a,y)$. But since we assumed that $\pi_G(g) = 0$, we get a contradiction with (9). We have thus proved relation (8).

Observe that

$$\liminf_{n\to\infty} \int_F \|x^n(a)\| \mu(da) \leqslant \liminf_{n\to\infty} \mathbf{1}_{G\times B} \cdot \int_A x^n d\mu = \mathbf{1}_{G\times B} \cdot \omega = \|\omega\|.$$

Applying Fatou's lemma, this yields a contradiction with (8). □

[9] If there exists $b \in B$ such that $\pi_B(b) < 0$ then $\pi \cdot e(a) > \pi \cdot [e(a) + (0, \mathbf{1}_{\{b\}})] \geqslant \inf \pi \cdot X$ and therefore $E = A$. If $\pi_B = 0$, then since $\pi \in \Pi$, there exists $g \in G$ such that $\pi_G(g) > 0$. From Assumption S.4 it follows that $\pi_G \cdot \omega_G > 0$ and therefore $\pi \cdot \omega > \pi \cdot (0, \omega_B) \geqslant \inf \pi \cdot X$. It then follows that $\mu(E) > 0$.

From Claim A.2 there exists $\nu \in \mathbb{N}$ such that

$$\forall n \geqslant \nu, \quad \pi^n_G \geqslant 1/(\nu+1)\mathbf{1}_G. \tag{10}$$

Claim A.3. *For each $n \geqslant \nu$, the function x^n_B is integrably bounded by the function r_ν, i.e.*

$$\forall n \geqslant \nu, \quad \|x^n_B(a)\| \leqslant r_\nu(a) \quad a.e. \tag{11}$$

Proof of Claim A.3. Indeed, fix $n \geqslant \nu$ and assume that there exists a measurable set $E \in \mathscr{A}$ with $\mu(E) > 0$ such that

$$\forall a \in E, \quad \|x^n_B(a)\| > r_\nu(a).$$

Using (10) together with Assumption M, we get that there exists a measurable subset $E' \subset E$ with $\mu(E') > 0$ and such that for every $a \in E'$,

$$\exists z(a) \in B(a, \pi^n), \quad u(a, z(a)) > u(a, x^n(a)) \quad \text{and} \quad \|z_B(a)\| \leqslant \|x^n_B(a)\|.$$

It follows[10] that for every $a \in E'$,

$$z(a) \in X^n(a), \quad \pi^n \cdot z(a) \leqslant \pi^n \cdot e(a) \qquad \text{and} \qquad u(a, z(a)) > u(a, x^n(a))$$

which contradicts (6). □

Applying a multidimensional version of Fatou's Lemma (see e.g., Schmeidler (1970), Artstein (1979) or Balder (1984)), there exists an allocation x such that

$$\int_A x_G(a)\mu(da) \leqslant \int_A e_G(a)\mu(da), \quad \int_A x_B(a)\mu(da) = \int_A e_B(a)\mu(da), \tag{12}$$

[10]Observe that since $\|z_B(a)\| \leqslant \|x^n_B(a)\|$, the vector $z(a)$ belongs to $X^n(a)$.

and

$$x(a) \in \mathrm{Ls}\{x^n(a)\} \quad \text{a.e.} \tag{13}$$

where $\mathrm{Ls}\{x^n(a)\}$ is the set of limit points of the sequence $\{x^n(a)\}$.

We claim that (π, x) is a competitive equilibrium. Indeed, passing to the limit in (2) we obtain that $\pi \cdot x(a) \leqslant \pi \cdot e(a)$ for almost every a, i.e.

$$x(a) \in B(a, \pi) \quad \text{a.e.} \tag{14}$$

Passing to the limit in (3) we obtain that for almost every a,

$$\forall z \in X, \quad u(a,z) > u(a,x(a)) \Longrightarrow \pi \cdot z \geqslant \pi \cdot e(a). \tag{15}$$

Since for every $\alpha > 0$, we have $u(a, x(a) + \alpha \mathbf{1}_G) > u(a, x(a))$, we deduce from relation (15) that budget constraints are binding, i.e.

$$\pi \cdot x(a) = \pi \cdot e(a) \quad \text{a.e.}$$

It then follows that

$$\pi \cdot \int_A [x(a) - e(a)]\mu(da) = 0$$

but from (12) we have

$$\int_A [x(a) - e(a)]\mu(da) \in -\mathbb{R}^G_+ \times \{0\}.$$

Since $\pi_G \in \mathbb{R}^G_{++}$ we actually obtain that

$$\int_A x d\mu = \int_A e d\mu. \tag{16}$$

Moreover since $\pi_G \in \mathbb{R}^G_{++}$ we have $\pi \cdot e(a) > \pi \cdot (0, e_B(a)) \geqslant \inf \pi \cdot X$. Therefore following standard arguments, we can deduce from (15) that for almost every a,

$$\forall z \in X, \quad u(a,z) > u(a,x(a)) \Longrightarrow \pi \cdot z > \pi \cdot e(a). \tag{17}$$

It follows from relations (14), (16) and (17) that (π, x) is a competitive equilibrium.

Appendix B: Proof of Proposition 5.1

Fix $\varepsilon > 0$, a price $\pi \in \Pi(\varepsilon)$ and let ρ be defined by (1). We consider the integrable function $r \in L^1(\mu, \mathbb{R})$ defined by $r(a) = \max(\rho(a), e_b(a))$. Let now $x = (x_g, x_b)$ in $B(a, \pi)$ such that $x_b > r(a)$.

If $x_g < e_g(a)$ then posing $y = e(a)$ we get that

$$y \in B(a, \pi) \cap P(a, x) \qquad \text{and} \qquad y_b < x_b,$$

proving that Assumption M is satisfied.

If $x_g \geqslant e_g(a)$ then for each $t > 0$, we pose

$$y^t = (y^t_g, y^t_b) \quad \text{where} \quad y^t_b = x_b - t \quad \text{and} \quad y^t_g = x_g + t\frac{\pi_b}{\pi_g}.$$

Observe that there exists $\tau > 0$ small enough, such that for all $0 < t \leqslant \tau$, we have $y^t \in \mathbb{R}^2_+$. Moreover $\pi \cdot y^t = \pi \cdot x$ which implies that $y^t \in B(a, \pi)$. It is straightforward to check that

$$\lim_{t \to 0^+} \frac{1}{t}\{u(a, y^t) - u(a, x)\} = \frac{\pi_b}{\pi_g} v'_g(a, x_g) + v'_b(a, x_b) \geqslant -\frac{1}{\varepsilon} v'_g(a, x_g) + v'_b(a, x_b).$$

Since the function $v_g(a, .)$ is concave, we have $v'_g(a, x_g) \leqslant v'_g(a, e_g(a))$. Since the function $v_b(a, .)$ is convex, we have $v'_b(a, x_b) \geqslant v'_b(a, \rho(a))$. Applying (1) we have

$$\lim_{t \to 0^+} \frac{1}{t}\{u(a, y^t) - u(a, x)\} \geqslant -\frac{1}{\varepsilon} v'_g(a, e_g(a)) + v'_b(a, \rho(a)) > 0.$$

Therefore there exists $t > 0$ small enough such that $u(a, y^t) > u(a, x)$, proving that Assumption M is satisfied.

Appendix C: Proof of Proposition 5.2

Since the initial endowment function e is bounded, there exists $M > 0$ such that for almost every $a \in A$ we have $\max(e_g(a), e_b(a)) \leqslant M$. It follows from the first condition in Proposition 5.2 that there exists $K_g > 0$ and $K_b > 0$ such that uniformly on A,

$$\forall x_g \geqslant K_g, \quad v_g(a, x_g) \leqslant \frac{3}{2}\gamma(a)x_g^\theta \qquad \text{and} \qquad \forall x_b \geqslant K_b, \quad v_b(a, x_b) \geqslant \frac{1}{2}\beta(a)x_b^\phi.$$

We fix $\varepsilon > 0$ and pose

$$r(a) = K_1(\varepsilon) + K_2(\varepsilon)\left(\frac{\gamma(a)}{\beta(a)}\right)^{\frac{1}{\phi-\theta}}$$

where $K_1(\varepsilon) := \max\{M, 1, K_b, (\varepsilon K_g + M)/(1-\varepsilon)\}$ and

$$K_2(\varepsilon) := 3\left[\frac{1-\varepsilon}{\varepsilon} + \frac{M}{\varepsilon}\right]^{\frac{\theta}{\phi-\theta}}.$$

Observe that the function r is integrable.

Claim C.1. *For every $\pi \in \Pi(\varepsilon)$, for every $a \in A$, if $x \in B(a, \pi)$ and $x_b > r(a)$ then $u(a, x) \leqslant 0$.*

Proof of Claim C.1. If $x \in B(a, \pi)$ then

$$x_g \leqslant \frac{R(a) + (1-\varepsilon)x_b}{\varepsilon}$$

where $R(a) = \pi \cdot e(a)$. Observe that since $\pi \in \Pi(\varepsilon)$, we have $|\pi \cdot e_g(a)| \leqslant M$. Therefore, using the fact that $K_1(\varepsilon) \geqslant (\varepsilon K_g + M)/(1-\varepsilon)$ we have

$$v_g(a, x_g) \leqslant v_g\left(a, \frac{R(a) + (1-\varepsilon)x_b}{\varepsilon}\right) \leqslant \frac{3}{2}\gamma(a)\left[\frac{M + (1-\varepsilon)x_b}{\varepsilon}\right]^\theta.$$

Moreover, using the fact that $K_1(\varepsilon) \geqslant K_b$ we have that

$$-v_b(a, x_b) \leqslant -\frac{1}{2}\beta(a)x_b^{\phi}.$$

Adding the last two inequalities, we obtain

$$v(a,x) \leqslant \frac{1}{2}\beta(a)x_b^{\theta}\left[3\frac{\gamma(a)}{\beta(a)}\left[\frac{1-\varepsilon}{\varepsilon}+\frac{M}{\varepsilon x_b}\right]^{\theta} - x_b^{\phi-\theta}\right].$$

Since $K_1(\varepsilon) \geqslant 1$ and following the choice of $K_2(\varepsilon)$, we have

$$3\frac{\gamma(a)}{\beta(a)}\left[\frac{1-\varepsilon}{\varepsilon}+\frac{M}{\varepsilon x_b}\right]^{\theta} - x_b^{\phi-\theta} \leqslant 3^{\phi-\theta}\frac{\gamma(a)}{\beta(a)}\left[\frac{1-\varepsilon}{\varepsilon}+\frac{M}{\varepsilon}\right]^{\theta} - x_b^{\phi-\theta} \leqslant 0$$

which implies that $u(a,x) \leqslant 0$. □

Let $\pi \in \Pi(\varepsilon)$ and $x : A \to \mathbb{R}^2_+$ be a feasible allocation. There exists a measurable set $E \in \mathscr{A}$ with $\mu(E) = \mu(A)$ such that for every $a \in E$, the utility $u(a, e(a))$ is strictly positive. Let $a \in E$ and assume that $x_b(a) > r(a)$. It follows from the previous claim that $u(a, x(a)) \leqslant 0$. In particular we have $e(a) \in P(a, x(a)) \cap B(a, \pi)$. Since $r(a) \geqslant K_1(\varepsilon) \geqslant M \geqslant e_b(a)$, it follows that $e_b(a) \leqslant x_b(a)$. This proves that Assumption M is satisfied.

References

ARAUJO, A., V. F. MARTINS-DA-ROCHA, AND P. K. MONTEIRO (2004): "Equilibria in reflexive Banach lattices with a continuum of agents," *Econom. Theory*, 24(3), 469–492.

ARROW, K. J., AND G. DEBREU (1954): "Existence of an equilibrium for a competitive economy," *Econometrica*, 22, 265–290.

ARTSTEIN, Z. (1979): "A note on Fatou's lemma in several dimensions," *J. Math. Econom.*, 6(3), 277–282.

AUMANN, R. J. (1964): "Markets with a continuum of traders," *Econometrica*, 32, 39–50.

——— (1966): "Existence of competitive equilibria in markets with a continuum of traders," *Econometrica*, 34, 1–17.

BALDER, E. J. (1984): "A unifying note on Fatou's lemma in several dimensions," *Math. Oper. Res.*, 9(2), 267–275.

BERGSTROM, T. C. (1976): "How to discard "free disposability"—at no cost," *J. Math. Econom.*, 3(2), 131–134.

CORNET, B., M. TOPUZU, AND A. YILDIZ (2003): "Equilibrium theory with a measure space of possibly satiated consumers," *J. Math. Econom.*, 39(3-4), 175–196.

DEBREU, G. (1962): "New Concepts and Techniques for Equilibrium Analysis," *International Economic Review*, 3(3), 257–273.

GAY, A. (1979): "A note on Lemma 1: "How to discard 'free disposability'—at no cost" (J. Math. Econom. **3** (1976), no. 2, 131–134) by T. C. Bergstrom," *J. Math. Econom.*, 6(2), 215–216.

HARA, C. (2005): "Existence of equilibria in economies with bads," *Econometrica*, 73(2), 647–658.

HART, O. D., AND H. W. KUHN (1975): "A proof of the existence of equilibrium without the free disposal assumption," *J. Math. Econom.*, 2(3), 335–343.

HILDENBRAND, W. (1970): "Existence of equilibria for economies with production and a measure space of consumers," *Econometrica*, 38, 608–623.

——— (1974): *Core and equilibria of a large economy*. Princeton University Press, Princeton, N.J.

MCKENZIE, L. W. (1959): "On the existence of general equilibrium for a competitive market," *Econometrica*, 27, 54–71.

——— (1961): "On the existence of general equilibrium: some corrections," *Econometrica*, 29, 247–248.

——— (1981): "The classical theorem on existence of competitive equilibrium," *Econometrica*, 49(4), 819–841.

OSTROY, J., AND W. ZAME (1994): "Nonatomic economies and the boundaries of perfect competition," *Econometrica*, 62, 593–633.

PODCZECK, K. (1997): "Markets with infinitely many commodities and a continuum of agents with non-convex preferences," *Econom. Theory*, 9(3), 385–426.

POLEMARCHAKIS, H. M., AND P. SICONOLFI (1993): "Competitive equilibria without free disposal or nonsatiation," *J. Math. Econom.*, 22(1), 85–99.

SCHMEIDLER, D. (1969): "Competitive equilibria in markets with a continuum of traders and incomplete preferences," *Econometrica*, 37, 578–585.

——— (1970): "Fatou's lemma in several dimensions," *Proc. Amer. Math. Soc.*, 24, 300–306.

WALD, A. (1936): "Über einige Gleichungssystem der matematischen Ökonomie," *Zeitschrift für Nationalökonomie*, 7, 637–670.

——— (1951): "On some systems of equations of mathematical economics," *Econometrica*, 19, 368–403.

WALRAS, L. (1877): *Éléments d'économie politique pure*. Lausanne, Corbaz.