Rational Expectations Equilibrium in Economies with Uncertain Delivery[⋆]

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Abstract. This paper studies general equilibrium with differential information in which trade is made *ex ante*, under private and incomplete state verification. Objects of choice are lists of bundles such that the agent has the right to receive one of them. The 'market', interpreted as being composed by competitive brokers, delivers the cheapest of the possible alternatives. Knowledge of the selection mechanism, and observation of the prevailing prices, allows agents to correctly predict the bundle that is to be delivered in each state of nature, and thus equate the utility of a list with the utility of the cheapest bundle included in the list. A perfectly informed agent with an arbitrarily small endowment is introduced in the economy in order to guarantee existence of a *rational expectations equilibrium.*

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1 Introduction

In chapter 7 of his “Theory of Value”, Debreu (1959) showed how the general equilibrium model could be extended to the case of uncertainty under public state verification. All that is needed is to consider a generalized notion of a commodity that includes the state of nature in which it is delivered (Arrow, 1953). The model becomes equivalent to the model without uncertainty: prices for the contingent commodities are announced, and agents choose the vector of contingent claims (specifying a bundle to be consumed in each of the possible states of nature) that they prefer, among those that satisfy their budget restriction. Then, the state of nature is publicly announced and agents receive the bundle that they chose to consume in the announced state.

We are interested in studying the implications of differential information, in the form of private state verification. While keeping the basic structure of the model, we assume that each agent will only be able to verify (in a court of law, for the contracts to be enforced) that the state of nature belongs to a certain set. Our basic assumption is that if an agent bought different bundles for delivery in two states and is not able to verify whether the true state is one or the other, then the agent has to accept delivery of any of these two bundles.

For example, consider an agent that cannot verify in a court of law whether the true state is A or B, but nevertheless buys A1 (delivery of good A in state 1) and B2 (delivery of good B in state 2). Then: if state 1 occurs, the agent can receive A or B; and if state 2 occurs, the agent can also receive A or B. Observe that the ‘set of alternatives that may be delivered’ is always the same in states that an agent cannot distinguish (a condition that is usually known as ‘measurability’).

This context may be dealt with by considering that objects of choice are lists such that the agents have the right to receive one of the bundles in the list. Notice that this concept extends Arrow’s (1953) notion of contingent goods. Without loss of generality, we can restrict our attention to lists that are ‘measurable’ with respect to each agent’s private information, since, as above, any non-measurable choice can be converted into a measurable one. It is important to understand that this measurability restriction (on lists) is not an actual restriction on trade, but a way to formalize the consequences of incomplete state verification.

This general market structure was introduced in two previous papers (2006a, 2006b). Prices of the contingent lists are announced, and agents choose the vector of contingent lists that they prefer in their budget set. Then, the agents are able to verify in which set of their information partition lies the true state of nature, and receive one of the alternatives in the list that they bought for delivery in these states. Notice that it is the 'market' that selects one of the bundles in the list for delivery.

The 'market' can be interpreted as consisting of many competitive brokers, who offer lists in exchange for the agent's endowments, and then trade among themselves in conditions of perfect information (a state of nature is announced to the brokers). In this internal market, an Arrow-Debreu equilibrium is generated, consisting of an allocation between brokers, together with equilibrium prices for the contingent goods. Brokers are profit maximizers, therefore, each broker buys (in the internal market) the cheapest of the alternatives in the list selected by the agent that the broker is dealing with. Another *no arbitrage* condition is that the price of a list should be equal to the price of the delivered bundle (brokers have no profits). The (internal) equilibrium defines, then, (1) the equilibrium prices of the lists; and (2) which of the alternative bundles is delivered to each agent.

In previous papers, we have studied prudent expectations equilibrium (2006a) and subjective expectations equilibrium (2006b). The meaning of prudent expectations is that agents expected to receive the worst possible bundle in a list. Agents with subjective expectations are more sophisticated. Their beliefs on the probabilities of delivery of the different alternatives depend on the prices that they observe (perfectly or imperfectly), and on the alternatives specified in the list. In this paper, we study the existence of a rational expectations equilibrium.

We assume that agents observe prices for all the contingent commodities, without any informational restriction. Knowing the selection mechanism, they can predict which bundle is going to be selected for delivery (the cheapest) in each state of nature. In this setting, the market appears as a friendly, truthful, mechanism. Prudent expectations were associated to the opposite paradigm, as the market was expected to deliver the worst alternative bundle to the agent.

Since, by observing prices, agents can calculate what will be the delivered bundle (the cheapest), they know which consumption bundle results from each of the lists. Therefore, instead of choosing among lists, they can choose directly among these resulting consumption bundles. These bundles are those that satisfy a sort of endogenous incentive compatibility restrictions. Considering an agent who does not distinguish between states s and t, for a bundle, (x^s, x^t), to be delivered, it must be such that $p^s \cdot x^s \leq p^s \cdot x^t$ and $p^t \cdot x^t \leq p^t \cdot x^s$. If these incentive compatibility conditions are not satisfied, the agent will not receive x^s in state s and x^t in state t. An agent with rational expectations chooses among bundles which are incentive compatible in this sense.

The choice set depends, therefore, on the equilibrium prices and on each agent's private information. If the correspondence between prices and incentive compatible bundles was continuous, then equilibrium existence would be guaranteed. However, lower hemi-continuity fails when prices in some state are null or when prices in different states are collinear (it is easy to see that the correspondence is upper hemi-continuous). Collinearity does not interfere with equilibrium existence because, in this case, it can be shown that (having convex preferences) agents choose the same bundle for delivery in both states, and the incentive compatibility restrictions can be removed.

We give a simple example of non-existence of equilibrium caused by null prices. Prices may be null, in spite of the monotonicity of preferences, because there may be some state in which resources are abundant, but such that no agent can verify that it has occurred. As a result, no agent is willing to buy commodities contingent on its occurrence.

Introducing a perfectly informed agent removes this problem, because this agent can verify the occurrence of any state. We do this in a way that imposes a lower bound in prices, simply by assuming this agent's preferences to be linear. The main result in this paper establishes existence of equilibrium in an economy with this additional trader.

The rest of the paper is organized as follows: in section 2 the model is presented; in section 3 an example of non-existence is given; in section 4 a small but perfectly informed trader is defined; and in section 5 existence of equilibrium is established (with this additional trader in the economy).

2 The model

We consider a finite number of agents ($i = 1, ..., n$), a finite number of possible states of nature ($s = 1, ..., \Omega$), and a finite number of commodities, ($j = 1, ..., l$). The private information of agent i is represented by a partition of the set of states of nature such that agent i can distinguish states that belong to different sets of the partition P_i. The set of states that agent i does not distinguish from s is denoted $P_i(\omega_s)$.[1] A function that is constant across elements of P_i is said to be P_i-measurable. Consumption of agent i in state s is $x_i^s \in \mathbb{R}_+^l$, and the contingent consumption plan of agent i is $x_i = (x_i^s)_{s\in\Omega} \in \mathbb{R}_+^{\Omega l}$.

A list is a finite set of bundles, indexed by $k = 1, ..., K$.[2] The list selected by agent i for delivery in state s is denoted $\tilde{x}_i^s \in \mathbb{R}_+^{Kl}$, with the k^{th} alternative being $\tilde{x}_i^{sk} \in \mathbb{R}_+^{\Omega l}$. The contingent list plan of agent i is $\tilde{x}_i \in \mathbb{R}_+^{\Omega K l}$.

The economy extends over two time periods. In the first, taking prices as given, agents trade their state-contingent endowments for P_i-measurable vectors of state-contingent lists, $\tilde{x}_i = (\tilde{x}_i^1, \tilde{x}_i^2, ..., \tilde{x}_i^\Omega)$, specifying the bundles that the market may deliver in each state of nature. In the second period, agents receive their information, and consume one of the bundles in the list that corresponds to the state of nature that occurs. If state s occurs, then agent i receives one of the bundles, $\tilde{x}_i^{sk}$, in the list $\tilde{x}_i^s$.

With the delivery of a bundle $x^s \in \mathbb{R}_+^l$ in state s, the market keeps the contract for delivery of any list in $\tilde{X}^s(x^s)$, defined as:

$$\tilde{X}^s(x^s) = \{\tilde{x}^s = (\tilde{x}^{s1}, ..., \tilde{x}^{sK}) \in \mathbb{R}_+^{Kl} : \ \exists k \ s.t. \ \tilde{x}^{sk} \leq x^s\}.$$

Each agent chooses a P_i-measurable vector of contingent lists, $\tilde{x}_i = (\tilde{x}_i^1, ..., \tilde{x}_i^K)$, so it makes sense to extend this correspondence to the whole set of states of nature.

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[1] This kind of information setting corresponds to what Laffont (1986) calls *fixed information structures without noise.*

[2] If an agent wants to guarantee delivery of a precise bundle, all the selected alternatives must be equal.

Delivery of $x = (x^1, ..., x^\Omega) \in \mathbb{R}_+^{\Omega l}$ keeps the contract for delivery of any list in $\tilde{X}(x)$, defined as:

$$\tilde{X}(x) = \tilde{X}^s(x^1) \times \tilde{X}^s(x^2) \times ... \times \tilde{X}^s(x^\Omega).$$

A more explicit definition of the same correspondence is:

$$\tilde{X}^s(x_i^s) = \cup_{k=1}^K \{(\mathbb{R}_+^l)^{k-1} \times [0, x_i^s] \times (\mathbb{R}_+^l)^{K-k}\}.^3$$

The price of list $\tilde{x}$ is the price of the cheapest bundle, x, that keeps the contract for the delivery of $\tilde{x}$. It is enough to determine the prices of the contingent goods (primitives). The prices of lists (derivatives) follow as a consequence.

As usual, prices of the contingent commodities are normalized to the simplex:

$$p \in \Delta_+^{\Omega l} = \left\{ p \in \mathbb{R}_+^{\Omega l} : \sum_{s=1}^{\Omega} \sum_{j=1}^{l} p^{sj} = 1 \right\}.$$

The price of a list, $\tilde{p}(\tilde{x}_i)$, is:

$$\tilde{p}^s(\tilde{x}_i^s) = \min_k \{p^s \cdot \tilde{x}_i^{sk}\};$$

$$\tilde{p}(\tilde{x}_i) = \sum_{s=1}^{\Omega} \tilde{p}^s(\tilde{x}_i^s) = \sum_{s=1}^{\Omega} \min_k \{p^s \cdot \tilde{x}_i^{sk}\}.$$

Therefore, the budget restriction faced by agent i is:

$$\tilde{B}_i(e_i, p) = \{\tilde{x}_i \in \mathbb{R}_+^{\Omega K l} : \sum_{s=1}^{\Omega} \min_k \{p^s \cdot \tilde{x}_i^{sk}\} \leq p \cdot e_i\}.$$

In an economy with uncertain delivery, $\mathcal{E} \equiv (e_i, u_i, P_i, q_i)_{i=1}^n$:

- A partition of Ω, P_i, represents the private information of agent i. The set of states that agent i does not distinguish from state s is denoted $P_i(s)$.
- Agents assign subjective probabilities to the different states of nature. To each state s, corresponds a prior probability q_i^s, with $\sum_{s=1}^{\Omega} q_i^s = 1$.

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[3] In this definition, $[0, x^s]$ denotes the set of bundles y^s such that $0 \leq y^s \leq x^s$. For example, with two alternatives and a single commodity: $x = 1$ implies $\tilde{X}^s(x) = \{[0,1] \times \mathbb{R}_+\} \cup \{\mathbb{R}_+ \times [0,1]\}$. This formulation makes it clear that $\tilde{X}$ is a continuous correspondence, because it is defined by finite unions and finite products of continuous correspondences.

- For each state s, a rational expectations function, $R_i^s(\tilde{x}_i^s, p) : \; [0,T]^{Kl} \times \Delta^{\Omega l} \rightarrow [0,T]^l$, returns the bundle that agent i expects to receive, which is simply the cheapest bundle among the alternatives in the list.[4]

- Preferences are the same in undistinguished states, represented by a vector of Von Neumann-Morgenstern (1944) utility functions $u_i^s : \mathbb{R}_+^l \rightarrow \mathbb{R}_+$, which are assumed to be continuous, weakly monotone and concave. The objective function combines beliefs with preferences for consumption: $\tilde{U}_i(\tilde{x}_i, p) = \sum_{s=1}^{\Omega} q_i^s u_i^s(R_i^s(\tilde{x}_i^s, p))$.

- The initial endowments are constant across undistinguished states, and strictly positive: $e_i^s \gg 0$ for all $s = \{1, ..., \Omega\}$.

The problem of agent i is to maximize the expected utility function, restricted to the budget set.

$$\max_{\tilde{x}_i \in \tilde{B}_i(e_i,p)} \tilde{U}_i(\tilde{x}_i, p) = \max_{\tilde{x}_i \in \tilde{B}_i(e_i,p)} \sum_{s=1}^{\Omega} q_i^s \; u_i^s(R_i^s(\tilde{x}_i^s, p)).$$

When several alternatives have the same price, the market is assumed to deliver the alternative that the agent prefers. Since the agent can calculate which alternative is delivered in each state of nature, the problem of the consumer can be written as a decision among bundles instead of lists, under a set of incentive compatibility conditions:

$$\max_{x_i \in \Phi_i(p) \cap B_i(e_i,p)} U_i(x_i) = \max_{x_i \in \Phi_i(p) \cap B_i(e_i,p)} \sum_{s=1}^{\Omega} q_i^s \; u_i^s(x_i^s).^5$$

A rational expectations equilibrium of the economy with uncertain delivery is a pair, (x^*, p^*), composed by a price system p^* and an allocation $x^* = (x_1^*, ..., x_n^*)$. These are such that, for every agent i:

(1) The bundle x_i^* maximizes expected utility, $U_i(x_i^*)$, in the choice set, $\Phi_i(p^*) \cap B_i(e_i, p^*)$.

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[4] In case of a tie, we assume that the market delivers the bundle that the agent prefers. As a result, it is enough to satisfy the endogenous incentive compatibility in equality.

[5] Remember that the price of the list and the price of the delivered bundle coincide.

(2) The bundles selected for delivery, x_i^*, do not violate the endogenous incentive compatibility restrictions. That is, for all s and $t \in P_i(s)$, $p^{*s} \cdot x_i^{*s} \leq p^{*s} \cdot x_i^{*t}$.

(3) The allocation, x^*, is feasible. That is, $\sum_i x_i^* \leq \sum_i e_i = e_T$.

Taking prices as given, each agent trades its initial endowments, e_i, for a vector of bundles that maximizes expected utility, $U_i(x_i)$, belonging to the endogenous incentive compatible set, $\Phi_i(e_i, p^*)$, and to the budget set.[6]

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[6] Remember that if state s occurs, agent i can only claim the right to receive one of the bundles x_i^t with $t \in P_i(s)$. Possible deliveries from a vector of state-contingent lists, $\tilde{x}_i$, are given by a P_i-measurable vector of lists, with the set of alternatives in each state s being $\tilde{x}_i^s = \cup_{t \in P_i(s)} \{\tilde{x}_i^t\}$.

3 Non-existence of equilibrium - an example

Consider an economy in which two agents trade a single good under uncertainty. There are three states of nature, and their future endowments depend on the state of nature that occurs:

$e_A = (100, 100, 1)$ and $e_B = (1, 100, 100)$.

Agents observe only their endowments.

$P_A = \{\{s_1, s_2\}; \{s_3\}\}$ and $P_B = \{\{s_1\}; \{s_2, s_3\}\}$.

Consumption must be positive, and a significant level of risk aversion induces agents to trade *ex ante*.

$U_i : \mathbb{R}_+^{\Omega l} \rightarrow \mathbb{R}$;

$U_i(x_i^s) = \sum_s q_i^s \sqrt{x_i^s}$.

For simplicity, assume that the different states occur with objective and publicly known probabilities:

$q = (q^1, q^2, q^3) = (0.45, 0.1, 0.45)$.

Prices in states s_1 and s_3 must be strictly positive, or else the demands of agent B and A would be infinite for the corresponding contingent goods.

With strictly positive prices for all the contingent goods, if agents selected different consumption levels in states that they did not distinguish, then, they would end up receiving the cheapest of the alternatives, which would be the lowest consumption level. In this case, we must have:

$x_A = (x_A^{12}, x_A^{12}, x_A^3)$ and $x_B = (x_B^1, x_B^{23}, x_B^{23})$.

Since agents are at the frontier of their budget sets:

$$\begin{cases} (p^1 + p^2)x_A^{12} + p^3 x_A^3 = 100(p^1 + p^2) + p^3; \\ p^1 x_B^1 + (p^2 + p^3)x_B^{23} = p^1 + 100(p^2 + p^3). \end{cases}$$

Adding the two:

$$p^1(x_A^{12}+x_B^1)+p^2(x_A^{12}+x_B^{23})+p^3(x_A^3+x_B^{23})=101p^1+200p^2+101p^3.$$

For this to be an equilibrium, the allocation must be feasible:

$$\begin{cases} x_A^{12}+x_B^1 \leq 101; \\ x_A^{12}+x_B^{23} \leq 200; \\ x_A^3+x_B^{23} \leq 101. \end{cases}$$

With strictly positive prices, the conditions are verified in equality. This implies that the allocation is of the form:

$$\begin{cases} x_A=(x_A^{12},x_A^{12},x_A^3)=(x_A^3+99,x_A^3+99,x_A^3); \\ x_B=(x_B^1,x_B^{23},x_B^{23})=(x_B^1,x_B^1+99,x_B^1+99). \end{cases}$$

The only individually rational allocation of this form corresponds to the initial endowments. There is no trade. But are agents maximizing their utility levels?

$$\begin{cases} x_A=(100,100,1); \\ x_B=(1,100,100). \end{cases} \Rightarrow \begin{cases} U(x_A)=0.45*10+0.1*10+0.45*1=5.95; \\ U(x_B)=0.45*1+0.1*10+0.45*10=5.95. \end{cases}$$

Suppose that $p^1=p^3$. Agent 1 can trade consumption in s_1 for consumption in s_3. But consuming less in s_1 implies that delivery in s_2 will also be of this lower quantity. In any case, the agent can select:

$$x_1'=(x_1'^{12},x_1'^{12},x_1'^3)=(81,81,20).$$

The corresponding utility level is:

$$U(x_1')=0.45*9+0.1*9+0.45*4.47=6.96.$$

In the case with asymmetric prices ($p^1 \neq p^3$), the same trade is even more favorable for one of the agents. We reached a contradiction, implying that there is no equilibrium with strictly positive prices.

With $p_2=0$, an alternative bundle can be big enough to violate feasibility and still be incentive compatible. There isn't, in fact, an incentive compatibility restriction because it is of the form $0 \cdot x^2 \leq 0 \cdot x^s$. Agents can choose a consumption level for s_2 that is big enough to violate feasibility and still desire to increase it. There cannot be a rational expectations equilibrium with $p_2=0$.

4 A catalyzer

To avoid the existence of zero prices, suppose that there is an agent that has a very small endowment, but that is perfectly informed and has a linear utility function. This agent will induce a lower bound in prices. Below a certain price level, the agent would select a consumption level that violated feasibility for the whole economy.

Let the information partition of agent $n+1$ be:

$$P_{n+1} = \{\{s_1\}, \{s_2\}, ..., \{s_\Omega\}\}.$$

The utility function of this small agent is:

$$U_{n+1}(x^s_{n+1}) = \sum_{s=1}^{\Omega} q^s_{n+1} v^s_{n+1} \cdot x^s_{n+1} = \sum_{s=1}^{\Omega} q^s_{n+1} \sum_{j=1}^{l} v^s_{n+1,j} x^s_{n+1,j}.$$

Its endowments can be arbitrarily small.

This agent will use all its endowments to buy a single contingent good, the one with the highest ratio between utility and price: $q^s_{n+1} v^s_{n+1,j} / p^s_j$. The quantity that the agent will buy is:

$$x^s_{n+1,j} = \frac{p \cdot e_{n+1}}{p^s_j}.$$

It is obvious that a sufficiently small p^s_j will induce $x^s_{n+1,j} > e^s_{T,j}$, violating feasibility. The demand of this agent will exceed the endowments of all the agents taken together. In practice, the effect of introducing this agent is to impose a strictly positive lower bound on equilibrium prices.

5 Existence of equilibrium

5.1 A sequence of economies

In order to establish existence of equilibrium, we construct a sequence of economies. In these economies, the choice set is not constrained to satisfy the endogenous incentive compatibility restrictions. Utility penalties are imposed instead to consumption choices that violate these restrictions. The penalty depends on the violation, in value, of the inequalities that guarantee incentive compatibility. In the economy $\mathcal{E}^k$, the utility penalty is equal to:

$$U^k_{pen}(x_i, p) = k \max_{t \in P_i(s)} \{p^s \cdot x^s_i - p^s \cdot x^t_i\}.$$

The penalties are never negative (from $s \in P_i(s)$ it follows that the maximum is at least zero), and increase along the sequence of economies. For sufficiently high k, the penalty is big enough for a bundle to have less utility than the initial endowments.

In the economy $\mathcal{E}^k$, the utility functions of the agents are:

$$U^k_i(x_i, p) = U_i(x_i) - k \sum_{s=1}^{\Omega} q^s_i \max_{t \in P_i(s)} \{p^s \cdot x^s_i - p^s \cdot x^t_i\} =$$

$$= \sum_{s=1}^{\Omega} \{q^s_i \ u^s_i(x^s_i) - k \max_{t \in P_i(s)} \{p^s \cdot x^s_i - p^s \cdot x^t_i\}\}.$$

It is clear that, for any $k \in \mathbb{N}$, the utility functions are continuous in prices and bundles. The maximum of linear functions is a convex function, and multiplying a convex function by a negative constant $(-k)$ yields a concave function. Thus, the objective function $U^k_i(x_i, p)$ is also concave. The fact that the utility functions are also continuous functions of prices does not interfere with existence of equilibrium.[7]

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[7] With price dependent preferences, it is known that equilibrium exists (Arrow and Hahn, 1971). Economies with price-dependent preferences were recently studied by Balasko (2003). See also our previous paper (2006b).

The utility functions are the only thing that varies along the sequence of economies. The sequence of economies has a sequence of equilibria, $(x^k, p^k)_{k=1}^{\infty}$, in the compact set that contains the total endowments of the economy, $[0, e_T]^n \times \Delta^{\Omega l}$, with $e_T = \sum_i e_i$.[8] There exists a subsequence that converges, but for the limit, (x^*, p^*), to be a "Rational Expectations Equilibrium", the following conditions must be satisfied:

(1) Feasibility: $\sum_i x_i^* \leq \sum_i e_i = e_T$;

(2) Budget restriction: $\forall i: \ p^* \cdot x_i^* \leq p^* \cdot e_i$;

(3) Incentive compatibility: $\forall i: \ x_i^* \in \Phi_i(p^*)$;

(4) Optimality: $\forall i: \ x_i \in B(p^*, e_i) \cap \Phi_i(p^*) \Rightarrow U_i(x_i^*) \geq U_i(x_i)$.

5.2 The first three conditions

Conditions (1) and (2) are easy consequences of the fact that (x^*, p^*) is the limit of a sequence of equilibria.

(1) The set of feasible allocations is closed, and the limit allocation, x_i^*, is the limit of a sequence of feasible allocations, therefore it is feasible.

(2) The limit allocation, x_i^*, is the limit of a sequence of allocations in the sequence of budget sets. It is straightforward to see that it also belongs to the limit budget set.

Suppose that x_i^* does not satisfy agent i's budget restriction. Let $\alpha = 3\|e_T\| + 1$, and select $\epsilon > 0$ such that $p^* \cdot x_i^* - p^* \cdot e_i = \alpha\epsilon$. Choosing a sufficiently high k, we can guarantee that $d(x^*, x^k) < \epsilon$ and $d(p^*, p^k) < \epsilon$. With $p^k = p^* + dp$, $x^k = x_i^* + dx_i$, and manipulating:

$$(p^*+dp)\cdot(x_i^*+dx_i)-(p^*+dp)\cdot e_i = p^*\cdot x_i^*-p^*\cdot e_i+p^*\cdot dx_i+dp\cdot x_i^*+dp\cdot dx_i-dp\cdot e_i =$$

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[8] The equilibrium allocation is always in $[0, e_T]^n$ because it is feasible.

$$= \alpha\epsilon + (p^* + dp) \cdot dx_i + dp \cdot (x_i^* - e_i) > \alpha\epsilon - \epsilon - \epsilon \cdot 3\|e_T\| = 0.$$

This means that x^k does not satisfy the budget restriction of $\mathcal{E}^k$. Contradiction.

(3) The limit allocation, x^*, satisfies the incentive compatibility restrictions in the limit economy. To see this, suppose that x^* violated one of the restrictions by some value $\mu > 0$, then, for a sufficiently high k, x^k would also violate the same restriction by at least $\mu/2$. That is, for $t \in P^s$:

$$p^{s*} \cdot x^{s*} > p^{s*} \cdot x^{t*} + \delta \Rightarrow p^{sk} \cdot x^{sk} > p^{sk} \cdot x^{tk} + \delta/2, \text{ for all } k > k_0.$$

Utility among feasible allocations is bounded by $U_i(e_T)$, so we can consider a k that is sufficiently high for $k\mu/2 > U_i(e_T) - U_i(e_i)$. It would follow that $U_i^k(x^k) < U_i(x^k) - k\mu/2 < U_i(x^k) - U_i(e_T) + U_i(e_i) < U_i(e_i) = U_i^k(e_i)$, which is a contradiction.

5.3 The fourth condition

The most difficult part of the proof is to verify that the limit, (x^*, p^*), maximizes the utility (4) of the agents in the incentive compatible budget set, $B(p^*, e_i) \cap \Phi_i(p^*)$. The fact that $\Phi_i(p)$ is not lower hemicontinuous could prevent (x^*, p^*) from being optimal. There could be an incentive compatible bundle $x' \in B(p^*, e_i) \cap \Phi_i(p^*)$ that is not even nearly incentive compatible in any of the economies in the sequence. In spite of having a low utility level for high k, this bundle may be optimal in the original economy, and, in this case, (x^*, p^*) would not be an equilibrium.

The strategy to prove that (4) holds is to pick a point, x' in $B(p^*, e_i) \cap \Phi_i(p^*)$, that is a possible optimum (being preferred to x^*) and find that a neighbor of x' also belongs to $B(p^k, e_i) \cap \Phi_i(p^k)$, for large k. This would contradict that (x^k, p^k) is an equilibrium, because the neighbor of x' would also be preferred to x^k in the economy $\mathcal{E}^k$.

Restricting the candidates x'

It is not possible to do this for any candidate for an optimum. The first part of the proof is to show that, without loss of generality, we pick candidates of a particular kind, described below.

Observe that if there are parallel prices for delivery in different states (that agent i does not distinguish), $p^{*s} = ap^{*t}$, two of the incentive compatibility inequalities originate equalities:

$$\begin{cases} p^{*s} \cdot x'^s \leq p^{*s} \cdot x'^t \\ p^{*t} \cdot x'^t \leq p^{*t} \cdot x'^s \end{cases} \Rightarrow \begin{cases} ap^{*t} \cdot x'^s \leq ap^{*t} \cdot x'^t \\ p^{*t} \cdot x'^t \leq p^{*t} \cdot x'^s \end{cases} \Rightarrow \begin{cases} p^{*s} \cdot x'^s = p^{*s} \cdot x'^t \\ p^{*t} \cdot x'^t = p^{*t} \cdot x'^s \end{cases}$$

The two consumption vectors cost the same in both states. Preferences are also the same in both states, because they belong to the same element of the agent's partition of information. If $u^s(x'^s) > u^s(x'^t)$, the agent would be better off selecting x'^s for consumption in both states. Thus, we must have $u^s(x'^s) = u^s(x'^t)$. Since utility functions are concave, the agent is not worse off consuming the average bundle in both states. Notice that if the original vector satisfied the incentive compatibility conditions, then this average vector also does.

We want to take this reasoning a bit further. Whenever there are two symmetric incentive compatibility conditions (as above) satisfied in equality (even if this is not caused by collinearity of state contingent prices), then an agent can choose a consumption vector x'_i such that $x'^s_i = x'^t_i$.

This may allow us to eliminate some inequalities from the original system of incentive compatibility conditions. But we must be careful when choosing a neighbor of x'. It is crucial to consider displacements from x' such that $dx'^s = dx'^t$, for the redundancy of the eliminated inequalities to be preserved.

An auxiliary lemma

We suppose, then, that there exists a $x''_i \in B(p^*, e_i) \cap \Phi_i(p^*)$ such that $U_i(x''_i) >$

$U_i(x_i^*)$, and that it equal in undistinguished states whenever two symmetric incentive compatibility conditions are satisfied in equality. The neighbor $x_i' = (1-\epsilon_1)x_i''$ is still preferred to x_i^*, also belongs to $\Phi_i(p^*)$, and has the advantage of belonging to the interior to the budget set.

$U_i(x_i') = U_i(x_i^*) + \upsilon$, with $\upsilon > 0$.

Since the utility functions are continuous, there exists $\epsilon_2 > 0$ such that (recall that there are no utility penalties in U_i):

$x_i \in B(x_i', \epsilon_2) \Rightarrow U_i(x_i) > U_i(x^k)$, for sufficiently high k.

We are assuming that the bundle x_i' satisfies the incentive compatibility conditions for the equilibrium prices p^*. Those that correspond to an element of the agent's information partition, $A_i^j = \{j1, ..., jJ\}$, are (omitting the subscripts $_i$ and $_j$):

$$\left\{\begin{array}{l}
p^{*1} \cdot x'^1 \leq p^{*1} \cdot x'^2; \\
... \\
p^{*1} \cdot x'^1 \leq p^{*1} \cdot x'^J; \\
p^{*2} \cdot x'^2 \leq p^{*2} \cdot x'^1; \\
... \\
p^{*2} \cdot x'^2 \leq p^{*2} \cdot x'^J; \\
... \\
... \\
p^{*J} \cdot x'^J \leq p^{*J} \cdot x'^1; \\
... \\
p^{*J} \cdot x'^J \leq p^{*J} \cdot x'^{J-1}.
\end{array}\right.
\Leftrightarrow
\left\{\begin{array}{l}
p^{*1} \cdot x'^2 - p^{*1} \cdot x'^1 = k^{12} \geq 0; \\
... \\
p^{*1} \cdot x'^J - p^{*1} \cdot x'^1 = k^{1J} \geq 0; \\
p^{*2} \cdot x'^1 - p^{*2} \cdot x'^2 = k^{21} \geq 0; \\
... \\
p^{*2} \cdot x'^J - p^{*2} \cdot x'^2 = k^{2J} \geq 0; \\
... \\
... \\
p^{*J} \cdot x'^1 - p^{*J} \cdot x'^J = k^{J1} \geq 0; \\
... \\
p^{*J} \cdot x'^{J-1} - p^{*J} \cdot x'^J = k^{J,J-1} \geq 0.
\end{array}\right.$$

It should be clear that this reasoning extends to all the sets $A_i^j \in P_i$. Keep in mind that we seek a neighbor of x_i' that is in $\Phi_i(p^k)$ and that therefore contradicts the fact that x^k is an equilibrium of $\mathcal{E}^k$. This would prove (4) by contradiction.

Let $d(x, x') < \epsilon_2$. For sufficiently high k, we know that $U_i(x_i) > U_i(x^k)$ and also that $d(p^k, p^*) < \epsilon_2$. Let $dx = x - x'$ and $dp = p^k - p^*$. Manipulating the first condition:

$$p^{*1}\cdot x'^2 - p^{*1}\cdot x'^1 = (p^{k1}-dp^1)\cdot(x^2-dx^2)-(p^{k1}-dp^1)\cdot(x^1-dx^1)=k^{12}\Leftrightarrow$$

$$\Leftrightarrow p^{k1}\cdot x^2 - p^{k1}\cdot x^1 = k^{12}+p^{k1}\cdot dx^2+dp^1\cdot(x^2-dx^2)-p^{k1}\cdot dx^1-dp^1\cdot(x^1-dx^1)\Leftrightarrow$$

$$\Leftrightarrow p^{k1}\cdot x^2 - p^{k1}\cdot x^1 > k^{12}-\epsilon_2-\epsilon_2(\|e_T\|+\epsilon_2)-\epsilon_2-\epsilon_2(\|e_T\|+\epsilon_2)\Leftrightarrow$$

$$\Leftrightarrow p^{k1}\cdot x^2 - p^{k1}\cdot x^1 > k^{12}-2\epsilon_2-2\epsilon_2(\|e_T\|+\epsilon_2)=k^{12}-2\epsilon_2(\|e_T\|+1+\epsilon_2).$$

Let $\epsilon_3 = 2\epsilon_2(\|e_T\|+1+\epsilon_2) > 0$. We have:

$$p^{k1}\cdot x^2 - p^{k1}\cdot x^1 > k^{12}-\epsilon_3.$$

Let k^{min} be minimum over the set of strictly positive k^{ab}. Choosing ϵ_2 small enough to make $\epsilon_3 < k^{min}$ guarantees that the strict inequalities for x' and p^* remain strict for any $x \in B(x', \epsilon_2)$ and p^k (with k large enough).

If all inequalities still held, then $U_i^k(x_i) = U_i(x_i) > U_i^k(x_i^k)$ and we would have a contradiction (the element of the equilibrium sequence, x^k, would not be a maximizer of U_i^k, because x_i would be preferred). The difficulty lies in checking that the inequalities which are not strict at (x', p^*) are satisfied at (x, p^k) (for sufficiently high k).

Our restriction of the candidates x' allows us to consider a reduced system of inequalities without any $k_{ab} = k_{ba} = 0$. That is, such that $k_{ab} = 0 \Rightarrow k_{ba} > 0$. But remember that when choosing a neighbor of x', the displacements must be such that $dx'^{j1} = dx'^{j2}$, for the redundancy of the eliminated inequalities to be preserved.

$$\begin{cases} p^{*j1}\cdot x'^{j2} - p^{*j1}\cdot x'^{j1} = k^{12}; \\ \ldots \\ p^{*j1}\cdot x'^{jJ} - p^{*j1}\cdot x'^{jJ} = k^{1J}; \\ p^{*j2}\cdot x'^{j1} - p^{*j2}\cdot x'^{j2} = k^{21}; \\ \ldots \\ \ldots \\ p^{*jJ}\cdot x'^{jJ-1} - p^{*jJ}\cdot x'^{jJ} = k^{JJ-1}. \end{cases}$$

Denote $\gamma > \min_{a,b}(1 - \frac{p^a \cdot p^b}{\|p^a\|\|p^b\|})\|p^a\| > 0$. Since we have a lower bound on prices, this minimum is strictly positive, unless prices were parallel in two states (remember that there are no parallel prices in this reduced system of inequalities). Keeping x sufficiently close to x' in order to preserve the strict inequalities, and selecting displacements parallel to prices: $dx^a = -\frac{\epsilon_2}{2} \frac{p^{ka}}{\|p^{ka}\|}$. Consider also a small deviation in prices, that is, a k that is large enough: $d(p^k, p^*) < \epsilon_5 = \frac{\epsilon_2 \gamma}{8\|e_T\|}$.

Given an inequality that is not strict, $k^{12} = 0$:

$$
\begin{aligned}
& p^{k1} \cdot x^2 - p^{k1} \cdot x^1 = \\
& = p^{*1} \cdot (x'^2 + dx^2) + dp^1 \cdot (x'^2 + dx^2) - p^{*1} \cdot (x'^1 + dx^1) - dp^1 \cdot (x'^1 + dx^1) = \\
& = p^{*1} \cdot dx^2 + dp^1 \cdot (x'^2 + dx^2) - p^{*1} \cdot dx^1 - dp^1 \cdot (x'^1 + dx^1) > \\
& > p^{*1} \cdot dx^2 - \epsilon_5(\|e_T\| + \epsilon_2) - p^{*1} \cdot dx^1 - \epsilon_5(\|e_T\| + \epsilon_2) = \\
& = p^{*1} \cdot dx^2 - p^{*1} \cdot dx^1 - 2\epsilon_5(\|e_T\| + \epsilon_2) > \\
& > -p^{*1} \cdot \frac{\epsilon_2}{2} \frac{p^{*2}}{\|p^{*2}\|} + p^{*1} \cdot \frac{\epsilon_2}{2} \frac{p^{*1}}{\|p^{*1}\|} - 4\epsilon_5 \|e_T\| = \\
& = -\frac{\epsilon_2}{2} \frac{p^{*1} \cdot p^{*2}}{\|p^{*1}\|\|p^{*2}\|}\|p^{*1}\| + \frac{\epsilon_2}{2} \frac{p^{*1} \cdot p^{*1}}{\|p^{*1}\|\|p^{*1}\|}\|p^1\| - 4\epsilon_5\|e_T\| = \\
& = \frac{\epsilon_2}{2} \frac{p^{*1} \cdot p^{*1}}{\|p^{*1}\|\|p^{*1}\|}\|p^1\| - \frac{\epsilon_2}{2} \frac{p^{*1} \cdot p^{*2}}{\|p^{*1}\|\|p^{*2}\|}\|p^{*1}\| - \frac{\epsilon_2}{2}\gamma = \\
& = \frac{\epsilon_2}{2}(1 - \frac{p^{*1} \cdot p^{*2}}{\|p^{*1}\|\|p^{*2}\|})\|p^{*1}\| - \frac{\epsilon_2}{2}\gamma \geq 0
\end{aligned}
$$

In sum, we have found a displacement dx such that:

$$p^{k1} \cdot x^2 - p^{k1} \cdot x^1 > 0.$$

This consumption bundle, $x_i = x'_i + dx$, would prevent x^k from being equilibrium in the economy k of the sequence, which is a contradiction.

Existence of a rational expectations equilibrium is, therefore, guaranteed if there is a lower bound on the prices of the contingent goods. For this bound to appear, we included an arbitrarily small, but perfectly informed trader in the economy.

Appendix: The endogenous incentive compatible set correspondence

The set of bundles satisfying the incentive compatibility restrictions depends on the prevailing prices. Consider the correspondence from prices to the set of incentive compatible bundles:

$$\Phi_i : \Delta_+^{\Omega l} \rightarrow \mathbb{R}_+^{\Omega l};$$

$$\Phi_i(p) = \left\{ x \in \mathbb{R}_+^{\Omega l} : \forall \omega_s,\ p^s \cdot x^s = \min_{t \in P_i(s)} \{p^s \cdot x^t\} \right\}.$$

If this correspondence were continuous, we could apply the theorem of existence of social equilibrium, yielding the result we seek: existence of rational expectations equilibrium in economies with uncertain delivery.

In finite dimensional Euclidean spaces, upper hemicontinuity of Φ_i at p_0 means that, given an arbitrary open set, V, containing $\Phi_i(p_0)$, there exists $\delta > 0$ such that for all $p \in B(p_0, \delta)$, we have $\Phi_i(p) \subseteq V$.

The correspondence is closed-valued since all the restrictions are inequalities which are not strict. With a compact range space, that is, in a bounded economy (for example, by the total initial endowments in the economy) a correspondence is upper hemicontinuous if and only if it has closed values. Therefore, Φ_i is upper hemicontinuous.

In finite dimensional Euclidean spaces, lower hemicontinuity of Φ_i at p_0 means that given an arbitrary open set, V, intersecting $\Phi_i(p_0)$, there exists $\delta > 0$ such that for all $p \in B(p_0, \delta)$, the image $\Phi_i(p)$ also intersects V.

The correspondence under study, Φ_i, is not lower hemicontinuous. Below is an example where this type of continuity fails.

Consider an economy with two goods, A and B, and two states of nature, s and t. Let $p_0 = (p_0^s, p_0^t) = (p_0^{As}, p_0^{Bs}; p_0^{At}, p_0^{Bt}) = (\frac{1}{4}, \frac{1}{4}; \frac{1}{4}, \frac{1}{4})$. The bundle $x_0 = (1, 0; 0, 1)$ belongs to the incentive compatible set, since:

$$p_0^s \cdot x_0^s \leq p_0^s \cdot x_0^t \Leftrightarrow \tfrac{1}{4} \leq \tfrac{1}{4}, \text{ and}$$

$$p_0^t \cdot x_0^t \leq p_0^t \cdot x_0^s \Leftrightarrow \tfrac{1}{4} \leq \tfrac{1}{4}.$$

Delivering $(1, 0)$ in state s and $(0, 1)$ in state t does not violate incentive compatibility because both bundles have the same price in both states.

A small perturbation in prices can make $(0,1)$ cheaper in state s and $(1,0)$ cheaper in state t. Consider an open ball around x_0 with radius $0 < \epsilon < \frac{1}{10}$. After a perturbation in prices to $p = (\frac{1}{4} + \delta, \frac{1}{4} - \delta, \frac{1}{4} - \delta, \frac{1}{4} + \delta)$, this ball does not intersect the incentive compatible set.

Suppose that there existed a vector $dx = (\epsilon^{As}, \epsilon^{Bs}, \epsilon^{At}, \epsilon^{Bt})$ such that $x = (1 + \epsilon^{As}, \epsilon^{Bs}; \epsilon^{At}, 1 + \epsilon^{Bt})$ is inside that open ball and belongs to the incentive compatible set:

$$(1)\quad (\tfrac{1}{4} + \delta, \tfrac{1}{4} - \delta) \cdot (1 + \epsilon^{As}, \epsilon^{Bs}) \leq (\tfrac{1}{4} + \delta, \tfrac{1}{4} - \delta) \cdot (\epsilon^{At}, 1 + \epsilon^{Bt}) \Leftrightarrow$$

$$\Leftrightarrow (\tfrac{1}{4} + \delta)(1 + \epsilon^{As}) + (\tfrac{1}{4} - \delta)\epsilon^{Bs} \leq (\tfrac{1}{4} + \delta)\epsilon^{At} + (\tfrac{1}{4} - \delta,)(1 + \epsilon^{Bt}) \Leftrightarrow$$

$$\Leftrightarrow \tfrac{1}{4} + \tfrac{1}{4}\epsilon^{As} + \delta + \delta\epsilon^{As} + \tfrac{1}{4}\epsilon^{Bs} - \delta\epsilon^{Bs} \leq \tfrac{1}{4}\epsilon^{At} + \delta\epsilon^{At} + \tfrac{1}{4} + \tfrac{1}{4}\epsilon^{Bt} - \delta - \delta\epsilon^{Bt} \Leftrightarrow$$

$$\Leftrightarrow \tfrac{1}{4}(\epsilon^{As} + \epsilon^{Bs} - \epsilon^{At} - \epsilon^{Bt}) + \delta(\epsilon^{As} - \epsilon^{Bs} - \epsilon^{At} + \epsilon^{Bt}) \leq -2\delta;$$

$$(2)\quad (\tfrac{1}{4} - \delta, \tfrac{1}{4} + \delta) \cdot (\epsilon^{At}, 1 + \epsilon^{Bt}) \leq (\tfrac{1}{4} - \delta, \tfrac{1}{4} + \delta) \cdot (1 + \epsilon^{As}, \epsilon^{Bs}) \Leftrightarrow$$

$$\Leftrightarrow (\tfrac{1}{4} - \delta)\epsilon^{At} + (\tfrac{1}{4} + \delta)(1 + \epsilon^{Bt}) \leq (\tfrac{1}{4} - \delta)(1 + \epsilon^{As} + (\tfrac{1}{4} + \delta)\epsilon^{Bs}) \Leftrightarrow$$

$$\Leftrightarrow \tfrac{1}{4}(\epsilon^{At} + 1 + \epsilon^{Bt} - 1 - \epsilon^{As} - \epsilon^{Bs}) + \delta(-\epsilon^{At} + 1 + 1 + \epsilon^{Bt} + \epsilon^{As} - \epsilon^{Bs} \leq 0 \Leftrightarrow$$

$$\Leftrightarrow \tfrac{1}{4}(\epsilon^{At} + \epsilon^{Bt} - \epsilon^{As} - \epsilon^{Bs}) + \delta(-\epsilon^{At} + \epsilon^{Bt} + \epsilon^{As} - \epsilon^{Bs}) \leq -2\delta.$$

Adding the two inequalities, we obtain:

$$(1+2)\quad \delta(\epsilon^{As} - \epsilon^{Bs} - \epsilon^{At} + \epsilon^{Bt}) \leq -2\delta \Leftrightarrow$$

$$\epsilon^{As} - \epsilon^{Bs} - \epsilon^{At} + \epsilon^{Bt} \leq -2.$$

Which is impossible, because $\epsilon^{As} - \epsilon^{Bs} - \epsilon^{At} + \epsilon^{Bt} \geq -4\epsilon > -\frac{4}{10}$.

This is an example of the failure of lower hemicontinuity of the incentive compatible set correspondence. When prices are equal in two states, a small perturbation may induce a significant change in the incentive compatible set.

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