

INTERGENERATIONAL EQUITY AND THE DISCOUNT RATE FOR COST-BENEFIT ANALYSIS

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ABSTRACT. Current Office of Management and Budget (OMB) guidelines use the interest rate as a basis for the discount rate, and have nothing to say about an intergenerationally fair discount rate. A traditional approach leads to too high values for the latter, and in a wide range. We propose to apply Relative Utilitarianism to derive the discount rate, and find it should equal the growth rate of real per-capita consumption, independent of the interest rate.

1. INTRODUCTION

Most of public policy decisions — whether to build a housing project in a city, how to manage extraction of natural resources, how to implement a pension reform, etc. — typically require trading-off economic costs and benefits, that are spread over time. Crucial, then, becomes the appropriate choice of the discount rate, or the way to translate benefits and costs into present (consumption) terms (either explicitly stated as a part of regulatory principles or implicitly embedded in a specific policy analysis). This choice should, naturally, be based on well-defined normative principles, i.e., for individuals making decisions on behalf of future generations, it has to be objective and justified. In this paper we use relative utilitarianism to derive an intergenerationally fair discount rate to evaluate benefits accruing from a public project in a general equilibrium model with overlapping generations.

We are not concerned with implementing decision rules based on this discount rate, rather, taking existing practices (summarized in the OMB Circulars, for example) as given, we suggest a way to think about the underlying principles behind these practices.

Circular A-4 of the U.S. Office of Management and Budget (2003) mandates that all executive agencies and establishments conduct a “regulatory analysis” for any new proposal, and more specifically (pp. 33–36), a cost-benefit analysis, at the rates of both 3% and 7%. Both rates are rationalized there as the interest rate: the first one relative to private savings, the second one relative to capital formation and/or displacement, i.e., as the gross return on capital.

The OMB circular does refer explicitly to the requirement of equity vis-à-vis of future generations, and acknowledges it by requiring, for

projects that might have substantial long-term impact, a further analysis at a “lower but positive” discount rate (p. 36) — but more specific suggestions are hard to find.¹ This is the question we want to address.

The issue of discounting, and — more broadly — intergenerational justice, has been controversial in the literature since, probably, Sidgwick (1874).² Ramsey (1928) (p. 543) presents discounting future utility (‘enjoyments’) as a “practice which is ethically indefensible and arises merely from the weakness of the imagination,” and suggests a way to overcome ‘technical’ difficulties of constructing a discount-free utilitarian social welfare criterion, (based on the difference between actual and ‘bliss’ level of utility) later referred to as “Ramsey criterion.” Koopmans (1960) provides axiomatization of ‘social impatience,’ i.e., discounted utilitarianism. A growing literature in social choice and welfare economics is concerned with incorporating intergenerational justice principles in developing a social welfare criterion (among the most recent contributions, see Asheim, Mitra, and Tungodden (2006) who demonstrates existence of welfare functions satisfying some of Koopmans (1960) postulates and principles of intergenerational equity, in particular, the axioms of ‘sustainable development’ by Chichilnisky (1996)); in addition a number of contributions are devoted to characterizing ethically acceptable (just) consumption allocations (Asheim (1991), Fleurbaey and Michel (2003) among others).

Treating different generations equally is embodied in the social welfare criterion that we use in this paper to evaluate an ultimate impact of a public project. Importance of using an explicit criterion for this purposes was stressed in Drèze and Stern (1987), this approach is opposed to a examining a “potential improvement” that a project might create.³ The authors argue in favor of transparency of cost-benefit analysis that formulating social welfare function offers along with consistency of the related choices and avoidance of special preference for

¹Other practical recommendations take a similar stand. “Morally speaking, there is no difference between current and future risk. Theories which, for example, attempt to discount effects on human health in twenty years to the extent that they are equivalent to only one-tenth of present-day effects in cost-benefit considerations are not acceptable.” Wildi, Appel, Buser, F.Dermange, Eckhardt, Hufschmied, and Keusen (2000)

² “How far we are to consider the interests of posterity when they seem to conflict with those of existing human beings? It seems, however, clear that the time at which a man exists cannot affect the value of his happiness from a universal point of view; and that the interests of posterity must concern a Utilitarian as much as those of his contemporaries, except in so far as the effect of his actions on posterity — and even the existence of human beings to be affected — must necessarily be more uncertain.” (p. 414)

³See Mishan (1976) for an in-depth discussion of “potential Pareto improvements” (traced back to Pigou (1932)) and their application to cost-benefit analysis. For a more recent overview of cost-benefit criteria see Coate (2000).

inaction.⁴ To properly tackle questions of intergenerational equity we suggest to use Relative Utilitarianism, a welfare criterion introduced in Dhillon and Mertens (1999), that allows for a meaningful comparison of well-being across individuals born at different times and faced with different consumption and policy choices.⁵ The analysis is focused on “small projects” viewed as “...a disturbance to the economy, displacing it from some initial equilibrium to a new one...” (Bell and Devarajan (1983), pp. 457-8) in order to abstract away potential effects of the size of the project on the discount rate. In other words, the approach is to rely on ‘shadow prices’ for project evaluation. In this case the discount rate (roughly speaking) should indicate how fast the impact of a project on social welfare decreases (or increases) with time.

Bell and Devarajan (1983) raised a concern that the shadow prices might depend on the way the perturbed economy (including the government) re-adjusts. One way to escape this ambiguity is to translate the effect of a public project (and all its consequences) into its consumption equivalent for individuals. Viewing public projects this way, we have no reason to introduce public goods into the model, which makes the analysis more transparent. Moreover, this representation is closer to the practical guidance for conducting cost-benefit analysis suggesting the impact of a public project to be monetized (see Circular A-4). Thus, the relevant shadow price becomes the marginal social value created by an additional unit of consumption.

It is not uncommon to use prevailing prices to represent the shadow prices, and thus, the interest rate (the monetized value of future consumption). However, if one is to formulate a welfare criterion supporting these shadow prices, the choice of the criterion will have to be ‘just right’. Let us look at this argument in a bit more detail.

The status-quo is a given competitive equilibrium. Construct a social welfare function (SWF), W , as a weighted sum of individual utilities, $\sum \lambda_n u_n$, such that the weights are chosen to equalize the individual marginal utilities of consumption *at the given equilibrium*, so $\lambda_n \nabla u_n = \mu p$,⁶ for the equilibrium price system p and some $\mu > 0$.⁷

⁴ “... a fundamental shortcoming of evaluation criteria based on Pareto improvements, whether actual or potential, is that, unless they are taken to imply that Pareto-improving changes are the only acceptable ones (a view which we regard as extremely unappealing and which attaches undue weight to the status quo), they provide no decision criterion for projects which cannot lead to Pareto improvements. It is difficult to overcome this problem without accepting the need to specify a social welfare function which embodies more definite judgements.” (p.49)

⁵Relative Utilitarianism is discussed in more detail in section 1.2.

⁶This condition is implied, for example, by one of the Samuelson (1954)’s optimality conditions, namely, his condition (3)

⁷Equivalently, assuming, e.g., concave utility functions, one can deduce from the First Welfare Theorem the existence of utility weights such that the given

Viewing projects as small perturbations of individual endowments, $\delta\omega_n$,⁸ we are interested in the induced variation of social welfare, δW ,

$$\begin{aligned} \delta W &= \sum \lambda_n \delta u_n = \sum \lambda_n \langle \nabla u_n, \delta c_n \rangle \\ (1) \quad &= \mu \langle p, \sum \delta c_n \rangle = \mu \langle p, \sum \delta \omega_n \rangle \end{aligned}$$

since $\langle p, \delta y \rangle = 0$, where y is the equilibrium production.

Thus, for the social welfare function so constructed prevailing prices are, indeed, the relevant ‘shadow prices’ that reflect the relative impact of varying different consumption goods on the social welfare function. In a dynamic interpretation, where the goods become dated goods, the equilibrium price system includes, in particular, the interest rate, as the price of tomorrow’s money in terms of today’s.

To sum up, if one is to accept the prevailing prices to reflect the relevant shadow prices for cost-benefit analysis — and that the correcting lump-sum transfers are unavailable — one has to admit that societal objective, or social welfare function, depends on equilibrium prices, i.e., welfare weights associated with individual utilities should change each time the prices do. Hardly a desirable approach.⁹

Next section shows that if one to use a traditional utilitarian criterion as a guide, the resulting the social discount rate is impractically imprecise and does not reflect the ‘true shadow price’ of future consumption, which, in a growing economy, should depend solely on the growth rate of individual consumption.

1.1. A simple computation using the traditional methodology.

To illustrate the problem of using the traditional methodology to derive the discount rate, let us start with a very simple model of an economy, in which individuals live for just one period, enjoying consumption $c_t > 0$ during their lifetime at t . Individual preferences over (lifetime) consumption are represented by a constant relative risk aversion utility function with coefficient, $\rho > 0$, so that $u(c) = c^{1-\rho}/(1-\rho)$; and suppose the economy is on a steady growth path with per-capita consumption growing exponentially at a rate $\gamma > 0$. Consider a policy that involves a variation in aggregate consumption of δC_t at each future date t and is to be evaluated at time 0. The status-quo per-capita consumption at time t is $c_0 e^{\gamma t}$, where c_0 is the initial (time

equilibrium maximizes the corresponding weighted sum of utilities over all feasible allocations; this yields the same weights.

⁸Since we omit for simplicity public goods and externalities from our formal model, the $\delta\omega_n$ are assumed to include, in addition to the direct effect of the project, also the appropriate compensating variation (in real terms) for the different external effects.

⁹Clearly, if a project generates a Pareto improvement, the choice of a social welfare function (provided it increases in individual utilities) is immaterial, it is only when some individuals might loose as a result of project implementation will the choice of the criterion matter.

0) per-capita consumption. Taking a traditional utilitarian criterion $W = \sum_t e^{-\beta t} N_t u(c_t)$ (N_t being the number of agents at time t) as a guide for evaluating this policy, the net (social) benefit equals

$$\begin{aligned} & \sum_t e^{-\beta t} N_t \left[u\left(c_0 e^{\gamma t} + \frac{\delta C_t}{N_t}\right) - u(c_0 e^{\gamma t}) \right] \\ &= \sum_t e^{-\beta t} N_t u'(c_0 e^{\gamma t}) \cdot \frac{\delta C_t}{N_t} = \sum_t c_0^{-\rho} e^{-(\rho\gamma + \beta)t} \delta C_t \end{aligned}$$

This means that future consumption is discounted at the rate $\rho\gamma + \beta$ under this criterion. Even if we are to follow Sidgwick (1874) and Ramsey (1928) and set $\beta = 0$, to write explicitly that we want to treat future generations equally, the magnitude of the suggested discount rate, $\rho\gamma$, is far above any rates applicable in practice; besides, the estimated values have an extremely wide range, as the next subsection demonstrates.

1.2. Orders of Magnitude for the Discount Rate. To estimate γ one may use a measure of growth of real per-capita GDP. Based on the data from the Bureau of Economic Analysis, over the past 70 years the average in the U.S. is estimated to be around 2–2.5% per annum (with averages over various decades since 1950 ranging from 3% to 1.8%).

In the above model, individuals live for 1 period, so the *only* role of ρ is to determine the individuals' attitudes towards risk. And consistency with, e.g., Harsanyi's axiomatization(s) of such additive SWFs forces then to interpret u as the individual's von Neumann-Morgenstern utility function, and hence ρ as his coefficient of relative risk aversion. One of the most recent overviews compiling various (micro) estimates of the risk aversion coefficients is contained in Einav and Cohen (2005). Remarkable is both the range as well as the magnitude of the suggested values, ranging from single- to three-digit values. They measure relative risk aversion coefficients from individual-level data on car insurance and annual income, obtaining *two-digit* estimates. Clearly, cost-benefit analysis will then only allow for very short-sighted policies. This remains true even with more conservative estimates, like, say, derived by Drèze (1981) ($\rho \sim 12$ – 15), or like those which seem accepted as corresponding to “representative” (instead of individual) behaviour in financial markets — say 3, leading to $\rho\gamma \sim 6$ – 7% , way too high.

In sum, it is impossible to view the traditional methodology described above as a correct interpretation of “treating future generations equally” — which is exactly what our SWF tried to embody, by using $\beta = 0$.

1.3. Discount Rate under Relative Utilitarianism. Since the traditional methodology failed so badly, producing unreasonably high discount factors within a wide range, let us now look at Relative Utilitarianism, introduced in Dhillon and Mertens (1999).

The axiomatisation consists basically of applying Arrow's axioms to preferences over lotteries, after “surgically removing” from them

everything which is clearly objectionable — i.e., which anyone would expect a good social welfare functional to violate: the implications that variations in the intensity of preference of x over y don't matter.

After this removal, one can add anonymity (implying here also that individuals of different generations are treated equally) to obtain an axiomatization of a unique social welfare functional,¹⁰ relative utilitarianism, that takes for each individual's preferences the unique von Neumann-Morgenstern representation having minimum 0 and maximum 1 over the feasible set, and sums those to obtain a representative of the corresponding social preferences.

It is stressed in that paper that this dependence on the feasible set implies that in actual use it should be applied with some universal feasible set, to quote “all alternatives that are feasible and just”. In particular, in the present situation, the feasible set should consist not only of the “baseline” and the different proposals under consideration, but of all policies and policy-changes that might be considered by any agency of the government.

In (exogenous) growth models, the rate of growth is unaffected by any policy variable: policies affect only the height of the growth path, which, in the simple setup described in subsection 1.1, translates into multiplying per-capita consumption by some constant along the growth path. Therefore, the set of feasible policies at time t can be viewed as a range of induced per-capita consumption levels $(1 - \eta)c_0e^{\gamma t}$ and $(1 + \zeta)c_0e^{\gamma t}$ for some constants η and ζ . Applying relative utilitarianism to the simple model, we have to normalize individual utility $u(c_t)$ on the set of feasible policies:

$$v(c_0e^{\gamma t} + \delta c_t) = \frac{u(c_0e^{\gamma t} + \delta c_t)}{u((1 + \zeta)c_0e^{\gamma t}) - u((1 - \eta)c_0e^{\gamma t})}$$

i.e., we divide by

$$\frac{e^{(1-\rho)\gamma t}}{c_0^\alpha} \left[\frac{-1}{(1 + \zeta)^{(\rho-1)}} + \frac{1}{(1 - \eta)^{(\rho-1)}} \right] \sim e^{(1-\rho)\gamma t}$$

So the variation of our SWF becomes

$$\sum_t e^{(\rho-1)\gamma t} \delta C_t u'(c_0 e^{\gamma t}) = \sum_t e^{(\rho-1)\gamma t} e^{-\rho\gamma t} \delta C_t = \sum_t e^{-\gamma t} \delta C_t$$

This implies that the previous discount rate of $\rho\gamma$ becomes now simply γ , $2 - 2\frac{1}{2}\%$, right in the ball-park of “positive and $< 3\%$ ”.

One could argue that the example is not representative; in particular, since individuals live only one period they have no incentive to save, so there can be no capital accumulation and growth. In a real model where there is growth and savings, there is also an interest rate — and individuals would smooth the shock over their lifetime using the going

¹⁰The axiomatization assumes a finite number of agents.

interest rate: so one would expect the result to be driven back to the interest rate, to a large extent at least.

We will show, nevertheless, that the result (as well as that of sect. 1.1) does remain valid in the much more general framework of the next section.

2. THE MODEL

We use a general-equilibrium overlapping generations model, cast in an exogenous growth framework.

2.1. The Consumption Sector.

2.1.1. *Population Dynamics.* Time is continuous, ranging from $-\infty$ to $+\infty$. There are several types of individuals. An individual of type τ lives up to age T_τ . The population dynamics is fully specified by non-decreasing right-continuous functions $P_{\tau,\tau'}$, defined on $[0, T_\tau]$ with $P_{\tau,\tau'}(s)$ being the number of children of type τ' an individual of type τ has at age s , and then specifying that we are looking at a corresponding invariant distribution.¹¹ But as long as we are not introducing bequest motives or the like, it is only this distribution that matters. It is such that, at time t , the number of individuals of type τ in the age-group $(s, s + ds)$ ($0 \leq s \leq T_\tau$) is given by $N_\tau e^{\nu(t-s)} ds$. So, population grows at rate $\nu > 0$, keeping the proportion of each age group of each type constant over time.

2.1.2. *Preferences and Endowments.* At each instant of his life, s , an individual of type τ born at time x consumes non-negative quantities of n goods, $c^{\tau,x}(s) \in \mathbb{R}_+^n$ and allocates fractions of his time to h types of labour, $z^{\tau,x}(s) \in \mathbb{R}_+^h$.¹²

His preferences over integrable life-time consumption-streams in \mathbb{R}^{n+h} are represented by a utility function U^τ (e.g., concave, differentiable, increasing in the goods, decreasing in labour).¹³

For balanced growth to be at all possible, we assume U^τ to be homogeneous, say of degree $1 - \rho^\tau$, in the n first coordinates (consumption-goods-streams).

There is no bequest motive.

Endowments are 0 — except for the “endowment of time,” which is unity at every instant (24h/day). This imposes an instantaneous

¹¹We keep everything deterministic here, just to avoid having to discuss irrelevant insurance markets for idiosyncratic risks.

¹²Sometimes we will use the notation $c^\tau(s, t)$ and $z^\tau(s, t)$ to stand for consumption and labor of an individual of type τ who is of age s at time t , so that $x = t - s$.

¹³We index consumption streams by age, in $[0, T_\tau]$, so all individuals of the same type have the same consumption set, $[0, T_\tau] \rightarrow \mathbb{R}_+^{n+h}$, and utility function, independently of their birth-date.

constraint on the individual feasible set requiring the sum of fractions of time devoted to all possible occupations to be always less than unity.

In what follows, a policy will be associated with a perturbation of endowments of consumption goods, $(\delta\omega)_i$ for $i = 1 \dots n$ — cf footnote 8.

2.2. Production.

2.2.1. Instantaneous production. Instantaneous production is described by a closed convex cone $Y \subset \mathbb{R}^{h+m+n+m}$, $t \in \mathbb{R}$, describing feasible production plans transforming $h + m$ inputs (h types of (effective) labour, $L(t) \in \mathbb{R}_+^h$, and m types of capital) into n consumption goods and m investment goods. Assume no free lunch, $Y \cap \mathbb{R}_+^n = \{0\}$.

Individuals supply labour (time) to the firms, and their productivity changes with time and age. The amount of effective labour of type i received at time t by a production firm from an individual of type τ and of age s is $e^{\gamma t} \varepsilon_i^\tau(s) z_i^\tau(s, t)$, where $\varepsilon_i^\tau(s)$ is this individual's life-cycle 'productivity' (in occupation i),¹⁴ and where γ is (labour-enhancing) technological progress. Recall $z_i^\tau(s, t)$ is the amount of labour (time) supplied by an individual (of type τ) born at time $t - s$.

Thus, (exogenous) growth in this model is driven by a steady increase in labour productivity.

2.2.2. Capital accumulation. There are m capital goods K^i ($i = 1 \dots m$), each with its corresponding investment good I^i , depreciation rate δ_i , and capital-accumulation equation $\frac{dK^i(t)}{dt} = I^i(t) - \delta^i K^i(t)$,¹⁵ together with the "initial condition" that $\limsup_{t \rightarrow -\infty} e^{-(\gamma+\nu)t} K^i(t) < \infty$.

Lemma 1. $K^i(t) = e^{-\delta^i t} \int_{-\infty}^t e^{\delta^i s} I^i(s) ds$ for all t , where the integral is a Lebesgue integral.

Proof. Continuity yields that K_t^i is bounded on any interval $(-\infty, t_0)$. The production technology implies a similar upper bound for I_t^i . In particular I_t^i is locally-integrable and thus K_t^i locally absolutely continuous. Letting $M_t = e^{\delta^i t} K_t^i$, the differential equation becomes $M_t' = e^{\delta^i t} I_t^i$, hence, by the local absolute continuity, $M_t = M_0 + \int_0^t e^{\delta^i s} I_s^i ds$. Therefore $K_t^i \geq 0$ yields $\int_{-T}^0 e^{\delta^i s} I_s^i ds \leq M_0 \forall T$, and $K_t^i \leq \bar{K} e^{(\gamma+\nu)t}$ for $t \leq 0$ yields $\int_{-T}^0 e^{\delta^i s} I_s^i ds \geq M_0 - \bar{K} e^{-(\gamma+\nu+\delta^i)T}$ for $T \geq 0$. In particular, the (locally integrable, as just seen) function $h(t) = e^{\delta^i t} I_t^i$ is such that $\int_{-T}^0 h(s) ds$ converges to M_0 when $T \rightarrow \infty$. Since our upper bound for I_t^i implies $h(t) \leq \bar{h} e^{(\gamma+\nu+\delta^i)t}$ for $t \leq 0$, we conclude that $h(t)$ is (absolutely) integrable on $(-\infty, t)$ for all t , and

¹⁴For example, in the textbook OLG models, ε would be 1 during the first half of life and 0 after.

¹⁵Assumed to hold a.e., and implying that K_t^i is assumed locally a Perron primitive and I_t^i locally Perron-integrable.

thus $K_t^i = e^{-\delta^i t} \int_{-\infty}^t e^{\delta^i s} I_s^i ds$ for all t , where the integral is a Lebesgue integral. \square

To ensure bounded production possibilities we need that capital can not reproduce itself (e.g., “rabbit economy.”)¹⁶ Lemma 1 ensures that $K^i(\cdot)$ is uniquely determined by $I^i(\cdot)$. However it might not be sufficient to guarantee that any investment policy (e.g., I is a function of current K instead of time) has a well-determined outcome, without either using of the full strength of the “initial condition” (Lemma 2 below), or, slightly strengthening the assumption that capital can not reproduce itself (Lemma ??).

Lemma 2. *Assume Y is such that no investment good can be produced without some form of labour input. Assume $R \equiv \gamma + \nu + \delta > 0$. Then the set of all feasible functions $K^i(t)$ and $I^i(t)$ is bounded above by $\bar{K}e^{(\gamma+\nu)t}$ for some \bar{K} .*

Proof. Clearly, we can set consumption to zero and assume that all agents work full-time. Fix a vector $L_0 \in \mathbb{R}^h$ such that any feasible vector of labour inputs $L(t) \leq L_0 e^{(\gamma+\nu)t}$. Enlarge the instantaneous production cone Y by allowing all investment goods to be perfect substitutes for each other and the same for the capital goods. Let $F: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ : (K, l) \mapsto \sup\{\sum_i I^i \mid \exists K^i \geq 0, \sum_i K^i \leq K, (-lL_0, -(K^i)_i, 0, (I^i)_i) \in Y\}$. The supremum is achieved, else with bounded inputs unbounded outputs would be feasible and, as Y is convex and closed, the same would be true for zero inputs, thus contradicting the assumption $Y \cap \mathbb{R}_+^n = \{0\}$. In particular, $F(K, l)$ is finite. Clearly, F is positively homogeneous of degree one, concave and continuous. Further, by the assumptions of the lemma, $F(K, 0) = 0$.

Let us, finally, improve the possibilities of capital accumulation by lowering each δ^i to $\delta = \min_i \delta^i$.

Then the capital accumulation equation becomes $K'(t) = F(K(t), e^{(\gamma+\nu)t}) - \delta K(t)$. Let $x(t) \equiv K(t)e^{-(\gamma+\nu)t}$, and $f(x) \equiv F(x, 1)$ — then $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous and concave. Then the differential equation becomes $x'(t) = f(x(t)) - Rx(t)$. As $\lim_{x \rightarrow \infty} \frac{f(x)}{x} = 0$ (because $F(1, 0) = 0$ and continuity), there is $\bar{x} \geq 0$ such that $f(x) - Rx \leq -1$ iff $x \geq \bar{x}$.

Let now $y(t) = e^{-(\gamma+\nu)t} \sum_i K_t^i$ along some feasible path in the original economy: a fortiori $y(t) \geq \bar{x}$ implies $y'(t) \leq -1$. Since, by the initial

¹⁶For instance, assume a policy where all agents work full-time and consume nothing (e.g., in order to get an upper bound on capital and investment). Assume a single good, a single type of labour and a CES production function $(AK^\alpha + BL^\alpha)^{1/\alpha}$. Assume further that $A^{1/\alpha} \geq R$ with $R = \gamma + \nu + \delta$. Note that $L_t = L_0 \exp(\gamma + \nu)t$ and let $D = BL_0^\alpha$, then $K'(t) = (AK^\alpha(t) + De^{(\gamma+\nu)t})^{1/\alpha} - \delta K(t)$; or with $x(t) = K(t)e^{-(\gamma+\nu)t}$, $x'(t) = (Ax^\alpha(t) + D)^{1/\alpha} - Rx(t) \geq D^{1/\alpha} > 0$. Since $x(t) \geq 0$, there is no solution, i.e., the upper bound of $K(t)$ is infinity. Note that even if $B = 0$, the solutions of the equation are $x(t) = Ce^{(A^{1/\alpha} - R)t}$, with $C \geq 0$ arbitrarily large, hence in this case $K(t)$ is unbounded as well.

condition, $\exists \bar{y}: y(t) \leq \bar{y}$ for $t < 0$, it follows that for all t , $y(t) \leq \bar{x}$. Hence our bound on each K_t^i , which themselves imply (via Y) a similar upper bound for the I_t^i . \square

In fact, at least under a slightly stronger condition on Y , the full strength of the initial condition is not needed, the result of lemma 1 suffices:

Lemma 3. *Assume $\exists \varepsilon > 0, A, B: (-L, -K, C, I) \in Y \implies \|I\| \leq A \|L\| + B \|K\|^{1-\varepsilon} \|L\|^\varepsilon$. Then the conclusions of lemma 2 hold, assuming just lemma 1, without the need for the “initial condition”.*

Proof. All norms on \mathbb{R}^n being equivalent, we can assume the ℓ_1 norm in the statement. The right hand member of the inequality is then concave, and we can proceed as in the proof of lemma 2: now $f(x) = A + Bx^{1-\varepsilon}$, and, since, as seen above, if $x(t) > \bar{x}$ then $x(s)$ must have been decreasing (and hence $x(s) > \bar{x}$) for all $s \leq t$, we can assume $A = 0$, by majorising f on $[\bar{x}, +\infty]$ by another such function (and if necessary increasing \bar{x} to an appropriate value for that new function). Thus we have to show that over all feasible paths (k_t, i_t) ($= e^{-(\gamma+\nu)t}(K_t, I_t)$) the k_t are uniformly bounded. And feasibility means $i_t \leq Bk_t^{1-\varepsilon}$ and $e^{Rt}k_t = \int_{-\infty}^t e^{Rs}i_s ds$ (and $k_t \geq 0$, $e^{Rs}i_s$ integrable on $[-\infty, t]$). I.e., letting $y_t = \int_{-\infty}^t e^{Rs}i_s ds$, we have $y_{-\infty} = 0$, $k_t = e^{-Rt}y_t$, $i_t = e^{-Rt}y_t'$, so our inequality becomes $y_t' \leq B e^{\varepsilon Rt} y_t^{1-\varepsilon}$, i.e., $\frac{dy_t^\varepsilon}{de^{\varepsilon Rt}} \leq \frac{B}{R}$. Since $y_{-\infty} = 0$, this yields $y_t^\varepsilon \leq \frac{B}{R} e^{\varepsilon Rt}$, i.e., $k_t \leq \left(\frac{B}{R}\right)^{1/\varepsilon}$. \square

3. EQUILIBRIA (SOLUTION CONCEPTS)

A natural solution concept for this economy — following the tradition in the OLG literature — is the Arrow-Debreu one. However, we can allow for a wider class of solutions, the only requirement being sustainability of economic growth driven by exponentially increasing labour productivity.

3.1. Time Invariance. The economy we have described possesses a convenient time-invariance property that will prove to be useful later. If we are to take a description of the economy at time t , i.e., the feasible consumption sets and production plans, as well as description of population, then we can use a transformation, which we will define below as growth-preserving, to get a description of the economy h units of time later.

Definition 4. *A transformation T_h of the economy at time t into the economy at time $t + h$ that*

- (1) *multiplies all non-labour individual quantities (endowments of goods, allocations of goods) in the economy at time t by $\exp(\gamma h)$,*

- (2) multiplies all aggregate quantities of such goods (capital, investment, total output of consumer-goods) in the economy at time t by $\exp((\gamma + \nu)h)$,
- (3) multiplies the aggregate quantities of population and labour in the economy at time t by $\exp(\nu h)$

is called ‘growth-preserving’.

Now we can claim that the economies we consider are time-invariant in the sense of the transformation defined above.

Lemma 5. *A growth-preserving transformation is an automorphism of the model:*

- it maps feasible production plans in a 1-to-1 way onto feasible production plans.
- it maps the preferences of each consumer between different consumption bundles (consumption=goods+labour) to the corresponding preferences of his image, born time h later. And his initial endowment is mapped as well to the initial endowment of his image.

Proof. The second part is obvious: for the endowments, it holds by definition of the transformation, and for the preferences, it follows because all agents of the same type have the same utility function over their consumption set, which is homogeneous in the goods: so multiplying the “goods-component” by a constant just multiplies to whole utility function by a constant, and hence doesn’t change preferences.

For the first part, note that the capital-accumulation equations are not affected, since they are linear and homogeneous in the aggregate goods. Remains to check for the “instantaneous production cone” Y that it too is preserved by the transformation. Assume thus for some t a vector $(-L, y)$ in Y_t — i.e., $(-\exp(\gamma t)L, y)$ in Y — before the transformation — where the coordinates of $y = (-K, C, I)$ are all aggregate consumption and investment outputs and capital inputs, and those of L are the aggregate labour input. Then, after the transformation, this vector becomes $[-\exp(\nu h)L, \exp((\gamma + \nu)h)y]$, and we have to show that this belongs to Y_{t+h} — i.e., that $[-\exp(\gamma(t+h))\exp(\nu h)L, \exp((\gamma + \nu)h)y]$ belongs to Y . Since this vector equals $\exp((\gamma + \nu)h)[-\exp(\gamma t)L, y]$, this follows straight from the fact that Y is a cone. \square

Remark 6. *A growth-preserving transformation induces a map from allocations in the initial economy to the allocations in the image economy.*

Definition 7. *A (set-valued) solution concept is time-invariant if the image of a solution by a growth-preserving transformation is a solution of the image economy.*

Next, we will provide several examples of time-invariant solution concepts: Arrow-Debreu equilibrium, Diamond (1965) equilibrium.

* Maximization of welfare with time-invariant tools * Case of fixed contracts * Choosing an equilibrium of the perturbed economy... The closest to the initial one.

3.2. The Arrow-Debreu Equilibrium. Next, to describe the Arrow-Debreu equilibrium for the our economy we have to define profits of a firm. It is convenient to think of two types of firms: a single firm that handles the instantaneous production and has Y as technology,¹⁷ and one firm per capital good that handles the corresponding investment and has the capital accumulation equation as technology.

Given $p_c(t) \in \mathbb{R}^n$, $p_I(t) \in \mathbb{R}^m$, the equilibrium prices for consumption and investment goods, and $p_k \in \mathbb{R}^m$, $p_l \in \mathbb{R}^h$, the equilibrium rental rates for capital and labour, the **production firm**¹⁸ chooses the amount of inputs to rent from the investment firms (aggregate capital, $K(t) \in \mathbb{R}_+^m$) and consumers (aggregate efficient labour, $L(t) \in \mathbb{R}_+^h$) as well as outputs of final (aggregate consumption, $C(t) \in \mathbb{R}_+^n$ and aggregate investment, $I(t) \in \mathbb{R}_+^m$) goods to maximize its profits, Π_C ,

$$\begin{aligned} \langle p_c(t), C(t) \rangle + \langle p_I(t), I(t) \rangle - \langle p_k(t), K(t) \rangle - \langle p_l(t), L(t) \rangle, \\ (-L(t), -K(t), C(t), I(t)) \in Y \end{aligned}$$

which are zero.¹⁹

The **investment firms** can choose a time-path of investment and rent out their accumulated capital (uniquely determined by Lemma (1)) to the production firm.

An investment firm i owns capital $K^i(t) = e^{-\delta^i t} \int_{-\infty}^t e^{\delta^i s} I^i(s) ds$ of type i and chooses an investment policy $I^i(\cdot)$ to maximize its profits

$$\Pi_I^i(I(\cdot)) \equiv \int_{-\infty}^{+\infty} I^i(t) [-p_I^i(t) + \int_0^{+\infty} e^{-\delta^i s} p_k^i(t+s) ds] dt$$

which should be zero.²⁰ This condition implies

$$(2) \quad p_I^i(t) = \int_0^{\infty} e^{-\delta^i s} p_k^i(t+s) ds$$

¹⁷A choice of production plan at any time t involves no implications for profits of the firm at later dates, so the profit maximization problem of a production firm is static

¹⁸One could introduce many production firms that have access to technology described by Y without changing the results. Indeed, the composition of the industry is irrelevant as long as the total production set is preserved.

¹⁹ We use the standard notation $\langle x, y \rangle$ for the inner product of two vectors, x and y .

²⁰JFM fill in.

Next, we have to define the life-time budget constraint of an individual of type τ born at time x .

$$(3) \quad \int_0^{T_\tau} \langle p_z(s+x), z^{\tau,x}(s) \rangle - \langle p_c(s+x), c^{\tau,x}(s) \rangle ds = 0$$

$$(4) \quad z^{\tau,x}(s), c^{\tau,x}(s) \geq 0, z^{\tau,x}(s) \leq 1$$

Clearly, the price for efficient unit of labour, $p_l(t)$, is proportional to the price of labour time, $p_z(t)$, at each instant t : $p_z(t) = e^{\gamma t} \langle \varepsilon^\tau(s), p_l(t) \rangle$.

Definition 8. *An Arrow-Debreu equilibrium for the OLG economy is a combination of*

- (1) *prices $(p_k(t), p_l(t), p_y(t)) \in \mathbb{R}^{m+h+n}$ for all t in \mathbb{R} , corresponding to h types of labor, m types of capital-investment goods and n commodities and*
- (2) *allocations of consumption and occupations for individuals of all ages s and all types τ at all points in time t $((z_j^\tau(s, t))_{j=1}^h)_\tau, ((c_i^\tau(s, t))_{i=1}^n)_\tau$, and capital goods of all types i and at all points in time t , $K^i(t)$*

such that

- (1) *all consumers maximize their utility by choosing consumption and occupations streams subject to the life-time budget constraints;*
- (2) *all firms maximize their profits;*
- (3) *markets for h types of labor, m types of capital-investment goods and n commodities clear at each time t*

3.3. Diamond Equilibrium. One could reproduce an equilibrium concept suggested by Diamond (1965) for this model. There are no investment firms, consumers hold the capital stock and lend it to the production firm. As in Arrow-Debreu equilibrium, consumers can also lend to each other. However, the value of the total savings of the consumers at each point in time (net savings) should equal to the total value of the accumulated capital. In case of one investment/capital good, this condition should replace equation (2).

4. CONSTRUCTING THE RELATIVE UTILITARIAN WELFARE FUNCTION

4.1. The Set of Alternatives. To formulate the social welfare function we need the feasible set, and the simplistic formulation used in section 1.3 is no longer adequate in the view of multiple goods, and several types of consumers. Ideally this should be defined in the space of policies, but since one of our aims is to prove that our result is completely independent of it, we will define it as the corresponding set in the space of (final — i.e., after all equilibrium readjustments) allocations.

The set of available allocations should be *time-invariant*, i.e., it should be mapped to itself by any growth-preserving transformation.

So, the time invariance is here to capture the previous idea that policies affect only the height of the growth path — while leaving the geometry of the feasible set completely arbitrary in all other respects.

Further, an obvious implication of ‘justice’ requirement on the feasible set is that each individual’s utility is bounded below.

4.2. The distribution of costs and benefits. We associate with any policy-change a corresponding perturbation of individual endowments of consumption goods over time. We want to evaluate the corresponding variation of social welfare, after individuals trade to a new equilibrium.

Let $\omega_y^\tau(t)$ be a perturbation of consumption (vector) of individual of type τ who was born at time y . It is clear that by just taking on a given day consumption away from the old and giving it to the young one could achieve artificial welfare increases: indeed, since their utilities at birth are weighted equally in the social welfare function, their own time-impatience will have for effect that the benefits of the transfer to the young is much greater than the disutility to the old.

Thus our variation of welfare will in general depend on the whole perturbation of endowments, not only on the aggregate.

It is clear that such things require much more work, and thought, and lead us astray from our subject — the discount rate for cost-benefit analysis. Hence, to be able to pursue our analysis, in a way unaffected by this problem, we will assume that somehow this problem is being taken care of by current policy, and that the aggregate perturbation $\Omega(t)$ gets distributed in a fixed (i.e., time- and commodity-independent) way across age groups and types. So the variation in welfare will be a function just of the aggregate Ω .

Let thus $\vartheta^\tau(s)$ be some integrable function, the distribution of endowments, with $\vartheta^\tau(s) = 0$ for $s < 0$ and $s > T_\tau$, and with $\sum_\tau \int_{-\infty}^{+\infty} \vartheta^\tau(s) ds = 1$. Then, a perturbation of consumption (vector) of individual of type τ who was born at time y is related to the aggregate perturbation $\Omega(t)$ in the following way,

$$\omega_y^\tau(t) = \vartheta^\tau(t - y) \frac{\Omega(t)}{N^\tau e^{\nu y}}$$

Recall, the population (within each type of individuals) grow at a constant rate ν , so total population of people of type τ who were born at time y is $N^\tau e^{\nu y}$ with N^τ being population of type τ born at time 0. Thus, we assume that the endowment is shared equally within each age-type category of individuals.

5. THE MAIN STATEMENT

Taking a balanced growth path as status-quo point, we can now view the social welfare function W as a real-valued function of aggregate endowments. Consider perturbation $\Omega(t)$ of consumption endowments. It is true that as a result of such a perturbation several equilibria might emerge. Out of these we choose the one closest to the initial stable growth path in terms of the distance $\sum_i \int |\ln p_i(t) - \ln p_i^\Omega(t)| dt$, where $p(t)$ is the price vector at time t prevailing at the initial equilibrium and $p^\Omega(t)$ is the price vector of a perturbed economy. Provided the perturbed economy has an equilibrium, the associated variation in W is well defined. We want to compute the differential of W at 0 (the status-quo point, the balanced growth path) for evaluating the effect of small perturbations, and to prove that whenever it exists it is of the form $\int \langle q, \Omega(t) \rangle e^{-\gamma t} dt$ for some $q \in \mathbb{R}^n$ — i.e., that the discount rate used equals γ .

To make the main statement as strong as possible, we need to use the weakest notion of differential, that of Gateaux-differential.²¹ We also need to specify the space of perturbations and its topology; we will use the space K (defined below), because that way the statement implies the same statement for about any other space of perturbations, since K embeds continuously as a dense subspace in about any other space.

We follow Gelfand and Shilov (1959) in defining K and the space K^* of continuous linear functionals on K (i.e., generalized functions).

Definition 9. *K is the space of infinitely differentiable functions with compact support, and a sequence of functions $\varphi_n \in K$ converges to zero if $\exists h \in \mathbb{R}: |x| \geq h \implies \varphi_n(x) = 0$ for all n , and φ_n and all its successive derivatives converge uniformly to zero.*

K^ is the space of linear functionals ψ on K s.t. $\psi(\varphi_n) \rightarrow 0$ whenever $\varphi_n \rightarrow 0$ in K .*

The economic meaning of $\Omega \in K^n$ is that the endowments are perturbed only over a bounded interval of time. Note that the status-quo (zero endowment) point also belongs to this space, so we can view the social welfare function W as being defined on K^n .

Next step is to define precisely the map from endowments to social welfare, given a solution concept that satisfies time-invariance. Let us consider a point-valued time-invariant solution concept ψ , which maps consumption endowments, $\Omega \in K^n$ to final allocations. Assume that its domain, D , contains zero, which corresponds to the economy

²¹A function f from a subset D of a topological vector space X to \mathbb{R} is Gateaux-differentiable at zero, if $\forall x \in X$ the set $\{t \in \mathbb{R}R | tx \in D\}$ is a neighbourhood of zero in $\mathbb{R}R$, say V_x , and if $t \mapsto f(tx)$ is differentiable at $t = 0$, say, with derivative d_x , and if $x \mapsto d_x$ is a continuous linear functional on X .

we described, in which individuals are born with no consumption endowments. As the solution concept is time-invariant, $\psi(0)$ describes a balanced growth path. Define the social welfare function W of relative utilitarianism by subtracting from each individual's normalised utility function its value at $\psi(0)$. (Thus, a constant is subtracted from each of the individual utilities to assure that welfare is well-defined on the growth path.) Denote by \aleph' the subset of the space \aleph of allocations where W is well defined (i.e., the integral converges). This set, for example, might include allocations that are not 'too different' from those on the balanced growth path $\psi(0)$, say, those that deviate from it over a bounded interval of time. Let us focus on the subset of consumption endowments, D' , for which W is well defined: $D' = \psi^{-1}(\aleph')$, note that $0 \in D'$, i.e., the 'no-endowment' (status-quo) belongs to this set, as by construction the welfare function is zero as evaluated at the initial balanced growth path, $\psi(0)$. Finally, we can define the map from endowments to the social welfare that corresponds to the chosen solution concept ψ , W_ψ is the composite map $W \circ \psi$ from D' to \mathbb{R} .

Now the main result can be represented in the following succinct form:²²

Theorem 10. *Consider a point-valued time-invariant solution concept ψ . If W_ψ is Gateaux-differentiable at 0, then its differential equals $\int \langle q, \Omega(t) \rangle e^{-\gamma t} dt$ for some $q \in \mathbb{R}^n$.*

This implies that the discount rate is the (per-capita) growth rate of output, γ . A rationale behind the proof is that the cost of consumption (in terms of inputs used in its production) becomes cheaper with time, due to the enhancement of labour productivity. Individual productivity grows at a rate γ , so this is exactly the rate of decrease in the real price of per-capita consumption.

Proof. By definition of Gateaux differential,

$$\begin{aligned} DW(\Omega^0) &= \lim_{\varepsilon \rightarrow 0} \frac{\delta_\varepsilon W(\Omega^0)}{\varepsilon}, \\ \delta_\varepsilon W(\Omega^0) &= W(\Omega^0 + \varepsilon \Omega) - W(\Omega^0) \end{aligned}$$

By assumption,

$$DW(0) = \langle \Omega, \mu \rangle$$

where $\mu \in (K^*)^n$, i.e., the differential at $\Omega^0 = 0$ is linear in Ω . It is sufficient for what follows to describe $\delta_\varepsilon W(\Omega^0)$, i.e., the change in the social welfare function caused by the perturbation of endowments, which amounts to subtracting a constant from each agent's utility, the

²²A similar result could be shown in the traditional set-up, provided (the multi-dimensional analog of) risk-aversion, ρ_τ , is independent of the type τ — giving then a discount factor of $\rho\gamma$, and hence showing the robustness of our conclusions from the mini-model in the introduction.

utility on the baseline, thus the criterion of interest is the difference δW .

To construct δW let us first normalise life-time utilities. Recall the set of available allocations is time-invariant. We have to compute w_t^τ , the difference between the sup and the inf over this set of the utility of an agent of type τ born at time t . By time-invariance, the set of consumption and labor allocations of this agent equals that for an agent of the same type born at time 0, except for rescaling the consumption component by $e^{\gamma t}$. Therefore, by the homogeneity of U^τ of degree $1 - \rho^\tau$ with respect to consumption, $w_t^\tau = e^{(1-\rho^\tau)\gamma t} w_0^\tau$. Let $w^\tau \equiv (w_0^\tau)^{-1}$. Then we get for normalised utility $U_t^{*\tau}$ (that enters the social welfare function)

$$U_t^{*\tau} = e^{(\rho^\tau - 1)\gamma t} w^\tau U^\tau$$

We, therefore, can write the social welfare function in the following form

$$(5) \quad \delta W(\cdot) \equiv \int_{-\infty}^{\infty} \sum_{\tau} N_t^\tau (\delta U_t^{*\tau}) dt$$

Let us define $V_t^\tau : \Omega_t \mapsto \mathbb{R}$ to be the utility level of individual of type τ born at time t , under an equilibrium with the perturbed endowments.

$$\begin{aligned} W(\Omega^0) &= \sum_{\tau} w^\tau W^\tau(\Omega^0) \\ W^\tau(\Omega^0) &\equiv \int_{-\infty}^{\infty} N_t^\tau e^{(\rho^\tau - 1)\gamma t} V_t^\tau(\Omega_t^0) dt \end{aligned}$$

Consider now the perturbation $\tilde{\Omega}_t$, where

$$(6) \quad \tilde{\Omega}_{t+h} = e^{(\gamma + \nu)h} \Omega_t$$

By Lemma 5 the corresponding ‘‘response’’ of the system is obtained from the response to Ω_t by delaying everything by h , multiplying all aggregate quantities of goods by $e^{(\gamma + \nu)h}$, and all per-capita quantities by $e^{\gamma h}$, and correspondingly for prices.

Hence, for utilities, by their homogeneity property in goods $1, \dots, n$,

$$V_{t+h}^\tau(\tilde{\Omega}) = e^{(1-\rho^\tau)\gamma h} V_t^\tau(\Omega)$$

and, in particular, when $\tilde{\Omega} = \Omega = 0$,

$$V_{t+h}^\tau(\tilde{\Omega}) - V_{t+h}^\tau(0) = e^{(1-\rho^\tau)\gamma h} (V_t^\tau(\Omega) - V_t^\tau(0))$$

Therefore,

$$\begin{aligned}
W^\tau(\tilde{\Omega}) - W^\tau(0) &= \\
&= \int_{-\infty}^{+\infty} N_0^\tau e^{\nu(t+h)} e^{(\rho^\tau-1)\gamma(t+h)} \left[V_{t+h}^\tau(\tilde{\Omega}) - V_{t+h}^\tau(0) \right] d(t+h) \\
&= \int_{-\infty}^{+\infty} N_0^\tau e^{\nu(t+h)+(1-\rho^\tau)\gamma h} e^{(\rho^\tau-1)\gamma(t+h)} \left[V_t^\tau(\Omega) - V_t^\tau(0) \right] dt \\
&= e^{\nu h} \int_{-\infty}^{+\infty} N_t^\tau e^{(\rho^\tau-1)\gamma t} \left[V_t^\tau(\Omega) - V_t^\tau(0) \right] dt = e^{\nu h} [W^\tau(\Omega) - W^\tau(0)]
\end{aligned}$$

(i.e., the factor $(1 - \rho^\tau)\gamma$ drops out). As a consequence, the total change in welfare is

$$\begin{aligned}
W(\tilde{\Omega}) - W(0) &= \sum_{\tau} w^\tau \left(W^\tau(\tilde{\Omega}) - W^\tau(0) \right) \\
&= e^{\nu h} [W(\Omega) - W(0)]
\end{aligned}$$

Therefore, applying the definition of the derivative, we get

$$\begin{aligned}
(7) \quad \langle \tilde{\Omega}, \mu \rangle &= \lim_{\varepsilon \rightarrow 0} \frac{W(\varepsilon \tilde{\Omega}) - W(0)}{\varepsilon} = \\
&= e^{\nu h} \lim_{\varepsilon \rightarrow 0} \frac{W(\varepsilon \Omega) - W(0)}{\varepsilon} = e^{\nu h} \langle \Omega, \mu \rangle
\end{aligned}$$

Define $T_h : t \mapsto \xi(t+h)$. By (6)

$$\tilde{\Omega} = e^{(\gamma+\nu)h} T_h \Omega$$

Combining with (7), we get

$$e^{(\gamma+\nu)h} \langle T_h(\Omega), \mu \rangle = \langle \tilde{\Omega}, \mu \rangle = e^{\nu h} \langle \Omega, \mu \rangle$$

and, due to arbitrariness of h and Ω , the following condition holds for all $h \in \mathbb{R}$ and all perturbations $\Omega \in K^n$,

$$\langle \Omega - e^{\gamma h} T_h(\Omega), \mu \rangle = 0$$

for $\mu \in (K^*)^n$. Dividing by h and taking limit as $h \rightarrow 0$, we get

$$\langle \gamma \Omega - (\Omega)', \mu \rangle = 0$$

Rearranging and using the definition of a derivative of a generalized function,

$$\langle \mu', f \rangle = - \langle \mu, f' \rangle, \quad f \in K, \quad \mu \in K^*$$

results in

$$\langle \gamma \mu + \mu', \Omega \rangle = 0, \quad \forall \Omega \in K^n$$

so we have to solve a differential equation $\gamma \mu + \mu' = 0$, which, by Lemma (11) has only the solutions of the form $\mu = q \otimes e^{-\gamma t}$ for some $q \in \mathbb{R}^n$,

therefore,

$$DW = e^{-\gamma t} \langle q, \Omega \rangle = \int_{-\infty}^{+\infty} e^{-\gamma t} \langle q, \Omega_t \rangle dt, \forall \Omega \in K^n.$$

□

Lemma 11. *Consider a homogeneous differential equation of the form*

$$(8) \quad y' = \lambda y,$$

for a given constant λ . Then every solution of that system in the class K^* of generalized functions is of the form

$$y = Ce^{\lambda t}, C \in \mathbb{R}$$

i.e., is a “classical solution”.

Proof. From (8) we have that for any $\varphi \in K$, $\langle y', \varphi \rangle = \lambda \langle y, \varphi \rangle$; by definition of the derivative of a generalized function this implies $\langle y, -\varphi' \rangle = \lambda \langle y, \varphi \rangle$, and so $\langle y, \lambda \varphi + \varphi' \rangle = 0$. Let $K_\lambda = \{\psi \in K \mid \int_{-\infty}^{\infty} e^{\lambda t} \psi(t) dt = 0\}$. Observe that $\forall \psi \in K_\lambda \exists \varphi \in K: \psi = \lambda \varphi + \varphi'$: take $\varphi(t) = \int_{-\infty}^t e^{\lambda s} \psi(s) ds$ (the converse is true as well, but we won't use it). So $y = 0$ on K_λ .

Note that any $\varphi \in K$ can be represented in the form $\varphi = \psi + c\varphi_0$, where $\psi \in K_\lambda$, c is a constant and $\varphi_0 \in K \setminus K_\lambda$ is fixed: choose $c = \frac{\int_{-\infty}^{\infty} e^{\lambda t} \varphi(t) dt}{\int_{-\infty}^{\infty} e^{\lambda t} \varphi_0(t) dt}$, then $\psi = \varphi - c\varphi_0 \in K_\lambda$.

Thus $\langle y, \varphi \rangle = c \langle y, \varphi_0 \rangle$, so, letting the constant $C = \frac{\langle y, \varphi_0 \rangle}{\int_{-\infty}^{\infty} e^{\lambda t} \varphi_0(t) dt}$, we get $\langle y, \varphi \rangle = C \int_{-\infty}^{\infty} e^{\lambda t} \varphi(t) dt, \forall \varphi \in K$, i.e., $y = Ce^{\lambda t}$. □

6. DISCUSSION OF THE MAIN RESULT

The Theorem relies on differentiability of the map from endowments to welfare. To demonstrate the non-vacuity of the statement, one has to show that (1) solution is non-empty valued, i.e., existence of a balanced growth path which belongs to the set of solutions; (2) the map from consumption endowments to allocations (under the solution concept) is differentiable; (3) the map from allocations to welfare is differentiable. Verifying each of the requirements (even in a model with fully specified preferences and technology) might not be a trivial — however tractable — problem, and lies beyond the scope of this paper.

6.1. The Value of a Human Life. We show now that the issue is not only that $\rho\gamma$ is not of the correct order of magnitude as compare to γ , but even that the former formula is conceptually wrong, and the latter exactly correct.

The value of life, according to any criteria [e.g., each of the four in Mishan's (1971) introduction, or even judicial criteria in assessing damages], is proportional to his life-time income, or to average life-time income at his time: anyway, proportional to $e^{\gamma t}$ in an exogenous

growth model. Further, one should note that if our full-fledged model further down were extended such as to allow for variable life-spans — so, individual “consumption-sets” are of the type $\cup_T C[0, T]$ —, then this conclusion would also formally follow from the model, given the homogeneity assumption on individual utilities (which is forced upon us to get a balanced growth model).

Hence, if we want 1 human life one generation down the road to count as much as 1 now, we must discount further consumption *exactly* by $e^{-\gamma t}$.

6.2. What makes the Traditional Approach fare so badly? If one accepts Harsanyi’s theory, that the social (von Neumann-Morgenstern) utility should be a positive linear combination of individual vNM utilities, by respect of unanimity, then the only arbitrary choice we have made here is to assign to all individuals the same weight, since they have the same utilities. This is standard in every application. And Harsanyi himself too argues that when two individuals have the same preferences, it is reasonable to assign them the same utilities. But that argument is for preferences and utilities over the (common) set of alternatives, while here — and in most applications — it is about preferences and utilities over an individual set of personal consequences. Even the sets are not directly comparable, since in economics goods are indexed by date and location — and there is even no economic reason to use the same physical units at different times or places.

This weakens substantially the argument for treating identically individuals having the same preferences — and that is what went wrong. It is also why relative utilitarianism fares much better, by deriving those different weights from a normalisation over *the common set of alternatives*.

Alternatively, if one were to reject Harsanyi’s theory, and be ready to live with social preferences which are completely irrational over lotteries, a welfarist approach is still useable, ignoring lotteries, and using some axiom system that relates individual *utilities*²³ to social utilities. So the cardinal representation of each individual’s preferences are left to the choice of the user, to represent what he thinks are or should be the preferences of society. In this spirit, one could arrive at the “correct” result in the following way:

- first apply a monotone transformation to each utility function, in such a way that they become all homogeneous of degree $1 - \rho$ in the consumption goods (it is crucial — for aggregation — to have the same ρ for all individuals, cf. footnote 22 below).
 ρ must be chosen here such as to induce the correct social preferences for equality within a generation.

²³A fiction in the “observing mathematician”’s (Rawls (1971)) mind, furthermore without any uniqueness property.

- next discount future utilities with the (negative, typically) discount rate $(1 - \rho)\gamma$.

But this procedure sounds a bit arbitrary (and looks suspiciously like a very ad-hoc social welfare functional): it may be better to use a social welfare functional from the outset.

Finally, a welfarist might want to defend that discount rates as high as $\rho\gamma$ do correctly represent “treating future generations equally”. While one can understand intellectually how that logic might lead to this, we just note that most people would agree with the OMB’s position that “treating future generations equally” requires a discount factor lower than the interest rate.

Note that the particular form that the welfare function takes depends on the time-invariant solution concept used, (in that after individual utilities are normalized between zero and unity, one’s utility on the chosen balanced growth path is subtracted) however all these functions give rise to the same discount rate, per-capita growth rate of consumption γ .

7. RELATED LITERATURE

In their fundamental work Arrow and Kurz (1970) offer a criterion, or a social welfare function, that has been widely used to evaluate public investments in the literature since then. Denote by N_t population at time t , let c_t be per-capita consumption and β be a (subjective) discount rate, then the criterion (in its simplest formulation)²⁴ is

$$(9) \quad \tilde{W}((c_t)_t) \equiv \int_0^\infty e^{-\beta t} N_t u(c_t) dt,$$

where u is a concave and increasing function of per-capita consumption. To put it in their own words,

The flow of felicity to society is the sum over individuals at a given time; the total utility from a policy is taken to be the sum over all time of the felicities of each time, discounted back to the present at a constant rate.

Criterion (9) can be presented as a true social welfare function, i.e., a function of individual (lifetime) utilities. Indeed, assume, for example, all individuals live for a fixed period of time (unity), and that an individual born at time t has a life-time utility of the form

$$U_t(c.) = \int_t^{t+1} e^{-\alpha(s-t)} u(c_s) ds,$$

where $c.$ is the time-path of consumption (as a function of age), and α is the individual time preference. Assume also that population grows

²⁴More generally, the utility, u , can depend directly on government investment, k_g in a given period.

exponentially at a rate ν . Then, aggregating over all individuals (integrate over t from $-\infty$ to $+\infty$) when discounting their life-time utilities at a rate β , one gets the following criterion:²⁵

$$(10) \quad W \equiv \int_{-\infty}^{\infty} N_t e^{-\beta t} U_t(c_t) dt$$

$$(11) \quad = \int_{-\infty}^{\infty} N_t e^{-\beta t} \int_t^{t+1} e^{-\alpha(s-t)} u(c_s) ds dt$$

$$(12) \quad = N_0 \int_{-\infty}^{\infty} e^{-\alpha s} u(c_s) \int_{s-1}^s e^{(\alpha+\nu-\beta)t} dt$$

$$(13) \quad = M \int_{-\infty}^{\infty} e^{-\beta t} N_t u(c_t) dt, \text{ where } M \equiv \int_0^1 e^{(\alpha+\nu-\beta)x} dx$$

Observe that, under the above assumptions, the two criteria (W and \tilde{W}) rank the policies that affect per-capita streams of consumption only after time zero in the same fashion. The advantage of using criterion W is its generality: it encompasses the Arrow and Kurz (1970) criterion and also allows for life-time utilities that are not necessarily time-separable. Finally, re-interpreting the criterion in this way allows to separate the individual time preference, α , from the social discount rate, β , and it allows as well to separate completely attitudes towards risk from the time preferences (and, in particular, from the inter-temporal substitution).

Interestingly, this social welfare function is identical to the “traditional utilitarian” criterion mentioned in the introduction and in footnote 22.

We do not attempt to provide an overview of the vast literature analyzing the overlapping generations model, referring instead to the work by Geanakoplos and Polemarchakis (1991). Kotlikoff (1998) in his survey of the literature stresses the importance of OLG modeling in analysing tax reform and privatizing social security; Erosa and Gervais (2001) compare policy implications derived from life-cycle models with those based on models with an infinitely-lived representative agent, and stress the importance of the former. The by-now classical two-period life-cycle OLG model and its policy implications are analyzed in detail in Kotlikoff (2002). In a more general set-up with age-related individual productivity, Erosa and Gervais (2002) offer an analysis of tax

²⁵The discount rate $\beta > 0$ is often introduced only for the ‘technical’ reason of making sure the social welfare function returns a finite number for strictly positive consumption profiles, given that $u(c) > 0$ if $c > 0$. Ramsey (1928) avoided this difficulty by suggesting to use a bounded function $u(c)$ and, then to minimize the difference between u and the ‘bliss’, B , or the highest attainable utility, using that as the criterion. Following Chiang (1992) we will refer to this substitution as “the Ramsey device”.

equivalence. In most of those models, however, there is no technological growth.

Restrictions on economic fundamentals to allow for a stable growth are summarized in King, Plosser, and Rebelo (2002), some of which are to be used in the current project, e.g., homogeneity of utility functions with respect to consumption goods, and constant returns to scale in production. Some other contributions in this vein are discussed in Arrow and Kurz (1970).

Indeterminacy is known to plague some classes of OLG models (Geanakoplos and Brown (1985)); hence the need to show that this problem is avoided in our case — in particular, making a policy change meaningful, in the sense that it generates predictable (determinate) changes in the economy.

8. CONCLUSIONS

The following is the message of the paper.

8.1. A policy re-interpretation. Real-life policies rarely involve direct consumption transfers (changes in endowments). The model can be re-interpreted to incorporate more realistic policies as follows. Assume a government has access to “stationary” policies, i.e., those that (along with some appropriate solution concept) generate a time-invariant path, for example, linear taxes — or non-linear (sales or income) tax-schedules indexed by average income. Then the result still holds provided a small change in such a policy generates a small change in resulting allocations, which is assured by the assumption of the main Theorem.

Ramsey (1928) conjectured that population growth and “future inventions and improvements in organisation” might have an effect on the trade-off between current and future consumption. We support this conjecture for an OLG economy that reached a stable growth path given Relative Utilitarian social welfare criterion. Moreover, the rate at which future (individual) consumption should be discounted is exactly the growth rate of per-capita consumption, so that (productivity-enhancing) technological improvement is the *only* factor that matters.

The answer is independent of individual impatience, the model does not require time-separable preferences. Crucial assumptions are those needed for stable growth to be feasible, i.e., (1) homogeneity of individual utility functions (over consumption/labour life-time streams); (2) constant returns to scale in production. Additional important assumption is implicit, it requires the social welfare function to be differentiable with respect to perturbations in consumptions endowments around the balanced growth path. We will describe a class of economies satisfying this restriction in the next paper.

8.2. Why not Interest Rate? Alternative cost of resources is a commonly accepted measure of inputs value used in regulatory analysis. One can hardly argue against using interest rate (say, the capital yield), if a capital used in a public project could, indeed, if used alternatively, generate that return. However, not all the projects that are subject to cost-benefit analysis according to the OMB regulations are of “public investment” nature. In fact, all federal agencies (except for Independent commissions and a small number of others) are required to analyse their proposals according to Circulars..... A-4 and A-94 are dealing with redistributive programs, such as Medicare, Medicaid, etc. as well as with management of natural resources (environmental programs, for example).²⁶ It is in those cases that the questions of intergenerational equity become apparent and require a sensible resolution, it is for those issues that the use of market interest rate (say, return on T-bonds) is completely irrelevant, and is hardly justified.

Besides, no matter which public project is evaluated, one can always use the Relative Utilitarian criterion, —which is consistent and embeds “equal treatment of different generations” — in addition to other criteria, including the alternative-cost one. After all, cost-benefit analysis is for suggesting evaluation criteria, not for dictating ultimate decisions.

**** Differentiability, further questions The approach can be extended to be directly applicable to the study of the marginal welfare impact of small policy variations, without having first to translate them into an equivalent flow of consumption goods, thus, enabling analytical evaluation of policies in this class of models.

The next step is to prove that the main statement is non-vacuous, as it would be, e.g., in case of indeterminacy. Based on our work in progress we can conjecture that the differentiability is not a very restrictive assumption in terms of underlying parameters of the model (γ, ρ, ν) at least for the case of Diamond-like economies with inelastic labor supply and time-separable constant relative risk aversion instantaneous utility function.

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²⁶reference***

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