Unawareness of Theorems

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Abstract

We provide a set theoretic model on knowledge and unawareness, explicitly employing in the construction reasoning through theorems. In particular, it is illustrated that unawareness of theorems not only constrains an agent's knowledge, but more importantly can impair his reasoning on what other agents know. For example, in contrast to Li [12], Heifetz, Meier and Schipper [10] and the standard model of knowledge, it is possible that two agents may disagree on whether another agent knows a particular event. The model follows Aumann [2] in defining common knowledge and characterizing it with respect to a self evident event, but departs in showing that the no trade theorems no longer hold.

1 Introduction

1.1 Motivation and outline

The usual assumption made in economics is that agents who participate in a model perceive the "world" the same way the analyst does. This does not imply that they have the same information, but that they understand how the model works, that is, they know all the relevant theorems and they do not miss any dimension of the problem they are facing. In essence, agents are as educated and as intelligent as the analyst and they can make the best decision, given the information and the preferences they have.

Modeling unawareness aims at relaxing this assumption, so that agents may perceive a more simplified version of the world. This is important since there are many instances where agents of different perception coexist in the same market. One such instance is the stock market, where one can find investors who are highly educated about how the stock market and the economy work, together with investors who understand only the basics.

The standard set theoretic model of knowledge without unawareness was first introduced into economics by Aumann [2], using partitions of a state space. Its simplicity and the fact

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that it did not rely on the theory of logic led to many economic applications.[1] However, it can be criticized on the grounds that it only models a highly sophisticated and rational agent, who is aware of everything, knows all the possible theorems that can be derived and has no constraints on the amount of calculations he can perform. A more realistic approach would permit a boundedly rational agent, who can make mistakes, not know some of the relevant theorems and be unaware of some of the dimensions of the world. This paper attempts to provide such a model, without departing from the set theoretic approach of Aumann [2] and its advantages.

Consider the following example, which has been cited numerous times in papers that model unawareness. Sherlock Holmes and Dr. Watson are investigating a crime where a horse was stolen from a stable and the keeper was killed. Holmes is the highly sophisticated and intelligent agent who has already solved the mystery, while Watson is the boundedly rational agent who struggles to keep up. After they conclude the interrogation, Watson has no suspects in his mind, so that he cannot even start his reasoning in order to find a solution. Hence, the first effect of unawareness is a limited state space, which restricts the agent's perception of the world, and subsequently the things that he can potentially know, or know that he does not know.

The first step in solving any problem, is being able to formulate it correctly, or ask the appropriate questions. Watson is unaware of the appropriate questions, and this unawareness limits his reasoning. Suppose that Holmes asks him: "Was there an intruder?". Before the question, Watson didn't know whether there was an intruder, and he didn't know that he didn't know. This possibility simply never crossed his mind, he was unaware of it.

Once he is aware of the new question, Watson can enlarge his state space and include the possibility of an intruder. Watson can now reason that if he is able to exclude the possibility of an intruder, he can at least formulate a list of suspects. But how can one answer this question, and how has Holmes already solved the mystery, when they conducted the interrogation together and therefore have the same information? Watson asks for Holmes' help:

> "Is there any other point to which you would wish to draw my attention?"
> "To the incident of the dog in the night-time."
> "The dog did nothing in the night-time."
> "That was the curious incident," remarked Sherlock Holmes.
> Doyle [4]

Watson is already aware of the possibility of an intruder, but he does not know the answer to this question. Although the information about the dog not barking is available to him, he is simply unaware of it, until Holmes points out that the incident of the dog in the night time is important. Once Watson becomes aware of this, he can collect the information that was always available to him, make the inference that no barking implies no intruder, and answer his question.

[1] An overview of the standard model of knowledge is given in Rubinstein [14]. A more philosophical treatment is given in Hintikka [11].

The second effect of unawareness is that readily available information cannot be used by an agent. More importantly, increased awareness can lead to answering questions that one was already aware of. Watson was able to answer the question he was already aware of, by becoming aware of another question (did the dog bark?), answering it and connecting the answers of the two questions with a theorem (no barking implies no intruder).

The third effect of unawareness is that it impairs the agent's ability to reason about the knowledge of others. Unawareness of the theorem "no barking implies no intruder" results in Watson not knowing whether there is an intruder, but also in reasoning that Holmes does not know either. In fact, Watson may be aware of many ways (or theorems) in which Holmes could have known (for example, because he asked a police officer), but Watson has correctly deduced that none of these ways were employed. He therefore inevitably concludes that Holmes does not know. In other words, Watson's state space is not rich enough to include Holmes' knowledge of no intruder through that specific theorem. Moreover, Watson is not making a mistake, as he is merely constrained in his reasoning by his unawareness. Since in the standard model partitions are common knowledge, a situation like that cannot be accommodated.

To conclude the example, Holmes and Watson are exposed to the same information and the standard model would predict that they would have the same state space and the same partition. However, Watson's reasoning is limited in three ways. Firstly, his state space is smaller than Holmes', limiting the events that he knows and the events he knows that he does not know. Secondly, information readily available to Watson is left unexploited, simply because he is unaware of its existence. Finally, his unawareness constrains his knowledge about questions that he is aware of, and his reasoning about Holmes' knowledge.

The paper builds a model around these observations. Firstly, following Li [12], awareness is expressed by a set of questions that the agent has in his mind. For example, the agent may be aware of the question "Is it raining?", so that this question enters his mind, and he is wondering about its answer. These questions and their possible answers construct the agent's subjective state space. A subjective state is a vector answering only some of the available questions, so that it provides an incomplete description of the world. A full state is a vector which provides an answer for all possible questions, and thus it is a complete description of the world. If an agent is aware of only some of the available questions, his subjective state space will provide an incomplete description of the world.

Secondly, the agent has some raw information. More specifically, for some questions he is aware of, he always knows the answer. An example of such questions are those that refer to his senses: sight, hearing, touch, smell and taste. These are the main tools he has in order to acquire his initial information about what has occurred in his environment, and there is simply no other means he can utilize. More importantly, he does not need to use any elaborate thinking, just to trust his senses. For example, if he is aware of the question "Do I see that it is raining?", then he is able to trust his sight and know whether he sees that it is raining or not. In the formal model it will also be assumed that the agent always knows the answer to questions that refer to his own awareness of questions and theorems.

Thirdly, the agent has some theorems in his mind. Theorems are just a way of connecting pieces of information together - they connect answers of different questions. For example, the theorem "If I see that it is raining, then *it is* raining", connects the answer "yes" to question A: "Do I see that it is raining?" with the answer "yes" to question B: "Is it raining?". We can conveniently model a theorem as an answer to another question. In this example, this question would be: "What states in the state space generated by questions A and B does the agent consider impossible?". The theorem would be the state that specifies "yes" to A and "no" to B, so that the agent is excluding any state that specifies that while he sees that it is raining, it is actually not raining. An agent may or may not be aware of that question, and may or may not know its answer; but if he does, then he can use the theorem in order to connect the answers of the questions "Do I see that it is raining?" and "Is it raining?".

Thus, the formal model works as follows: when state ω occurs, it prescribes what questions the agent is aware of, which are of his immediate perception, and which represent theorems that he knows. By combining these three elements we can construct the possibility correspondence $P(\omega)$, which represents the states that the agent considers possible at ω and consequently what the agent knows. The multi agent case is a natural extension. When i is reasoning about j, he first has to reason about what questions and what theorems j is aware of and what is j's immediate perception, in order to reason about what j knows. Consequently, i's unawareness can constrain his reasoning about what events j knows.

1.2 Related literature

One of the first attempts to model unawareness was by Geanakoplos [7], using non-partitional information structures. However, Dekel, Lipman and Rustichini [3] showed that modeling unawareness necessarily precludes the use of a standard state space. On the other hand, Halpern [9] and Modica and Rustichini [13] used syntactic models, relying on the theory of logic. The two main papers that try to circumvent the problem and provide a set theoretic generalization of the standard model are Li [12] and Heifetz, Meier and Schipper [10]. They depart from the standard model in that they use multiple state spaces. Feinberg [5] and Feinberg [6] model unawareness in the context of games.

The standard model includes a unique state space and the possibility correspondence P maps states (of that one state space) to subsets of the same state space. The interpretation is that for any $\omega \in \Omega$, the set $P(\omega)$ denotes the states that the agent considers possible, when ω has occurred. In contrast, modeling unawareness using multiple state space implies that the possibility correspondence maps states of any possible state space to subsets of possibly different state spaces. The reason is that awareness varies with the state. For example, suppose that state $\omega \in \Omega$ specifies that the agent's awareness is smaller, so that if ω occurs, the agent's state space is Ω' and not Ω. Then, the set of states that the agent considers possible, $P(\omega)$, is a subset of Ω' and not of Ω. In other words, while ω belongs to Ω, $P(\omega)$ is a subset of another state space, Ω'. As a result, a model with unawareness has to impose additional axioms on the possibility correspondence P, restricting what it prescribes across different state spaces.

One of the main differences between this model and the two set theoretic models which deal with unawareness, namely Li [12] and Heifetz, Meier and Schipper [10], is that weaker restrictions are imposed on what the possibility correspondence P prescribes across state spaces. This is achieved by explicitly modeling the use of theorems.

In particular, Li assumes a possibility correspondence P, just as in the standard model, which maps full states to subsets of the full state space Ω^*, which in Li's terminology is the most complete state space.[2] For each full state $\omega^* \in \Omega^*$, $P(\omega^*)$ denotes the set of full states that the agent would consider possible, if he were aware of all the possible questions. In Li's words, $P(\omega^*)$ denotes the agent's factual information. If the agent is not fully aware at ω^* (i.e. he is aware only of a subset of all the possible questions), so that his state space is different from the full state space, then what he actually perceives as possible is the projection of $P(\omega^*)$ to the set of questions that he is aware of. Similarly, when i is reasoning about j's knowledge, he projects j's full state partition to his state space. Heifetz, Meier and Schipper [10] follows a similar approach. Their property "Projections Preserve Knowledge" requires that if the agent considers subjective states in $P(\omega)$ as possible at subjective state ω, then at the projection of ω to a more limited set of questions V, he considers possible only the projection of $P(\omega)$ to the set of questions V. In essence, these two properties place a restriction on what the possibility correspondence can prescribe across different state spaces.

In order to illustrate why these two properties are restrictive, recall the Holmes example. Suppose that Watson is only aware of the question "Was there an intruder?", without being aware of the theorem that the dog not barking implies there is no intruder. Moreover, suppose that Holmes is fully aware and the only state that he considers possible is the one that specifies that there is no intruder. According to Li's approach, Watson's view of Holmes knowledge is the projection of Holmes' partition to Watson's state space. That is, Watson is able to reason that Holmes knows that there is no intruder, although Watson himself is unaware of any theorem that could lead to such a conclusion. Heifetz, Meier and Schipper [10] would make the same prediction, due to the "Projections Preserve Knowledge" property.

It is therefore argued that unawareness of theorems has a much more profound effect than just to limit one's knowledge. It also constrains the agent's reasoning on what others know, indicating a significant departure from the standard model, where partitions are common knowledge. In particular, if in the standard model an agent does not exclude the true state, (i.e. he is not making any mistakes), then he can never make a mistake about another agent's knowledge. A property of this model, however, is that even if a limitedly aware agent includes a constrained view of the true state, he may still reason that another agent cannot know an event, when in fact he does. This also implies that the standard model is not a good approximation of a situation where agents have a common state space that we wish to model, but nevertheless there is some background, unmodeled unawareness that influences their knowledge on that state space.

The paper proceeds as follows. Section 2 introduces the basic one agent model, while its main results are presented in Section 3. Section 4 includes the multi agent model, where common knowledge is defined and characterized with respect to a self evident event. Section 5 examines the so-called no-trade theorems. Proofs are contained in the Appendix.

[2] Since the full state space Ω^* is the most complete state space, only an agent who is aware of all possible questions, is also aware of Ω^*. A full state ω^*, is an element of the full state space.

2 The Model

2.1 Preliminaries

Consider a set of questions Q_0, and denote by A_q as the set of possible answers for question $q \in Q_0$. Everything that is relevant for the description of the physical world can be expressed using questions in Q_0. That is, a *state of the world* is a vector, specifying an answer for each question in Q_0. Refer to these as the "basic questions". An *epistemic state* is much bigger, as it contains a full description of the world, together with the awareness and the knowledge of the agent about the world and about his knowledge and awareness. If there is more than one agent, an epistemic state also specifies what each agent knows about the other agents' awareness and knowledge of the world, and of their own and others' knowledge and awareness, and so on. All these extra dimensions can be expressed by having more questions. Denote by Q the set of all such questions, including those in Q_0. Say that a *full state* is a vector, specifying an answer for each question in Q. More formally, the *full state space* Ω^* is a subset of the Cartesian product $\times_{q \in Q} A_q$, where A_q contains all possible answers for question $q \in Q$. A full state ω^* is an element of the full state space Ω^*. The set A_q can contain one, two, or more answers. The notion of awareness that will be defined in the following sections requires that if an agent is aware of a question, then he is aware of all possible answers.

Given any set of questions $V \subseteq Q$, a *subjective state space* Ω is the projection of Ω^* to the Cartesian product $\times_{q \in V} A_q$. A *subjective state* ω is an element of the subjective state space $\Omega \subseteq \times_{q \in V} A_q$ and it is a vector specifying an answer only for the questions that belong to V. Each subjective state ω belongs to only one subjective state space Ω. A collection of subjective states of a particular subjective state space Ω constitutes an *event* E. In other words, an event E is a subset of a subjective state space Ω (and given Ω^*, there is a unique subjective state space Ω satisfying this inclusion). Define $\mathcal{V}_E$ to be the unique set of questions V such that $E \subseteq \Omega \subseteq \times_{q \in V} A_q$. Define the *negation of* E to be the complement of E with respect to the subjective state space Ω of which it is a subset:

$$\neg E = \Omega \setminus E.$$

Note that $\mathcal{V}_E = \mathcal{V}_{\neg E}$, since they are both subsets of the same subjective state space, $\Omega \subseteq \times_{q \in V} A_q$. Denote the complement of Ω by the empty set associated with it, $\emptyset_{\mathcal{V}_\Omega}$. This empty set can be thought of as the contradiction, expressed with the vocabulary or questions available in $\mathcal{V}_\Omega$.

2.2 Restrictions and enlargements of events

Take two sets of questions, $V' \subseteq V \subseteq Q$, and define Ω to be the projection of the full state space Ω^* on the Cartesian product $\times_{q \in V} A_q$, and Ω' to be the projection of Ω^* on $\times_{q \in V'} A_q$. There exists a surjective projection $\Pi^V_{V'} : \Omega \to \Omega'$. For any subjective state $\omega \in \Omega$, $\Pi^V_{V'}(\omega)$ is the *restriction of* ω to the smaller set of questions V'. In other words, $\Pi^V_{V'}(\omega)$ is the unique subjective state $\omega' \in \Omega'$ that gives the same answers as ω, to all questions that belong to V'. Take an event $E \subseteq \Omega$, which is a collection of subjective states that belong to Ω. Its restriction to the smaller set of questions V' is denoted by $\Pi^V_{V'}(E)$. To save on notation and only when it is unambiguous of which state space E is a subset, we abbreviate $\Pi^V_{V'}(E)$ by $E_{V'}$.

For any subjective state $\omega' \in \Omega'$, its *enlargement* to the bigger set of questions V is the inverse image $(\Pi^V_{V'})^{-1}(\omega')$. This inverse image $(\Pi^V_{V'})^{-1}(\omega')$ is an event, a collection of subjective states $\omega \in \Omega$ that give the same answer as ω' to all questions in V'. Take an event $E' \subseteq \Omega'$, which is a collection of subjective states ω' that belong to Ω'. Its enlargement to the bigger set of questions V is denoted by $(\Pi^V_{V'})^{-1}(E)$. Again, when it is unambiguous of which state space E is a subset, we abbreviate $(\Pi^V_{V'})^{-1}(E)$ by E_V.

2.3 The full state space

For a subset of questions $V \subseteq Q_0$, where Q_0 is the set of basic questions, the resulting Cartesian product of their answers is $\underset{q\in V}{\times} A_q$. Define mV to be the question "What subjective states in $\underset{q\in V}{\times} A_q$ does the agent consider impossible?". The collection of possible answers for question mV is the collection of all proper subsets of $\underset{q\in V}{\times} A_q$. The questions mV capture the agent's knowledge of theorems, as shown in Section 2.5.

For each question q, where $q \in Q_0$, or $q = mV$, for $V \subseteq Q_0$, define aq to be the question "Is the agent aware of question q?". This question captures the agent's awareness of questions, as shown in Section 2.4. In a multi agent model, it will also capture the agent's knowledge about each agent's awareness. The possible answers for this type of questions are just two: "yes" and "no". Questions of the type aaq, $aaaq$, $aa\ldots aq$ are not defined. Justification for this restriction will be given in Section 2.4, where awareness of questions will be defined.

The set of all questions Q contains the basic questions Q_0, together with all questions mV, where $V \subseteq Q_0$, and all questions aq, where $q \in Q_0$, or $q = mV$, for $V \subseteq Q_0$. The full state space Ω^* is a subset of the Cartesian product of the answers of all questions in Q:

$$\Omega^* \subseteq \underset{q\in Q}{\times} A_q.$$

Define $\mathcal{S}$ to be the union of all state spaces:

$$\mathcal{S} = \bigcup \{\Pi^Q_V(\Omega^*) : \emptyset \neq V \subseteq Q\}.$$

A state space $\Omega \in \mathcal{S}$ is generated by a non empty set of questions $V \subseteq Q$, and it is the projection of the full state space Ω^* to V. The construction of the full state space in the multi agent case is more complicated, as an agent has to reason about other agents' reasoning as well. The details are given in Section 4.2.

2.4 Awareness

The awareness of an agent is given by W, which is a mapping from the union of all state spaces $\mathcal{S}$ to sets of questions. For any state $\omega \in \mathcal{S}$,

$$W(\omega) = \bigcup \left\{ \{q, aq\} \subseteq \mathcal{V}_{\{\omega\}} : \omega_{aq} = \text{“yes”} \right\}$$

denotes the questions, of which the agent is aware if $\omega \in \mathcal{S}$ occurs. If ω specifies “yes” to question aq, then the agent is aware of question q at ω. We then assume that he is also aware of question aq. In other words, the agent is always aware of pairs of questions, that is, q with its respective aq question. Recall that $\mathcal{V}_{\{\omega\}}$ is the set of questions V such that $\omega \in \Pi_V^Q(\Omega^*)$.

As stated in Section 2.3, questions of the type aaq, $aaaq$, $aa \ldots aq$ are not permitted by the model. The first reason for this restriction stems for the definition of W, which specifies that the agent is aware of q and aq if ω specifies “yes” to question aq. Therefore, question aaq, which would also specify whether the agent is aware of question aq, is not needed. Another reason why these higher orders of questions would seem necessary is to express that if an agent is aware of something, then he is aware that he is aware of it, he is aware that he is aware that he is aware of it, and so on. One of the results of Theorem 2 is exactly this property and it does not require these higher order questions. In the multi agent case, questions of the type $a^i a^j a^k q$ where $i \neq j$ and $j \neq k$ arise naturally when common knowledge is defined and thus they will be included in the formal model.

The agent's subjective state space at $\omega \in \mathcal{S}$ is

$$\Omega(\omega) = \Pi_{W(\omega)}^Q(\Omega^*),$$

which is the projection of the full state space Ω^* to the set of questions he is aware of at ω. [3]

Take an event E and define $U(E)$ to be the set of states $\omega \in \mathcal{S}$ that describe that the agent is unaware of it:

$$U(E) = \{\omega \in \mathcal{S} : \mathcal{V}_E \nsubseteq W(\omega)\}.$$

A state ω belongs to $U(E)$ if the agent's awareness at ω, $W(\omega)$, does not contain $\mathcal{V}_E$. In other words, the agent is not aware of all the questions for which each state in E gives an answer. Note that $U(E)$ is not an event, since it contains states of different state spaces.

Given a set of questions V that generate the state space Ω_V^*, we define $U_V(E)$ to be the states of that particular state space, which describe that the agent is unaware of E. Hence, $U_V(E)$ is an event. It is natural to require that V is big enough so that the generated state space Ω_V^* can adequately express E and the agent's awareness of it. Hence, we first require that V should contain all questions in $\mathcal{V}_E$. Secondly, we require that for each question $q \in \mathcal{V}_E$, V contains its respective counterpart aq. Denote this set of questions by $\alpha(V)$.[4] Then, the condition is that $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \subseteq V$.

[3] If $W(\omega) = \emptyset$, then define $\Omega(\omega) = \emptyset$. In that case, $\Omega(\omega)$ is not an event and carries no awareness.

[4] The respective counterpart of aq is aq itself, since question aaq is not allowed in the model. Formally, $\alpha(V) = \{aq : q \in V, q \neq aq' \text{ for all } q' \in Q\} \bigcup \{q \in V : q = aq', q' \in Q\}$.

The event

$$U_V(E) = \Omega_V^* \cap U(E),$$

contains all states $\omega \in \Omega_V^*$ where the agent is unaware of event E. The set of states in Ω_V^* that describe that the agent is aware of E is $\neg U_V(E)$, the complement of $U_V(E)$ with respect to state space Ω_V^*. Denote this event by $A_V(E)$.

2.5 Theorems and impossible states

A theorem of the form "A implies B" can equivalently be expressed as the impossibility of a particular state. To give an example, suppose that V contains the following two questions: "Is p true?" and "Is r true?", each with two answers: "yes" and "no". The state space generated from these two questions contains the following states:

$$(p_y, r_y) \qquad (p_y, r_n) \qquad (p_n, r_y) \qquad (p_n, r_n),$$

where for example, (p_y, r_y) is the state that specifies "yes" for both questions. Suppose that the following theorem is true: "p_y implies r_y". This can equivalently be expressed by characterizing the state (p_y, r_n) as impossible. Therefore, an agent who knows this theorem is able to exclude this state from happening, irrespective of how much information he has. The theorem "p_y if and only if r_y" would require that states (p_y, r_n) and (p_n, r_y) are impossible. Alternatively, the theorem "r_y implies p_y" would require that (p_n, r_y) is impossible.

More generally, consider the set of questions $V = \{p_1, p_2, \ldots, p_k\}$. The theorem "$p_{1y}$ and p_{2y} and $\ldots p_{k-1y}$ implies p_{ky}" can be expressed by requiring that the subjective state $(p_{1y}, p_{2y}, \ldots, p_{k-1y}, p_{kn})$ is impossible. That the agent knows the theorem can be expressed by including $(p_{1y}, p_{2y}, \ldots, p_{k-1y}, p_{kn})$ as an answer of the question mV. That the agent considers subjective states $E \subset \times_{q\in V} A_q$ to be impossible can be expressed by answering E to question mV. Note that the agent does not need to be aware of all questions in Q in order to know that theorem. He merely has to be aware of question mV and all questions in V. Moreover, knowledge of a theorem does not imply anything about its validity. The agent may very well have a wrong theorem.

The agent's knowledge of theorems is given by the function M, which maps $\mathcal{S}$ to subsets of $\mathcal{S}$. For any $\omega \in \mathcal{S}$,

$$M(\omega) = \left\{\omega_1 \in \Omega(\omega) : \Pi_V^{W(\omega)}(\omega_1) \in \omega_{mV}, \{mV\} \cup V \subseteq W(\omega)\right\}$$

denotes the set of subjective states that the agent considers impossible at ω, and expresses what theorems he knows at that state. An element $\omega_1 \in \Omega(\omega)$ of the agent's state space at ω, is considered impossible if two conditions are met. Firstly, at ω the agent is aware of question mV and all questions in V. Secondly, the projection of ω_1 to the set of questions V is contained in ω_{mV}, which is the answer that ω specifies for question mV: "What states in $\times_{q\in V} A_q$ does the agent consider impossible?" This answer, ω_{mV}, is an event, a subset of the Cartesian product $\times_{q\in V} A_q$.

To give a simple example, suppose that V contains two questions: "Is it raining?" and "Is it snowing?" and suppose that the agent knows the theorem "rain implies no snow". This knowledge of the theorem is modeled firstly by making him being aware of both questions in V and question mV. Secondly, by answering ω_2 to question mV, where ω_2 specifies that it is both raining and snowing. Note that ω_2 is an element of the state space $\underset{q\in V}{\times} A_q$. Since the agent will typically be aware of much more questions, ω_2 is not an element of his state space. But it is natural that any state in his state space that specifies that it rains and it snows, should be considered impossible. This is formally captured by requiring that if a state in his state space, when projected to V is equal to ω_2, it is considered impossible.

2.6 Immediate perception

For some questions it is the case that if the agent is aware of $q \in Q$, then he always knows the answer. For example, questions that describe what the agent sees, hears, touches, smells and tastes are questions that the agent can always answer for himself, when he is aware of them. Denote by X the set of all such questions. Note that X naturally depends on the agent, a feature which will be captured in the formalism.

The following Axiom is assumed throughout the paper. Define $\mathbb{E}$ to be the set that contains all epistemic questions $aq \in Q$ for $q \in Q$ and any $mV \in Q$, for $V \subseteq Q$:

$$\mathbb{E} = \{aq \in Q : q \in Q\} \cup \{mV \in Q : V \subseteq Q\}.$$

Axiom 1. $\mathbb{E} \subseteq X$.

The Axiom states that X contains at least all the epistemic questions that belong to Q. These epistemic questions are of the type aq for $q \in Q$ or of the type mV, for some set of questions $V \subseteq Q$. This implies that if the agent is aware of an aq or an mV question, then he always knows their answer, which in particular means that he knows whether he is aware of question q, and which subjective states in $\Pi_V^Q(\Omega^*)$ he considers impossible.[5] Of course, some basic questions may also be included in X, such as those referring to the agent's five senses, such as what he sees, hears, touches, tastes and smells.

2.7 The possibility correspondence

In the standard model without unawareness there is only one state space, Ω, and the possibility correspondence P maps elements of Ω into subsets of Ω. The interpretation is that if $\omega \in \Omega$ occurs, then the agent considers all states in $P(\omega)$ as possible. In a model with unawareness there are many state spaces, and the domain of the possibility correspondence P is the union of all state spaces, $\mathcal{S}$.

[5]Note that in the multi agent setting, Axiom 1 does not imply that for each question the agent is aware of, he also knows whether any other agent is aware of it as well. That is, uncertainty of unawareness is permitted.

As was indicated first in the introduction, the interpretation of $P(\omega)$ is that the agent comes to consider a set of states as possible by using three elements. The first is the set of questions that he is aware of. The projection of Ω^* on these questions constitutes his state space. Secondly, for some questions that he is aware of, he always knows the answer. These are the questions that refer to his immediate perception, such as what he sees, hears etc. Thirdly, the agent knows some theorems, which represent what states he considers impossible. Summarizing, the agent is characterized by the triple

$$(W, M, X),$$

where W is a mapping from $\mathcal{S}$ to subsets of Q, M maps $\mathcal{S}$ to subsets of $\mathcal{S}$, and $X \subseteq Q$ is a designated subset of questions. As was indicated in the previous sections, the function W models awareness of questions, M models knowledge of theorems, and X models the agent's immediate perception of the world. Combining the three elements in (W, M, X) we can describe what subjective states the agent will consider possible at each ω. This is captured by a possibility correspondence, which maps $\mathcal{S}$ to subsets of $\mathcal{S}$.

For any $\omega \in \mathcal{S}$,

$$P(\omega) = \left\{\omega' \in \Omega(\omega) : \omega'_q = \omega_q, q \in W(\omega) \cap X\right\} \setminus M(\omega)$$

denotes the subjective states the agent considers possible if ω occurs. More specifically, at ω the agent is aware of questions that belong to $W(\omega)$ and his subjective state space is $\Omega(\omega)$. For the questions in $W(\omega)$ that also belong to X, he knows the answer. This is the answer that ω specifies for that question. For all other questions in $W(\omega)$ he does not know the answer, but he can utilize his knowledge of theorems by excluding the impossible states $M(\omega)$. Note that although ω belongs to the state space $\Pi^Q_{\mathcal{V}_{\{\omega\}}}(\Omega^*)$, $P(\omega)$ is a subset of $\Omega(\omega) = \Pi^Q_{W(\omega)}(\Omega^*)$. These state spaces are different if and only if $\mathcal{V}_{\{\omega\}}$ is different from $W(\omega)$.

The formula above may be viewed as a description of the agent's reasoning. It is argued that there exist some questions that the agent is aware of and which he can always answer without invoking the use of any theorems he knows. Examples of such questions are those that refer to his senses: what he can see, hear, touch, taste and smell. Such questions represent the agent's immediate perception of the world and there is simply no other means of acquiring more raw information about what has occurred. In order to further refine the set of subjective states he considers possible, the agent has to rely on his thinking, or put it differently, on the theorems he knows, expressed by the set of subjective states he considers impossible.

Consider a brief example. If the agent is aware of the question "Do I see that it is snowing?", then we could argue that he will always be able to answer it with a "yes" or a "no". On the other hand, the answer to the question "Is it snowing?" is not immediate. If he sees that it is snowing, then he may use the theorem "if I see that it is snowing, then *it is* snowing ", but if he does not see that it is snowing, (for instance because he is in a room without any windows), then he may not be able to determine whether it is snowing.

Examining the formula for $P(\omega)$ we can suggest some further interpretations. Firstly, as it will be shown in Theorem 1 the agent considers the true state possible if and only if he has not excluded it through a theorem. That is, he will exclude the true state only if he knows a "wrong" theorem. Since most properties of the model do not require that the agent always includes the true state, we can model situations where agents are permitted to make mistakes. However, a certain degree of non delusion is already built in the model. Note that for the questions that belong to the agent's set of immediate perception, X, he is always able to answer them truthfully.

A more significant departure from the rational agent paradigm is when $P(\omega)$ is empty. This can happen if what the agent "sees" directly contradicts his theorems. For example, suppose that the agent sees that it is raining and through his theorems he deduces that it is actually raining and therefore the bus will be late. If on the contrary he also sees that the bus arrives early, then he has contradicted himself and $P(\omega)$ is empty. Hence, he understands that he has made a mistake, he knows nothing and he therefore needs to rethink some of his theorems. Although for most properties we do not need that $P(\omega)$ is nonempty, we state the axiom formally.

Axiom 2. *For all* $\omega^* \in \Omega^*$, $P(\omega^*) \neq \emptyset$.

It is straightforward to show that Axiom 2 implies that $P(\omega) \neq \emptyset$ for all $\omega \in \mathcal{S}$ such that $W(\omega) \neq \emptyset$. A stronger Axiom is that the agent never excludes the true state.

Axiom 3. *For all* $\omega^* \in \Omega^*$, $\Pi^Q_{W(\omega^*)}(\omega^*) \in P(\omega^*)$.

This is a generalization of the familiar Axiom of nondelusion. When ω^* occurs, the agent does not exclude the true state, but his unawareness allows him to perceive only a limited version of the truth. This is the restriction of the true state ω^* to his awareness $W(\omega^*)$. Similarly, this version implies that for all $\omega \in \mathcal{S}$ such that $W(\omega) \neq \emptyset$, $\Pi^Q_{W(\omega)}(\omega) \in P(\omega)$.

Finally, we make two remarks on the model. Firstly, it may seem restrictive that X is constant and does not depend on the full state. Someone could argue that there may be some full states where a question q belongs to X because the agent knows the answer without invoking the use of any theorems, but for some other full states this does not hold and q does not belong to X. So suppose that such a question q exists. The argument should then provide a reason, or a story, explaining what is happening in some states where q is included in X, which is not happening in other states where q excluded from X. In other words, a "theorem" is provided that explains why the agent is able to answer question q in some states, and that same theorem cannot be employed in other states, where he cannot answer question q. In that case, q can be excluded altogether from X and the agent can use his knowledge of theorems in order to answer it in some full states. Thus, it is without loss of generality to assume that X is a set and not a function with Ω^* as its domain.

Secondly, explicitly modeling the use of theorems means that awareness of a question does not imply anything about how well an agent "understands" the notions that this question uses. For example, if the agent is aware of the question "Is it snowing?", this does not imply that he also has a clear and correct understanding of the notion "snow". His understanding of this notion is captured solely by his knowledge of theorems. If he is also aware of the question "Is it cold?" and knows the theorem "Snowing implies being cold", then we can say that he has some understanding of what snowing means. Knowing more (correct) theorems, that connect the question "Is it snowing?" with other questions the agent is aware of, means that he has an even better understanding of the notion "snow". It is perhaps easier to understand what it means for an agent to be aware of a question, if we imagine that for him each question is represented by a number, and each possible answer is again represented by a number. This way, "understanding" a notion is solely captured by knowledge of theorems.

2.8 Knowledge operator

Take an event E and define $K(E)$ to be the set of states $\omega \in \mathcal{S}$ that describe that the agent knows it:

$$K(E) = \{\omega \in \mathcal{S} : \mathcal{V}_E \subseteq W(\omega) \text{ and } \emptyset \neq (P(\omega))_{\mathcal{V}_E} \subseteq E\}.$$

A state $\omega \in \mathcal{S}$ belongs to $K(E)$ if it describes that the agent is aware of E and that the set of states he considers possible at ω, restricted to the set of questions $\mathcal{V}_E$, is a subset of E. Note that $K(E)$ is not an event, since it contains states of different state spaces.

Given a set of questions V that generate state space Ω^*_V, we define $K_V(E)$ to be the states of that particular state space, which describe that the agent knows E. We impose the restriction $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \subseteq V$, which means that only a state space rich enough to express E and the agent's awareness of it, can contain such an event. The event

$$K_V(E) = \Omega^*_V \cap K(E),$$

contains all states $\omega \in \Omega^*_V$ which describe that the agent knows E. The set of states in Ω^*_V that describe that the agent is aware of event E but does not know it is $A_V(E) \cap \neg K_V(E)$.

Example

In this example we model the Sherlock Holmes and Dr. Watson story. Both are investigating a crime where a horse was stolen and its keeper was killed. After they both concluded the investigation, Holmes already knows who did it, while Watson is still wondering. There are two basic questions. That is, Q_0 consists of question I: "Was there an intruder?" and question B:"Did Watson hear the dog bark?", with two possible answers, "yes" and "no". Questions aI and aB refer to Watson's awareness. For example, aB is the question "Is Watson aware of question B?". Finally, if we set $V = \{B, I\}$, then the state space Ω defined by questions in V contains the following four states:

$$\omega_1 = (I_y, B_y) \quad \omega_2 = (I_n, B_y) \quad \omega_3 = (I_y, B_n) \quad \omega_4 = (I_n, B_n),$$

where, for example state $\omega_1 = (I_y, B_y)$ specifies that there was an intruder and that Watson heard the dog barking. Question mV is the question "What states in $\times_{q \in V} A_q$ does Watson consider impossible?", while amV is the question "Is Watson aware of question mV?".

The six questions, I, aI, B, aB, mV, amV constitute the set of all questions, Q.[6] By Axiom 1, questions aI, aB, mV and amV belong to the set of Watson's immediate perception, X. We also assume that question B belongs to X.[7] Define the most complete state space Ω^* to be the Cartesian product of the answers of these six questions. First, take state $\omega_1^* = (I_n, B_n, aI_y, aB_n, (amV)_n, (mV)_{\omega_3})$. This state specifies that there was no intruder and that Watson did not hear the dog barking, he is aware of question I but not of question B. Since he is not aware of all questions in V, he cannot express any theorems in that state space. Hence, what state ω_1^* specifies for questions mV and amV is irrelevant. Formally, we have

$$W(\omega_1^*) = \{I, aI\},$$
$$\Omega(\omega_1^*) = \{(I_y, aI_y), (I_y, aI_n), (I_n, aI_y), (I_n, aI_n), \}$$
$$M(\omega_1^*) = \emptyset,$$
$$X = \{aI, aB, mV, amV, B\},$$
$$P(\omega_1^*) = \{(I_y, aI_y), (I_n, aI_y)\}.$$

When ω_1^* occurs, Watson is aware of questions I and aI. Out of these questions, only aI belongs to his immediate perception, so that he knows that he is aware of I and aI, and he is able to exclude any state that answers "no" to question aI. Since he is unaware of any theorem, his set of impossible states is the empty set and he cannot further refine the set of states he considers possible. As a result, $P(\omega_1^*)$ consists of states (I_y, aI_y) and (I_n, aI_y).

Define $E = \{I_n\}$ to be the event "there was no intruder". Event E is a subset of the state space which contains only two states, I_y and I_n. The agent is aware of event E at ω_1^*, but he does not know it. In other words, he does not know whether there is an intruder, although he is aware of the possibility of an intruder. Formally, we have that $\omega_1^* \in A_Q(E) \cap \neg K_Q(E)$.

The second interesting full state is $\omega_2^* = (I_n, B_n, aI_y, aB_y, (amV)_y, (mV)_{\omega_3})$, which differs from ω_1^* in that now Watson is aware of all questions, and knows the theorem "not hearing the dog barking implies no intruder", represented by state ω_3. Formally, we have Q as above and

$$W(\omega_2^*) = \{I, B, aI, aB, mV, amV\},$$
$$\Omega(\omega_2^*) = \Omega^*$$
$$M(\omega_2^*) = \{\omega^* \in \Omega^* : \omega_I^* = \text{"yes" and } \omega_B^* = \text{"no"}\},$$
$$X = \{aI, aB, mV, amV, B\},$$
$$P(\omega_2^*) = \{(I_n, B_n, aI_y, aB_y, (amV)_y, (mV)_{\omega_3})\}.$$

[6] Formally, Q contains also questions $m\{I\}$, $am\{I\}$, $m\{B\}$ and $am\{B\}$. Since we do not consider states where the agent is aware of these questions, we have excluded them from Q in order to save on notation.

[7] Including question B in X slightly distorts Doyle's original story, in favor of making the example simple and concise. The original story is that the crime took place in the previous night, while Watson was absent from the stable. Information about the dog not barking is gathered by residents of the house, during the interrogations.

When ω_2^* occurs, Watson's state space is the full state space, Ω^*. Firstly, because questions aI, aB, amV and mV belong to X, he considers a full state as possible only if it specifies that he is aware of all questions and that he considers ω_3 to be impossible. Since he is aware of question B which belongs to X, he is also able to exclude any state that specifies that he heard the dog barking. Finally, considering ω_3 as impossible means that he is able through $M(\omega_2^*)$ to exclude any state that specifies that there was an intruder and he heard no barking. As a result, at ω_2^* the only state Watson considers possible is ω_2^* itself, which specifies that there was no intruder and he heard no barking. Summarizing, he is able to reason that since he heard no barking, there cannot be an intruder. Therefore, at ω_2^* he is aware of the event E that there was no intruder, and he also knows it. Formally, $\omega_2^* \in K_Q(E)$. This reasoning was impossible at ω_1^*, when he was only aware of questions I and aI. In other words, *what Watson is unaware of at* ω_1^* *hurts his knowledge about a question he is aware of at* ω_1^*.

The events $K_Q(E)$ and $U_Q(E)$ are subsets of the full state space Ω^*, so that only an agent who is aware of all questions in Q is aware of them. Hence, only for an agent who is fully aware (like Holmes) do these events represent his reasoning about what Watson knows and is aware of.[8] Suppose that another agent is only aware of a subset of Q, namely $V' = \{I, aI\}$. His reasoning about Watson's unawareness and knowledge of event $E = \{I_n\}$, "there was no intruder", will be expressed in state space $\Omega' = \underset{q \in V'}{\times} A_q$ and it is represented by events $U_{V'}(E) \subseteq \Omega'$ and $K_{V'}(E) \subseteq \Omega'$, respectively. State space Ω' has only four states:

$$(I_y, aI_y) \quad (I_n, aI_y) \quad (I_y, aI_n) \quad (I_n, aI_n).$$

In state space Ω', the event "Watson is unaware of E" is $U_{V'}(E) = \{(I_y, aI_n), (I_n, aI_n)\}$, while the event "Watson knows E" is $K_{V'}(E) = \emptyset_{V'}$. Although state space Ω' is rich enough to include a description of E, it is not rich enough to include the possibility of Watson knowing event E. In other words, an agent who is only aware of V' would reason that it is simply impossible for Watson to know E.

3 Results

3.1 Overview of the properties of the standard model

Consider a state space Ω and a possibility correspondence $P : \Omega \to 2^{\Omega} \backslash \emptyset$. The interpretation is that when $\omega \in \Omega$ occurs, the agent learns that $P(\omega)$ has occurred. It is assumed that the possibility correspondence P satisfies the following properties:

P1 For any $\omega \in \Omega$, $\omega \in P(\omega)$.

P2 For any $\omega, \omega' \in \Omega$, $\omega' \in P(\omega)$ implies $P(\omega') \subseteq P(\omega)$.

P3 For any $\omega, \omega' \in \Omega$, $\omega' \in P(\omega)$ implies $P(\omega') \supseteq P(\omega)$.

[8] Since the analyst is always aware of all questions in Q, these two events represent his reasoning about Watson's unawareness and knowledge.

The first property says that the agent never excludes the true state from being possible. The second property states that if $\omega' \in P(\omega)$, then it cannot be that for some state ω'', $\omega'' \in P(\omega')$ but $\omega'' \notin P(\omega)$. If it were the case, a rational agent would argue the following at ω: Since ω'' is excluded, it cannot be that ω' has occurred, but this contradicts that $\omega' \in P(\omega)$. Similarly, the third property states that if $\omega' \in P(\omega)$, then it cannot be that $\omega'' \in P(\omega)$ but $\omega'' \notin P(\omega')$. A rational agent would argue at ω that since he cannot exclude ω'' and $\omega'' \notin P(\omega')$, it must be that ω' has not occurred, a contradiction of $\omega' \in P(\omega)$.[9]

An event E is just a subset of Ω. The set of states where the agent knows event E is given by the knowledge operator $K : 2^\Omega \to 2^\Omega$. In particular, for any event $E \subseteq \Omega$,

$$K(E) = \{\omega \in \Omega : P(\omega) \subseteq E\}.$$

The interpretation is that the agent knows event E at ω if in all the states he considers possible, E is true. Note that $K(E)$ is also an event, since it is a subset of Ω.

The following properties hold for the knowledge operator:

K1 Necessitation: $K(\Omega) = \Omega$.

K2 Monotonicity: $E \subseteq F \implies K(E) \subseteq K(F)$.

K3 Conjuction: $K(E) \cap K(F) = K(E \cap F)$.

K4 The Axiom of Knowledge: $K(E) \subseteq E$.

K5 The Axiom of Transparency: $K(E) \subseteq K(K(E))$.

K6 The Axiom of Wisdom: $\neg K(E) \subseteq K(\neg K(E))$.

Properties $K1, K2, K3$ are derived from the definition of the knowledge operator K, while property $P1$ implies $K4$, $P2$ implies $K5$ and $P3$ implies $K6$.

3.2 Results for the one agent model with unawareness

The following Theorem generalizes properties $P1, P2$ and $P3$ of the standard model without unawareness. All the results of this Section are valid for the multi agent case as well.

Theorem 1.

1. *$\{\omega\}_{W(\omega)} \notin M(\omega) \iff \{\omega\}_{W(\omega)} \in P(\omega)$.*[10]

2. *$\omega \in P(\omega_1)$ implies $P(\omega) = P(\omega_1)$.*

[9] The interpretations for P2 and P3 are taken from Rubinstein [14].

[10] The following property is also true. Suppose $M \subseteq \Pi_V^Q(\Omega^*)$ is a set of impossible states. Then, if $\omega \notin M$ and $(\Pi_{W(\omega)}^{\mathcal{V}_{\{\omega\}}})^{-1}(M(\omega)) \subseteq M$, then $\Pi_{W(\omega)}^{\mathcal{V}_{\{\omega\}}}(\omega) \in P(\omega)$.

The first part of Theorem 1 says that the agent includes the true state if and only if he does not consider it an impossible state through a theorem. In other words, he will exclude the true state only if he knows a wrong theorem. Note that state ω belongs to the state space generated by questions $\mathcal{V}_{\{\omega\}}$, but the agent's awareness at that state is $W(\omega)$, which a subset of $\mathcal{V}_{\{\omega\}}$. Therefore, the agent's perception of ω when ω occurs is the projection of ω to his state space.

The second part of Theorem 1 is the generalization of properties $P2, P3$, that if ω belongs to $P(\omega_1)$, then $P(\omega) = P(\omega_1)$. It states that if the agent considers ω to be possible when ω_1 has occurred, then the agent's view of what states he would consider possible if ω had occurred is the same as the states he considers possible now that ω_1 has occurred. In other words, if the agent considers states z and z' to be possible, then these two states should not, in his view, differ in their specification of what is the agent's knowledge of theorems and his awareness of questions.

It is important to emphasize that ω and ω_1 do not belong to the same state space. State ω_1 belongs to state space $\Pi^Q_{\mathcal{V}_{\{\omega_1\}}}(\Omega^*)$, while ω belongs to state space $\Omega(\omega_1) = \Pi^Q_{W(\omega_1)}(\Omega^*)$, which is the state space the agent has at ω_1.

The next property is the most important departure from other models dealing with unawareness, and stems from the explicit use of theorems in the construction.

Property 1. Awareness Leads to Knowledge
Suppose Axiom 2 holds. For any event E, if $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \subseteq V_2 \subseteq V_1$, then $K_{V_2}(E) \subseteq \Pi^{V_1}_{V_2}(K_{V_1}(E))$.

The condition $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \subseteq V_2, V_1$ ensures that $K_{V_2}(E)$, $K_{V_1}(E)$ are well defined, as explained in Section 2.8. The property says that state spaces which are generated by more questions give a more complete description of the agent's knowledge of an event E. Hence, if a more complete description of the world ω belongs to $K_{V_1}(E)$, its projection to a less complete state space generated by questions V_2, may not belong to the less complete description of the agent's knowledge, $K_{V_2}(E)$.

The intuition is that if a state space is generated by more questions, then it may also include more mV questions that represent theorems, and in effect contain more ways in which an agent can know the event. An implication is that an agent can gain knowledge of an event he was already aware of, by gaining more awareness and more theorems. Conversely, what an agent is unaware of hurts his knowledge about events he is aware of. Axiom 2 is needed in order to exclude the possibility that the agent knows nothing, because the theorems that he knows contradict each other. In the multi agent case an agent's limited awareness may lead to incomplete reasoning of other agents' knowledge. See Section 4.1 for further discussion and illustration of this property in the multi agent context.

For the one agent case, suppose that the agent's awareness is V_2, so in his state space the event "the agent knows E" is given by $K_{V_2}(E)$. If instead his awareness was increased to V_1, then in his view the event "the agent knows E" would be $K_{V_1}(E)$. Suppose now that $\omega \in K_{V_1}(E)$ has occurred, so that the agent with awareness V_1 knows event E. The agent with awareness V_2 though, is only aware that the projection of ω to V_2 has occurred.

The property says that this projection $\Pi^{V_1}_{V_2}(\omega)$, may not belong to $K_{V_2}(E)$, and therefore the agent with the more limited awareness will not know (but be aware of) event E.[11]

This feature is new, at least in the economics literature. On the one hand, the standard model assumes an agent who is aware of everything and knows all relevant theorems. On the other hand, the property Projections Preserve Knowledge of Heifetz, Meier and Schipper [10] implies that $\Pi^{V_1}_{V_2}(K_{V_1}(E)) = K_{V_2}(E)$, so that if a state ω specifies that the agent knows an event E, its projection ω' to a smaller state space will also specify that he knows it.[12] The one agent case of Li [12] is unrestricted, so that there is no implied relationship between $\Pi^{V_1}_{V_2}(K_{V_1}(E))$ and $K_{V_2}(E)$. In fact, one can construct examples where a state ω specifies that the agent does not know an event E, and yet its projection ω' to a smaller state space specifies that the agent knows event E. In that case, the less complete description of the world ω' would specify that the agent knows more, a possibility that is excluded by the present model. In Li's multi agent model, when i is reasoning about j, the equality holds: $\Pi^{V^i_1}_{V^i_2}(K^j_{V^i_1}(E)) = K^j_{V^i_2}(E)$.

Concluding, more awareness can lead to increased knowledge of events that we were already aware of. For example, becoming aware of (and knowing) Newton's theory enabled us to explain how the planets move, a question of which we were aware since ancient times. Equivalenlty, what we *are unaware of* hurts our knowledge about things we *are aware of*. The reason is that information that is readily available for us (for example, the distance between the planets and their size) is left unexploited, either because we are unaware of its existence, or because we don't know the theorems that can utilize it in order to provide answers. Aragones, Gilboa Postlewaite and Schmeidler [1] argues that these phenomena can be partly explained by computational complexity. An agent may learn something without getting new information, just by noticing certain regularities in the data he observes and by forming new theorems.

The next theorem groups properties that have been proposed in the literature, or are generalizations of the properties of the standard model.

Theorem 2. *Suppose* $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \cup \mathcal{V}_F \cup \alpha(\mathcal{V}_F) \cup \alpha(V) \subseteq V$. *Then:*

1. **Symmetry** $U_V(E) = U_V(\neg E)$.
2. **Plausibility** $U_V(E) \subseteq \neg K_V(E) \cap \neg K_V(\neg K_V(E))$.
3. **Strong Plausibility** $U_V(E) \subseteq \neg K_V(E) \cap \neg K_V(\neg K_V(E)) \cap \ldots \cap \neg K_V(\neg K_V(\ldots \neg K_V(E)))$.
4. **AU Introspection** $U_V(E) \subseteq U_V(U_V(E))$.

[11] More awareness leads to being aware of more events, and consequently knowing some of these. This feature is standard and covered by other models as well. The new feature is that of knowing an event that you were already aware of.

[12] In fact, this equality is essential in Heifetz, Meier and Schipper [10], since it ensures that $K(E)$, as defined in their paper, is an event. Their paper has a different definition of what an event is. In particular, an event is the union of a subset of a state space and all its enlargements. Therefore, an event contains states of different state spaces, which is not true in this paper.

5. **Subjective Necessitation** *Suppose Axiom 2 holds. Then, for all* $\omega \in \Omega_V^*$, $\omega \in K_V(\Omega(\omega))$.

6. **Generalized Monotonicity**$E_{\mathcal{V}_E \cup \mathcal{V}_F} \subseteq F_{\mathcal{V}_E \cup \mathcal{V}_F}$, $\mathcal{V}_F \subseteq \mathcal{V}_E \implies K_V(E) \subseteq K_V(F)$.[13]

7. **Conjunction** $K_V(E) \cap K_V(F) = K_V\left(E_{\mathcal{V}_E \cup \mathcal{V}_F} \cap F_{\mathcal{V}_E \cup \mathcal{V}_F}\right)$.

8. **AA-Self Reflection** $\omega \in A_V(E) \iff \omega \in A_V(A_{W(\omega)}(E))$.[14]

9. **AK-Self Reflection** $\omega \in A_V(E) \iff \omega \in A_V(K_{W(\omega)}(E))$.[15]

10. **The Axiom of Knowledge** *Suppose Axiom 3 holds. Then,* $K_V(E) \subseteq E_V$.

11. **KU Introspection** $K_V(U_V(E)) = \emptyset_V$.[16]

12. **A-Introspection** *Suppose Axiom 2 holds. Then,* $\omega \in A_V(E) \iff \omega \in K_V(A_{W(\omega)}(E))$.[17]

13. **The Axiom of Transparency** $\omega \in K_V(E) \iff \omega \in K_V(K_{W(\omega)}(E))$.

14. **The Axiom of Wisdom** *Suppose Axiom 2 holds. Then,* $\omega \in A_V(E) \cap \neg K_V(E) \iff \omega \in K_V(A_{W(\omega)}(E) \cap \neg K_{W(\omega)}(E))$.

The condition $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \cup \mathcal{V}_F \cup \alpha(\mathcal{V}_F) \cup \alpha(V) \subseteq V$ only ensures that the events $U_V(E)$, $K_V(E)$, $U_V(U_V(E))$ are well defined. Symmetry, AA-Self Reflection, AK-Self Reflection and A-Introspection have been proposed by Modica and Rustichini [13] and Halpern [9]. Plausibility, AU Introspection, and KU Introspection have been proposed by Dekel, Lipman and Rustichini [3], while Strong Plausibility has been proposed by Li [12]. The remaining properties are generalizations of the six properties of the standard model, cited in Section 3.1. Some of these generalizations are proposed by Li [12].

Symmetry states that if an agent is unaware of an event, then he is also unaware of its negation. Strong Plausibility states that if the agent is unaware of an event, then he does not know it, he does not know that he does not know it, and so on for any higher order of not knowing that he does not know. AU Introspection says that if an agent is unaware of an event, then he is unaware that he is unaware of it.

Subjective necessitation states that at any state ω, the agent knows his state space, which is $\Omega(\omega)$. For this property to hold we need that the agent considers at least one state as possible. Generalized monotonicity says that if at ω the agent knows event E, he is aware of F and E implies F, then he knows F. These two events may be subsets of different state spaces, so the usual notion of implication, $E \subseteq F$, is not defined. Li [12] has proposed a generalized version of implication: The event E implies the event F if the enlargement of

[13] A variant of this property states that if $\omega \in K_V(E)$, $\mathcal{V}_F \subseteq W(\omega)$ and $E_{\mathcal{V}_E \cup \mathcal{V}_F} \subseteq F_{\mathcal{V}_E \cup \mathcal{V}_F}$, then $\omega \in K_V(F)$.

[14] A variant of this property is $\omega \in A_V(E) \iff \{\omega\}_{W(\omega)} \in A_{W(\omega)}(A_{W(\omega)}(E))$.

[15] A variant of this property is $\omega \in A_V(E) \iff \{\omega\}_{W(\omega)} \in A_{W(\omega)}(K_{W(\omega)}(E))$.

[16] A variant of this property is $\omega \notin K_V(U_{W(\omega)}(E))$ for all $\omega \in \Omega_V^*$.

[17] A variant of this property is $\omega \in A_V(E) \iff \{\omega\}_{W(\omega)} \in K_{W(\omega)}(A_{W(\omega)}(E))$.

E to the set of questions $\mathcal{V}_E \cup \mathcal{V}_F$ is a subset of the enlargement of F to the same set of questions. Conjunction states that the agent knows events E and F if and only if he knows that E and F have occurred. Since E and F may be subsets of different state spaces, their intersection is not defined. Li [12] has proposed a generalized version of intersection to be the intersection of their enlargements to the set of questions $\mathcal{V}_E \cup \mathcal{V}_F$.

KU Introspection states that the agent cannot know that he is unaware of an event E. Note that for this property to hold we do not require that the agent does not exclude the true state, as in Li [12] and Heifetz, Meier and Schipper [10]. Properties AA-Self Reflection, AK-Self Reflection and A-Introspection say that equivalent conditions for an agent to be aware of an event is that he is aware that he is aware of it, he knows that he is aware of it and he is aware that he knows it.

The last two properties generalize the axioms of transparency and wisdom. The axiom of transparency states that the agent knows an event E at ω if and only if he knows that he knows it at ω. Note that $K_{W(\omega)}(E)$ is the event "the agent knows event E", expressed in the awareness of the agent at ω. The axiom of wisdom is similar. The agent is aware of but does not know event E if and only if he knows that he is aware of and does not know it. For this property to hold, we need Axiom 2 to exclude the possibility that the agent has an empty $P(\omega)$, and therefore knows nothing.

4 Multi agent model

Extending the model to the multi agent case is straightforward - the main difficulty arises from the use of slightly more involved notation. But the mechanics are the same. Consider a set I of agents, with generic elements i, j and k. As in the one agent case, agent i is able to "construct" his possibility correspondence P^i by singling out the epistemic questions that he is aware of and refer to him ($m^i V$, $a^i q$) and combine them with the basic questions and the set of his immediate perception X^i. When i is reasoning about j's possibility correspondence P^j, he analogously has to single out the epistemic questions that i is aware of and refer to j, ($m^j V$, $a^j q$) and combine them with the basic questions and j's set of immediate perception X^j. Analogously, when i is reasoning about j's reasoning about k, he first has to determine which of the questions in X^k and of the type ($m^k V$, $a^k q$) is j aware of. In essence, i only needs to determine j's subjective state space, since this will be the most complete description of the world according to j. Naturally, i's own unawareness prevents him from having a complete image of j's state space.

In the one agent case we constructed $K^i_V(E)$, $U^i_V(E)$, where V is any set of questions that is big enough to express E and agent i's awareness of it. Note that V generates the state space Ω^*_V. If i's awareness is given by V and he is reasoning about j's knowledge and unawareness, he will use events $K^j_V(E)$, $U^j_V(E)$, since these events give the most complete description of j's knowledge and unawareness according to i. If i's view of j's awareness is V', then i's view of j's reasoning about k is given by events $K^k_{V'}(E)$, $U^k_{V'}(E)$, since in i's view, V' generates j's most complete description of the world. This can be extended for any sequence of agents. Once we have determined that i's view of j's view of ... of m's awareness is V'', then $K^n_{V''}(E)$, $U^n_{V''}(E)$ represent i's view of j's view of ... of m's reasoning for any agent n.

Note that by definition, for i to be aware of question $a^j q$, he also has to be aware of question $a^i a^j q$. Higher orders of reasoning require more complicated questions of the type $a^i a^j a^k a^l a^m q$. Moreover, although agent i always knows the answer to questions of the type $a^i q, m^i V$ when he is aware of them, this does not necessarily hold for questions that refer to other agents, such as $a^j q, m^j V$, which are basic questions from i's perspective. Therefore, uncertainty of other agents' awareness is permitted.

4.1 Unawareness and reasoning about others

In a multi agent context, the Awareness Leads to Knowledge Property implies that i's limited awareness may also impair his reasoning about j's knowledge. For example, it may be that while i is aware of E, he wrongly deduces that j does not know it, exactly because i is unaware of the theorem that led j to know E. This clearly distinguishes the present approach from that of Li [12] and Heifetz, Meier and Schipper [10], which do not allow for such situations. In this paper though, the knowledge operator K_V is indexed by a set of questions V, and the Awareness Leads to Knowledge Property shows that a bigger set of questions V' gives a more complete description of one's knowledge.

To illustrate, suppose that agent i has awareness V^i, agent j has awareness V^j and $V^j \subseteq V^i$. They are both reasoning whether agent k knows event E. Suppose that both i and j are informed that the true state has occurred, namely ω for i and its projection to questions V^j for j, $\Pi^{V^i}_{V^j}(\omega)$. Moreover, suppose that $\omega \in K^k_{V^i}(E)$ but $\Pi^{V^i}_{V^j}(\omega) \notin K^k_{V^j}(E)$. In that case, i says that k knows E, while j says that k does not. More importantly, since i knows that ω has occurred and j knows that $\Pi^{V^i}_{V^j}(\omega)$ has occurred, it is also the case that i knows that k knows event E, while j knows that k does not know event E! In other words, i and j disagree on what k knows.

This feature is certainly not standard, since in the usual model we can never have two agents who disagree on what another agent knows. To be more precise, it can never be that i knows that k knows an event, while j knows that k does not know this event.[18] Li [12] and Heifetz, Meier and Schipper [10] also exclude such a possibility, because they assume that i's view of j's knowledge is the projection of P^j to i's state space.

It is important to emphasize that this inability of j to know that k knows E is not because j has made some mistake (like excluding the true state from being possible), but because he is unaware of something k is aware of. In effect, one of the main postulates of the standard model, namely that partitions are common knowledge, is not true with unawareness of theorems.

[18]In the standard model, the partitions of the agents are common knowledge. This means that the agents not only agree on what every other agent knows at each state ω, but this is also common knowledge. In a model with unawareness this property does not hold, mainly because the agents have different state spaces. However, the "Awareness Leads to Knowledge" property is needed in order for agents to disagree on what another agent knows.

Consider the following example which illustrates how two agents can disagree on what a third agent knows. Suppose that agent k is inside a basement with no windows, and that it is raining. Agent j is informed that k is inside the basement, so he reasons that because k cannot see what is happening outside, he does not know that it is raining, and j knows that this is the case. On the other hand, agent i is aware of and knows the existence of a computer in the basement, connected with a camera outside the building. If he is informed that k is also aware of and knows this, then he can reason that k can see whether it is raining by checking the computer. Moreover, he knows that this is the case. Concluding, the more aware agent i knows that k knows that it is raining, while the less aware agent j knows that k does not know whether it is raining.

It is important to note that the source of the two agents' *disagreement stems from their different awareness, not from their different information.* Had j been aware of the possibility of a computer in the basement, even if he didn't know whether it is connected with a camera or whether k was aware of it, would enable him to say that he didn't know whether k knows that it is raining. In that case, i and j would not disagree, but i would have more information. It is precisely the fact that j is unaware of the possibility of the computer that makes him know that k does not know that it is raining. Moreover, j is not making any mistakes, because it is true that with this limited awareness, k would not know whether it rained. Finally, this disagreement can only occur if what one agent is unaware of, hurts his knowledge about what he is aware of, so that the " Awareness Leads to Knowledge" property is necessary.

As an epilogue to this example, suppose that agent k performs a specific action if and only if he knows that it is raining. Moreover, suppose that agent j knows this and k's action is visible to him. Since j knows that k does not know that it is raining, he reasons that k should not perform this action. Nevertheless, he observes him performing it. If agent j excludes the possibility that he has made some mistake in his reasoning, then he can only conclude that k is aware of something that j is not aware of, that led him to know that it is raining. In other words, *agent j understands that he is unaware of something that he cannot specify.*

4.2 The full state space

This section gives a detailed construction of the full state space, which is the state space of the analyst or of a fully aware agent. The construction is similar to that of a beliefs space: starting from an initial state space S, define each player's first order beliefs on S, then each player's second order beliefs on S and all other players first order beliefs, and so on. The difference with this formulation is that instead of beliefs we have the epistemic questions $a^i q$ and $m^i V$, that describe the awareness of questions and knowledge of theorems for each agent i.

For any state space Ω, the set of epistemic questions $\mathcal{E}^i(\Omega)$ of agent i about Ω consists of the questions $a^i q$ and $m^i V$ about Ω. In particular, suppose $\Omega = \times_{q \in V} A_q$ is generated from a set of questions V. The set of all questions of the type $m^i V_1$, for all nonempty subsets V_1

of V is

$$\left\{m^i V_1 : \emptyset \neq V_1 \subseteq V\right\}. \tag{1}$$

These questions represent all the theorems that agent i can potentially have about state space Ω.

The set

$$\left\{a^i q : q \in V \cup \left\{m^i V_1 : \emptyset \neq V_1 \subseteq V\right\}\right\} \tag{2}$$

contains all the $a^i q$ questions, for all questions in V and $\{m^i V_1 : \emptyset \neq V_1 \subseteq V\}$. Denote the union of the two sets of questions in (1) and (2) by $\mathcal{E}^i(\Omega)$. An element that gives an answer to all questions in $\mathcal{E}^i(\Omega)$ describes agent i's awareness of questions and knowledge of theorems, about state space Ω.

To construct the full state space Ω^*, we begin with an initial state space $S = \times_{q \in Q_0} A_q$, which is generated from a set of basic questions Q_0. A state of nature $s \in S$ gives a detailed description of the world, but not what agents are aware of or know. Let $\Omega_1^i = S$ be each agent i's first order state space. Questions in $\mathcal{E}^i(\Omega_1^i)$ describe agent i's awareness of questions and knowledge of theorems about state space Ω_1^i. Define the set of all combinations of answers for these questions to be T_1^i:

$$T_1^i = \times_{q \in \mathcal{E}^i(\Omega_1)} A_q,$$

which we interpret as the first order type of agent i. The second order state space for agent i is

$$\Omega_2^i = S \times \times_{j \neq i} T_1^j.$$

An element in Ω_2^i describes the state of nature $s \in S$, together with the awareness of questions and knowledge of theorems about S, for all agents besides i. The set $\mathcal{E}^i(\Omega_2^i)$ contains all the epistemic questions of agent i about state space Ω_2^i. Note that there are some questions in $\mathcal{E}^i(\Omega_2^i)$ that also belong to $\mathcal{E}^i(\Omega_1^i)$. For example, if q is a basic question and belongs to Q_0, then $a^i q$ belongs to $\mathcal{E}^i(\Omega_1^i) \cap \mathcal{E}^i(\Omega_2^i)$. To avoid any duplication of questions, we define the second order type of agent i to be

$$T_2^i = \times_{q \in \mathcal{E}^i(\Omega_2^i) \setminus \mathcal{E}^i(\Omega_1^i)} A_q.$$

An element in $T_1^i \times T_2^i$ specifies the questions the agent is aware of and the theorems he knows in state space Ω_2^i. Accordingly, the third order state space of agent i is

$$\Omega_3^i = \Omega_2^i \times_{j \neq i} T_2^j.$$

Continuing inductively, we define for all $k \geq 1$,

$$\Omega_{k+1}^i = \Omega_k^i \times_{j \neq i} T_k^j,$$

$$T_{k+1}^i = \times_{q \in \mathcal{E}^i(\Omega_{k+1}^i) \setminus \mathcal{E}^i(\Omega_k^i)} A_q.$$

Define T^i to be the Cartesian product $\bigtimes_{n=1}^{\infty} T_n^i$. An element in T^i contains an answer for all epistemic questions about agent i. In particular, it gives an answer to only questions of the type $a^i q$, or of the type $m^i V$, where q can be either a basic question or an epistemic question about another agent (e.g. $q = a^j a^k a^i q'$), while V can contain both basic and epistemic questions for all other agents. Note that questions of the type $a^i a^i \dots a^i q$ are not created. Summarizing, an element in T^i describes agent i's awareness of questions and knowledge of theorems for each successively bigger state space Ω_k, where $k \geq 1$.

Interpreting T^i as the set of all types for agent i, we can define the full state space Ω^* to specify a state of nature $s \in S$, together with a type for each player $i \in I$:

$$\Omega^* \subseteq S \times \bigtimes_{i \in I} T^i.$$

The set of all questions that generate the full state space Ω^* is denoted by Q. Formally, $\mathcal{V}_{\Omega^*} = Q$ and $\Omega^* \subseteq \bigtimes_{q \in Q} A_q$.

4.3 Common knowledge

An event E is common knowledge if everyone knows it, everyone knows that everyone knows it and so on, ad infinitum. The extra complication that arises when defining common knowledge in a setting with unawareness is that every agent has a possibly different state space. More importantly, agents have to reason about other agents' awareness, before they reason about their knowledge. For example, before i claims that he knows that j knows that k knows event E, he first has to reason which questions he knows that j is aware of.

To give a simple example, suppose that full state ω^* occurs and agent i has awareness $W^i(\omega^*)$ and state space $\Omega^i(\omega^*)$. Agent i's reasoning about whether agent j knows event E is represented by event $K^j_{W^i(\omega^*)}(E)$, which is a subset of his own state space $\Omega^i(\omega^*)$. Similarly, when i is reasoning about j's reasoning about whether agent k knows E, he first has to specify what is his view of j's awareness. If we denote this view by set $V^{ij}(\omega^*)$, then i's view of j's view of k's knowledge of E would be the event $K^k_{V^{ij}(\omega^*)}(E)$, which is a subset of $\Pi^Q_{V^{ij}(\omega^*)}(\Omega^*)$, i's view of j's state space.

What remains to be determined is the set $V^{ij}(\omega^*)$, i's view of j's awareness at ω^*. Firstly, any state $\omega \in \Omega^i(\omega^*)$ specifies exactly what is i's view of j's awareness at that state. But i does not necessarily know what state has occurred, he only knows that one state in $P^i(\omega^*)$ has occurred. In other words, it may be that for some questions that i is aware of, he does not know whether j is aware of them as well. The set $V^{ij}(\omega^*)$ denotes the questions that i knows that j is aware of, at ω^*:

$$V^{ij}(\omega^*) = \bigcap_{\omega \in P^i(\omega^*)} W^j(\omega).$$

More formally, $V^{ij}(\omega^*)$ is the set of questions that i and j are aware of in all the states that i considers possible, when full state ω^* has occurred. Note that this set depends only on full state ω^* which determines i's awareness and knowledge, and not on any state of i's subjective state space.

If we set $V^i(\omega^*) = W^i(\omega^*)$, then i's view of the event "j knows that k knows E", is the set $K^j_{V^i(\omega^*)} K^k_{V^{ij}(\omega^*)}(E)$. The analyst's view that i knows that event is the set

$$K^i_Q K^j_{V^i(\omega^*)} K^k_{V^{ij}(\omega^*)}(E).$$

Note that $K^i_Q K^j_{V^i(\omega^*)} K^k_{V^{ij}(\omega^*)}(E)$ is a subset of the full state space Ω^*. This event depends on full state ω^*, which determines i's awareness, and therefore i's view of j's awareness.

Continuing recursively, we can define the set of questions that i knows that j knows that k is aware of, in order to construct events of higher order reasoning of i knowing that j knows that k knows that l knows event E. To facilitate such an extension, we need to define, for any agent $i \in I$ and any event E, the set $P^i(E)$, which denotes the set of states that agent i considers possible when the true state lies somewhere in E.

Definition 1. *For any event E, let $\mathcal{V}^i_E = \bigcap_{\omega \in E} W^i(\omega)$.*
If $\mathcal{V}^i_E \neq \emptyset$, then $P^i(E) = \bigcup_{\omega \in E} \prod^{W^i(\omega)}_{\mathcal{V}^i_E}(P^i(\omega))$. Otherwise, $P^i(E) = \emptyset_{\mathcal{V}_E}$.

The definition is analogous to the definition of $P^i(E)$ in the standard model. The extra complication is that different states in E may describe different awareness for agent i, and therefore the sets $P^i(\omega)$ and $P^i(\omega')$ for $\omega, \omega' \in E$ may be subsets of different state spaces. If there is a nonempty set of questions $\mathcal{V}^i_E$ that i is aware of for any $\omega \in E$, then we project $P^i(\omega)$ and $P^i(\omega')$ to that set of questions. If $\mathcal{V}^i_E$ is empty, then we define $P^i(E)$ to be the empty set. The set of questions that at ω^*, i knows that j knows that k is aware of is

$$V^{ijk}(\omega^*) = \bigcap_{\omega \in P^j(P^i(\omega^*))} W^k(\omega).$$

The event "at ω^*, i knows that j knows that k knows that l knows event E" is

$$K^i_Q K^j_{V^i(\omega^*)} K^k_{V^{ij}(\omega^*)} K^l_{V^{ijk}(\omega^*)}(E).$$

Adding more agents to the sequence can easily be accommodated. For any $k \geq 2$, define

$$V^{i_1 i_2 \ldots i_k}(\omega^*) = \bigcap_{\omega \in P^{i_{k-1}}(\ldots(P^{i_2}(P^{i_1}(\omega^*))))} W^{i_k}(\omega)$$

to be the questions that i_1 knows that i_2 knows that ...that i_k is aware of at ω^*. The definition of common knowledge is analogous to that of the standard model, except for the subscripts on each knowledge operator.

Definition 2. *Event E is common knowledge among agents $i = 1, \ldots, I$ at ω^* if and only if for any $n \in \mathbb{N}$ and any sequence of agents $i_1, \ldots, i_n$,*
$\omega^* \in K^{i_1}_Q(K^{i_2}_{V^{i_1}(\omega^*)}(K^{i_3}_{V^{i_1 i_2}(\omega^*)} \ldots (K^{i_n}_{V^{i_1 i_2 \cdots i_{n-1}}(\omega^*)}(E))))$. [19]

[19] $K^{i_1}_Q(K^{i_2}_{V^{i_1}(\omega^*)}(K^{i_3}_{V^{i_1 i_2}(\omega^*)} \ldots (K^{i_n}_{V^{i_1 i_2 \cdots i_{n-1}}(\omega^*)}(E))))$ is well defined if $V^{i_1 \cdots i_n}(\omega^*) \cup \alpha^n(V^{i_1 \cdots i_n}(\omega^*)) \subseteq V^{i_1 \cdots i_{n-1}}(\omega^*)$ for any $n \geq 1$. This is true because $\alpha^{i_n}(V^{i_1 \cdots i_n}(\omega^*)) \subseteq V^{i_1 \cdots i_n}(\omega^*)$ for any $n \geq 2$.

Just as in the standard model, there is an equivalent definition of common knowledge, which employs the possibility correspondences P^i, instead of the knowledge operators K^i.

Proposition 1. *Event E is common knowledge among agents $i = 1, \dots, I$ at ω^* if and only if for any $n \in \mathbb{N}$ and any sequence of agents $i_1, \dots, i_n$, $\mathcal{V}_E \subseteq V^{i_1 \dots i_n}(\omega^*)$ and*

$$\emptyset \neq P^{i_n}(\dots(P^{i_2}(P^{i_1}(\omega^*)))) \subseteq E_{V^{i_1 \dots i_n}(\omega^*)}.$$

4.4 Common knowledge of awareness

Define $V^*(\omega^*)$ to be the intersection of all sets $V^{i_1 \dots i_n}(\omega^*)$, for any sequence $i_1, i_2, \dots, i_n$, $n \in \mathbb{N}$:

$$V^*(\omega^*) = \bigcap_{\substack{i_1 \dots i_n \\ n \in \mathbb{N}}} V^{i_1 \dots i_n}(\omega^*).$$

As the next Proposition shows, this is the largest set of questions that is commonly known at ω^* that everyone is aware of. In particular, for a set of questions $V \subseteq Q$, define the event "everyone is aware of all questions in V":

$$EA(V) = \{\omega \in \Omega^*_V : \forall i \in I, V \subseteq W^i(\omega)\}.$$

Note that this event is a subset of state space Ω^*_V. The following proposition states that if $V^*(\omega^*)$ is nonempty, then the event "everyone is aware of $V^*(\omega^*)$" is common knowledge at ω^* . Moreover, $V^*(\omega^*)$ is the largest set of questions that it is commonly known at ω^* that everyone is aware of.

Proposition 2. *Suppose that $V^*(\omega^*) \neq \emptyset$. Then, the event "everyone is aware of $V^*(\omega^*)$", $EA(V^*(\omega^*))$, is common knowledge at ω^*. Moreover, if $EA(V)$ is common knowledge at ω^*, then $V \subseteq V^*(\omega^*)$.*

Note that if an event E is common knowledge at ω^*, then by Proposition 1 $\mathcal{V}_E \subseteq V^*(\omega^*)$, which implies that $V^*(\omega^*)$ is nonempty.

4.5 Characterizing common knowledge

In the standard model, an event E^* is common knowledge at ω^* if and only if there is an event E which is self evident for all agents, it contains ω^* and is a subset of E^*. The following two theorems provide a characterization of common knowledge along these lines. The definition of a self evident event is given below, and it is a direct analog of the standard definition.

Definition 3. *Event E is self evident for $i \in I$ if $E \subseteq K^i_{\mathcal{V}_E}(E)$.*

An event E is self evident for agent $i \in I$ if whenever it happens, the agent knows it. If an event E is self evident for all $i \in I$, then it is called a public event.

The first theorem gives sufficient conditions for event E^* to be common knowledge. In particular, suppose that there is a public event E, whose enlargement to the set of questions Q contains full state ω^*, it is a subset of the enlargement of event E^* to Q and the awareness of E^* is contained in that of E (that is, $\mathcal{V}_{E^*} \subseteq \mathcal{V}_E$). Then E^* is common knowledge at ω^*.

Theorem 3. *Suppose Axiom 2 holds. Moreover, E is a public event and for some event E^* such that $\mathcal{V}_{E^*} \subseteq \mathcal{V}_E$, $\omega^* \in E_Q \subseteq E^*_Q$. Then, E^* is common knowledge at ω^*.*

The second theorem gives necessary conditions for an event E^* to be common knowledge. There is one added assumption, namely that there exists a finite sequence $j_1 \ldots j_k, k \in \mathbb{N}$, of agents, for which the set of questions that j_1 knows that j_2 knows that ...that j_{k-1} knows that j_k is aware of, which is $V^{j_1 \cdots j_k}(\omega^*)$, is equal to the largest set of questions that is commonly known at ω^* that everyone is aware of, $V^*(\omega^*)$.

Theorem 4 shows that if this assumption, together with Axiom 3 (nondelusion) hold and event E^* is common knowledge at ω^*, then there is a public event E whose enlargement to the set of questions Q contains ω^* and it is a subset of the enlargement of E^* to Q.

Theorem 4. *Suppose Axiom 3 holds. Moreover, E^* is common knowledge at ω^*, and for some sequence $j_1 \ldots j_k$ of agents, $V^{j_1 \cdots j_k}(\omega^*) = V^*(\omega^*)$. Then, there exists a public event E such that $\mathcal{V}_{E^*} \subseteq \mathcal{V}_E$ and $\omega^* \in E_Q \subseteq E^*_Q$.*

5 No trade theorems

The literature on no trade theorems stems from the well known result by Aumann [2] that if a group of agents has a common prior and their posteriors about an event E is common knowledge, then these posteriors are identical. Dealing with the same problem in a model with unawareness gives rise to the following complications. The first is that agents typically have different subjective state spaces, since they are unaware of questions that others are aware of. It is not therefore clear on what state space this "common" prior should be applied. Secondly, the multiplicity of state spaces also implies that each state space defines a different posterior for agent i about event E. A result which has a similar flavor to Property 1 is that state spaces generated by more awareness will give a more complete specification of an agent's posterior.

This section presents a result analogous to that of Aumann [2], where common knowledge of the posteriors implies they are identical. Nevertheless, as was shown in the previous section, the largest state space whose subset can be common knowledge, is the state space generated by the questions that is commonly known that everyone is aware of. Consequently, the result can only refer to posteriors defined on that state space. If an agent's awareness is bigger than what is commonly known that everyone is aware of, then his actual posterior may be different from the common one and there may be incentive to trade.

Recall that in the standard model without unawareness and given a prior μ on the unique state space Ω, we can define i's posterior of event E at $\omega \in \Omega$ using Bayes' law:

$$q^i(\omega) = \frac{\mu(P^i(\omega) \cap E)}{\mu(P^i(\omega))}. \tag{3}$$

Every state in Ω specifies a posterior of i about event E, so that it is meaningful to talk about the posteriors being common knowledge, as in Aumann's theorem.

In a model with unawareness, a posterior can be defined for each state of each state space. Take an arbitrary set of questions V and consider a prior μ on the state space Ω^*_V. Suppose that at $\omega \in \Omega^*_V$, the agent's awareness is $W^i(\omega) = V$. Then, for any event $E \subseteq \Omega^*_V$, define i's posterior of E at ω to be given by Equation (3). Note that we are defining a posterior only in the special case where the agent's awareness at $\omega \in \Omega^*_V$ contains all questions in V. For cases where ω specifies that the agent's awareness is a strict subset of V, a more complicated definition is needed. Since such a case is not considered in the theorem below, we omit the details.

Let $I = \{i, j\}$ and recall from the previous section that $V^*(\omega^*)$ is the largest set of questions that it is commonly known at ω^* that both agents are aware of. Assume that $V^*(\omega^*)$ is nonempty and define a prior μ on the state space $\Omega = \Pi^Q_{V^*(\omega^*)}(\Omega^*)$, generated by these questions. Take a state $\omega \in \Omega$ where both agents' awareness is $V^*(\omega^*)$ and define their posteriors about event $E \subseteq \Omega$ at ω to be $q^i(\omega) \equiv q^i$ and $q^j(\omega) \equiv q^j$, respectively. Consider the event

$$E^* = \left\{\omega \in \Omega : q^i(\omega) = q^i, q^j(\omega) = q^j\right\}.$$

Event E^* contains all the states in Ω where both agents are aware of $V^*(\omega^*)$ and i's posterior about E is q^i, while j's posterior about E is q^j. The following theorem states that if E^* is common knowledge, then $q^i = q^j$.

Theorem 5. *Suppose that Axiom 3 holds. Moreover, E^* is common knowledge at ω^*, so that it is commonly known that i's posterior is q^i and j's posterior is q^j. If for some sequence of agents $j_1, \ldots, j_k$, $V^{j_1 \cdots j_k}(\omega^*) = V^*(\omega^*)$, then $q^i = q^j$.*

Although Theorem 5 specifies that common knowledge of the posteriors implies they are identical, it should be stressed that these posteriors are defined on the state space generated by the questions that is commonly known that everyone is aware of. If the state spaces of i and j are generated by more questions, then their posteriors may be different and there may be incentive to trade. This is illustrated in the following example.

Example

Let $I = \{i, j\}$ and suppose there are two basic questions, p: "Will the prices be high tomorrow?" and r: "Are the interest rates low today?", each with two possible answers, "yes" and "no". If we set $V = \{p, r\}$, then $m^i V$ is asking what theorems in $\times_{q \in V} A_q$ i knows. Define $\omega_1 \in \times_{q \in V} A_q$ to be the state "low interest rates today and low prices tomorrow", so that if ω_1 is the answer to $m^i V$, then i knows that low interests rates today imply high prices tomorrow.

At $\omega^* \in \Omega^*$ agent i is aware of both p and r, together with the epistemic questions $a^i r$, $m^i V$, $a^i m^i V$ and all the questions of the type $a^{i_1} a^{i_2} \ldots a^{i_n} p$, where $n \in \mathbb{N}$ and $i_k \neq i_{k+1}$ for all $k \leq n$. Examples of these types of questions are $a^i a^j a^i a^j p$ and $a^j a^i a^j a^i p$. Denote this set of epistemic questions of the two agents about p by V_1:

$$V_1 = \{a^{i_1} a^{i_2} \ldots a^{i_n} p : n \in \mathbb{N}, i_k \neq i_{k+1} \text{ for all } k \leq n\}.$$

Agent j's awareness at ω^* contains p and all the questions in V_1. In other words, j is unaware of question r and he therefore cannot express any theorem between interest rates and prices, nor can he reason that i has formulated such a theorem. Summarizing,

$$W^i(\omega^*) = \{p, r, a^i r, m^i V, a^i m^i V, V_1\},$$

$$W^j(\omega^*) = \{p, V_1\}.$$

Agent i's set of immediate perception X^i contains questions r, $a^i r$, $m^i V$, $a^i m^i V$ and all questions in V_1. Note that Axiom 1 only requires that the epistemic questions about agent i should be included in X^i. Adding the epistemic questions of j about p greatly simplifies the example, but in general is not required. Agent i cannot immediately "see" whether the prices will be high tomorrow, but he can infer it at ω^* with his theorem that connects interest rates today with the prices tomorrow, since r is in his immediate perception X^i. On the other hand, agent j's immediate perception X^j contains just set V_1, so it is impossible for him to infer whether the prices will be high or low tomorrow. Moreover, he is unaware of any possibility that i will know.

Suppose that ω^* answers ω_1 to question $m^i V$, so that i knows that low interest rates imply high prices tomorrow. Moreover, assume that ω^* specifies that the answer to both r and p is "yes", so that there are indeed low interest rates and high prices. Combining these elements we can infer that at ω^*, agent i considers possible only state $\omega_2 \in \Omega^i(\omega^*)$, which specifies "yes" to questions p, r, $a^i r$, $a^i m^i V$ and all questions in V_1, while it answers ω_1 to question $m^i V$. On the other hand, agent j knows that the answer to all questions in V_1 is "yes", but he does not know whether the prices tomorrow will be high or low. That is, he considers as possible two states, ω_3, $\omega_4 \in \Omega^j(\omega^*)$, where ω_3 specifies that the prices are high. More formally, $P^i(\omega^*) = \{\omega_2\}$ and $P^j(\omega^*) = \{\omega_3, \omega_4\}$. Note that i has a different state space than j and that the projection of ω_2 to j's awareness is ω_3. Moreover, since j's awareness does not include the theorem about r, his view of i's knowledge is identical to his own knowledge. That is, $P^j(\omega_3) = P^i(\omega_3) = P^j(\omega_4) = P^i(\omega_4) = \{\omega_3, \omega_4\}$. In fact, the event $E^* = \{\omega_3, \omega_4\}$ is common knowledge at ω^*.

Although we omit the details here, it is straightforward to show that $V^*(\omega^*)$, the set of questions that is commonly known at ω^* that both agents are aware of, is equal to j's awareness $W^j(\omega^*)$. Suppose that we define a common prior μ on the state space $\Omega^*_{V^*(\omega^*)}$, such that $\mu(\omega_3) = \mu(\omega_4)$. Define event $E = \{\omega_3\}$ to specify that the prices will be high. Agent i's posteriors about E at ω_3 and at ω_4 are equal, and the same holds for agent j:

$$q^i(\omega_3) = q^i(\omega_4) \equiv q^i,$$

$$q^j(\omega_3) = q^j(\omega_4) \equiv q^j.$$

Therefore, the event $E^* = \{\omega_3, \omega_4\}$ specifies that i's posterior about E is q^i, while j's posterior is q^j. Since E^* is common knowledge at ω^*, Theorem 5 implies that $q^i = q^j$ and in fact they are both equal to one half. A high price tomorrow is equally likely with a low price.

For agent j this is the end of the story, since he is unaware of questions outside of $V^*(\omega^*)$ and cannot reason beyond q^j and q^i, which represent for him the most "accurate" posteriors on E. On the other hand, agent i is more aware. His state space is generated from questions $W^i(\omega^*)$, and the enlargement of E to that state space is $E_{W^i(\omega^*)}$. At ω_2, his posterior on $E_{W^i(\omega^*)}$ is 1, since he is certain that ω_2 has occurred.[20] Agent i is able to reason that prices will be high tomorrow because he is aware of question r, he knows that interest rates are low today and he knows the theorem that connects them with high prices.

Discussion

Theorem 5 shows that whenever the posteriors defined on the common state space of the two agents are common knowledge, they are identical. On the other hand, the example showed that if i is aware of more questions, his "true" posterior may be different and beyond the other agent's reasoning. Hence, there may be incentive for trade.

Intuition for this result can be obtained if we interpret the equality of the posteriors as the outcome of the following procedure, described in the context of the standard model by Geanakoplos and Polemarchakis [8]. Suppose that initially i and j have different posteriors about E, and in particular i has a posterior above a half and wants to buy, while j has a posterior below a half, and wants to sell. Suppose that they meet and they announce their posteriors and their willingness to trade. Then, i can use j's announcement in order to further refine what he knows, by taking the intersection of his own information with the set of states that describe a posterior below a half for j. Agent i can now announce a possibly different posterior which reflects his new information, which now can be used by j to refine his. Geanakoplos and Polemarchakis [8] shows that if the partitions are finite, then the agents will agree on the posterior, after finitely many steps.

A necessary condition for this result is that each agent knows the other agent's partition, which is certainly true in the standard model. This is also true in this model but only for the state space which is generated from the questions that it is commonly known that everyone is aware of. Therefore, the updating of the posteriors that was described above, can only refer to this common state space. If agent i is aware of more questions, then announcing his "true" posterior will be of no value to the unaware agent j, because he is simply unaware of the states that would enable i to announce such a posterior. As a result, j cannot further refine his own knowledge. Concluding, it is possible that two agents with different awareness will engage in trade, simply because the revealment of new information by the different posteriors is constrained by the agents' common awareness.

[20] A technical detail is that we have not defined a prior μ' on i's state space at ω^*, generated by questions $W^i(\omega^*)$. As long as this prior assigns a positive probability to ω_2, his posterior will be 1, so we omit the details. Moreover, there is nothing special with a posterior of 1. An alternative example would produce a posterior higher than 1/2 but lower than 1.

A Appendix

Proof of Theorem 1.

1(a). The proof is immediate from the definition of $P(\omega)$.

1(b). For footnote 10 we have that $\omega \notin M \implies \omega \notin (\Pi^V_{W(\omega)})^{-1}(M(\omega)) \implies \Pi^V_{W(\omega)}(\omega) \notin M(\omega) \implies \Pi^V_{W(\omega)}(\omega) \in P(\omega)$.

2. First, we prove the following proposition.

Proposition 3. $\omega \in P(\omega_1)$ *implies*

i) $W(\omega_1) = W(\omega)$.

ii) $M(\omega_1) = M(\omega)$.

Proof.

i) Suppose $q \in W(\omega_1)$. There are two cases. Either $q \neq aq'$ for any $q' \in Q$, or $q = aq'$ for some $q' \in Q$. In the first case, we have that $\omega_{1aq} =$ "yes" and $aq \in W(\omega_1)$. In the second case, $\omega_{1aq'} =$ "yes" and $aq' \in W(\omega_1)$. The proof is identical in both cases, so we just illustrate the first case. From Axiom 1, $aq \in X \cap W(\omega_1)$. Since $\omega \in P(\omega_1)$, we have $\omega_{aq} =$ "yes", which, together with $\{q, aq\} \subseteq W(\omega_1) = \mathcal{V}_\omega$ implies $\{q, aq\} \subseteq W(\omega)$. The other direction is immediate since $\mathcal{V}_\omega = W(\omega_1)$.

ii) Suppose $\omega_2 \in M(\omega_1)$. Then, there exist $\{mV\}$, V such that $\{mV\} \cup V \subseteq W(\omega_1)$ and $\Pi^{W(\omega_1)}_V(\omega_2) \in \omega_{1mV}$. From *i)* we have $W(\omega_1) = W(\omega)$, which implies $\{mV\} \cup V \subseteq W(\omega)$. Moreover, from Axiom 1 we have that $mV \in X \cap W(\omega_1)$. Thus, $\omega \in P(\omega_1)$ implies $\omega_{mV} = \omega_{1mV}$ and therefore $\omega_2 \in M(\omega)$. The other direction is identical.

□

Sets $P(\omega_1)$ and $P(\omega)$ are repeated below:

$$P(\omega_1) = \{\omega_2 \in \Omega(\omega_1) : \omega_{2q} = \omega_{1q}, q \in W(\omega_1) \cap X\} \setminus M(\omega_1),$$

$$P(\omega) = \{\omega_2 \in \Omega(\omega) : \omega_{2q} = \omega_q, q \in W(\omega) \cap X\} \setminus M(\omega).$$

From Proposition 3 we have $W(\omega_1) = W(\omega)$ and $M(\omega_1) = M(\omega)$. Since $\omega \in P(\omega_1)$ implies that $\omega_q = \omega_{1q}$ for all $q \in W(\omega_1) \cap X = W(\omega) \cap X$, we have that $P(\omega_1) = P(\omega)$.

□

Proof of Property 1.

First we prove that if $V_2 \subseteq V_1$, then $(K_{V_2}(E))_{V_1} \subseteq K_{V_1}(E)$. Suppose $\omega \in (K_{V_2}(E))_{V_1}$. Then, $\{\omega\}_{V_2} \in K_{V_2}(E)$, which implies that $\emptyset \neq P(\{\omega\}_{V_2}) \subseteq E_{W(\{\omega\}_{V_2})}$ and $\mathcal{V}_E \subseteq W(\{\omega\}_{V_2})$. We have to show that $\mathcal{V}_E \subseteq W(\omega)$ and $\emptyset \neq P(\omega) \subseteq E_{W(\omega)}$. Firstly, since $V_2 \subseteq V_1$ we also have $W(\{\omega\}_{V_2}) \subseteq W(\omega)$. Therefore, $\mathcal{V}_E \subseteq W(\omega)$. Non emptiness of $P(\omega)$ is guaranteed by Axiom 2.

Secondly, we show that $(P(\omega))_{W(\{\omega\}_{V_2})} \subseteq P(\{\omega\}_{V_2})$. Suppose that $\omega' \in (P(\omega))_{W(\{\omega\}_{V_2})}$. Then, there exists $\omega_1 \in P(\omega)$ such that $\{\omega_1\}_{W(\{\omega\}_{V_2})} = \omega'$. Moreover, $\omega_1 \in P(\omega)$ implies that $\omega_{1q} = \omega_q$ for all $q \in W(\omega) \cap X$, hence $\omega'_q = \omega_q$ for all $q \in W(\{\omega\}_{V_2}) \cap X$. Next, we need to show that $\omega' \notin M(\{\omega\}_{V_2})$. Suppose that $\omega' \in M(\{\omega\}_{V_2})$. Then, there exist V and mV such that $V \cup \{mV\} \subseteq W(\{\omega\}_{V_2})$ and $\{\omega'\}_V \in \omega_{mV}$. Since $\{\omega_1\}_{W(\{\omega\}_{V_2})} = \omega'$ and $V \cup \{mV\} \subseteq W(\{\omega\}_{V_2}) \subseteq W(\omega)$, we have that $\{\omega_1\}_V \in \omega_{mV}$, which implies that $\omega_1 \in M(\omega)$ and $\omega_1 \notin P(\omega)$, a contradiction.

We have shown that $(P(\omega))_{W(\{\omega\}_{V_2})} \subseteq P(\{\omega\}_{V_2}) \subseteq E_{W(\{\omega\}_{V_2})}$, and $\mathcal{V}_E \subseteq W(\{\omega\}_{V_2}) \subseteq W(\omega)$. Therefore, $P(\omega) \subseteq E_{W(\omega)}$, which implies that $(K_{V_2}(E))_{V_1} \subseteq K_{V_1}(E)$. Finally, since $V_2 \subseteq V_1$, we also have that $K_{V_2}(E) \subseteq (K_{V_1}(E))_{V_2}$.

□

Proof of Theorem 2.

1. **Symmetry** Follows from $\mathcal{V}_E = \mathcal{V}_{\neg E}$.
3. **Strong Plausibility** By assumption, $\mathcal{V}_E \subseteq V = \mathcal{V}_{\neg K_V(E)} = \mathcal{V}_{\neg K_V(\neg K_V(E))} = \mathcal{V}_{\neg K_V(\neg K_V(\ldots \neg K_V(E)))}$. Suppose $\omega \in U_V(E)$. Then, $\mathcal{V}_E \not\subseteq W(\omega)$ and therefore $V \not\subseteq W(\omega)$. Hence, $\omega \in \neg K_V(E) \cap \neg K_V(\neg K_V(E)) \cap \ldots \cap \neg K_V(\neg K_V(\ldots \neg K_V(E)))$.
4. **AU Introspection** Suppose $\omega \in U_V(E)$, Then, $\mathcal{V}_E \not\subseteq W(\omega)$ and since $\mathcal{V}_E \subseteq V = \mathcal{V}_{U_V(E)}$, we have $\mathcal{V}_{U_V(E)} \not\subseteq W(\omega)$, which implies $\omega \in U_V(U_V(E))$.
5. **Subjective Necessitation** First, note that $K_V(\Omega(\omega))$ is well defined becasue $W(\omega) \cup \alpha(W(\omega)) \subseteq V$. Subjective necessitation then follows from $\mathcal{V}_{\Omega(\omega)} = W(\omega)$ and $\emptyset \neq P(\omega) \subseteq \Omega(\omega)$.
6. **Generalized Monotonicity** Suppose $\omega \in K_V(E)$. Then, $\mathcal{V}_E \subseteq W(\omega)$ and $\emptyset \neq P(\omega) \subseteq E_{W(\omega)}$. Also, $\mathcal{V}_F \subseteq W(\omega)$ which implies that $E_{W(\omega)} \subseteq F_{W(\omega)}$. Therefore, $\omega \in K_V(F)$.
7. **Conjunction** We have that $\mathcal{V}_E \subseteq W(\omega)$ and $\mathcal{V}_F \subseteq W(\omega)$ if and only if $\mathcal{V}_E \cup \mathcal{V}_F \subseteq W(\omega)$. Also, $\emptyset \neq P(\omega) \subseteq E_{W(\omega)}$ and $\emptyset \neq P(\omega) \subseteq F_{W(\omega)}$ if and only if $\emptyset \neq P(\omega) \subseteq E_{W(\omega)} \cap F_{W(\omega)} = (E_{\mathcal{V}_E \cup \mathcal{V}_F} \cap F_{\mathcal{V}_E \cup \mathcal{V}_F})_{W(\omega)}$. The latter equality follows because $\omega_1 \in (E_{\mathcal{V}_E \cup \mathcal{V}_F} \cap F_{\mathcal{V}_E \cup \mathcal{V}_F})_{W(\omega)} \iff \{\omega_1\}_{\mathcal{V}_E \cup \mathcal{V}_F} \in E_{\mathcal{V}_E \cup \mathcal{V}_F} \cap F_{\mathcal{V}_E \cup \mathcal{V}_F} \iff \omega_1 \in E_{W(\omega)} \cap F_{W(\omega)}$.
8. **AA-Self Reflection** $\omega \in A_V(E)$ implies $W(\omega) \cup \alpha(W(\omega)) \subseteq V$ and $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \subseteq W(\omega)$. Therefore, $A_V(A_{W(\omega)}(E))$ is well defined and $\omega \in A_V(A_{W(\omega)}(E))$. For the other direction, suppose that $\omega \in A_V(A_{W(\omega)}(E))$. Since $A_{W(\omega)}(E)$ is defined only if $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \subseteq W(\omega)$, we have that $\omega \in A_V(E)$.

9. **AK-Self Reflection** The proof is similar.

10. **The Axiom of Knowledge** $\omega \in K_V(E)$ implies $\mathcal{V}_E \subseteq W(\omega)$ and $\emptyset \neq P(\omega) \subseteq E_{W(\omega)}$. Axiom 3 implies $\{\omega\}_{W(\omega)} \in P(\omega)$. Hence, $\{\omega\}_{W(\omega)} \in E_{W(\omega)}$, which implies $\omega \in E_V$.

11. **KU Introspection** Suppose $\omega \in K_V(U_V(E))$. Then, $W(\omega) = V$ and there exists $\omega_1 \in P(\omega) \subseteq U_V(E)$. From Proposition 3 we have that $W(\omega_1) = W(\omega) = V$. Moreover, the definition of $U_V(E)$ implies that $\mathcal{V}_E \subseteq V$. Therefore, $\mathcal{V}_E \subseteq W(\omega_1)$. But $\omega_1 \in U_V(E)$ implies that $\mathcal{V}_E \nsubseteq W(\omega_1)$, a contradiction.

    The proof of footnote 16 is identical. Suppose $\omega \in K_V(U_W(\omega)(E))$. Then, there exists $\omega_1 \in P(\omega) \subseteq U_{W(\omega)}(E)$. From Proposition 3, we have that $W(\omega_1) = W(\omega)$. Moreover, the definition of $U_{W(\omega)}(E)$ implies that $\mathcal{V}_E \subseteq W(\omega)$. Therefore, $\mathcal{V}_E \subseteq W(\omega_1)$. But $\omega_1 \in U_{W(\omega)}(E)$ implies that $\mathcal{V}_E \nsubseteq W(\omega_1)$, a contradiction.

12. **A-Introspection** Suppose $\omega \in A_V(E)$. Then, $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \subseteq W(\omega) \subseteq V$ and $W(\omega) = \mathcal{V}_{A_{W(\omega)}(E)}$, so we just have to show that $\emptyset \neq P(\omega) \subseteq A_{W(\omega)}(E)$. That $P(\omega)$ is non empty follows from Axiom 2. Suppose that $\omega_1 \in P(\omega)$. From Proposition 3, we have $W(\omega) = W(\omega_1)$ which implies $\mathcal{V}_E \subseteq W(\omega_1)$ and $\omega_1 \in A_{W(\omega)}(E)$. For the other direction, suppose that $\omega \in K_V(A_{W(\omega)}(E))$. This implies that $\omega \in A_V(A_{W(\omega)}(E))$ and $\omega \in A_V(E)$ follows from AA-Self Reflection.

13. **The Axiom of Transparency** Suppose $\omega \in K_V(E)$. Then, $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \subseteq W(\omega)$ and $\emptyset \neq P(\omega) \subseteq E_{W(\omega)}$. We have to show that $\emptyset \neq P(\omega) \subseteq K_{W(\omega)}(E)$, or that $\omega_1 \in P(\omega)$ implies $\mathcal{V}_E \subseteq W(\omega_1)$ and $\emptyset \neq P(\omega_1) \subseteq E_{W(\omega_1)}$. From Proposition 3, we have that $\omega_1 \in P(\omega)$ implies $W(\omega_1) = W(\omega)$. Hence, $\mathcal{V}_E \subseteq W(\omega_1) = W(\omega)$. From Theorem 1 we have that $\omega_1 \in P(\omega)$ implies $P(\omega) = P(\omega_1)$. Thus, $\emptyset \neq P(\omega_1) \subseteq E_{W(\omega)} = E_{W(\omega_1)}$.

    Suppose $\omega \in K_V K_{W(\omega)}(E)$, which implies that $\emptyset \neq P(\omega) \subseteq K_{W(\omega)}(E)$. Hence, for all $\omega_1 \in P(\omega)$, we have that $\omega_1 \in K_{W(\omega)}(E)$, $W(\omega) = W(\omega_1)$, $P(\omega) = P(\omega_1)$ and $\emptyset \neq P(\omega_1) \subseteq E_{W(\omega)}$. Therefore, $\emptyset \neq P(\omega) \subseteq E_{W(\omega)}$ and $\omega \in K_V(E)$.

14. **The Axiom of Wisdom** Suppose $\omega \in A_V(E) \cap \neg K_V(E)$. Then, $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \subseteq W(\omega)$ and either $P(\omega) = \emptyset$ or $\emptyset \neq P(\omega) \nsubseteq E_{W(\omega)}$. Axiom 2 implies that $P(\omega) \neq \emptyset$, so we just have to show that $P(\omega) \subseteq A_{W(\omega)}(E) \cap \neg K_{W(\omega)}(E)$. Suppose $\omega_1 \in P(\omega)$. Proposition 3 implies that $W(\omega_1) = W(\omega)$. Hence, $\mathcal{V}_E \subseteq W(\omega_1)$ and $\omega_1 \in A_{W(\omega)}(E)$. Theorem 1 implies that $P(\omega) = P(\omega_1)$. Thus $P(\omega_1) \nsubseteq E_{W(\omega)} = E_{W(\omega_1)}$ and $\omega_1 \in \neg K_{W(\omega)}(E)$.

    Suppose $\omega \in K_V(A_{W(\omega)} \cap \neg K_{W(\omega)}(E))$. Then, $\emptyset \neq P(\omega) \subseteq A_{W(\omega)} \cap \neg K_{W(\omega)}(E)$. Since $A_{W(\omega)}(E)$ is defined only if $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \subseteq W(\omega)$, we have that $\omega \in A_V(E)$. It remains to show that $\omega \in \neg K_V(E)$, or that $P(\omega) \nsubseteq E_{W(\omega)}$. We know that for all $\omega_1 \in P(\omega)$, $\omega_1 \in \neg K_{W(\omega)}(E)$, which implies that $P(\omega_1) \nsubseteq E_{W(\omega)}$. Since $P(\omega) = P(\omega_1)$, we have that $P(\omega) \nsubseteq E_{W(\omega)}$.

$\square$

The proof of Proposition 1 is an immediate consequence of the following Proposition.

Proposition 4. *For any sequence $i_1 \dots i_n$, the following two statements are equivalent:*

- $\mathcal{V}_E \subseteq V^{i_1 \dots i_n}(\omega^*)$ *and* $\emptyset \neq P^{i_n}(\dots(P^{i_2}(P^{i_1}(\omega^*)))) \subseteq E_{V^{i_1 \cdots i_n}(\omega^*)}$,
- $\omega^* \in K_Q^{i_1}(K_{V^{i_1}(\omega^*)}^{i_2}(\dots K_{V^{i_1 \cdots i_{n-1}}(\omega^*)}^{i_n}(E)))$.

Proof. First, we show that $\mathcal{V}_E \subseteq V^{i_1 \dots i_n}(\omega^*)$ implies $\mathcal{V}_E \cup \alpha^{i_n}(\mathcal{V}_E) \subseteq V^{i_1 \dots i_{n-1}}(\omega^*)$, so that $K_Q^{i_1}(K_{V^{i_1}(\omega^*)}^{i_2}(\dots K_{V^{i_1 \cdots i_{n-1}}(\omega^*)}^{i_n}(E)))$ is well defined. Since $V^{i_1 \dots i_n}(\omega^*) \subseteq V^{i_1 \dots i_{n-1}}(\omega^*)$, we have $\mathcal{V}_E \subseteq V^{i_1 \dots i_{n-1}}(\omega^*)$. Take $\alpha^{i_n} q \in \alpha^{i_n}(\mathcal{V}_E)$. If $\alpha^{i_n} q \in \mathcal{V}_E$, we are done. Suppose $\alpha^{i_n} q \notin \mathcal{V}_E$. Then, $q \in \mathcal{V}_E$, which implies that $q \in V^{i_1 \dots i_n}(\omega^*)$. But then, $\alpha^{i_n} q \in V^{i_1 \dots i_n}(\omega^*)$ and $\alpha^{i_n} q \in V^{i_1 \dots i_{n-1}}(\omega^*)$. The rest of the proof is by induction:

- For $n = 1$ and since $V^{i_1}(\omega^*) = W^{i_1}(\omega^*)$, by definition we have that $\omega^* \in K_Q^{i_1}(E)$ if and only if $\mathcal{V}_E \subseteq V^{i_1}(\omega^*)$ and $\emptyset \neq P^{i_1}(\omega^*) \subseteq E_{V^{i_1}(\omega^*)}$.
- For $n = k$, suppose that $\mathcal{V}_E \subseteq V^{i_1 \dots i_k}(\omega^*)$ and $P^{i_k}(\dots(P^{i_1}(\omega^*))) \subseteq E_{V^{i_1 \cdots i_k}(\omega^*)}$ if and only if $\omega^* \in K_Q^{i_1}(\dots K_{V^{i_1 \cdots i_{k-1}}(\omega^*)}^{i_k}(E))$.
- For $n = k + 1$, we need to show that $\mathcal{V}_E \subseteq V^{i_1 \dots i_{k+1}}(\omega^*)$ and $\emptyset \neq P^{i_{k+1}}(\dots(P^{i_1}(\omega^*))) \subseteq E_{V^{i_1 \cdots i_{k+1}}(\omega^*)}$ if and only if $\omega^* \in K_Q^{i_1}(\dots K_{V^{i_1 \cdots i_k}(\omega^*)}^{i_{k+1}}(E))$.

Suppose that $\mathcal{V}_E \subseteq V^{i_1 \dots i_{k+1}}(\omega^*)$ and $\emptyset \neq P^{i_{k+1}}(\dots(P^{i_1}(\omega^*))) \subseteq E_{V^{i_1 \cdots i_{k+1}}(\omega^*)}$. This implies that $(P^{i_{k+1}}(\omega))_{V^{i_1 \cdots i_{k+1}}(\omega^*)} \subseteq E_{V^{i_1 \cdots i_{k+1}}(\omega^*)}$, for all $\omega \in P^{i_k}(\dots(P^{i_1}(\omega^*))) \neq \emptyset$. We want to show that $\emptyset \neq P^{i_k}(\dots(P^{i_1}(\omega^*))) \subseteq K_{V^{i_1 \cdots i_k}(\omega^*)}^{i_{k+1}}(E)$ and $\mathcal{V}_{K_{V^{i_1 \cdots i_k}(\omega^*)}^{i_{k+1}}(E)} \subseteq V^{i_1 \dots i_k}(\omega^*)$, which from the induction hypothesis implies that
$\omega^* \in K_Q^{i_1}(\dots(K_{V^{i_1 \cdots i_{k-1}}(\omega^*)}^{i_k}(K_{V^{i_1 \cdots i_k}(\omega^*)}^{i_{k+1}}(E))))$. The second claim is true by definition, so for the first claim suppose that $\omega \in P^{i_k}(\dots(P^{i_1}(\omega^*)))$. Then, $(P^{i_{k+1}}(\omega))_{V^{i_1 \cdots i_{k+1}}(\omega^*)} \subseteq E_{V^{i_1 \cdots i_{k+1}}(\omega^*)}$. Since $\mathcal{V}_E \subseteq V^{i_1 \dots i_{k+1}}(\omega^*) \subseteq W^{i_{k+1}}(\omega)$, we also have $(P^{i_{k+1}}(\omega))_{\mathcal{V}_E} \subseteq E$. Together, they imply $\omega \in K_{V^{i_1 \cdots i_k}(\omega^*)}^{i_{k+1}}(E)$.

For the other direction, suppose that $\omega^* \in K_Q^{i_1}(\dots K_{V^{i_1 \cdots i_k}(\omega^*)}^{i_{k+1}}(E))$. From the induction hypothesis, this implies that $\emptyset \neq P^{i_k}(\dots(P^{i_1}(\omega^*))) \subseteq K_{V^{i_1 \cdots i_k}(\omega^*)}^{i_{k+1}}(E)$. This means that for all $\omega \in P^{i_k}(\dots(P^{i_1}(\omega^*)))$, we have $\omega \in K_{V^{i_1 \cdots i_k}(\omega^*)}^{i_{k+1}}(E)$ and therefore $\mathcal{V}_E \subseteq W^{i_{k+1}}(\omega)$ and $\emptyset \neq (P^{i_{k+1}}(\omega))_{\mathcal{V}_E} \subseteq E$. Hence,

$$\mathcal{V}_E \subseteq \bigcap_{\omega \in P^{i_k}(\dots(P^{i_1}(\omega^*)))} W^{i_{k+1}}(\omega) = V^{i_1 \dots i_{k+1}}(\omega^*) \text{ and}$$

$$\emptyset \neq P^{i_{k+1}}(\dots(P^{i_1}(\omega^*))) = \bigcup_{\omega \in P^{i_k}(\dots(P^{i_1}(\omega^*)))} \left(P^{i_{k+1}}(\omega)\right)_{V^{i_1 \cdots i_{k+1}}(\omega^*)} \subseteq E_{V^{i_1 \cdots i_{k+1}}(\omega^*)}.$$

□

Proof of Proposition 2. For the first claim we are using the definition of Proposition 1. We need to show that for any sequence $i_1, \dots i_n$, $\mathcal{V}_{EA(V^*(\omega^*))} = V^*(\omega^*) \subseteq V^{i_1 \cdots i_n}(\omega^*)$ and

$$\emptyset \neq P^{i_n}(\dots(P^{i_1}(\omega^*))) \subseteq \Big(EA(V^*(\omega^*)\Big)_{V^{i_1 \cdots i_n}(\omega^*)}.$$

The first part is true by definition. Also, note that if $P^{i_n}(\dots(P^{i_1}(\omega^*)))$ is empty, then $V^{i_1 \cdots i_n i}(\omega^*)$ is empty for any $i \in I$, which implies that $V^*(\omega^*)$ is empty, a contradiction.

For the second part, suppose that $\omega \in P^{i_n}(\dots(P^{i_1}(\omega^*)))$. Then, for any $i \in I$, we have that $V^*(\omega^*) \subseteq V^{i_1 \cdots i_n i}(\omega^*) \subseteq W^i(\omega)$. We will show that $V^*(\omega^*) \subseteq W^i(\{\omega\}_{V^*(\omega^*)})$, which implies that $\{\omega\}_{V^*(\omega^*)} \in EA(V^*(\omega^*))$. Take $q \in V^*(\omega^*)$ and consider the following three cases: i) $q = a^i q'$, for some $q' \in Q$. If $\omega_q =$ "no", then $q \notin W^i(\omega)$, which contradicts the fact that $V^*(\omega^*) \subseteq W^i(\omega)$. Therefore, $\omega_q =$ "yes" and $q \in W^i(\{\omega\}_{V^*(\omega^*)})$. ii) $q \neq a^i q'$ for any $q' \in Q$ and $a^i q \notin V^*(\omega^*)$. This implies that for some sequence $j_1 \dots j_k$ of agents, $a^i q \notin V^{j_1 \cdots j_k}(\omega^*)$. But then, $q \notin V^{j_1 \cdots j_k i}(\omega^*)$, which implies that $q \notin V^*(\omega^*)$, a contradiction. iii) $q \neq a^i q'$ for any $q' \in Q$ and $a^i q \in V^*(\omega^*)$. This is identical to case i) and is omitted.

For the second claim of the Proposition, suppose that $EA(V)$ is common knowledge at ω^*. Since $\mathcal{V}_{EA(V)} = V$, we have that for any sequence $i_1 \dots i_n$ of agents, $V \subseteq V^{i_1 \cdots i_n}(\omega^*)$. Therefore, $V \subseteq V^*(\omega^*)$. □

Proof of Theorem 3. First, we prove the following Lemma.

Lemma 1. *Event E is common knowledge at ω^*.*

Proof. Using Proposition 1, we just need to show that for any sequence $i_1 \dots i_n$ of agents, $\mathcal{V}_E \subseteq V^{i_1 \cdots i_n}(\omega^*)$ and $\emptyset \neq P^{i_n}(\dots(P^{i_1}(\omega^*))) \subseteq E_{V^{i_1 \cdots i_n}(\omega^*)}$. The proof is by induction:

- For $n = 1$, since E is self evident for i_1 and from the proof of Property 1 we have $\omega^* \in E_Q \subseteq \big(K^{i_1}_{\mathcal{V}_E}(E)\big)_Q \subseteq K^{i_1}_Q(E)$. Note that we need Axiom 2 in order to use Property 1. Hence, $\mathcal{V}_E \subseteq W^{i_1}(\omega^*) = V^{i_1}(\omega^*)$ and $\emptyset \neq P^{i_1}(\omega^*) \subseteq E_{V^{i_1}(\omega^*)}$.
- Suppose that for $n = k$, $\mathcal{V}_E \subseteq V^{i_1 \cdots i_k}(\omega^*)$ and $\emptyset \neq P^{i_k}(\dots(P^{i_1}(\omega^*))) \subseteq E_{V^{i_1 \cdots i_k}(\omega^*)}$.
- For $n = k+1$, we need to show that $\mathcal{V}_E \subseteq V^{i_1 \cdots i_{k+1}}(\omega^*)$ and $\emptyset \neq P^{i_{k+1}}(\dots(P^{i_1}(\omega^*))) \subseteq E_{V^{i_1 \cdots i_{k+1}}(\omega^*)}$. By definition,

$$P^{i_{k+1}}(\dots(P^{i_1}(\omega^*))) = \bigcup_{\omega \in P^{i_k}(\dots(P^{i_1}(\omega^*)))} \big(P^{i_{k+1}}(\omega)\big)_{V^{i_1 \cdots i_{k+1}}(\omega^*)}.$$

From the induction hypothesis, for any $\omega \in P^{i_k}(\dots(P^{i_1}(\omega^*)))$ we have

$$\omega \in E_{V^{i_1 \cdots i_k}(\omega^*)} \subseteq \Big(K^{i_{k+1}}_{\mathcal{V}_E}(E)\Big)_{V^{i_1 \cdots i_k}(\omega^*)} \subseteq K^{i_{k+1}}_{V^{i_1 \cdots i_k}(\omega^*)}(E).$$

Hence, $\mathcal{V}_E \subseteq W^{i_{k+1}}(\omega)$ and $\emptyset \neq P^{i_{k+1}}(\omega) \subseteq E_{W^{i_{k+1}}(\omega)}$. Therefore, $\mathcal{V}_E \subseteq V^{i_1 \cdots i_{k+1}}(\omega^*)$, and $\emptyset \neq P^{i_{k+1}}(\dots(P^{i_1}(\omega^*))) \subseteq E_{V^{i_1 \cdots i_{k+1}}(\omega^*)}$.

$\square$

Since $\mathcal{V}_{E^*} \subseteq \mathcal{V}_E$ and $E_Q \subseteq E^*_Q$, we have that $E \subseteq E^*_{\mathcal{V}_E}$. Fix a sequence $i_1 \dots i_n$ of agents. From Generalized Monotonicity and the fact that $\mathcal{V}_{E^*} \subseteq \mathcal{V}_E \subseteq V^{i_1 \cdots i_{n-1}}(\omega^*)$ we have $K^{i_n}_{V^{i_1 \cdots i_{n-1}}(\omega^*)}(E) \subseteq K^{i_n}_{V^{i_1 \cdots i_{n-1}}(\omega^*)}(E^*_{\mathcal{V}_E}) \subseteq K^{i_n}_{V^{i_1 \cdots i_{n-1}}(\omega^*)}(E^*)$. By applying Generalized Monotonicity recursively we have that $K^{i_1}_Q(\dots(K^{i_n}_{V^{i_1 \cdots i_{n-1}}(\omega^*)}(E))) \subseteq K^{i_1}_Q(\dots(K^{i_n}_{V^{i_1 \cdots i_{n-1}}(\omega^*)}(E^*)))$. Therefore, $\omega^* \in K^{i_1}_Q(\dots(K^{i_n}_{V^{i_1 \cdots i_{n-1}}(\omega^*)}(E^*)))$ and since this holds for all sequences $i_1, \dots, i_n$, E^* is common knowledge at ω^*.

$\square$

Proof of Theorem 4. Since E^* is common knowledge at ω^*, we have that for any sequence $i_1 \dots i_n, n \in \mathbb{N}$, $\mathcal{V}_{E^*} \subseteq V^{i_1 \cdots i_n}(\omega^*)$. Therefore, $\mathcal{V}_{E^*} \subseteq V^*(\omega^*)$ and the following event is well defined:

$$E = \bigcap_{\substack{i_1,\dots,i_n \\ n \in \mathbb{N}}} K^{i_1}_{V^*(\omega^*)} K^{i_2}_{V^*(\omega^*)} \dots K^{i_n}_{V^*(\omega^*)}(E^*).$$

Since $\mathcal{V}_E = V^*(\omega^*)$, we have that $\mathcal{V}_{E^*} \subseteq \mathcal{V}_E$. It remains to show that $\omega^* \in E_Q \subseteq E^*_Q$ and that E is a public event.

- $\omega^* \in E_Q$

  Take any sequence $i_1 \dots i_n$, $n \in \mathbb{N}$. We want to show that $\{\omega^*\}_{V^*(\omega^*)} \in K^{i_1}_{V^*(\omega^*)} K^{i_2}_{V^*(\omega^*)} \dots K^{i_n}_{V^*(\omega^*)}(E^*)$. To ease on the notation, set

  $$F = K^{i_1}_{V^*(\omega^*)} K^{i_2}_{V^*(\omega^*)} \dots K^{i_n}_{V^*(\omega^*)}(E^*).$$

  Then, we need to show that $\{\omega^*\}_{V^*(\omega^*)} \in F$.

  By assumption, for some sequence $j_1 \dots j_k$ of agents, $V^{j_1 \cdots j_k}(\omega^*) = V^*(\omega^*)$. We will show that

  $$K^{j_1}_Q K^{j_2}_{V^{j_1}(\omega^*)} \dots K^{j_k}_{V^{j_1 \cdots j_{k-1}}(\omega^*)}(F) \subseteq (F)_Q. \tag{4}$$

  Axiom 3 implies that for any $\omega \in \mathcal{S}$, we have $\{\omega\}_{W^{j_k}(\omega)} \in P^{j_k}(\omega)$. The axiom of knowledge then implies that

  $$K^{j_k}_{V^{j_1 \cdots j_{k-1}}(\omega^*)}(F) \subseteq (F)_{V^{j_1 \cdots j_{k-1}}(\omega^*)}.$$

  From generalized monotonicity we have that

  $$K^{j_{k-1}}_{V^{j_1 \cdots j_{k-2}}(\omega^*)} K^{j_k}_{V^{j_1 \cdots j_{k-1}}(\omega^*)}(F) \subseteq K^{j_{k-1}}_{V^{j_1 \cdots j_{k-2}}(\omega^*)}\left((F)_{V^{j_1 \cdots j_{k-1}}(\omega^*)}\right).$$

  The definition of the knowledge operator implies that $K^{j_{k-1}}_{V^{j_1 \cdots j_{k-2}}(\omega^*)}\left((F)_{V^{j_1 \cdots j_{k-1}}(\omega^*)}\right) \subseteq K^{j_{k-1}}_{V^{j_1 \cdots j_{k-2}}(\omega^*)}(F)$. Applying the axiom of knowledge we have that $K^{j_{k-1}}_{V^{j_1 \cdots j_{k-2}}(\omega^*)}(F) \subseteq (F)_{V^{j_1 \cdots j_{k-2}}(\omega^*)}$. Combining these results we have that

  $$K^{j_{k-1}}_{V^{j_1 \cdots j_{k-2}}(\omega^*)} K^{j_k}_{V^{j_1 \cdots j_{k-1}}(\omega^*)}(F) \subseteq (F)_{V^{j_1 \cdots j_{k-2}}(\omega^*)}.$$

Continuing recursively, we obtain (4). Note that $V^{j_1 \cdots j_k}(\omega^*) = V^*(\omega^*)$ implies that for any agent $i \in I$, $V^{j_1 \cdots j_k i}(\omega^*) = V^*(\omega^*)$. Since E^* is common knowledge at ω^*, ω^* belongs to the left hand side of (4), and therefore also to the right hand side of (4). Hence, $\{\omega^*\}_{V^*(\omega^*)} \in K^{i_1}_{V^*(\omega^*)} K^{i_2}_{V^*(\omega^*)} \ldots K^{i_n}_{V^*(\omega^*)}(E^*)$. Since this holds for any sequence $i_1 \ldots i_n$, we have that $\{\omega^*\}_{V^*(\omega^*)} \in E$, which implies that $\omega^* \in E_Q$.

- $E_Q \subseteq E^*_Q$

  Take any sequence $i_1 \ldots i_n$. Axiom 3 and the axiom of knowledge imply that $K^{i_n}_{V^*(\omega^*)}(E^*) \subseteq E^*_{V^*(\omega^*)}$. From generalized monotonicity and the axiom of knowledge, $K^{i_{n-1}}_{V^*(\omega^*)} K^{i_n}_{V^*(\omega^*)}(E^*) \subseteq K^{i_{n-1}}_{V^*(\omega^*)}(E^*_{V^*(\omega^*)}) \subseteq K^{i_{n-1}}_{V^*(\omega^*)}(E^*) \subseteq E^*_{V^*(\omega^*)}$. Continuing recursively we have that $K^{i_1}_{V^*(\omega^*)} K^{i_2}_{V^*(\omega^*)} \ldots K^{i_n}_{V^*(\omega^*)}(E^*) \subseteq E^*_{V^*(\omega^*)}$. Since this holds for any sequence $i_1 \ldots i_n$, we have that $E \subseteq E^*_{V^*(\omega^*)}$. Hence, $E_Q \subseteq E^*_Q$.

- E is a public event. That is, for any $i \in I$, $E \subseteq K^i_{V^*(\omega^*)}(E)$.
  Fix $i \in I$ and suppose $\omega \in E$. We need to show that $\omega \in K^i_{V^*(\omega^*)}(E)$. By conjuction,

$$K^i_{V^*(\omega^*)}(E) = \bigcap_{\substack{i_1 \cdots i_n \\ n \in \mathbb{N}}} K^i_{V^*(\omega^*)} K^{i_1}_{V^*(\omega^*)} K^{i_2}_{V^*(\omega^*)} \ldots K^{i_n}_{V^*(\omega^*)}(E^*).$$

  Hence, we just need to show that for any sequence $i_1 \ldots i_n$, $\omega \in K^i_{V^*(\omega^*)} K^{i_1}_{V^*(\omega^*)} K^{i_2}_{V^*(\omega^*)} \ldots K^{i_n}_{V^*(\omega^*)}(E^*)$. This is true since for any $i \in I$, $E \subseteq K^i_{V^*(\omega^*)} K^{i_1}_{V^*(\omega^*)} K^{i_2}_{V^*(\omega^*)} \ldots K^{i_n}_{V^*(\omega^*)}(E^*)$.

□

Proof of Theorem 5. From Theorem 4, there exists a nonempty public event E' such that $\mathcal{V}_{E^*} \subseteq \mathcal{V}_{E'}$ and $\omega^* \in E'_Q \subseteq E^*_Q$. Its proof also shows that $\mathcal{V}_{E'} = V^*(\omega^*)$, which implies that $E' \subseteq E^*$. We need to show that $E' = \bigcup_{\omega \in E'} P^i(\omega)$. IF $\omega \in E'$, Axiom 3 implies $\omega \in \bigcup_{\omega \in E'} P^i(\omega)$. For the opposite direction, since E' is a public event, $\omega \in E'$ implies $P^i(\omega) \subseteq E'$. Therefore, $E' = \bigcup_{\omega \in E'} P^i(\omega)$, and by symmetry $E' = \bigcup_{\omega \in E'} P^j(\omega)$.

The next step is to show that E' is partitioned by P^i. Firstly, since E' is public, for any $\omega \in E'$, $W^i(\omega) = \mathcal{V}_{E'}$. Secondly, Axiom 3 implies nondelusion while Theorem 1, together with E' being public imply that if $\omega, \omega_1 \in E'$ then either $P^i(\omega) = P^i(\omega_1)$ or $P^i(\omega) \cap P^i(\omega_1) = \emptyset$. The rest of the proof is identical to that of Aumann [2].

Agent i's posterior at $\omega \in E'$ is

$$q^i(\omega) = \frac{\mu(P^i(\omega) \cap E)}{\mu(P^i(\omega))}.$$

Since $q^i(\omega) = q^i$ for all $\omega \in E'$ we can sum over the disjoint partition cells of E' and derive $\mu(E')q^i = \mu(E' \cap E)$. Similarly for agent j we have $\mu(E')q^j = \mu(E' \cap E)$ and therefore $q^i = q^j$.

□

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