# The von Neumann–Gale Growth Model: New Applications and Solutions to Old Problems

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Von Neumann's (1937) model of an expanding economy, generalized by Gale (1956), was one of the first models in Mathematical Economics that served as the basis for a rich and interesting theory. This theory was developed for the most part in the 1950s and 1960s. Substantial contributions to it were made by Gale, Nikaido, McKenzie, Radner, Rockafellar and others (for an introductory exposition see e.g. Nikaido 1968).

The theory of the von Neumann–Gale model, in its original form, was purely deterministic. It did not reflect the influence of random factors on economic growth. The importance of taking these factors into account in the basic models of economic dynamics was realized early on. First attempts to build stochastic analogues of the von Neumann–Gale model were undertaken in the 1970s by Dynkin, Radner and their research groups. However, the initial attack on the problem left many questions unanswered, as the studies in this field faced serious mathematical difficulties. To overcome the difficulties new methods were required, which were developed only in 1990s. These methods made it possible to obtain stochastic generalizations of central results of the theory. For an account of these achievements see the survey by Evstigneev and Schenk-Hoppé (2006).

Originally, the focus of studies related to the von Neumann–Gale model was on economic growth. It has recently been observed (Dempster, Evstigneev and Taksar 2005) that a stochastic version of the von Neumann–Gale framework can be fruitfully applied to the analysis of fundamental problems in Mathematical Finance (asset pricing under transaction costs, capital growth theory). This observation gave a new momentum to work in this area, raised new questions and revived interest to long-standing open problems.

In spite of the current progress achieved, a substantial gap remained. The theory lacked quite satisfactory results on the existence of equilibrium in the stochastic von Neumann–Gale model. The results available established the existence of equilibrium under certain conditions that also guaranteed its stability (the turnpike property). These conditions seemed to be too restrictive in the context of economic models and did not cover a number of new examples arising in financial applications. Furthermore, the conditions were substantially distinct from those conventionally imposed in the deterministic case.

Quite recently, new results have been obtained (Evstigneev and Schenk-Hoppé 2006a) making it possible to fill the gap. An existence theorem for equilibrium has been proved under assumptions fully analogous to the standard deterministic ones. The problem of obtaining a result of this kind was posed by Eugene B. Dynkin in the early 1970s and remained open for more than

three decades. The theorem established is applicable to both old models—those related to Economics, and new ones, coming from Finance.

To prove the existence theorem we employ the classical strategy going back to Dvoretzky, Wald and Wolfowitz (1950). It is based on the idea of "elimination of randomization". We first construct, using an additional source of randomness (a "sunspot" process), an appropriate extension of the given model where an equilibrium exists. Then, by exploiting some subtle properties of convexity, we eliminate randomization and establish the existence of equilibrium in the original model.

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