Subjective Expectations Equilibrium in Economies with Uncertain Delivery*

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September 21st, 2006

Abstract. We develop a model of general equilibrium with trade *ex ante* in a context of private and incomplete state verification. Instead of choosing bundles, agents choose *lists* of bundles out of which the market then selects one bundle for delivery. With agents having *subjective expectations* about the bundle that will be delivered, we study existence of a *subjective expectations equilibrium.*

Keywords: General equilibrium, Private information, Incomplete information, Uncertain delivery, Lists of bundles.

JEL Classification Numbers: C62, D51, D82.

*We are grateful to Jacques Drèze, Juan Pablo Torres-Martínez, Nicholas Yannelis and two referees for useful comments and suggestions. João Correia-da-Silva (joao@fep.up.pt) acknowledges support from CEMPRE, Fundação para a Ciência e Tecnologia and FEDER (SFRH/BPD/24450/2005). Carlos Hervés-Beloso (cherves@uvigo.es) acknowledges support from Ministerio de Ciencia y Tecnología and FEDER (HP2003-0002 and BEC2003-09067-C04-01).

1 Introduction

In the theory of general equilibrium under uncertainty, besides being defined by their physical properties and location in space and time, commodities are also defined by the state of nature in which they are made available.[1] With this extension of the commodity space, the model of Arrow and Debreu (1954) covers the case of trade *ex ante* with public state verification. First, agents make contracts to exchange their state-contingent endowments for a state-contingent consumption plan, that specifies a consumption bundle for each possible state of nature. After the state of nature is publicly announced, trade takes place and each agent consumes the bundle that corresponds to the announced state.

With private state verification, trade becomes more complicated. Agents may not be interested in buying good $A1$ (delivery of good A in state 1) because they may fear that, even if state 1 occurs, they may not be able to verify (for example, to prove in a court of law) that state 1 occurred, and that they are, therefore, entitled to receive good A. Suppose that an agent cannot verify whether the true state is 1 or 2. The traditional Walrasian approach would be to restrict the agent to consumption plans that deliver the same bundle in the two states.[2] We suggest a weaker restriction: the agent has to accept any of the bundles contracted for delivery in state 1 and state 2. Equivalently, the market has two alternatives: to deliver the bundle that corresponds to state 1, or the bundle that corresponds to state 2. The agent, in turn, has no alternative other than to accept the bundle that is delivered.[3]

This relaxes, in a natural way, the measurability restriction. An agent may select different bundles, A and B, for delivery, respectively, in states 1 and 2, that he does not distinguish. But, in any of these states, the market may choose to deliver A or B. It is as if the agent had bought the same *list*, $A \vee B$, for delivery in both states. In a previous paper (2006), we introduced this conceptual model of an *economy with uncertain delivery*, where objects of choice are lists of bundles out of which the market has discretion to select a bundle for delivery. For example, the list $A \vee B$

[1] For example, instead of talking about good A in state 1, or good B in state 2, we talk about good $A1$ or good $B2$. These goods are known as contingent consumption claims (Arrow, 1953).

[2] See Radner (1968), Yannelis (1991) and the volume on 'differential information economies' edited by Glycopantis and Yannelis (2005).

[3] For example, suppose that what is decisive for the agent is not to *know* whether the true state is 1 or 2, but to have the ability to *prove* it in a court of law.

gives an agent the right to receive bundle A or bundle B.

Defined in terms of lists instead of bundles, 'measurability' is not an exogenous restriction on trade. It is rather a way to formalize an actual enforceability issue. Consider an agent that cannot verify, for example, in a court of law, whether the true state is 1 or 2. If the agent buys $A1$ (delivery of good A in state 1) and $B2$, the market may deliver $A1$ and $B2$, but may also deliver $A1$ and $A2$, $B1$ and $A2$ or $B1$ and $B2$. In sum: the agent receives 'good A or good B' in states '1 and 2', that is, $(A \vee B)1$ and $(A \vee B)2$. Observe that an agent can buy something that is not 'measurable', but, in practice, the 'set of alternatives that may be delivered' is measurable - in this case it is 'A or B' in both states that the agent does not distinguish. Since, without loss of generality, we can convert (as above) any non-measurable choice into a measurable one, this 'measurability' restriction (on 'lists') does not restrict trade agreements. It describes the consequences of private information in terms of actual outcomes.

It should be clear that we are handling a weaker informational restriction, than the usual measurability. An agent always has the possibility of buying lists with only one bundle in order to guarantee consumption of the same bundle in states of nature that he does not distinguish. Therefore, efficiency of trade frequently improves (and never diminishes) relatively to the Walrasian Expectations Equilibrium solution.

A possible interpretation of the model of an economy with uncertain delivery is the following. Each agent deals with a broker, who offers plans of state-contingent lists in exchange for the agent's state-contingent endowments. The broker takes the state-contingent endowments to an 'internal market' and trades them for a state-contingent consumption plan (the cheapest) satisfying the requirements of the plan of lists selected by the agent. Among themselves, then, brokers trade state-contingent commodities in an 'internal' Arrow-Debreu market. We assume that brokers always keep their contracts and that they make no profits. As a result, the price charged for a list is equal to the price (in the internal market) of the cheapest bundle that satisfies the requirements of the list. Otherwise there would be opportunities for arbitrage. The price of a list is a linear function, and there is no price discrimination as introduced by Aliprantis, Tourky and Yannelis (2001).[4]

An example is helpful in understanding this interpretation. Consider three states of nature, and a broker offering a contingent plan of lists, $\tilde{x} = [(a \vee b \vee c), (a \vee b \vee$

[4]If all alternatives in a list vary proportionally, the cheapest alternative remains the same, and its price (which is equal to the price of the list) varies in the same proportion.

$c), (d \vee e)]$. The broker has many ways of keeping the contract. One is to buy (in the internal market) the state-contingent bundle (a, b, d), another is to buy (c, c, e), or (a, c, e), etc. In any case, in state 1 and state 2 (first and second coordinates) delivery must be of a, b or c, and in state 3 (third coordinate) delivery has to be of d or e. Delivering one of these bundles, the broker keeps the contract. The choice of the broker will surely be the cheapest of the alternatives. Thus, this cheapest bundle has two fundamental characteristics: (1) its price is the price charged for the list (the competitive brokers have no profit); and (2) it is the bundle that will be delivered. In this way, the prices of lists are uniquely determined by the prices of the contingent commodities in the internal market. Selection of bundles to be delivered is also determined internally. In sum, the 'internal market mechanism' is responsible for price-setting and for the selection of the bundles to be delivered to each agent in each state of nature, among the possibilities specified in the lists.

In this paper we study the case in which agents have subjective beliefs. Facing a list and the prevailing prices, agents construct subjective expectations on the probabilities of receiving each of the different bundles in the list. These expectations, in turn, determine preferences over lists. The main objective is to give conditions that guarantee existence of equilibrium.

We remark that, in this model, the preferences of the agents for lists depend on their subjective expectations, and therefore may be a function of prices.[5] It is not that agents prefer to consume expensive or cheap bundles. What happens is that agents take the prices of the different bundles as a signal of the probability of delivery of each of the bundles in a list, and this implies that their preferences for lists are price-dependent. This makes sense because prices are related to the economic difficulty to deliver the goods.

The paper is organized as follows. In section 2, the main section, preferences and prices are extended from bundles to lists, the model is presented, and existence of equilibrium is established. In section 3 we make some remarks and conclude the paper with an example.

[5]With price-dependent preferences, it is known that equilibrium exists (Arrow and Hahn, 1971). Existence of equilibrium in economies with price-dependent preferences was recently studied by Cornet and Topuzu (2005).

2 The economy with uncertain delivery

An economy with uncertain delivery is a *differential information economy* in which objects of choice are state-contingent plans of consumption lists instead of state-contingent plans of consumption bundles. A *list* is a set of bundles such that the market delivers one of the bundles in the list.

2.1 Basic setup

We consider a finite number of agents ($i = 1, ..., n$), a finite number of possible states of nature ($s = 1, ..., S$), a finite number of commodities, ($j = 1, ..., l$), and a finite number of alternatives in a list ($k = 1, ..., K$). The private information of agent i is represented by a partition of the set of states of nature such that agent i can distinguish states that belong to different sets of the partition P_i. The set of states that agent i does not distinguish from s is denoted $P_i(s)$. A function that is constant across elements of P_i is said to be P_i-measurable. Consumption of agent i in state s is $x_i^s \in \mathbb{R}_+^l$, and the contingent consumption plan of agent i is $x_i \in \mathbb{R}_+^{Sl}$. The list selected by agent i for delivery in state s is denoted $\tilde{x}_i^s \in \mathbb{R}_+^{Kl}$, with the k^{th} alternative being $\tilde{x}_i^{sk} \in \mathbb{R}_+^l$. The state-contingent plan of lists selected by agent i is $\tilde{x}_i \in \mathbb{R}_+^{SKl}$.

The economy extends over two time periods. In the first, agents observe prices and trade their state-contingent endowments for P_i-measurable vectors of state-contingent lists, $\tilde{x}_i = (\tilde{x}_i^1, \tilde{x}_i^2, ..., \tilde{x}_i^S)$, specifying the bundles that the market may deliver in each state of nature. In the second period, agents receive their information, and consume one of the bundles in the list that corresponds to the state of nature that occurs. If the state of nature is s, agent i receives one of the bundles $\tilde{x}_i^{sk}$ in the list $\tilde{x}_i^s$.

For example, suppose that the set of possible states of nature is $\Omega = \{1, 2, 3\}$, the private information of agent i is $P_i = \{\{1, 2\}, \{3\}\}$, and the (measurable) vector of consumption lists is $\tilde{x}_i = (\tilde{x}_i^1, \tilde{x}_i^2, \tilde{x}_i^3) = [(a \vee b \vee c), (a \vee b \vee c), (d \vee e \vee e)]$. In the second period, if the state of nature is 1 or 2, agent i receives a, b or c; if it is 3, then the agent receives d or e.

2.2 From bundles to lists

There is a correspondence from bundles to lists playing a fundamental role in the model. For example, delivery of the bundle x keeps the contract for delivery of the list $(x \vee y)$, but would not keep the contract for delivery of $(2x \vee x + y)$, because both alternatives exceed x (given an x and y that are not null). With the delivery of a bundle $x^s \in \mathbb{R}^l_+$ in state s, the market can keep the promise of delivery of any list in $\tilde{X}(x^s)$, defined as:

$$\tilde{X}(x^s) = \{\tilde{x}^s = (\tilde{x}^{s1}, ..., \tilde{x}^{sK}) \in \mathbb{R}^{Kl}_+ : \exists k \ s.t. \ \tilde{x}^{sk} \leq x^s\}.$$

Each agent chooses a P_i-measurable vector of contingent lists, $\tilde{x}_i = (\tilde{x}^1_i, ..., \tilde{x}^K_i)$, so it makes sense to extend the correspondence to the whole set of states of nature. Delivery of $x_i = (x^1_i, ..., x^S_i) \in \mathbb{R}^{Sl}_+$ keeps the contract for delivery of any list in $\tilde{X}^S(x)$, defined as:

$$\tilde{X}^S(x_i) = \tilde{X}(x^1_i) \times \tilde{X}(x^2_i) \times ... \times \tilde{X}(x^S_i).$$

A more explicit definition of the same correspondence is:

$$\tilde{X}(x^s_i) = \cup^K_{k=1}\{(\mathbb{R}^l_+)^{k-1} \times [0, x^s_i] \times (\mathbb{R}^l_+)^{K-k}\}.$$

In this definition, $[0, x^s]$ denotes the set of bundles y^s such that $0 \leq y^s \leq x^s$. For example, with two alternatives and a single commodity: $x = 1$ implies that $\tilde{X}(x) = \{[0, 1] \times \mathbb{R}_+\} \cup \{\mathbb{R}_+ \times [0, 1]\}$. This formulation makes it clear that $\tilde{X}$ is a continuous correspondence (lower and upper hemicontinuous), because it is a finite union of a finite product of continuous correspondences.

2.3 Prices of lists

We seek equilibrium prices defined on lists, restricting our search to prices that satisfy a *no arbitrage* condition: the price of a list $\tilde{x}$ is equal to the price of the cheapest bundle, x, that keeps the contract for delivery of $\tilde{x}$.

To see that this is (in its essence) a no arbitrage assumption, suppose that there is an intermediary (broker) between an agent and the market. The intermediary promises to deliver a state-contingent list in exchange for the agent's state-contingent endowments. In the market, the intermediary trades the agent's endowments for a bundle, x, that satisfies the requirements of the list, that is, for an x such that $\tilde{x} \in \tilde{X}^S(x)$. Finally, the intermediary delivers x to the agent, keeping the promise to deliver a bundle in $\tilde{x}$. If $\tilde{p}(\tilde{x}) > p \cdot x$, another intermediary is willing to offer the list at a lower price. If $\tilde{p}(\tilde{x}) < p \cdot x$, no intermediary is willing to make this trade. In sum, lists are traded at a price equal to the price of the cheapest bundle satisfying $\tilde{x} \in \tilde{X}(x)$, and, therefore, it is enough to determine the prices of the contingent goods (primitives). The prices of lists (derivatives) follow as a consequence.

As usual, prices of the contingent commodities are normalized to the unit simplex of $\mathbb{R}^{Sl}_{+}$:

$$p \in \Delta^{Sl}_{+} = \left\{ p \in \mathbb{R}^{Sl}_{+} : \sum_{s=1}^{S} \sum_{j=1}^{l} p^{sj} = 1 \right\}.$$

The price of a list, $\tilde{p}(\tilde{x}_i)$, is:

$$\tilde{p}^s(\tilde{x}^s_i) = \min_k \{p^s \cdot \tilde{x}^{sk}_i\};$$

$$\tilde{p}(\tilde{x}_i) = \sum_{s=1}^{S} \tilde{p}^s(\tilde{x}^s_i) = \sum_{s=1}^{S} \min_k \{p^s \cdot \tilde{x}^{sk}_i\}.$$

Therefore, the budget restriction faced by agent i is:

$$\tilde{B}_i(e_i, p) = \{\tilde{x}_i \in \mathbb{R}^{SKl}_{+} : \sum_{s=1}^{S} \min_k \{p^s \cdot \tilde{x}^{sk}_i\} \leq \sum_{s=1}^{S} p^s \cdot \tilde{x}^s_i = p \cdot e_i\}.$$

2.4 Subjective expectations

When selecting a list, agents form ‘subjective expectations’ on the probability of receiving each of the bundles in the list. These beliefs depend on the observation of prices, which signal the cost of delivery of each alternative. The only common information that we assume is that agents know that the total resources in the economy belong to the set $[0, T)^{Sl}$. Consequently, they attribute a null probability of delivery to alternatives which are outside this set. This allows us to write lists in a compact set, $\tilde{X}^S = [0, T]^{SKl}$.[6]

Let $E^{sk}_i(\tilde{x}^s_i, p)$ represent the subjective probability of receiving the k^{th} alternative in the list $\tilde{x}^s_i$, given prices p. Let also $E^s_i = (E^{s1}_i, E^{s2}_i, ..., E^{sK}_i)$, with $\sum_{k=1}^{K} E^{sk}_i = 1$, and $E_i = (E^1_i, E^2_i, ..., E^S_i)$.

[6] A list with less than K alternatives can be represented in the same space by completing the remaining coordinates with the bundle $(T, ..., T)$ (an irrelevant alternative), or with repeated alternatives.

Given prices, p, and a list $\tilde{x}_i^s$ for delivery in state s, agent i has subjective expectations regarding the probabilities of delivery of each of the bundles in the list, given by the vector:

$$E_i^s: \ [0,T]^{Kl} \times \Delta_+^{Sl} \to \Delta_+^K;$$

$$E_i^s(\tilde{x}_i^s, p) = (E_i^{s1}(\tilde{x}_i^s, p), E_i^{s2}(\tilde{x}_i^s, p), ..., E_i^{sK}(\tilde{x}_i^s, p)).$$

Preferences in the different states of nature are represented by a P_i-measurable vector of Von Neumann-Morgenstern utility functions, $u_i^s : \mathbb{R}_+^l \to \mathbb{R}$, assumed to be continuous, weakly monotone and concave. The function $\tilde{u}_i^s : \mathbb{R}_+^{Kl} \to \mathbb{R}^K$ returns the vector $\tilde{u}_i^s(\tilde{x}_i^s) = (u_i^s(\tilde{x}_i^{s1}), ..., u_i^s(\tilde{x}_i^{sK}))$. Agents combine subjective expectations with preferences for consumption to maximize a subjective expected utility function:

$$\tilde{U}_i(\tilde{x}_i, p) = \sum_{s=1}^{S} q_i^s E_i^s(\tilde{x}_i^s, p) \cdot \tilde{u}_i^s(\tilde{x}_i^s) = \sum_{s=1}^{S} q_i^s \sum_{k=1}^{K} E_i^{sk}(\tilde{x}_i, p) u_i^s(\tilde{x}_i^{sk}).$$

To guarantee existence of equilibrium, an hypothesis of continuity is needed. A small change in prices or in the alternative bundles in the list must imply only a small change in the subjective probabilities of delivery of the different alternatives in the list. It may seem natural to assume continuity of E_i, but this would be too restrictive. For example, suppose that there are only two alternatives in the list and they are equal, $\tilde{x}^s = (a, a)$. If the first diminishes a little, to $(a-\delta, a)$, then the agent may expect the market to deliver this smallest bundle with certainty, $E_i^s = (1, 0)$. While if it is the second alternative that diminishes a little, to $(a, a-\delta)$, then the expectations may be $E_i^s = (0, 1)$. In a previous paper (2006), we assumed these 'prudent expectations', which are incompatible with assuming continuity of E_i.

We will allow failures of continuity, but only of a precise nature. Jumps of probability beliefs can only occur between alternatives that are equal or that at least have the same utility.

Before stating the hypothesis, we aggregate alternatives in a given contingent list according to their utility: $G(k) = \{k' : \ u_i^s(\tilde{x}_i^{sk'}) = u_i^s(\tilde{x}_i^{sk})\}$. Similarly, let $G(k, \epsilon) = \{k' : \ |u_i^s(\tilde{x}_i^{sk'}) - u_i^s(\tilde{x}_i^{sk})| < \epsilon\}$, and notice that for small ϵ the sets defined are the same. The continuity condition on expectations is the following:

Assumption 1

Consider a given $(\tilde{x}_i^s, p)$, and let $\epsilon > 0$ be small enough for $G(k, \epsilon) = G(k), \ \forall k$. Let $F_i^{sk}(\tilde{x}_i^s, p; \epsilon) = \sum_{k' \in G(k,\epsilon)} E_i^{sk'}(\tilde{x}_i^s, p)$. Then, $F_i^{sk}(\tilde{x}_i^s, p; \epsilon)$ is continuous at $(\tilde{x}_i^s, p)$.

Observe that, from continuity of u_i^s, small changes in $(\tilde{x}_i^s, p)$ preserve the sets G. Therefore, in a neighborhood of $(\tilde{x}_i^s, p)$, we have the following equality that shows continuity of subjective expected utility:

$$\tilde{U}_i(\tilde{x}_i, p) = \sum_{s=1}^{S} q_i^s \sum_{k=1}^{K} E_i^{sk}(\tilde{x}_i, p) u_i^s(\tilde{x}_i^{sk}) = \sum_{s=1}^{S} q_i^s \sum_{G(k,\epsilon)} F_i^{sk}(\tilde{x}_i^s, p, \epsilon) u_i^s(\tilde{x}_i^{sk}).$$

This continuity condition is strong enough to imply continuity of subjective expected utility and weak enough to allow 'subjective expectations' to encompass 'prudent expectations' as a particular case. In the case of 'prudent expectations', only the worst alternatives have positive probabilities (in this set: $F_i^{sk} = 1$) and the only kind of continuity failure in 'prudent expectations' are transferences of probability between equally worse alternatives, which do not violate assumption 1.

We also make an assumption related to *no satiation*. There exists a small positive vector that, when added to the vector of lists, leaves expectations unchanged.[7]

Assumption 2

Consider $(\tilde{x}_i, p)$ *and* $\epsilon > 0$. *There exists* $0 < z \leq \epsilon$ *such that:*

$E_i(\tilde{x}_i + z, p) = E_i(\tilde{x}_i, p)$.

This implies that with additional resources, z, it is possible to design a list such that $\tilde{U}_i(\tilde{x}_i + z, p) > \tilde{U}_i(\tilde{x}_i, p)$. This, in turn, implies that the brokers use all the value of the agent's endowments. Otherwise, the preferred list, $\tilde{x}_i^s$, could be one that implied delivery of x_i with $\tilde{p}(\tilde{x}_i) = p \cdot x_i < p \cdot e_i$.

A convexity condition is also necessary. It would be enough to assume that $\tilde{U}_i(\tilde{x}_i, p)$ is quasi-concave. But this restriction would be too strong. To see this, consider two commodities and linear utility: $u(x, y) = x + y$. We have $u(2, 0) = u(0, 2) = 2$ and $u(2a, 0) = u(0, 2a) = 2a$. Let $a > 1$. How much is $\tilde{U}((2, 0) \vee (0, 2a))$? And $\tilde{U}((0, 2) \vee (2a, 0))$? Both lists imply consumption of bundles with utility of either 2 or $2a$. Suppose that agents look beyond the worst outcome: $\tilde{U}((2, 0) \vee (0, 2a)) > 2$ and $\tilde{U}((0, 2) \vee (2a, 0)) > 2$. But, looking at the average allocation, $[(1, 1) \vee (a, a)]$, a 'prudent' agent would expect the market to deliver $(1, 1)$, and not (a, a), because $(1, 1) < (a, a)$. This violates quasi-concavity: $\tilde{U}((1, 1) \vee (a, a)) = u(1, 1) = 2$.

[7] For alternatives with some coordinate already at the bound T, the increase may be null. This is only necessary for the new list to remain in $[0, T]^{Sl}$. With positive or null increase, the expectation that corresponds to such alternatives is zero, because agents know that total endowments are in $[0, T)^{Sl}$.

We impose a weaker convexity assumption: given prices p, if the bundle $x \in \mathbb{R}^{Sl}_+$ allows the broker to offer a list $\tilde{x}$, and $y \in \mathbb{R}^{Sl}_+$ allows the offer of a list $\tilde{y}$ with the same subjective utility, then, any convex combination $z = \lambda x + (1-\lambda)y$, with $\lambda \in [0,1]$, allows the broker to offer a list $\tilde{z}$ with at least the same subjective utility as $\tilde{x}$ and $\tilde{y}$.

Assumption 3

Given prices p, consider two lists, $\tilde{x} \in \tilde{X}^S(x)$ and $\tilde{y} \in \tilde{X}^S(y)$, with $\tilde{U}(\tilde{x}, p) = \tilde{U}(\tilde{y}, p)$. Then, for any convex combination $z = \lambda x + (1-\lambda)y$, with $\lambda \in [0,1]$, there exists a list $\tilde{z} \in \tilde{X}(z)$ with $\tilde{U}(\tilde{z}, p) \geq \tilde{U}(\tilde{x}, p)$.

An example of expectations that induce preferences which satisfy assumption 3 are the *prudent expectations* (*minimax* preferences), according to which agents expect to receive the worst alternative with certainty.

2.5 The model

The economy with uncertain delivery is defined by $\mathcal{E} \equiv (e_i, u_i, P_i, q_i, E_i)_{i=1}^n$, where, for each agent i:

- A partition of the set of possible states of nature, P_i, represents private information. The set of states that agent i does not distinguish from state s is denoted $P_i(s)$.

- A vector q_i represents the subjective prior probabilities on the occurrence of the different states of nature. To each state s corresponds the subjective probability $q_i^s \geq 0$, with $\sum_{s=1}^S q_i^s = 1$.

- For each state s, a subjective expectations vector function, $E_i^s(\tilde{x}_i^s, p) : [0,T]^{Kl} \times \Delta_+^{Sl} \to \Delta_+^K$, gives the subjective probabilities of delivery of each of the K bundles in the list $\tilde{x}_i^s$. These functions are constant across undistinguished states, that is, the vector $E_i = (E_i^1, ..., E_i^S)$ is P_i-measurable.

- Preferences in the different states are represented by a P_i-measurable vector of Von Neumann-Morgenstern utility functions, $u_i^s : \mathbb{R}^l_+ \to \mathbb{R}$, assumed to be continuous, weakly monotone and concave. The function $\tilde{u}_i^s : \mathbb{R}^{Kl}_+ \to \mathbb{R}^K$ returns the vector $\tilde{u}_i^s(\tilde{x}_i^s) = (u_i^s(\tilde{x}_i^{s1}), ..., u_i^s(\tilde{x}_i^{sK}))$. The subjective expected utility function combines beliefs with preferences for consumption: $\tilde{U}_i(\tilde{x}_i, p) = \sum_{s=1}^{S} q_i^s E_i^s(\tilde{x}_i^s, p) \cdot \tilde{u}_i^s(\tilde{x}_i^s) = \sum_{s=1}^{S} q_i^s \sum_{k=1}^{K} E_i^{sk}(\tilde{x}_i, p) u_i^s(\tilde{x}_i^{sk})$.

- The initial endowments are P_i-measurable and strictly positive: $e_i^s \gg 0$ for all $s = \{1, ..., S\}$.

The problem of agent i is to maximize subjective expected utility, restricted to the budget set:

$$\max_{\tilde{x}_i \in \tilde{B}_i(e_i, p)} \tilde{U}_i(\tilde{x}_i, p) = \max_{\tilde{x}_i \in \tilde{B}_i(e_i, p)} \sum_{s=1}^{S} q_s \sum_{k=1}^{K} E_i^{sk}(\tilde{x}_i, p)\ \tilde{u}_i^s(\tilde{x}_i^{sk}).$$

A *subjective expectations equilibrium* of the economy with uncertain delivery is a triple, $(\tilde{x}^*, x^*, p^*)$, composed by a price system p^*, an allocation $x^* = (x_1^*, ..., x_n^*)$, and P_i-measurable vectors of lists $\tilde{x}_i^*$. These are such that, for every agent i:

(1) The list $\tilde{x}_i^*$ maximizes subjective expected utility, $\tilde{U}_i(\tilde{x}_i^*, p^*)$, in the agent's budget set, $\tilde{B}_i(e_i, p^*)$.

(2) The bundles selected for delivery, x_i^*, belong to the set of alternatives defined in the lists $\tilde{x}_i^*$, that is, $\tilde{x}_i^* \in \tilde{X}^S(x_i^*)$.

(3) The allocation, x^*, is feasible. That is, $\sum_i x_i^* \leq \sum_i e_i$.

Taking prices as given, each agent trades its initial endowments, e_i, for a P_i-measurable vector of state-contingent lists, $\tilde{x}_i^*$, that maximizes subjective expected utility, $\tilde{U}_i(\tilde{x}_i, p^*)$, in the budget set, $\tilde{B}_i(e_i, p^*)$.[8] The brokers take the endowments of the agents to an internal market for contingent goods, where they trade among themselves, seeking to buy bundles that satisfy the requirements of the lists that they promised to deliver to the agents. Brokers should buy the cheapest of the bundles that keep their promises, and, in this case, the price that they pay for these bundles is equal to the price that they charged for the list. These are the bundles that the agents actually receive for consumption, and obviously must constitute a feasible allocation.

[8]The information of the agents is such that, if state s occurs, they can only claim the right to receive a bundle that is in one of the lists $\tilde{x}_i^t$ with $t \in P_i(s)$. This way, any vector of state-contingent lists, $\tilde{y}_i$, can be substituted by one that is P_i-measurable, with the set of alternatives in state s being $\tilde{x}_i^s = \cup_{t \in P_i(s)} \{\tilde{y}_i^t\}$.

2.6 Existence of equilibrium

To establish existence of equilibrium, it is useful to define first a sort of 'perceived utility' or 'value function', $V_i(x_i, p)$, as the maximum expected utility of a list that the bundle x_i can deliver.

$$V_i(x_i, p) = \max_{\tilde{x}_i \in \tilde{X}^S(x_i)} \tilde{U}_i(\tilde{x}_i, p).$$

Lists have a maximum of K alternatives, and bounded coordinates, that is, $\tilde{x}_i \in [0, T]^{SKl}$. This way, the correspondence from bundles to sets of lists, $\tilde{X}^S(x)$, is continuous with non-empty compact values. Since the objective function, $\tilde{U}_i(\tilde{x}_i, p)$, is continuous, we can apply Berge's Maximum Theorem to find that the value function, $V_i(x_i, p)$, is continuous. Assumption 3 implies that the set of bundles x_i that maximizes $V(x_i, p)$ in the budget set is convex.

A hidden budget restriction can be defined in terms of bundles instead of lists. The cost of a list equals the cost of the cheapest alternative. Thus, lists in the budget of an agent must belong to some $\tilde{X}^S(x)$, with $x \in B_i(e_i, p)$.

$$B_i(e_i, p) = \left\{ x_i \in [0, T]^{Sl}, \text{ such that } \sum_{s=1}^{S} p^s x_i^s \leq \sum_{s=1}^{S} p^s e_i^s \right\}.$$

The problem of the consumer can be solved in two steps:

$$x_i = \text{argmax}_{x \in B_i(e_i, p)} V_i(x, p);$$

$$\tilde{x}_i = \text{argmax}_{\tilde{x} \in \tilde{X}(x_i)} \tilde{U}_i(\tilde{x}, p).$$

The idea of the proof is to find (x^*, p^*) using a classical fixed-point argument, and then determine $\tilde{x}^*$ solving the second step of the consumer's problem.

Theorem 1

Let $\mathcal{E} \equiv (e_i, u_i, P_i, q_i, E_i)_{i=1}^n$ be an economy with uncertain delivery satisfying assumptions 1, 2 and 3.

There exists a triple $(\tilde{x}^*, x^*, p)$ that is an equilibrium of $\mathcal{E}$.

Proof.

Consider correspondences, ψ_i, which assign to given prices, p, bundles, x_i', that maximize $V_i(x_i, p)$ in the budget set, $B_i(e_i, p)$.

$$\psi_i : [0, T]^{nSl} \times \Delta_+^{Sl} \to [0, T]^{Sl};$$

$x_i' \in \psi_i(x,p) \Leftrightarrow x_i' = \text{argmax}_{x_i \in B_i(e_i,p)}\{V_i(x_i,p)\}$, $\forall i$.

Consider also a correspondence, ψ_p, that assigns to the total demand, $\sum_i x_i$, the prices, p', which maximize the value of excess demand:

$\psi_p : [0,T]^{nSl} \times \Delta_+^{Sl} \to \Delta_+^{Sl}$;

$p' \in \psi_p(x,p) \Leftrightarrow p' = \text{argmax}_{p \in \Delta_+^{Sl}}\{p \cdot \sum_i (x_i - e_i)\}$.

We know that the objective functions, V_i and $V_p(x,p) = p \cdot \sum_i (x_i - e_i)$, are continuous, and that $B_i(e_i,p)$ is a continuous correspondence. We are, therefore, in the conditions of application of Berge's Maximum Theorem, which shows that each of the correspondences ψ_i and ψ_p is upper hemicontinuous with non-empty and compact values. They also have convex values because the objective functions are quasi-concave. The product correspondence retains these properties and maps a compact set into itself:

$\psi \equiv \prod_{i=1}^{n} \psi_i \times \psi_p$;

$\psi : [0,T]_+^{nSl} \times \Delta_+^{Sl} \to [0,T]_+^{nSl} \times \Delta_+^{Sl}$;

$(x',p') \in \psi(x,p) \Leftrightarrow x_i' \in \psi_i(x,p)$, $\forall i$ and $p' \in \psi_p(x,p)$.

Existence of a fixed-point, (x^*, p^*), follows from Kakutani's Theorem.

The fact that p^* maximizes the value of excess demand implies that:

$p' \cdot \sum_i (x_i^* - e_i) \leq p^* \cdot \sum_i (x_i^* - e_i) \leq 0$, for all $p' \in \Delta_+^{Sl}$.

Making $p' = e^j = (0, ..., 1, ..., 0)$, for each j, shows that x^* is a feasible allocation: $\sum_i (x_i^* - e_i) \leq 0$.

Observe also that x_i^* solves the first step of the consumer's problem.

Finding the lists that are offered to the agents solves the problem of the consumer, and completes the triple of equilibrium $(\tilde{x}^*, x^*, p^*)$. By continuity and compacity, we can find the lists that maximize $\tilde{U}_i(\tilde{x}_i^*, p^*)$ among those that can be offered with resources x_i^*:

$\tilde{x}_i^* = \text{argmax}_{\tilde{x}_i \in \tilde{X}^S(x_i^*)} \tilde{U}_i(\tilde{x}_i, p^*)$.

This completes the proof. QED

In equilibrium, the delivered bundles, x_i^*, have two important properties, which support the interpretation of the market mechanism as being driven by informed brokers trading in an internal market:

(1) The Law of Walras is satisfied;[9]

(2) The delivered bundles are the cheapest among the alternatives in the list.[10]

3 Concluding Remarks

In economies with uncertain delivery, objects of choice are lists of bundles and the market selects one of the bundles in the chosen list for delivery. Economic equilibrium can be seen (but not necessarily) as the result of the interaction between agents and brokers. Brokers offer lists in exchange for the agent's endowments, and then trade among themselves in order to obtain a bundle that keeps the contract for delivery of the list that they promised to the agent.

Agents can only guess the probabilities of receiving each of the bundles in a list. Existence of equilibrium with subjective expectations is conditional on the specific expectations (see assumptions 1, 2 and 3). In essence, the expectations functions, $E_i(\tilde{x}_i, p)$, must imply continuity, quasi-concavity and monotonicity of the perceived utility, $V_i(x_i, p)$, of the bundles used by the brokers to design lists for the agents.

In this model, agents trade before receiving their private information. Accordingly, the solution it is more comparable with the concept of Walrasian Expectations Equilibrium (WEE - Radner, 1968), which can also be seen as an *ex-ante* notion, than with the concepts of Rational Expectations Equilibrium (REE - Radner, 1979) and Bayesian Walrasian Equilibrium (BWE - Balder and Yannelis, 2005), which are *interim* notions. Notice that, with trade being made *ex ante*, the state of nature cannot be revealed by prices, because it still did not occur.

An assumption made here that is common to the BWE is that agents are not assumed to know all the primitives in the economy (the random initial endowments, random utility functions and private information sets of all the agents). On the

[9] It is easy to see that assumption 2 implies weak monotonicity of $V_i(x_i, p)$. As a result, maximizers are in the frontier of the budget set. This, in turn, implies that Law of Walras is satisfied: $p^* \cdot \sum_i x_i^* = p^* \cdot \sum_i e_i = 0$.

[10] To see this, notice that if this was not the case, a small positive z could be added to the cheapest alternative, x_i', to construct a deliverable list, $\tilde{U}_i(\tilde{x}_i^* + z, p^*)$, with more utility (as a consequence of assumption 2), and still with $p^* \cdot (x_i' + z) \leq p^* \cdot e_i$. This would imply that $V_i(x_i' + z, p^*) > V_i(x_i^*, p^*)$, a contradiction.

other hand, our measurability restriction is weaker, as it is imposed on lists and not on the actual allocation.

In economies with uncertain delivery, agents are not restricted to consume the same bundle in states that they are not able to distinguish (in the sense of verifying the difference, for example, in a court of law). Agents buy lists of bundles, having the right to receive one of the bundles in the list. As a consequence, they may receive different alternatives in states that they do not distinguish.

Efficiency of trade is improved in a certain sense. For given prices, the maximizing choice of agents yields higher utility for everyone than under the restriction of equal consumption in undistinguished states (increase in indirect utility is a natural consequence of the extension of the consumption set). Yet, this does not imply a Pareto improvement of welfare because equilibrium prices will be different from those that constitute the WEE. Obviously, the new equilibrium prices will be more favorable to some agents, but may be more adverse to others. Some agents may not benefit with the opening of markets for lists.

The expansion of trade possibilities does not create problems of incentive compatibility. As in differential information economies, the P_i-measurability restriction (that here is in terms of lists) implies Bayesian incentive compatibility (Koutsougeras and Yannelis, 1993). On the other hand, 'coalitional Bayesian incentive compatibility' is not guaranteed in general, since we assumed "free disposal" (Glycopantis, Muir and Yannelis, 2002).

An example

To clarify the equilibrium concept presented in this paper, we recast the example of Kreps (1977), as did Balder and Yannelis (2005). Consider an economy with two agents, A and B, two equally probable states of nature, $\Omega = \{s, t\}$, and two goods, 1 and 2, in each state of nature. The informed agent, A, distinguishes the two states: $P_A = \{\{s\}; \{t\}\}$; while the uninformed agent, B, does not: $P_B = \{s, t\}$. The agents make trade agreements before receiving their contingent endowments and their information. Endowments are given by:

$$e_A(s) = e_A(t) = e_B(s) = e_B(t) = (1.5, 1.5).$$

Their state-dependent utility functions are:

$$u_A^s(x(s)) = ln(x_1(s)) + x_2(s); \quad u_A^t(x(t)) = 2ln(x_1(t)) + x_2(s);$$

$$u_B^s(x(s)) = 1.5ln(x_1(s)) + x_2(s); \quad u_B^t(x(t)) = 1.5ln(x_1(t)) + x_2(t).$$

To have a specific form for the expectations functions, $E_i(\tilde{x}_i, p)$, we assume that the agents have *prudent expectations*. This is a particular case of subjective expectations where agents expect to receive the bundle with the lowest utility (independently of prices, p). In this case, adding alternatives to a list does not increase its utility. Anyway, an agent may choose a list with several alternatives if it makes the list cheaper - this occurs when prices are different in states that the agent does not distinguish (Correia-da-Silva and Hervés-Beloso, 2006).[11]

The informed agent, A, distinguishes everything, and, therefore, simply maximizes expected utility:

$$U_A(x_A) = 0.5ln(x_{A1}(s)) + 0.5x_{A2}(s) + ln(x_{A1}(t)) + 0.5x_{A2}(t).$$

Agent B may gain by choosing a list with two alternatives, in spite of not knowing which alternative will be delivered in each state. Assuming 'prudent expectations':

$$U_B(x_B) = min_{j=s,t}\{1.5ln(x_{B1}(j)) + x_{B2}(j)\}.$$

Equating marginal rates of substitution to price ratios, we obtain:

$$\frac{1}{x_{A1}(s)} = \frac{p_1(s)}{p_2(s)};\ \ \frac{2}{x_{A1}(t)} = \frac{p_1(t)}{p_2(t)};\ \ 1 = \frac{p_2(s)}{p_2(t)};\ \ \frac{1.5}{x_{B1}(s)} = \frac{p_1(s)}{p_2(s)};\ \ \frac{1.5}{x_{B1}(t)} = \frac{p_1(t)}{p_2(t)}.$$

An alternative price normalization makes calculations easier: set $p_2(s) = p_2(t) = 1$. This yields:

$$\frac{1}{x_{A1}(s)} = \frac{1.5}{x_{B1}(s)} = p_1(s);\ \ \frac{2}{x_{A1}(t)} = \frac{1.5}{x_{B1}(t)} = p_1(t).$$

Solving:

$$p_1(s) = \tfrac{5}{6};\ \ p_1(t) = \tfrac{7}{6};\ \ p \cdot e_A = p \cdot e_B = 6;$$

$$x_A(s) = (1.20, 1.75);\ x_A(t) = (1.71, 1.25);\ u_A^s = 1.935;\ u_A^t = 2.326;\ U_A = 2.130;$$

$$x_B(s) = (1.80, 1.25);\ x_B(t) = (1.29, 1.75);\ u_B^s = 2.129;\ u_B^t = 2.129;\ U_B = 2.129.$$

In this example, the WEE and the private core are the initial endowments (Glycopantis, Muir and Yannelis, 2005). There is no trade.

$$x_A(s) = (1.5, 1.5);\ x_A(t) = (1.5, 1.5);\ u_A^s = 1.905;\ u_A^t = 2.311;\ U_A = 2.108;$$

$$x_B(s) = (1.5, 1.5);\ x_B(t) = (1.5, 1.5);\ u_B^s = 2.108;\ u_B^t = 2.108;\ U_B = 2.108.$$

It is straightforward to interpret the welfare gains generated by uncertain delivery. Buying a list with two alternatives, agent B guaranteed the right to receive

[11] Remember that the price of a list is equal to the price of the cheapest bundle that can be delivered to keep the contract for the delivery of the list.

$(1.80, 1.25)$ or $(1.29, 1.75)$. This flexibility of agent B relatively to the allocation is welfare enhancing. Good 1 is more useful to agent A in state t than in state s, thus the market 'appreciates' the possibility of delivering more of this good to agent B in state s than in state t (in return, the market delivers more of good 2 in state t than in state s).[12] Agent B prefers any of the alternatives in the list, $(1.80, 1.25)$ and $(1.29, 1.75)$, to the autarky solution, $(1.50, 1.50)$. Moreover, agent B is indifferent between the two alternatives. Therefore, the possibility of being deceived does not worry the uninformed agent.

Our result differs from that given by BWE, which is an *interim* notion. The BWE allocation is (Balder and Yannelis, 2005):

$x_A(s) = (1.45, 1.54)$; $x_A(t) = (1.46, 1.55)$; $u_A^s = 2.277$; $u_A^t = 1.931$; $U_A = 2.104$;

$x_B(s) = x_B(t) = (1.54, 1.46)$; $u_B^s = 2.110$; $u_B^t = 2.110$; $U_B = 2.110$.

[12]It may be verified that the cheapest way for a broker to deliver the list bought by agent B is to deliver $(1.80, 1.25)$ in state s and $(1.29, 1.75)$ in state t.

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