The Topology of Fear*

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November 2005 revised August 2006

Abstract

Many years of experimental observations have raised questions about the *rationality* of economic agents - for example, in the Allais Paradox or the Equity Premium Puzzle. The problem is a narrow notion of rationality that fails to account how we respond to fear. In this article we extend the notion of rationality with new axioms of choice under uncertainty and the decision criteria they imply [9] [10]. In the absence of catastrophes, the old and the new approach coincide, and both lead to standard expected utility. A sharp difference emerges when facing catastrophes. We found that one classic axiom of choice postulates that rational behavior should be 'insensitive' to rare events with major consequences (Theorem 2), which does not seem right. This by itself can lead to paradoxes. We replace this with an axiom that allows extreme responses to extreme events. The implied decision criteria are a combination of expected utility with extremal responses. They offer a new understanding of rationality that seems consistent with previously unexplained observations, including the Equity Premium Puzzle, 'jump diffusion' processes and 'heavy tails', and agrees also with Gerard's Debreu's 1953 formulation of market behavior and Adam Smith's Invisible Hand [13].

1 Introduction

Catastrophes are rare events with major consequences. They play a special role in our decision processes. Using magnetic resonance experimental psychologists have observed different brains' reaction to situations involving extreme fear. Certain regions of the brain - for example, the *amygdala* - often light up when a person makes decisions while confronted with events that inspire extreme

*This research is part of the Columbia's Program on Information and Resources, and the Columbia Consortium for Risk Management (CCRM). A first version of this paper, without Theorem 2, was presented at the NBER Conference *Mathematical Economics: The Legacy of Gerard Debreu* at UC Berkeley, October 21, 2005. I am grateful to Kenneth Arrow, Bernard Cornet, Jean Michel Grandmont, Peter Hammond, Marcel Richter and other participants for valuable discussions.

fear[1] [18]. Neurologists believe that such events alter cognitive processes and the behavior that could be otherwise expected. These observations seem relevant to the issue of rationality in decision making under uncertainty, and inspired the results presented in this article.

Allais [1],Tversky [29], and Kahneman [16] have all shown that the predictions of standard economic models based on rational behavior conflict with the experimental evidence on how we make decisions.[2] In this article we argue that the problem lies in the standard definition of rationality, which is narrow and exclusively based on testing whether or not we optimize *expected utility.* Expected utility has known limitations. It underestimates our responses to rare events no matter how catastrophic they may be [9]. This insensitivity has unintended consequences. It creates an illusion of 'irrational behavior' since what we anticipate does not agree with what we observe.

We argue that the insensitivity of expected utility to rare events, and the attendant inability to explain responses to events that invoke fear, are the source of many of the experimental paradoxes and failures of rationality that have been found over the years. This paper traces the problem to one classic axioms of choice under uncertainty that was developed half a century ago by von Neumann, Morgenstern, Arrow, Hernstein and Milnor. Arrow calls it Monotone Continuity (MC) [2] and Hernstein and Milnor [15] call it Axiom 2. The axiom requires that nearby stimuli should lead to nearby responses, which is reasonable enough. However the notion of 'nearby' that is used in these classic pieces is not innocent: it implies *insensitivity to rare events with major consequences* (Theorem 2). This underestimates our responses to catastrophes, creating an impression of irrationality that is not entirely reasonable. One solution is to negate the suspect axiom and require *sensitivity to rare events.* This is the approach followed here. In the absence of rare events however both approaches coincide. Therefore this work provides an extension of classic decision theory.

In the face of catastrophes our approach suggests new ways of evaluating risks. As an illustration consider the rule of thumb 'maximize expected utility while minimizing the worst losses in the event of a catastrophe'. This rule is *inconsistent* with expected utility. Therefore any observer that anticipates expected utility optimization will be disappointed, and will believe that there is irrationality at stake. But this is not true. The rule is rational once we take into account rational responses to extreme events. It is consistent with what people do on an everyday basis, and with the experimental evidence on 'jump - diffusion processes' and 'heavy tails' distributions that are not well explained by standard theory.[3]

[1] Not all rare events invoke fear, but situations involving extreme fear are generally infrequent.

[2] These works focus on the predictions of *expected utility* in general terms, and their findings led to the creation of *behavioral economics.*

[3] In practical terms the criteria we propose can help explain experimental observations that conflict with the standard notions of rationality: the Allais paradox (Chichilnisky [8], [9] [10]), the Equity Premium and the Risk Free Rate Puzzles (Mehra and Prescott [26], Mehra [22], Weil [33], Chichilnisky, Dasol and Wu, [12]), and the prevalence of *heavy tails* and jump-diffusion processes in financial markets (Chichilnisky and Shmatov [11]).

The key to the new decision criteria is a 'topology of fear,' that can be sharply sensitive to catastrophes no matter how infrequent they may be.[4] A similar topology was used in Debreu's 1953 formulation [13] of Adam Smith's Invisible Hand theorem.[5] The rankings of lotteries that satisfy the new axioms for choice under uncertainty are a mix of 'countably additive measures' with 'purely finitely additive measures' with both parts present. This combination has not been used before except in [7] [9] [10] [8] [11] [12]. However purely finitely additive measures could play an important role in explaining how our brains respond to extreme risks.

2 Definitions and Examples

Uncertainty is described by a system that is in one of several **states**, indexed by the real numbers endowed with a standard Lebesgue measure. In each state a **utility function** $u : R^n \to R$ ranks the **outcomes**, which are described by vectors in R^n.[6] When the probability associated with each state is given, a description of the utility achieved in each state is called a **lottery.**[7] In our context a **lottery** is a function $f : R \to R$ and the space of all lotteries is a function space L that we take to be the space of measurable and essentially bounded functions $L = L_\infty(R)$ with the standard norm $\| f \| = ess \sup_{x \in R} | f(x) |$.[8] This is the same $L_\infty(R)$ space used in 1953 by Debreu and others to identify commodities in infinitely many states [13], [4]. **Choice under uncertainty** means the ranking of lotteries. An **event** E is a set of states. E^c is the set of states of the world not in E. A **monotone decreasing** sequence of events $\{E^i\}_{i=1}^\infty$, is a sequence for which for all i, $E^{i+1} \subset E^i$. If there is no state on the world common to all members of the sequence, $\bigcap_{i=1}^\infty E^i = \varnothing$, $\{E^i\}$ is called a **vanishing sequence**. **Axioms** for choice under uncertainty describe natural and self evident properties of choice. Continuity is a standard requirement, it captures the notion that nearby stimuli give raise to nearby responses. However continuity depends on the notion of 'closeness' that is used. In Arrow [2] P., 48, two lotteries[9] are **close** to each other when they have different consequences in **small** events, which Arrow defines as follows "An event that is far out on a

[4] This refers to the sup norm in L_∞.

[5] Debreu's article was submitted to the National Academy of Sciences by John Von Neumann, but its formulation goes much beyond Von Neumann's own axioms and his own formulation of expected utility.

[6] When more is better, the function $u : R^n \to R$ is a monotonically increasing continuous function.

[7] The definition of a *lottery* can take several forms, all leading to similar results. It can be defined by probability distributions given the outcomes in each state, or by outcomes rather than by the utility of these outcomes as we do here. We adopt this definition to simplify the presentation.

[8] *Boundedness* is required to fit observed behavior, as explained by Arrow [2] in particular the St. Peterburg's Paradox, discovered by Bernoulli, where people in St. Peterburg only wished to pay a finite amount of money to play a lottery that had an infinite expected value.

[9] The equivalent to the notion of "lotteries" in our frameowork is the notion of "actions" in Arrow [2].

vanishing sequence is '**small**' by any reasonable standards" (Arrow, [2] P. 48). Our definition, using the L_∞ norm, is quite different,[10] some catastrophic events are small under Arrow's definition but not necessarily under ours, see Example 5 in the Appendix.

Axiom of Monotone Continuity (MC) *(Arrow [2], p.48) Given a and b, where $a \succ b$, a consequence c, and a vanishing sequence $\{E^i\}$, suppose the sequences of actions $\{a^i\}, \{b^i\}$ satisfy the conditions that (a^i, s) yield the same consequences as (a, s) for all s in $(E^i)^c$ and the consequence c for all s in E^i, while (b^i, s) yields the same consequences as (b, s) for all s in $(E^i)^c$ and the consequence c for all s in E^i. Then for all i sufficiently large, $a^i \succ b$ and $a \succ b^i$.*

A ranking function $W : L_\infty \to R$ is called **insensitive to rare events** when it neglects small probability events; formally if given two lotteries (f, g) there exists $\epsilon = \epsilon(f, g) > 0$, such that $W(f) > W(g)$ if and only if $W(f') > W(g')$ for all f', g' satisfying $f' = f$ and $g' = g$ a.e. on $A \subset R$ when $\mu(A^c) < \epsilon$.[11] Similarly, $W : L \to R$ is said to be **insensitive to frequent events** when for every two lotteries f, g there exists such $\epsilon(f, g) > 0$ that $W(f) > W(g)$ if and only if $W(f') > W(g')$ for all f', g' such that $f' = f$ and $g' = g$ a.e. on $A \subset R : \mu(A^c) > 1 - \epsilon$.

We say that W is **sensitive to rare events,** when W is **not** insensitive to small probability events, and we say that W is **sensitive to frequent events** when W is **not** insensitive to frequent events. The ranking W is called **continuous and linear** when it defines a linear function on the utility of lotteries that is continuous with respect to the norm in L_∞. Here are the new axioms introduced in [9] [10]:

Axiom 1: The ranking $W : L_\infty \to R$ is linear and continuous on lotteries

Axiom 2: The ranking $W : L_\infty \to R$ is sensitive to rare events

Axiom 3: The ranking $W : L_\infty \to R$ is sensitive to frequent events

The **expected utility** of a lottery f is a ranking defined by $W(f) = \int_{x \epsilon R} f(x) d\mu(x)$ where μ is a measure with an integrable density function $\phi(.)$ $\epsilon\ L_1$[12] so $\mu(A) = \int_A \phi(x) dx$, where dx is the standard Lebesgue measure on R. Expected utility satisfies Axioms 1 and 3, but not Axiom 2:

[10] When the set of states R is endowed with a standard Lebesgue measure the notion of 'smallness' according to Arrow's definition implies that a small event has a small Lebesgue measure. In Arrow's definition of closeness, two lotteries are close when they differ in sets of small probability. Infrequent events matter little under his definition. In our case this may not not true in some cases, as seen in the definitions and Theorems below.

[11] A^c denoted the complement of the set A.

[12] $L_1(R)$ is the space of all measurable and integrable functions on R, $f \epsilon L_1 \Leftrightarrow \| f \| = \int_R | f(x) | dx < \infty$.

Proposition 1: *Expected utility is insensitive to rare events.*

For a proof see (Chichilnisky [9] [8] [10])

The following theorem identifies all the decision criteria that satisfy Axioms 1, 2 and 3. Examples are provided in the Appendix.

Theorem 1 *A ranking of lotteries $W : L_\infty \to R$ satisfies all three axioms 1, 2 and 3, if and only if there exist two continuous linear functions on L_∞, ϕ_1 and ϕ_2 and a real number λ, $0 < \lambda < 1$, such that:*

$$W(f) = \lambda \int_{x \epsilon R} f(x)\phi_1(x)dx + (1-\lambda)\langle f, \phi_2 \rangle \tag{1}$$

where $\int_R \phi_1(x)dx = 1$, while ϕ_2 is a purely finitely additive measure.[13]

For a proof see *Chichilnisky* [9] [8] [10].

The first term in (1) is similar to expected utility and the density function could be for example $\phi_1(x) = e^{-\delta x}$. This density defines a countably additive measure that is absolutely continuous with respect to the Lebesgue measure.[14] The second term of (1) is of a different type: the operator $\langle f, \nu_2 \rangle$ represents the action of a measure $\phi_2 \epsilon L^*_\infty$ that is **singular** with respect to the Lebesgue measure, giving all weight to rare events. The following *rules of thumb* provide an intuitive description of the decision criteria that satisfy all three axioms, and actual examples are provided in the Appendix:

Example 1 Choosing a portfolio: Maximize expected utility while seeking to minimize total value losses in the event of a catastrophe.

Example 2 Network Optimization: Maximize expected through-put of electricity while seeking to minimize the probability of a black - out.

Example 3 Heavy Tails: The following function illustrates the singular measure that appears in the second term in (1) a special case, for those lotteries in L_∞ that have limiting values at infinity, $L'_\infty = \{f \epsilon L_\infty : \lim_{x\to\infty}(x) < \infty\}$. Define

$$\Psi(f) = \lim_{x\to\infty} f(x) \tag{2}$$

Ψ is a continuos linear function on L_∞ that is not representable by an L_1 function as in (1); it is also insensitive to events that are frequent with respect to the Lebesgue measure because it only takes into consideration limiting behavior, and does not satisfy Axiom 3. This asymptotic behavior tallies with the observation of *heavy tails* in financial markets [11]. Observe that the function Ψ is only defined on a subspace of L_∞; to define a purely finitely additive measure on all of L_∞ one seeks to extend Ψ to all of L_∞. The last section of this article describes what is involved in obtaining such an extension.

[13] Definitions of 'countably additive' and 'purely finitely additive' measures are in the Appendix under the heading *The dual space L^*_∞*. Observe that ϕ_2 cannot be represented by an L_1 function.

[14] A measure is called *absolutely continuous* with respect to the Lebesgue measure when it asigns zero measure to any set of Lebesgue measure zero; otherwise the measure is called *singular*.

3 Two Approaches to Decision Theory

This section compares our approach with the classic theory of choice under uncertainty. Both approaches postulate that nearby stimuli lead to nearby responses or choices. There are however different views of what constitutes 'nearby.' The classic theory considers two lotteries to be close when they differ in events of small measure, while our notion is stricter, requiring that the lotteries be close almost everywhere, which implies sensitivity to rare events. See Example 5 in the Appendix.

The best way to compare the two approaches is to put side by side the decision criteria that each implies. The classic theory of choice under uncertainty, as presented e.g. in Arrow ([2]) and Hernstein and Milnor ([15]) shows that, on the basis of standard axioms of choice, the ranking of lotteries $W(f)$ is an *expected utility function.*[15]

Our decision criteria are quite different. Expected utility rankings do not satisfy our crucial sensitivity Axiom 2, as shown in Proposition 1.[16] We need to modify expected utility adding another component called 'purely finitely additive' elements of L_∞^*.[17] The latter embody the notion of sensitivity for rare events. Purely finitely additive measures are the main difference between our decision crieteria and the classic expected utility criterion.

Where exactly does this difference originate? It is possible to trace the difference between the two approaches to a single classic axioms of choice, the Axiom of Monotone Continuity (MC) showing that it contradicts our sensitivity Axiom 2:

Theorem 2: *A ranking of lotteries $W(f) : L_\infty \to R$ satisfies the Monotone Continuity Axiom if and only if it is insensitive to rare events.*

Proof:

A formal statement of the theorem is $MC \Leftrightarrow \sim Axiom\ 2$. A simple example shows why the MC axiom leads to insensitivity to rare events.

Assume that the Monotone Continuity Axiom MC is satisfied. By definition, this implies for every two lotteries $f \succ g$, every outcome c and every vanishing

[15] This means that there exists an utility function $u : R \to R$ and a probability density $\delta(t) : R \to R$, $\int_R \delta(t)dx = 1$ such that f is preferred over g if and only if $W(f) > W(g)$, where $W(f) = \int_R u(f(t), t)d\delta(t)$. In our framework, the classic axioms imply that the ranking $W(.)$ is a continuous linear function on L_∞ that is defined by a countably additive measure with an integrable density function $\delta(t)$ i.e. δ is in $L_1(R)$.

[16] The space L_∞^* is called the 'dual space' of L_∞, and is known to contain two different types of rankings $W(.)$, (i) integrable functions in $L_1(R)$ that can be represented by countably additive measures on R, and (ii) 'purely finitely additive measures'which are not representable by functions in L_1[9], cf. the Appendix.

[17] Indeed, there is an entire subspace of L_∞^* that consists of functions that do not have a representation as L_1 functions in R. This subspace consists of 'purely finitely additive measures' that are defined in the Appendix, with examples provided there. Purely finitely additive measures are not representable by countably additive measures, they are not continuous with respect to the Lebesgue measure of R, and they cannot be represented as functions in $L_1(R)$.

sequence of events $\{E^i\}$ there exists N such that altering arbitrarily the outcomes of the lotteries f and g on the event E^i, where $i > N$, does not alter the ranking namely $f' \succ g'$, where f' and g' are the altered versions of lotteries f and g respectively.[18] In particular since, for any given f and g, Axiom MC applies to **every** vanishing sequence of events $\{E^i\}$, we may choose a sequence of events consisting of open intervals $I = \{I^i\}_{i=1}^{\infty}$ such that $I^i = \{x \in R : x > i\}$ and another $J == \{J^i\}_{i=1}^{\infty}$ such that $J^i = \{x \in R : x < -i\}$. Consider the sequence $K = \{K^i\}$ where $K^i = I^i \bigcup J^i$. The sequence K is a vanishing sequence by contruction. Therefore there exists an $i > 0$ such that for all $N > i$, any alternations of the lotteries f and g over K^N, denoted f^N and g^N respectively, leave the ranking unchanged i.e. $f^N \succ g^N$. Therefore Axiom of MC implies insensitivity of the ranking W in unbounded sets of events such as $\{E^i\}$.

More generally, assume that the ranking satisfies Axiom 2, and is therefore sensitive to rare events. We will show that W cannot satisfy MC. Axiom 2 implies that there at least are two lotteries f, g with $f \succ g$, and for these two lotteries a sequence of measurable sets $U = \{U^n\}_{n=1}^{\infty}$ such that for all $n = 1, 2...$, (i) $\mu(U^n) < 1/n$, where μ is the Lebesgue measure on R, and (ii) altering the lotteries f and g on the set U^n reverses their ranking namely $f^n \prec g^n$, where f^n and g^n are the lotteries resulting from the alteration of f and g on the set U^n. Now, without loss of generality we may assume for every $n = 1, ...$, the sets in the sequence U satisfy $U^{n+1} \subset U^n$. Otherwise, one can construct another sequence $V = \{V^n\}$, where $V^n = \bigcup_{j=1}^{\infty} U^j - \bigcup_{j=i}^{n} U^j$, such that V has the same ranking reversal property (ii), and by choosing appropriately a subsequence of $\{V^n\}$, denoted also $\{V^n\}$, we can ensure that $\mu(V^n) < 1/n$. Observe that this new family V satisfies $(i), (ii)$, and by construction for all n, $V^{n+1} \subset V^n$. Furthermore, without loss of generality we may assume also that (iii) the family of sets $\{U^n\}$ has empty intersection, i.e. $\bigcap_{n=1}^{\infty} U^n = \varnothing$. This is because the intersection of the family $\{U^n\}$, $\bigcap_{n=1}^{\infty} U^n$, has Lebesgue measure zero since, by construction, for all n, $\mu(U^n) < 1/n$. But the lotteries f and g are both elements of L_∞, and are therefore defined almost everywhere in the Lebesgue measure in R, so we may therefore consider the new family $\{W^n\}$ such that $W^n = U^n - (\bigcap_{n=1}^{\infty} U^n)$ without changing the lotteries f and g in any way, and in particular keeping the properties (i) and (ii). Because the family U satisfies (ii) and (iii) it is, by definition, a vanishing family. If the Axiom of Monotone Continuity were satisfied, then for some n, and all $N > n$, altering the lotteries f and g on the set U^N leads to $f^N \succ g^N$, a contradiction with (ii). Therefore whenever Axiom 2 is satisfied, the Axiom of Monotone Continuity does not hold. This completes the first part of the proof.

It remains to establish, conversely, that the negation of Axiom MC implies Axiom 2. Assume that Axiom MC is not satisfied. Then for some pair of

[18] For simplicity, one may consider alterations in those lotteries that involve the 'worst' outcome $c = \inf_R \mid f(x), g(x) \mid$, which exists because f and g are bounded a.e. on R by assumption.

lotteries f, g and outcome c there exists a vanishing sequence $\{E^i\}$ such that for every $N > 0$, there is an $i > N$ such that $f \succ g$, but arbitrary alterations of f and g on E^i reverse the ranking, i.e. $g^i \succ f^i$. Observe that since $\{E^i\}$ is a vanishing sequence and μ a Lebesgue measure on R, $lim_{N\to\infty}\mu(E^N) = 0$. The negation of Axiom MC implies therefore that altering f and g on sets of arbitrarily small measure reverses the ranking, i.e. $g^i \succ f^i$. Since this holds for all $N > 0$, there is no $\varepsilon = \varepsilon(f, g) > 0$ such that $f \succ g \Leftrightarrow f^i \succ g^i$ and thus by definition, the ranking W is sensitive to rare events or equivalently, Axiom 2 is satisfied. We have therefore shown that the negation of Axiom MC implies Axiom 2.

The Monotone Continuity Axiom is therefore equivalent to insensitivity to rare events, as we wished to prove. ■

If one considers a family of subsets of events containing no rare events, for example when the Lebesgue measure of all events contemplated is bounded below, then the two approaches are identical:

Theorem 3: Absent rare events, Axioms 1, 2, and 3 are consistent with Expected Utility Theory.

Proof: Axiom 2 is an empty requirement when there are no rare events, and Axioms 1 and 3 are consistent with Expected Utility.■

4 The Topology of Fear and the Value of Life

Our axioms and Debreu's 1953 work [13] use the same notion of continuity or 'closeness' of lotteries. **Proximity** of two lotteries requires that the **supremum** of their distance across all states a. e. should be small, and distance is therefore measured by extremals. The formal description of this topology is the standard sup norm of L_∞. Since the topology focuses on situations involving extremal events, such as catastrophes, this may be called the **topology of fear**.

Our Axiom 1 requires continuity of the ranking,[19] and it **is** satisfied by expected utility functions. However our Axiom 2 requires a further condition, that W be sensitive to rare events with major consequences. Expected utility does not satisfy this condition as shown in Proposition 1. As shown in Theorem 1, the only acceptable rankings W under our axioms are a convex combination of L_1 function *plus a* purely finitely additive measure that put all weight on extreme or rare events. Both parts must be present. By contrast, under the classic axioms of Arrow [2], Hernstein and Milnor [15] the rankings W are expected utilities.[20]

[19] With respect to the sup norm.

[20] Arrow [2], p. 257, introduces the axiom of *Monotone Continuity* attributing it to C. Villegas [32],1964, p. 1789. It requires that modifying an action in events of small probabilities should lead to similar rankings. At the same time Hernstein and Milnor [15] p. 293 require a form of continuity in their Axiom 2 that is similar to Arrow's *Monotone Continuity* and leads to their *Continuity Theorem* on p. 293. The axioms of continuity required by Arrow and by Hernstein and Milnor are quite different from the type of continuity that we require here, see the Appendix Example 7.

We saw that one classic requirements of the classic theory, Axiom MC, is key because it creates insensitivity to rare events. We can illustrate how this works in the following situation that was pointed out by Arrow [2] about how people value their lives. If a is an action that involves receiving one cent, b is another that involves receiving zero cents, and c is a third action involving receiving one cent and facing a small probability of death, Arrow's *Monotone Continuity* requires that the third action involving death and one cent should be preferred to the second involving zero cents when the probability of death is small enough. Even Kenneth Arrow says of his requirement 'this may sound outreageous at first blush...' [2]. Outrageous or not, we saw in Theorem 2 that MC leads to the neglect of rare events with major consequences, like death.

Theorem 2 shows that our Axiom 2 rules out these examples that Arrow calls outrageous. We can also exemplify how Axiom 2 provides a reasonable resolution to the problem, as follows. Axiom 2 implies that there exist catastrophic outcomes such as the risk of death, so terrible that one is unwilling to face a small probability of death to obtain one cent versus half a cent, no matter how small the probability may be. Indeed, according to our sensitivity Axiom 2, no probability of death is acceptable when one cent and half a cent are involved. However according to Axiom 2, in other cases, there may be a small enough probability that the lottery involving daeath may be acceptable. It all depends on what are the other outcomes involved. For example, if instead of one cent and half a cent one considers one billion dollars and half a cent - as Arrow suggests in [2] - under certain conditions we may be willing to take the lottery that involves a small probability of death and one billion dollars over the one that offers half a cent. More to the point, a small probability of death caused by a medicine that can cure an incurable cancer may be preferable to no cure. This seems a reasonable solution to the issue that Arrow raises. Sometimes one is willing to take a risk with a small enough probability of a catastrophe, in other cases one is not. It all depends on what else is involved. This is the content of our Axiom 2.

In any case, absent rare events with major consequences, our Axiom 2 is voided of meaning. In this case our theory of choice under uncertainty is consistent with or collapses to the standard expected utility theory. Therefore this work can be viewed as an extension of classic decision theory.

5 Gerard Debreu and the Invisible Hand

Gerard Debreu [13] was not explicitly concerned with rare or extremal events; the goal of his article was to show that any Pareto efficient allocation can be decentralized as the equilibrium of a competitive market. He wanted to prove Adam Smith's Invisible Hand Theorem. In the infinite dimensional case, Debreu proposed that a natural space of commodities would be the space of essentially bounded measurable functions L_∞ that we use in this article to describe our lotteries, with the same *sup norm* that we use here. Debreu chose this space because it is alone among all infinite dimensional L_p spaces in having a positive quadrant with a non - empty interior. This property is crucial to his proof of decentralization by market prices, which relies on Hahn - Banach's separating hyperplane theorem, a theorem that requires the existence of interior or internal points, see Chichilnisky [3], and Chichilnisky and Kalman [4].

Yet Debreu's choice of commodity space leads naturally to Pareto efficient allocations where prices are in the dual of L_∞ namely in L_∞^* which, as we saw above, contain both standard functions in L_1 as well as purely - finitely - additive measures that are not representable by standard functions, see Example 7 in the Appendix. In the latter case, one may lose the ability of assigning any price to a commodity within a given period of time, or within a given state of uncertainty. A non - zero price can give rise to zero value in each period and each state. Examples of this nature were constructed by Malinvaud [24] and later on by Mc Fadden [25] and were extensively discussed in the literature [28] [23] [4]. In 1980 this author and Peter Kalman provided a necessary and sufficient condition to overcome the problem, a 'cone condition' that is necessary and sufficient for the prices that support Pareto Efficient allocations to be always in L_1, Chichilnisky and Kalman [4] and Chichilnisky [5]. A few years later, this result was followed by a similar *uniform properness* condition used in Mas Collel [21]. The results in [4] [5] and [21] modified the original L_∞ model introduced by Debreu and ensured that all Pareto efficient allocations would have associated market prices that are well defined in every state and every time period. They resolved the paradoxical behavior of the infinite dimensional model that Debreu had introduced in 1953.

Debreu's original 1953 formulation was however prescient.

Fifty years later, we find that the general L_∞ framework that he proposed in 1953 is consistent with the decision criteria emerging from the new axioms for choice under uncertainty, giving rise to prices in L_∞^* that are partly in L_1 and partly not, as shown in the representation Theorem 1 above. The L_1 part overcomes the concerns expressed above, since it creates non - zero value in each period and in each state. Yet the purely finitely part allows sensitivity to *rare events.* This solution was certainly not contemplated by the literature that followed Debreu's 1953 work, which focused instead on eliminating the purely finitely additive parts. This elimination is not necessary if one takes into account - as shown above - that the criteria may have two different parts, and that one of them (in L_1) suffices to define non - zero prices in all states and periods. In any case, Gerard Debreu's 1953 theorem on the Invisible Hand is compatible with the decision criteria postulated in this paper. And while such decision criteria could have appeared to be paradoxical at the time, it seems now that they may be better suited to explain the experimental evidence and the observations of behavior under uncertainty than expected utility when rare events are at stake.

6 The Axiom of Choice and Rare Events

The Axiom of Choice postulates that there exists a universal and consistent fashion to select an element from every set. This section illustrates how it is connected with the axioms of choice under uncertainty that are proposed here.

The best way to describe the situation is by means of an example, see also [30], [31] [4].

Representing a Purely Finitely Additive Measure

Consider the purely finitely measure ρ defined in the Appendix Example 6: for every Borel measurable set $A \subset R$, $\rho(A) = 1$ whenever $A \supseteqq \{r : r > a,$for some $a \in R\}$, and $\rho(A) = 0$ otherwise. It is easy to show that ρ is not countably additive.

Consider a family of countably many disjoint sets $\{V_i\}_{i=0,1,...}$ defined as follows

$$\text{for } i = 0, 1, ..., \qquad V_i = (i, i+1) \bigcup (-i-1, -i]$$

Observe that any two sets in this family are disjoint namely

$$V_i \bigcap V_i = \varnothing \text{ when } i \neq j,$$

and that the union of the entire family covers the real line:

$$\bigcup_{i=0}^{\infty} V_i = \bigcup_{i=0}^{\infty} (i, i+1) \bigcup (-i-1, -i] = R.$$

Since the family $\{V_i\}_{i=0,1,...}$ covers the real line, the measure of its union is one, i.e.

$$\rho(\bigcup_{i=0}^{\infty} V_i) = 1. \tag{3}$$

Yet since every set V_i is bounded, each set has ρ - measure zero. If ρ was countably additive we would have

$$\rho(\bigcup_{i=0}^{\infty} V_i) = \sum_{i=0}^{\infty} V_i = 0, \text{ which contradicts (3)}$$

Since the contradiction arises from assuming that ρ is countably additive, ρ must be purely finitely additive, as we wished to prove.

Observe that ρ assigns zero measure to bounded sets, and a positive measure only to unbounded that contain a '*neighborhood of* $\{\infty\}$[21]. We can define an explicit function on L_∞ that represents the action of this purely finitely additive

[21] By a 'neighborhood of ∞' we mean the sets of the form

$$\{x \epsilon R : x > a \text{ for some } a \epsilon R\}.$$

measure ρ if we restrict our attention to the closed subspace L'_∞ of L_∞ consisting of those functions $f(x)$ in L_∞ that have a limit when $x \to \infty$, by the formula

$$\rho(f) = \lim_{x\to\infty} f(x) \tag{4}$$

When restricted to functions f in the subspace L'_∞

$$\rho(f) = \int f(x)d\rho(x) = \lim_{x\to\infty} f(x).$$

Observe that one can describe the function ρ as a limit of a sequence of delta functions whose support increases without bound:

$$\rho(f) = \lim_{N\to\infty} \int f(x).\Delta_N = \lim_{N\to\infty} f(N)$$

where $\{N\}_{N=1,2,...}$ is the sequence of natural numbers, and where

Δ_N is a 'delta' measure on R supported on the set $\{N\}$,

defined by $\Delta_N(A) = 1$ when $A \supset (N-\varepsilon, N+\varepsilon)$ for some $\varepsilon > 0$,

and $\Delta_N(A) = 0$ otherwise.

The problem is now to extend the function ρ to a function that is defined on the entire space L_∞. This could be achieved in various ways but as we will see, each of them requires the Axiom of Choice.

One can use Hahn - Banach's theorem to extend the function ρ from the closed subspace $L'_\infty \subset L_\infty$to the entire space L_∞ while preserving its norm[22]. However, in its general form Hahn - Banach's theorem requires the Axiom of Choice. Alternatively, one can extend the notion of a *limit* in (4) to encompass all functions in L_∞ including those that have no standard limit. This can be achieved by using the notion of convergence along a *free ultrafilter* arising from compactifying the real line R as in ([6]). However the existence of a *free ultrafilter* requires the Axiom of Choice. This illustrates why the attempts to construct *purely finitely additive measures* that are representable as functions on L_∞, require the Axiom of Choice.

Since our criteria require purely finitely additive measures this illustrates a connection between the Axiom of Choice and our axioms for choice under uncertainty. This connection is not entirely surprising since both sets of axioms are about the ability to choose universally and consistently. What is interesting, however, is that the consideration of rare events that are neglected in standard decision theory conjures up the Axiom of Choice.

[22] The norm of the function $\lim f(x)$ defined above is one, because by definition

$$\| \lim(.) \| \sup_{f\epsilon L, \|f\|=1} | \lim f(x) |= 1.$$

.

7 Appendix

Example 4: A ranking that is insensitive to frequent events

Consider $W(f) = \liminf_{x\epsilon R}(f(x))$. This ranking is insensitive to frequent events of arbitrarily large Lebesgue measure (see [9]) and therefore does not satisfy our Axiom 3.

Example 5: Two approaches to the 'closeness' of lotteries

This example provides two sequence of lotteries that converge to each other according to the notion of closeness defined in Arrow [2], but not according to our notion of closeness using the sup norm. Our notion of closeness is thus more demanding, implying our requirement of sensitivity to rare events.

Arrow's Axiom MC requires that if a lottery is derived from another by *altering its outcomes* on events that are *sufficiently small* (as defined in ([2]) P. 48), the preference relation of that lottery with respect to any other given lottery should remain unaltered.[23] To define 'small' differences between two lotteries, or equivalently to define 'closeness', Arrow considers two lotteries (f, g) to be 'close' when there is a vanishing sequence $\{E^i\}$, such that f and g differ only in sets of events E^i for large enough i ([2]) P. 48). This notion of closeness is based on the standard Lebesgue measure. Observe that for any vanishing sequence of events $\{E^i\}$, as i becomes large enough, the Lebesgue measure μ of the set E^i becomes small, formally $\lim_{i\to\infty} \mu(E^i) = 0$. Therefore in Arrow's framework, two lotteries that differ in sets of small enough Lebesgue measure are very close to each other. Our framework is different since two lotteries f and g are 'close' when they are uniformly close almost everywhere, i.e. when $\sup_R \mid f(t) - g(t) \mid < \varepsilon$ *a.e.*for a suitably small $\varepsilon > 0$. The difference between the two conceptsof 'closenes' becomes sharpest when considering vanishing sequences of sets. In our case, two lotteries that differ in sets of events along a vanishing sequence E^i may never get close to each other. Consider as an example the family $\{E^i\}$ where $E^i = [i, \infty)$, $i = 1, 2, ...$ This is a vanishing family of events, because $E^i \subset E^{i+1}$ and $\bigcap_{i=1}^{\infty} E^i = \varnothing$. Consider now the lotteries $f^i(t) = K$ when $t \in E^i$ and $f^i(t) = 0$ otherwise, and $g^i(t) = 2K$ when $t \in E^i$ and $g^i(t) = 0$ otherwise. Then for all i, $\sup_{E^i} \mid f^i(t) - g^i(t) \mid = K$. In our topology this implies that f^i and g^i are **not** 'close' to each other, as the difference $f^i - g^i$ does not converge to zero. No matter how far along we are along the vanishing sequence E^i the two lotteries f^i, g^i differ by K. Yet since the lotteries f^i, g^i differ from $f \equiv 0$ and $g \equiv 0$ respectively only in the set E^i, and $\{E^i\}$ is a vanishing sequence, for large enough i they are as 'close' as desired according to Arrow's [2] definition. But They are not 'close' according to our notion of closeness, which is more demanding, as we wished to show.

The dual space L_∞^*

[23] This statement is a direct quote Arrow's own explanation of the Axiom of Monotone Continuity, see [2] p. 48, para 2., once one translates the notion of 'choosing an action' to the notion of 'choosing a lottery'.

Countably and purely finitely additive measures

The space of continuous linear functions on L_∞ is a well known space called the "dual' of L_∞, and is denoted L^*_∞. This dual space has been fully characterized e.g. in [30] [31]. Its elements are defined by integration with respect to measures on R. The dual space L^*_∞ consists of (i) L_1 functions g that define countably additive measures μ on R by the rule

$$\mu(A) = \int_A g(x)dx$$

where $\int_R \mid g(x) \mid dx < \infty$ and therefore μ is *absolutely continuous* with respect to the Lebesgue measure, namely it gives measure zero to any set with Lebesgue measure zero, and (ii) a 'non - L_1 part' consisting of purely finitely additive measures ρ that are 'singular' with respect to the Lebesgue measure and give positive measure to sets of Lebesgue measure zero; these measures ρ are finitely additive but they are *not* countably additive. A measure η is called *finitely additive* when for any family of pairwise disjoint measurable sets $\{A_i\}_{i=1,...N}$ $\eta(\bigcup_{i=1}^{N} A_i) = \sum_{i=1}^{N} \eta(A_i)$. The measure η is called *countably additive* when for any family of pairwise disjoint measurable sets $\{A_i\}_{i=1,...\infty}$ $\eta(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \eta(A_i)$. The countably additive measures are in a one to one correspondence with the elements of the space $L_1(R)$ of integrable functions on R. However, purely finitely additive measures cannot be identified by such functions. Nevertheless, purely finitely additive measures play an important role, since they ensure that the ranking criteria are 'sensitive to rare events' (Axiom 2). These measures define continuous linear real valued functions on L_∞, thus belonging to the dual space of L_∞ [31], but cannot be represented by functions in L_1.

Example 6:

A purely finitely additive measure that is not countably additive

The following is a measure that is finitely additive but not countably additive and therefore cannot be represented by an integrable function in L_1: for every Borel set $A \subset R$, define $\rho(A) = 1$ whenever $A \supseteqq \{r : r > a,\text{for some } a \in R\}$, and $\rho(A) = 0$ otherwise. ρ is a finitely additive set function but it is not countably additive, since R can be represented as a disjoint union of countably many bounded intervals, $R = \bigcup_{i=1}^{\infty} U_i$ each of which has zero measure $\rho(U_i) = 0$, so the sum of their measures $\sum \rho(U_i) = 0$ is zero, while by definition $\rho(R) = 1$. This measure ρ cannot be represented by an L_1 function.

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