Universal theory of equilibrium in models with complementarities

By

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Abstract

We develop a universal theory of equilibrium for models with complementarities on partially ordered sets (posets), unifying lattice-based theories used widely in economics and other disciplines and poset-based theories useful to study stochastic systems in many settings. Our theorems for extremal equilibria, structure of equilibrium set, and monotone comparative statics (MCS) of equilibrium generalize both types of theories in a unified manner. This extends to new theorems for MCS of the infimum equilibrium set, supremum equilibrium set, full equilibrium set, and isotone equilibrium set, and to a universal theory of order approximation of equilibria as well. As an application, we show new, deeper structural features of equilibrium in the canonical isotone stochastic dynamic economy with correlated shocks due to Hopenhayn and Prescott (1992).

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1 Introduction

We develop a universal theory of equilibrium for models with complementarities on partially ordered sets (posets), unifying lattice-based theories used widely in economics and other disciplines and poset-based theories useful to study stochastic systems in many settings.

Our theory includes as a special case Sabarwal (2023b)'s general theory of equilibrium in models with complementarities, which unifies and generalizes the theory of equilibrium in lattice-based models prevalent in the literature, such as Tarski (1955), Topkis (1978), Topkis (1979), Milgrom and Roberts (1990), Shannon (1990), Vives (1990), Milgrom and Shannon (1994), Zhou (1994), Quah and Strulovici (2009), Prokopovych and Yannelis (2017), Che, Kim, and Kojima (2021), and others.

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Our theory includes large classes of stochastic phenomena typically excluded from lattice-based models, because the space of measures on a poset is not a lattice in general, but is a poset in the stochastic order (also known as first order stochastic dominance). Our results unify the study of equilibrium in stochastic systems with complementarities, which include isotone kernel systems, stochastic dynamical systems, stochastic dynamic economies, and Markov decision processes, expanding the scope of the theory greatly.

As a concrete example, consider the model with complementarities in Hopenhayn and Prescott (1992) based on the canonical stochastic dynamic economy with correlated shocks from Stokey and Lucas (1989). Let $X \subseteq \mathbb{R}^\ell$ be a convex Borel set of actions, $Z \subseteq \mathbb{R}^k$ a compact set of realizations of uncertainty, q the structure of serially correlated shocks, $\Gamma : X \times Z \rightrightarrows X$ a feasibility correspondence, A the graph of Γ, $F : A \to \mathbb{R}$ a one-period return function, $\beta \in (0,1)$ the constant discount rate. Using standard assumptions, let $v : X \times Z \to \mathbb{R}$ be the unique value function associated with this problem, given by $v(x,z) = \sup_{x' \in \Gamma(x,z)} \{F(x',x,z) + \beta \int v(x',z') q(z,dz')\}$, and $\gamma(x,z) = \{x' \in \Gamma(x,z) \mid v(x,z) = F(x',x,z) + \beta \int v(x',z') q(z,dz')\}$ the policy correspondence. A policy g (a measurable selection from γ) and serially correlated shocks q determine an associated stochastic kernel p on the state space $X \times Z$ which governs the stochastic evolution of the economy over time. An *equilibrium* in a Stokey-Lucas economy is a pair (g,μ) where g is a measurable selection from γ and μ is a stationary distribution of the associated stochastic process governed by p. Equivalently, μ is a fixed point of the adjoint $\mathcal{T}_p$ of p on the set of probability measures on $X \times Z$. When γ has more than one measurable selection, there are multiple possible evolutions of the economy giving rise to a system of stochastic processes. Letting $\mathcal{G}$ be the set of all measurable selections from γ and $\mathcal{X}$ be the set of probability measures on $X \times Z$, the *equilibrium set* is $\mathcal{E} = \{(g,\mu) \in \mathcal{G} \times \mathcal{X} \mid \mu = \mathcal{T}_p(\mu)\}$. Complementarities are included as follows. Using the natural partial order on $\mathbb{R}^n$, X is subcomplete, F is supermodular on $X \times X$ for each z and has increasing differences in $(x',x;z)$, Γ has strict complementarity, graph of $\Gamma(\cdot,\cdot,z)$ is a sublattice for each z, Γ is ascending, and q is an isotone kernel. With these assumptions, an *HP model* is given by $(X, Z, q, \Gamma, F, \beta)$. Hopenhayn and Prescott (1992) show that γ has an isotone selection that is the lowest policy $\underline{g}$ ($\underline{g}$ is an isotone function and $\forall g \in \mathcal{G}, \underline{g} \preceq g$, pointwise) and an isotone highest policy $\overline{g}$, and use these to prove that an equilibrium always exists in their model. They show that several foundational problems such as investment theory, stochastic growth theory, and industry equilibrium may be modeled using their framework.

We show that the HP model is a special case of our formulation of a stochastic dynamic economy. There are many questions of interest in addition to existence of equilibrium in the HP model. Consider an isotone smallest policy $\underline{g}$ ($\underline{g}$ is an isotone function and $\forall g \in \mathcal{G}, \underline{g} \preceq g$, pointwise) or an isotone largest policy $\overline{g}$, a typical feature of models with complementarities. What does this imply for the structure of the equilibrium set $\mathcal{E}$ beyond existence of equilibrium? How does $\mathcal{E}$ relate to the infimum equilibrium set $\underline{\mathcal{E}} = \{(\underline{g}, \mu) \in \mathcal{G} \times \mathcal{X} \mid \mu = \mathcal{T}_{\underline{p}}(\mu)\}$ or the supremum equilibrium set $\overline{\mathcal{E}} = \{(\overline{g}, \mu) \in \mathcal{G} \times \mathcal{X} \mid \mu = \mathcal{T}_{\overline{p}}(\mu)\}$? What do $\underline{\mathcal{E}}$ or $\overline{\mathcal{E}}$ tell us about an arbitrary equilibrium in $\mathcal{E}$? What is the structure of the set of isotone equilibria, $\mathcal{E}^{iso} = \{(g, \mu) \in \mathcal{E} \mid g \text{ is isotone}\}$?

Additional questions arise about how equilibria change when parameters of the economy change. We include parameters as a poset $(T, \preceq_T)$, leading to the parametric policy correspondence $(x, z, t) \mapsto \gamma(x, z, t)$ and the correspondence $t \mapsto \mathcal{G}(t)$, where $\mathcal{G}(t)$ is the set of measurable selections from the t-section of γ. The *equilibrium set at* t is $\mathcal{E}(t) = \{(g, \mu) \in \mathcal{G}(t) \times \mathcal{X} \mid \mu = \mathcal{T}_p(\mu)\}$, where p is derived from $g \in \mathcal{G}(t)$ and $q(t)$, and the *equilibrium correspondence* is the mapping $t \mapsto \mathcal{E}(t)$. Similarly, there is the *infimum equilibrium correspondence*, $t \mapsto \underline{\mathcal{E}}(t)$, the *supremum equilibrium correspondence*, $t \mapsto \overline{\mathcal{E}}(t)$, and the *isotone equilibrium correspondence*, $t \mapsto \mathcal{E}^{iso}(t)$. Hopenhayn and Prescott (1992) show that a higher shock (in terms of stochastic order) implies that the smallest and largest policies move up. What are the implications of the higher policies for the entire steady state distribution at the higher shocks? More generally, what are the impacts of parametric isotone selections from γ on selections from the different equilibrium correspondences? When do we have monotone comparative statics (MCS) of equilibrium (an isotone selection from the equilibrium correspondence)? Do we have MCS in isotone equilibria (an isotone selection from the isotone equilibrium correspondence)? For parameters $\hat{t} \preceq_T \tilde{t}$, how do the equilibrium sets $\mathcal{E}(\hat{t})$, $\underline{\mathcal{E}}(\hat{t})$, $\overline{\mathcal{E}}(\hat{t})$, and $\mathcal{E}^{iso}(\hat{t})$ relate to $\mathcal{E}(\tilde{t})$, $\underline{\mathcal{E}}(\tilde{t})$, $\overline{\mathcal{E}}(\tilde{t})$, and $\mathcal{E}^{iso}(\tilde{t})$? Are there natural relations under which we have MCS of equilibrium correspondences (rather than MCS in terms of selections from correspondences), that is, when t goes up, the entire corresponding equilibrium set goes up as well?

We provide widely applicable answers to these questions by abstracting core features shared by all these models and showing their common implications for the theory of equilibrium. For this purpose, we define a *poset model* as $(X, \preceq_X, \Phi)$ where $(X, \preceq_X)$ is a poset and $\Phi : X \rightrightarrows X$ is a correspondence on X. An *equilibrium* is a fixed point of Φ, and the *equilibrium set* of the model is $\mathcal{E}(\Phi) = \{x \in X \mid x \in \Phi(x)\}$. A *universal model with complementarities* is a poset model in which $(X, \preceq_X)$ is either chain sup-complete with smallest element or chain inf-complete with largest element and Φ has an isotone selection. Parameters are included as a poset $(T, \preceq_T)$ along with a correspondence $\Phi : X \times T \rightrightarrows X$, and the equilibrium set at t is the fixed point set of the t-section of Φ, denoted $\mathcal{E}(\Phi_t)$. The equilibrium correspondence is $\mathcal{E} : T \rightrightarrows X$, $t \mapsto \mathcal{E}(\Phi_t)$.

As a first contribution, we show that both the variety of situations studied in Sabarwal (2023b) and the variety of stochastic systems included here are unified in terms of the same properties on their associated poset models. Sabarwal (2023b) shows that the theory of equilibrium in lattice-based models is unified by studying their associated lattice models $(X, \preceq_X, \Phi)$, where $(X, \preceq_X)$ is a nonempty complete lattice and Φ has an isotone, isotone infimum, or isotone supremum selection. These are special cases of poset models. We show that all the stochastic systems studied here are unified in terms of the same properties on their associated measure theory model $(\mathcal{X}, \preceq_s, \Phi)$, a poset model in which $\mathcal{X}$ is the set of probability measures on an underlying state space, $\preceq_s$ is the stochastic order on measures, and Φ is the adjoint correspondence associated with the stochastic system. Therefore, the theory of equilibrium in all these classes of models is unified by the study of the associated poset models with these common properties. The study of stochastic systems is more complex because isotone properties are affected by the additional interdependence among the multitude of policy functions, the kernels they induce, their effects on the associated adjoint correspondence, and the associated steady state distributions. Moreover, isotone equilibria arise naturally in stochastic systems, opening up additional avenues of study. Including parameters compounds all these effects.

As a second contribution, we show that the main benefits of different models with complementarities such as existence of equilibrium, existence of extremal equilibrium, and MCS of extremal equilibria generalize to universal models in a natural manner using only isotone, isotone infimum, and isotone supremum selections from Φ. In addition, stochastic systems have MCS of isotone equilibrium. No other conditions are imposed on Φ. The images $\Phi(x)$ are not assumed to have any additional structure such as chain inf-complete, chain sup-complete, complete lattice, compact or convex. The correspondence Φ is not assumed to have any continuity properties, maintaining a benefit of monotone methods in studying models with discontinuities or non-convexities where the use of standard tools from topology and convex analysis may be limited. We allow for cases not covered by Smithson (1971), using Abian and Brown (1961) in that case. We strengthen results for existence of maximal or minimal equilibria in Li (1984) and Che, Kim, and Kojima (2021) by proving existence of largest or smallest equilibria (without using continuity conditions on Φ), and strengthening a result due to Markowsky (1976) as well.

As a third contribution, we provide conditions under which the full equilibrium set in a poset model is a chain sup-complete set with smallest equilibrium, a chain inf-complete set with largest equilibrium, or a chain complete set with extremal equilibria. Our theorem goes beyond existing results for structure of the fixed point set of correspondences on posets. Markowsky (1976)'s result for isotone functions is a special case when Φ is singleton valued. Our theorem also goes beyond existence of maximal elements and inductive set structure of the fixed point set in chain sup-complete posets in Li (1984), and goes beyond existence of maximal or minimal fixed points in Che, Kim, and Kojima (2021). Our theorem also generalizes the complete lattice structure theorem for fixed points of correspondences in lattice models proved in Sabarwal (2023b), which, in turn, generalized the well-known lattice-based theorems proved in Tarski (1955), Vives (1990), and Zhou (1994). We use this theorem to prove that the set of isotone equilibria in every HP model is chain complete, a new result for the HP model.

As a fourth contribution, we define the star chain complete set order, a new relation on nonempty subsets of a poset that is helpful to compare equilibrium sets in poset models, including stochastic systems. For nonempty $A, B \subseteq X$, A is lower than B in the *star chain complete set order*, denoted $A \sqsubseteq^{*cc} B$, if for every nonempty chain $C \subseteq A$, $\sup_B C$ exists (in B), and for every nonempty chain $C \subseteq B$, $\inf_A C$ exists (in A). To define supremum and infimum for arbitrary subsets of a poset, we follow Sabarwal (2023b): For nonempty subsets D and E of poset X, $\sup_D E$ is an element of D that is an upper bound of E and is the smallest upper bound of E among elements of D, and $\inf_D E$ is defined similarly.

As a fifth contribution, we provide a universal theory of order approximation of equilibria using the star chain complete set order. We prove that *in every universal model* $(X, \preceq_X, \Phi)$ *with an isotone infimum selection* $\underline{\Phi}$*, it must be that* $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*cc} \mathcal{E}(\Phi)$. Therefore, for every nonempty chain $C \subseteq \mathcal{E}(\Phi)$, $\inf_{\mathcal{E}(\underline{\Phi})} C \in \mathcal{E}(\underline{\Phi})$. In other words, if a nonempty chain C of equilibria formalizes a specialized equilibrium notion of interest, it is uniquely and tightly approximated from below in an order theoretic manner using equilibria from the infimum selection. In the special case that $C = \{e^*\}$ is a singleton, this proves that *every equilibrium* $e^* \in \mathcal{E}(\Phi)$ *is uniquely and tightly order approximated from below by an equilibrium using the infimum selection.* This is particularly useful if the infimum selection is easier to work with or has some useful computational, dynamic, or theoretical properties. Our result requires very little structure for the universal model (only isotone infimum selection). Similarly, *in every universal isotone supremum model,* $\mathcal{E}(\Phi) \sqsubseteq^{*cc} \mathcal{E}(\overline{\Phi})$, and in particular, *every equilibrium* $e^* \in \mathcal{E}(\Phi)$ *is uniquely and tightly order approximated from above by an equilibrium using the supremum selection.* Moreover, *these results apply to every stochastic system.* This implies the following new results for the HP model. In every HP model, for every equilibrium (g, μ) in the model, there is a unique equilibrium in $\underline{\mathcal{E}}$ (set of equilibria associated with the smallest policy $\underline{g}$) closest to it from below and a unique equilibrium in $\overline{\mathcal{E}}$ (equilibria associated with $\overline{g}$) closest to it from above, from among all equilibria in those respective sets.

As a sixth contribution, we use the star chain complete order to prove MCS of entire equilibrium sets associated with different equilibrium correspondences. We prove that *every universal parametric isotone infimum model has MCS of the infimum equilibrium set in star chain complete set order* (that is, for every $\hat{t}, \tilde{t} \in T, \hat{t} \preceq_T \tilde{t} \Rightarrow \mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*cc} \mathcal{E}(\underline{\Phi}_{\tilde{t}})$). This implies that $\forall e^* \in \mathcal{E}(\underline{\Phi}_{\hat{t}})$, $\exists$ unique $\tilde{e} \in \mathcal{E}(\underline{\Phi}_{\tilde{t}})$ higher than e^* and closest to it among all equilibria in $\mathcal{E}(\underline{\Phi}_{\tilde{t}})$, and $\forall e^* \in \mathcal{E}(\underline{\Phi}_{\tilde{t}})$, $\exists$ unique $\hat{e} \in \mathcal{E}(\underline{\Phi}_{\hat{t}})$ lower than e^* and closest to it among all equilibria in $\mathcal{E}(\underline{\Phi}_{\hat{t}})$. Similarly, *every universal parametric isotone supremum model has MCS of the supremum equilibrium set in star chain complete set order* ($\hat{t} \preceq_T \tilde{t} \Rightarrow \mathcal{E}(\overline{\Phi}_{\hat{t}}) \sqsubseteq^{*cc} \mathcal{E}(\overline{\Phi}_{\tilde{t}})$). Moreover, *if the poset model at every t satisfies our conditions to have a nonempty chain complete poset of equilibria, then the model has MCS of the full equilibrium set in star chain complete set order* ($\hat{t} \preceq_T \tilde{t} \Rightarrow \mathcal{E}(\Phi_{\hat{t}}) \sqsubseteq^{*cc} \mathcal{E}(\Phi_{\tilde{t}})$). This implies that $\forall e^* \in \mathcal{E}(\Phi_{\hat{t}})$, $\exists$ unique $\tilde{e} \in \mathcal{E}(\Phi_{\tilde{t}})$ higher than e^* and closest to it among all equilibria at $\tilde{t}$, and $\forall e^* \in \mathcal{E}(\Phi_{\tilde{t}})$, $\exists$ unique $\hat{e} \in \mathcal{E}(\Phi_{\hat{t}})$ lower than e^* and closest to it among all equilibria at $\hat{t}$. In the special case when Φ is singleton valued, these conditions are satisfied in every universal parametric model with complementarities. Therefore, *when* Φ *is singleton valued, every universal parametric model has MCS of the full equilibrium set in star chain complete set order. All these results apply to every stochastic system* as well. Moreover, *under similar conditions, every stochastic system has MCS of the full isotone equilibrium set in star chain complete set order* ($\hat{t} \preceq_T \tilde{t} \Rightarrow \mathcal{E}^{iso}(\hat{t}) \sqsubseteq^{*cc} \mathcal{E}^{iso}(\tilde{t})$). Furthermore, we prove that *every HP model has MCS of the infimum equilibrium set, supremum equilibrium set and the full isotone equilibrium set in star chain complete set order.* In particular, $\forall (g^*, \mu^*) \in \mathcal{E}(\hat{t})$ that is isotone, $\exists$ unique $(\tilde{g}, \tilde{\mu}) \in \mathcal{E}(\tilde{t})$ that is isotone, higher than (g^*, μ^*) and closest to it among all isotone equilibria at $\tilde{t}$, and $\forall (g^*, \mu^*) \in \mathcal{E}(\tilde{t})$ that is isotone, $\exists$ unique $(\hat{g}, \hat{\mu}) \in \mathcal{E}(\hat{t})$ that is isotone, lower than (g^*, μ^*) and closest to it among all isotone equilibria at $\hat{t}$. These results can help in policy analysis by proving existence of order-nearest equilibria before or after a policy change, not only in terms of optimal actions g but also for the entire steady state distribution μ in the economy. These results are not true for the strong set order (or the uniform set order) even in the canonical standard cases, as shown in Sabarwal (2023b).

As a seventh contribution, we focus on the theory of equilibrium in universal models with complementarities. That is, we take the underlying behavioral description as given, with minimal conditions consistent with our core understanding of interdependent situations with complementarities, and investigate systemic influences and equilibrium impact of such behavior. The underlying behavior can be the solution to an optimization problem, as is often the case, but we do not require that in the general case. Anything that is an accurate description of the situation being studied is permissible as long as it satisfies our weak conditions. This has several benefits. First, it provides unified, off-the-shelf theorems that apply regardless of the manner in which individual choices are made as long as they satisfy weak conditions. Second, our conditions are naturally satisfied in many models, they are intuitively easy to check, and they allow for new situations. Third, it separates the study of optimization from the theory of equilibrium. This means that our results can guide new research lines to discover more general behavioral properties that do not fall under the purview of the existing optimization models with complementarities but satisfy our weaker conditions. Fourth, our study isolates salient properties of equilibrium that unify and generalize important and large classes of applications in lattice-based models and in stochastic systems. A hallmark of the analysis here is that we identify and prove the role that particular isotone selections play in a unified theory of equilibrium in a transparent manner using deeper and more foundational order theoretic arguments.

We do not make any topological assumptions in developing the universal theory of equilibrium, maintaining a benefit of monotone methods in studying models with discontinuities or non-convexities where the use of standard tools from topology and convex analysis may be limited. A strand of the theory considers monotone sequences that arise from iterating the smallest best response and using particular versions of continuity, shows that such sequences converge to the smallest equilibrium. Examples can be seen in Milgrom and Roberts (1990), Milgrom and Shannon (1994), Amir (1996), Echenique (2002), Roy and Sabarwal (2012), Balbus, Reffett, and Woźny (2014), Sabarwal (2021), Balbus, Dziewulski, Reffett, and Woźny (2022), and others. Balbus, Olszewski, Reffett, and Woźny (2023) study strong set order increasing (respectively, strongly monotone) upper order hemicontinuous correspondences on complete (respectively, σ-complete) lattices. Sabarwal (2023a) studies the poset-based theory. Our results may open the door to study additional models with complementarities, for example directional MCS as in Quah (2007), Barthel and Sabarwal (2018), and Paul and Sabarwal (2018), or dynamic supermodular games as in Echenique (2004) and Feng and Sabarwal (2020).

The paper is organized as follows. Section 2 defines universal models with complementarities, including a variety of stochastic systems, and proves the main results for existence of equilibrium, existence of extremal equilibrium, and structure of the equilibrium set. Section 3 formulates the star chain complete set order, proves its properties, provides comparisons among different set orders, and provides relations among the infimum equilibrium set, the supremum equilibrium set, and the full equilibrium set using this relation. It also formalizes the theory of order approximation of equilibria in universal models. Section 4 defines universal parametric models with complementarities, proves MCS of particular equilibrium selections and includes theorems about MCS of different equilibrium correspondences. Section 5 concludes.

2 Universal models with complementarities

A *partial order* on a set X is a binary relation $\preceq$ that is reflexive, antisymmetric, and transitive. A ***partially ordered set***, or ***poset***, is a set X along with a partial order $\preceq$ on it, denoted $(X, \preceq)$. For a poset $(X, \preceq)$ and subset A of X, the *relative partial order* on A is the usual one: for every $x, x' \in A$, $x \preceq_A x' \Leftrightarrow x \preceq x'$, and in this case, $(A, \preceq_A)$ is a poset in the relative partial order. For posets $(X, \preceq_X)$ and $(Y, \preceq_Y)$, the Cartesian product $X \times Y$ is a poset under the *product partial order* given by $(x, y) \preceq (x', y') \Leftrightarrow x \preceq_X x'$ and $y \preceq_Y y'$. For posets $(X, \preceq_X)$ and $(Y, \preceq_Y)$, a function $f : X \to Y$ is ***isotone*** if for every $\hat{x}$ and $\tilde{x}$ in X, $\hat{x} \preceq_X \tilde{x} \implies f(\hat{x}) \preceq_Y f(\tilde{x})$. The partial order on the set of all functions from X to poset $(Y, \preceq_Y)$ is the *product (or pointwise)* partial order: $f \preceq g \Leftrightarrow (\forall x)\ f(x) \preceq_Y g(x)$.

Two points x, y in a poset $(X, \preceq)$ are *comparable* (or *ordered*), if $x \preceq y$ or $y \preceq x$. In this case, we say that x is lower than y when $x \preceq y$, or x is higher than y when $y \preceq x$. A ***chain*** is a subset $C \subseteq X$ in which every pair of points is comparable. The empty set is trivially a chain. Points x, y are *strictly comparable* (or *strictly ordered*), if they are comparable and $x \neq y$. In this case, we say x is strictly lower than y, denoted $x \prec y$, or x is strictly higher than y, denoted $y \prec x$, as the case may be. Two points x, y are *incomparable* (or *noncomparable*, or *unordered*), if they are not comparable, that is, $x \npreceq y$ and $y \npreceq x$.

Let X be a poset and $E \subseteq X$. An *upper bound for* E is an element $x \in X$ such that for every $e \in E$, $e \preceq x$. The ***sup of*** E ***in*** X, denoted $\sup_X E$, is an element $\overline{e} \in X$ such that (1) $\overline{e}$ is an upper bound for E and (2) for every $x \in X$ that is an upper bound for E, $\overline{e} \preceq x$. A *lower bound for* E is an element $x \in X$ such that for every $e \in E$, $x \preceq e$. The ***inf of*** E ***in*** X, denoted $\inf_X E$, is an element $\underline{e} \in X$ such that (1) $\underline{e}$ is a lower bound for E and (2) for every $x \in X$ that is a lower bound for E, $x \preceq \underline{e}$. When convenient, we denote $\underline{x} = \inf_X X$ and $\overline{x} = \sup_X X$.

A poset $(X, \preceq)$ is ***chain sup-complete (respectively, inf-complete)*** if for every nonempty chain $C \subseteq X$, $\sup_X C \in X$ (respectively, $\inf_X C \in X$). A poset X is ***chain complete*** if it is chain inf-complete and chain sup-complete.

A ***lattice*** is a poset $(X, \preceq)$ in which for every $x, y \in X, x \wedge y := \inf_X\{x, y\} \in X$ and $x \vee y := \sup_X\{x, y\} \in X$. Subset A of poset $(X, \preceq)$ with the relative partial order is a *complete lattice* if for every nonempty $E \subseteq A$, $\inf_A E \in A$ and $\sup_A E \in A$. It follows that if A is a nonempty complete lattice, then $\inf_A A \in A$ and $\sup_A A \in A$. Subset A of poset X is *subcomplete* if for every nonempty $B \subseteq A$, $\inf_X B \in A$ and $\sup_X B \in A$.

Let A, B be subsets of poset X. A ***is lower than*** B ***in the strong set order (SSO), denoted*** $A \sqsubseteq^s B$, if $\forall a \in A$, $\forall b \in B$, $\inf_X\{x, y\} \in A$ and $\sup_X\{x, y\} \in B$. A ***is lower than*** B ***in the weak set order***, $A \sqsubseteq^w B$, if (1) $\forall x \in A$, $\exists y \in B$, $x \preceq y$, and (2) $\forall y \in B$, $\exists x \in A$, $x \preceq y$. If condition (1) (respectively, (2)) is satisfied, we say that A *is lower than* B *in the upper (respectively, lower) weak set order*, $A \sqsubseteq^{uw} B$ (respectively, $A \sqsubseteq^{lw} B$). A ***is lower than*** B ***in the uniform set order***, $A \sqsubseteq^u B$, if $\forall x \in A$, $\forall y \in B$, $x \preceq y$. It follows immediately that for nonempty subsets A, B of lattice X, $A \sqsubseteq^u B \Rightarrow A \sqsubseteq^s B \Rightarrow A \sqsubseteq^w B$.

For arbitrary sets X and Y, a ***correspondence from*** X ***to*** Y, denoted $\Phi : X \rightrightarrows Y$, is a function from X to the power set of Y, $\Phi : X \to \mathcal{P}(Y)$. It is nonempty valued if for every $x \in X$, $\Phi(x) \neq \emptyset$. It is singleton valued if for every x in X, $\Phi(x)$ is a singleton subset of Y. A function $f : X \to Y$ is viewed as a correspondence that is singleton valued, and conversely (and in this case, we'll use either notation without further mention). A *selection from correspondence* Φ is a function $f : X \to Y$ such that $f(x) \in \Phi(x)$ for every $x \in X$. For correspondence $\Phi : X \rightrightarrows X$, a point $x \in X$ is ***a fixed point of*** Φ if $x \in \Phi(x)$, and the ***fixed point set*** of Φ is $\mathcal{E}(\Phi) = \{x \in X \mid x \in \Phi(x)\}$.

A ***poset model*** is a triple $(X, \preceq, \Phi)$, where $(X, \preceq)$ is a poset and $\Phi : X \rightrightarrows X$ is a correspondence. An ***equilibrium*** in the poset model is a fixed point of Φ. The ***equilibrium set*** of the poset model is the fixed point set $\mathcal{E}(\Phi)$.

A poset model $(X, \preceq, \Phi)$ is ***isotone*** if Φ has an isotone selection. A poset model $(X, \preceq, \Phi)$ is ***isotone infimum*** if the infimum selection exists in Φ and is isotone, that is, $\forall x \in X, \underline{\Phi}(x) \coloneqq \inf_{\Phi(x)} \Phi(x)$ exists and $x \mapsto \underline{\Phi}(x)$ is isotone. This is equivalent to Φ is isotone in lower weak set order ($\hat{x} \preceq_X \tilde{x} \Rightarrow \Phi(\hat{x}) \sqsubseteq^{lw} \Phi(\tilde{x})$) and $\forall x, \underline{\Phi}(x)$ exists. The ***infimum equilibrium set*** is $\mathcal{E}(\underline{\Phi}) = \{x \in X \mid x = \underline{\Phi}(x)\}$. A poset model $(X, \preceq, \Phi)$ is ***isotone supremum*** if the supremum selection exists in Φ and is isotone, that is, $\forall x \in X, \overline{\Phi}(x) \coloneqq \sup_{\Phi(x)} \Phi(x)$ exists and $x \mapsto \overline{\Phi}(x)$ is isotone. This is equivalent to Φ is isotone in upper weak set order and $\forall x, \overline{\Phi}(x)$ exists. The ***supremum equilibrium set*** is $\mathcal{E}(\overline{\Phi}) = \{x \in X \mid x = \overline{\Phi}(x)\}$. These definitions do not require $\Phi(x)$ to be a complete lattice, or lattice, or chain subcomplete, or chain complete, or to satisfy any topological properties such as closedness or compactness. (The definitions here use infimum or supremum over $\Phi(x)$ not X. It is easy to check that $\inf_{\Phi(x)} \Phi(x)$ exists $\Rightarrow \inf_{\Phi(x)} \Phi(x) = \inf_X \Phi(x)$, and the same holds if $\inf_X \Phi(x) \in \Phi(x)$, and similarly for supremum. Therefore, in these definitions we can use either version. This is not true more generally: For arbitrary $E \subseteq \Phi(x)$, $\inf_{\Phi(x)} E$ exists $\not\Rightarrow \inf_{\Phi(x)} E = \inf_X E$, and similarly for supremum. The weaker definition using a subset of X rather than X over which infimum or supremum is taken is more relevant for this paper, and therefore, we use $\inf_{\Phi(x)} \Phi(x)$ or $\sup_{\Phi(x)} \Phi(x)$ to keep the notation consistent with other parts of the paper where the distinction is important.)

A ***universal model with complementarities*** is an isotone poset model $(X, \preceq, \Phi)$ in which either X is chain sup-complete with $\inf_X X \in X$, or X is chain inf-complete with $\sup_X X \in X$. A ***universal isotone infimum model*** is an isotone infimum poset model $(X, \preceq, \Phi)$ in which X is chain sup-complete with $\inf_X X \in X$. A ***universal isotone supremum model*** is an isotone supremum poset model $(X, \preceq, \Phi)$ in which X is chain inf-complete with $\sup_X X \in X$.

Example 1 (General model with complementarities)**.** Following Sabarwal (2023b), a ***general model with complementarities*** (or ***general model***) is a poset model $(X, \preceq, \Phi)$ in which $(X, \preceq)$ is a nonempty complete lattice and $\Phi : X \rightrightarrows X$ is a correspondence with an isotone selection. A nonempty complete lattice is necessarily a chain complete poset with a smallest and a largest element, and therefore, *every general model with complementarities is a universal model with complementarities.* A ***general isotone infimum (respectively, supremum) model*** is a general model $(X, \preceq, \Phi)$ in which the infimum (respectively, supremum) selection exists and is isotone The class of general models with complementarities is very large. As shown in Sabarwal (2023b), every Topkis model (Topkis (1978), Topkis (1979)), every Vives model (Vives (1990)), every MR model

(Milgrom and Roberts (1990), every Zhou model (Zhou (1994)), every generalized MS model (Shannon (1990), Milgrom and Shannon (1994)), every CKK model (Che, Kim, and Kojima (2021)), and every PY model (Prokopovych and Yannelis (2017)) is a general model that is isotone infimum and/or isotone supremum in a natural manner. Moreover, the class of general models includes models not covered by any of these standard models.

Example 2 (Measure theory model)**.** Let X be a Polish space with a closed partial order $\preceq_X$ and $\mathcal{M}(X)$ be the set of finite measures on the Borel sets of X, denoted $\mathcal{B}(X)$. The ***stochastic order,*** $\preceq_s$**,** on $\mathcal{M}(X)$ is the usual one: $\mu \preceq_s \nu$ if for every increasing set $A \in \mathcal{B}(X)$, $\mu(A) \leq \nu(A)$. Set $A \subseteq X$ is *increasing* if $\forall a \in A$, $\forall x \in X$, $a \preceq_X x \Rightarrow x \in A$. As shown in Kamae, Krengel, and O'Brien (1977), this is equivalent to $\int f d\mu \leq \int f d\nu$ for every bounded, isotone, measurable, real-valued f on X. A ***measure theory model*** is $(\mathcal{X}, \preceq_s, \Phi)$, where $\mathcal{X} \subseteq \mathcal{M}(X)$, $\preceq_s$ is the stochastic order on $\mathcal{X}$, and $\Phi : \mathcal{X} \rightrightarrows \mathcal{X}$ is a correspondence. It follows immediately that every measure theory model is a poset model. An ***equilibrium*** of the measure theory model is a fixed point of Φ. The ***equilibrium set*** of the measure theory model is the set of fixed points of Φ, $\mathcal{E}(\Phi)$.

The measure theory model defined here is quite general. It includes models that may arise as a result of optimizing behavior in economic models of constrained optimization where the constraints are a set of measures or probabilities. It also arises as a result of optimal control for different models of stochastic dynamics, such as kernel systems, stochastic dynamical systems, stochastic dynamic economies, and Markov decision processes.

Complementarities in a measure theory model are included as follows. A measure theory model is ***isotone*** if either $\mathcal{X}$ is chain sup-complete with $\inf_{\mathcal{X}} \mathcal{X} \in \mathcal{X}$ or $\mathcal{X}$ is chain inf-complete with $\sup_{\mathcal{X}} \mathcal{X} \in \mathcal{X}$, and Φ has an isotone selection. A measure theory model is ***isotone infimum*** if $\mathcal{X}$ is chain sup-complete with $\inf_{\mathcal{X}} \mathcal{X} \in \mathcal{X}$ and Φ has an isotone infimum selection, and it is ***isotone supremum*** if $\mathcal{X}$ is chain inf-complete with $\sup_{\mathcal{X}} \mathcal{X} \in \mathcal{X}$ and Φ has an isotone supremum selection.

Example 3 (Kernel systems)**.** A ***kernel system*** is a collection $(X, \preceq_X, \mathcal{B}(X), \mathcal{P})$, where X is a Polish space, $\preceq_X$ is a closed partial order on X, $\mathcal{B}(X)$ are the Borel sets of X, and $\mathcal{P}$ is a subset of kernels on $X \times \mathcal{B}(X)$, denoted $\mathcal{P} \subseteq ker(X \times \mathcal{B}(X))$. A kernel on $X \times \mathcal{B}(X)$ is a function $p : X \times \mathcal{B}(X) \to [0, 1]$ that is measurable in x for every $A \in \mathcal{B}(X)$, and is a probability measure on $\mathcal{B}(X)$ for every $x \in X$. When convenient, we let $p(x)$ denote the measure $p(x, \cdot)$. The *associated measure theory model* is $(\mathcal{X}, \preceq_s, \Phi)$, where $\mathcal{X}$ is the set of probability measures on $\mathcal{B}(X)$, $\preceq_s$ is stochastic order, and Φ is the *adjoint correspondence for* $\mathcal{P}$ defined by $\Phi : \mathcal{X} \rightrightarrows \mathcal{X}$, $\Phi(\mu) = \{\mathcal{T}_p(\mu) \mid p \in \mathcal{P}\}$, where $\mathcal{T}_p$ is the adjoint of p defined by $\mathcal{T}_p : \mathcal{X} \to \mathcal{X}$, $\mu \mapsto \mathcal{T}_p(\mu)$, where $\mathcal{T}_p(\mu)(A) = \int_X p(x, A)d\mu(x)$, $\forall A \in \mathcal{B}(X)$. An ***equilibrium*** of kernel system $(X, \preceq_X, \mathcal{B}(X), \mathcal{P})$ is a pair (p, μ) such that $p \in \mathcal{P}$, $\mu \in \mathcal{X}$, and $\mu = \mathcal{T}_p(\mu)$. The ***equilibrium set*** is $\mathcal{E} = \{(p, \mu) \in \mathcal{P} \times \mathcal{X} \mid \mu = \mathcal{T}_p(\mu)\}$ with the product order. Isotone equilibria play an important role in theory and applications. Kernel p is ***isotone***, if $\forall \hat{x}, \tilde{x} \in X, \hat{x} \preceq_X \tilde{x} \Rightarrow p(\hat{x}) \preceq_s p(\tilde{x})$, where $\preceq_s$ is the stochastic order on $\mathcal{X}$. Equilibrium (p, μ) is ***isotone*** if p is isotone. The ***isotone equilibrium set*** is $\mathcal{E}^{iso} = \{(p, \mu) \in \mathcal{E} \mid p \text{ is isotone}\}$.

A kernel system with singleton $\mathcal{P} = \{p\}$ is the special case of a (discrete-time) Markov process, and in this case, an equilibrium is a stationary distribution of the Markov process. We do not impose uniqueness of equilibrium even in this case. With multiple kernels, a kernel system can be viewed as a system of Markov processes, and the equilibrium set is the collection of all processes along with their stationary distributions. We impose no constraint on the mechanism by which the kernels $p \in \mathcal{P}$ in a kernel system may arise.

Complementarities are included as follows. The ***kernel order,*** $\preceq_k$**,** on $ker(X \times \mathcal{B}(X))$ is defined as follows: $p \preceq_k q$ if $\forall x \in X, p(x) \preceq_s q(x)$. A kernel system is ***isotone*** if either $\inf_X X \in X$ or $\sup_X X \in X$, and there is $p \in \mathcal{P}$ such that p is isotone. It is ***isotone infimum*** if $\inf_X X \in X$ and there is $\underline{p} \in \mathcal{P}$ such that $\underline{p}$ is isotone and $\forall p \in \mathcal{P}, \underline{p} \preceq_k p$. The ***infimum equilibrium set*** is $\underline{\mathcal{E}} = \{(\underline{p}, \mu) \in \mathcal{P} \times \mathcal{X} \mid \mu = \mathcal{T}_{\underline{p}}(\mu)\}$. It is ***isotone supremum*** if $\sup_X X \in X$ and there is $\overline{p} \in \mathcal{P}$ such that $\overline{p}$ is isotone and $\forall p \in \mathcal{P}, p \preceq_k \overline{p}$. The ***supremum equilibrium set*** is $\overline{\mathcal{E}} = \{(\overline{p}, \mu) \in \mathcal{P} \times \mathcal{X} \mid \mu = \mathcal{T}_{\overline{p}}(\mu)\}$. No additional conditions (including topological conditions) are imposed on the structure of $\mathcal{P}$ or on kernels $p \in \mathcal{P}$.

Example 4 (Stochastic dynamical systems)**.** Many stochastically evolving phenomena including physical, human, animal, societal, and socioeconomic can be modeled using stochastic dynamics. An abstract model is formulated as follows. A ***stochastic dynamical system*** is a collection $((S, \preceq_S, \mathcal{B}(S)), (Z, \preceq_Z, \mathcal{B}(Z)), q, \mathcal{G})$, where S is a Polish space, $\preceq_S$ is a closed partial order on S, $\mathcal{B}(S)$ are the Borel sets of S, Z is a Polish space, $\preceq_Z$ is a closed partial order on Z, $\mathcal{B}(Z)$ are the Borel sets of Z, q is a kernel on $S \times \mathcal{B}(Z)$, and $\mathcal{G} \subseteq mbl(S \times Z, S)$ is a subset of measurable functions from $S \times Z$ to S with the pointwise partial order. At this stage, we impose no additional conditions (beyond measurability) on policies $g \in \mathcal{G}$. We view S as the state space of the dynamical system, Z as the set of possible realizations of exogenous shocks to the system (or exogenous uncertainty), q as the state dependent distribution of exogenous shocks, and $\mathcal{G}$ as a set of policies or update rules. For a given policy $g \in \mathcal{G}$, the dynamical system evolves as follows. In period t, if the state of the system is s_t and the shock to the system is z_t, the state next period is $s_{t+1} = g(s_t, z_t)$. Shocks z_t are state dependent and governed by $q(s_t, \cdot)$.

Equilibrium of a stochastic dynamical system is defined by transforming the system into an associated kernel system, then an associated measure theory system, then studying the equilibrium behavior of the associated adjoint correspondence and relating it back to the policies in the original system, as follows. The *associated kernel system* is $(S, \preceq_S, \mathcal{B}(S), \mathcal{P})$ with $\mathcal{P} = \{p \in ker(S \times \mathcal{B}(S)) \mid p(s, A) = q(s, [g^{-1}(A)]_s),\ g \in \mathcal{G}\}$, where $[g^{-1}(A)]_s$ is the s-section of $g^{-1}(A)$, and the *associated measure theory model* is $(\mathcal{X}, \preceq_s, \Phi)$, derived as in Example 3. An ***equilibrium*** of a stochastic dynamical system is a pair (g, μ) such that $\mu = \mathcal{T}_p(\mu)$, where p is derived from g as above, $p(s, A) = q(s, [g^{-1}(A)]_s)$. The ***equilibrium set*** is $\mathcal{E} = \{(g, \mu) \in \mathcal{G} \times \mathcal{X} \mid \mu = \mathcal{T}_p(\mu)\}$ with the product order. Equilibrium (g, μ) is ***isotone*** if g is isotone, and the ***isotone equilibrium set*** is $\mathcal{E}^{iso} = \{(g, \mu) \in \mathcal{E} \mid g \text{ is isotone}\}$.

A stochastic dynamical system with a single policy or update rule $\mathcal{G} = \{g\}$ gives rise to the Markov process governed by the associated kernel. With multiple policies, a stochastic dynamical system can have multiple dynamic evolutions based on different policies, thereby generating a system of Markov processes and each process may have its own collection of stationary distributions.

Complementarities are included as follows. A stochastic dynamical system is ***isotone*** if either $\inf_S S \in S$ or $\sup_S S \in S$, q is isotone, and there is isotone $g \in \mathcal{G}$. The definition of q is isotone follows the one in Example 3. It is ***isotone infimum*** if $\inf_S S \in S$, q is isotone, and there is isotone $\underline{g} \in \mathcal{G}$ such that $\forall g \in \mathcal{G}, \underline{g} \preceq g$. The ***infimum equilibrium set*** is $\underline{\mathcal{E}} = \{(\underline{g}, \mu) \in \mathcal{G} \times \mathcal{X} \mid \mu = \mathcal{T}_{\underline{p}}(\mu)\}$, where $\underline{p}$ is derived from $\underline{g}$ as above. It is ***isotone supremum*** if $\sup_S S \in S$, q is isotone, and there is isotone $\overline{g} \in \mathcal{G}$ such that $\forall g \in \mathcal{G}, g \preceq \overline{g}$. The ***supremum equilibrium set*** is $\overline{\mathcal{E}} = \{(\overline{g}, \mu) \in \mathcal{G} \times \mathcal{X} \mid \mu = \mathcal{T}_{\overline{p}}(\mu)\}$, where $\overline{p}$ is derived from $\overline{g}$ as above. No additional conditions (including topological conditions) are imposed on $\mathcal{G}$. No additional conditions (including continuity conditions) are imposed on policies $g \in \mathcal{G}$.

Example 5 (Stochastic dynamic economies)**.** Many dynamic economic systems have serially correlated shocks (weather shocks, technological advances, agglomeration, unemployment shocks, and so on). A general model incorporating this is formulated as follows. A ***stochastic dynamic economy*** is a collection $((X, \preceq_X, \mathcal{B}(X)), (Z, \preceq_Z, \mathcal{B}(Z)), q, \mathcal{G})$, where X is a Polish space, $\preceq_X$ is a closed partial order on X, $\mathcal{B}(X)$ are the Borel sets of X, Z is a Polish space, $\preceq_Z$ is a closed partial order on Z, $\mathcal{B}(Z)$ are the Borel sets of Z, $q \in ker(Z \times \mathcal{B}(Z))$ is a kernel on $Z \times \mathcal{B}(Z)$, and

$\mathcal{G} \subseteq mbl(X \times Z, X)$ is a subset of measurable functions from $X \times Z$ to X with the pointwise partial order. We impose no additional conditions (beyond measurability) on the policies $g \in \mathcal{G}$. Shocks to the economic system are serially correlated: The distribution of shocks $z_{t+1} \in B$ next period depends on the shock z_t today through the kernel $q(z_t, B)$. (IID shocks are a special case.) This formulation arises in the foundational and widely used model of a stochastic dynamic economy with correlated shocks formulated in Stokey and Lucas (1989), with their standard assumptions. Example 8 provides more details.

Stochastic dynamics in such economies are analyzed using their *associated stochastic dynamical system* $((S, \preceq_S, \mathcal{B}(S)), (Z, \preceq_Z, \mathcal{B}(Z)), q, \hat{\mathcal{G}})$, where $S = X \times Z$, and $\hat{\mathcal{G}}$ is the collection of extension of policies in $g \in \mathcal{G}$ given by $\hat{g} : X \times Z \times Z \to X \times Z$, $\hat{g}(x, z, z') = (g(x, z), z')$. A stochastic dynamic economy evolves as follows: In period t, if the state of the system is $s_t = (x_t, z_t)$ and the serially correlated shock to the system is z_{t+1}, governed by $q(z_t, \cdot)$, the state next period is $s_{t+1} = (g(x_t, z_t), z_{t+1})$. Including serially correlated shocks means that two components of the economy evolve in a correlated manner: x_{t+1} evolves with $g(x_t, z_t)$ and z_{t+1} evolves with $q(z_t, \cdot)$. The *associated kernel system* is $(S, \preceq_S, \mathcal{B}(S), \mathcal{P})$, where $S = X \times Z$ with product partial order $\preceq_S$, product sigma-algebra $\mathcal{B}(S)$, and $\mathcal{P} = \{p \in ker(S \times \mathcal{B}(S)) \mid p((x, z), A) = q(z, [\hat{g}^{-1}(A)]_{(x,z)}),\ g \in \mathcal{G}\}$, and the *associated measure theory model* is $(\mathcal{X}, \preceq_s, \Phi)$, where $\mathcal{X}$ is the set of probability measures on $\mathcal{B}(S)$, $\preceq_s$ is stochastic order, and $\Phi : \mathcal{X} \rightrightarrows \mathcal{X}$ is the adjoint correspondence given by $\Phi(\mu) = \{\mathcal{T}_p(\mu) \mid p \in \mathcal{P}\}$. An ***equilibrium*** of a stochastic dynamic economy is a pair (g, μ) such that $\mu = \mathcal{T}_p(\mu)$, where p is derived from g as above, $p((x, z), A) = q(z, [\hat{g}^{-1}(A)]_{(x,z)})$. The ***equilibrium set*** is $\mathcal{E} = \{(g, \mu) \in \mathcal{G} \times \mathcal{X} \mid \mu = \mathcal{T}_p(\mu)\}$ with the product order. Equilibrium (g, μ) is ***isotone*** if g is isotone. The ***isotone equilibrium set*** is $\mathcal{E}^{iso} = \{(g, \mu) \in \mathcal{E} \mid g \text{ is isotone}\}$.

In order to ensure a unique equilibrium, models in dynamic macroeconomics sometimes include additional assumptions such as smoothness, continuity, contractibility, and convexity. These may be violated due to nonconvex technologies, self-fulfilling behavior, and regime switching costs, leading to multiple equilibria and discontinuous behavior. A benefit of models with complementarities is that they can accommodate some of these situations. A stochastic dynamic economy is ***isotone*** if either $(\inf_X X, \inf_Z Z) \in X \times Z$ or $(\sup_X X, \sup_Z Z) \in X \times Z$, q is isotone, and there is $g \in \mathcal{G}$ such that g is isotone. It is ***isotone infimum*** if $(\inf_X X, \inf_Z Z) \in X \times Z$, q is isotone, there is $\underline{g} \in \mathcal{G}$ such that $\underline{g}$ is isotone, and $\forall g \in \mathcal{G}, \underline{g} \preceq g$. The ***infimum equilibrium set*** is $\underline{\mathcal{E}} = \{(\underline{g}, \mu) \in \mathcal{G} \times \mathcal{X} \mid \mu = \mathcal{T}_{\underline{p}}(\mu)\}$, where $\underline{p}$ is derived from $\underline{g}$ as above. It is ***isotone supremum*** if

$(\sup_X X, \sup_Z Z) \in X \times Z$, q is isotone, there is $\overline{g} \in \mathcal{G}$ such that $\overline{g}$ is isotone, and $\forall g \in \mathcal{G}, g \preceq \overline{g}$. The ***supremum equilibrium set*** is $\overline{\mathcal{E}} = \{(\overline{g}, \mu) \in \mathcal{G} \times \mathcal{X} \mid \mu = \mathcal{T}_{\overline{p}}(\mu)\}$, where $\overline{p}$ is derived from $\overline{g}$ as above. No additional conditions (including topological conditions) are imposed on the structure of $\mathcal{G}$ or on policies $g \in \mathcal{G}$. In particular, no policy in $\mathcal{G}$ is assumed to be upper or lower semicontinuous.

Example 6 (Markov decision processes)**.** Another large class of stochastically evolving phenomena is modeled using Markov decision processes, in which endogenous actions and exogenous uncertainty simultaneously affect the evolution of the process. A ***Markov decision process*** is a collection $((X, \preceq_X, \mathcal{B}(X)), (Z, \preceq_Z, \mathcal{B}(Z)), q, \mathcal{G})$, where X is a Polish space, $\preceq_X$ is a closed partial order on X, $\mathcal{B}(X)$ are the Borel sets of X, Z is a Polish space, $\preceq_Z$ is a closed partial order on Z, $\mathcal{B}(Z)$ are the Borel sets of Z, $q \in ker((X \times Z) \times \mathcal{B}(Z))$, and $\mathcal{G} \subseteq mbl(X \times Z, X)$ is a subset of measurable functions from $X \times Z$ to X with the pointwise partial order. We impose no additional conditions (beyond measurability) on the policies $g \in \mathcal{G}$. In this formulation, shocks to the stochastic system are simultaneously state dependent and serially correlated: The distribution of the shock $z_{t+1} \in B$ next period depends on both today's action x_t and today's shock z_t through the kernel $q((x_t, z_t), B)$.

Equilibrium in Markov decision processes is defined using their *associated stochastic dynamical system* $((S, \preceq_S, \mathcal{B}(S)), (Z, \preceq_Z, \mathcal{B}(Z)), q, \hat{\mathcal{G}})$, where $S = X \times Z$, and $\hat{\mathcal{G}}$ is the collection of extension of policies in $g \in \mathcal{G}$ given by $\hat{g} : X \times Z \times Z \to X \times Z$, $\hat{g}(x, z, z') = (g(x, z), z')$. The *associated kernel system* $(S, \preceq_S, \mathcal{B}(S), \mathcal{P})$ and *associated measure theory model* $(\mathcal{X}, \preceq_s, \Phi)$ are defined analogously. An ***equilibrium*** of a Markov decision process is a pair (g, μ) such that $\mu = \mathcal{T}_p(\mu)$, where p is derived from g analogously $p((x, z), A) = q((x, z), [\hat{g}^{-1}(A)]_{(x,z)})$. The ***equilibrium set*** is $\mathcal{E} = \{(g, \mu) \in \mathcal{G} \times \mathcal{X} \mid \mu = \mathcal{T}_p(\mu)\}$ with the product order. Equilibrium (g, μ) is ***isotone*** if g is isotone. The ***isotone equilibrium set*** is $\mathcal{E}^{iso} = \{(g, \mu) \in \mathcal{E} \mid g \text{ is isotone}\}$. The formulation here applies regardless of the mechanism generating the collection of admissible policies $\mathcal{G}$.

A Markov decision process is ***isotone*** if either $(\inf_X X, \inf_Z Z) \in X \times Z$ or $(\sup_X X, \sup_Z Z) \in X \times Z$, q is isotone, and there is isotone $g \in \mathcal{G}$. It is ***isotone infimum*** if $(\inf_X X, \inf_Z Z) \in X \times Z$, q is isotone, and there is isotone $\underline{g} \in \mathcal{G}$ such that $\forall g \in \mathcal{G}, \underline{g} \preceq g$. The ***infimum equilibrium set*** is $\underline{\mathcal{E}} = \{(\underline{g}, \mu) \in \mathcal{G} \times \mathcal{X} \mid \mu = \mathcal{T}_{\underline{p}}(\mu)\}$, where $\underline{p}$ is derived from $\underline{g}$ as above. It is ***isotone supremum*** if $(\sup_X X, \sup_Z Z) \in X \times Z$, q is isotone, and there is isotone $\overline{g} \in \mathcal{G}$ such that $\forall g \in \mathcal{G}, g \preceq \overline{g}$. The ***supremum equilibrium set*** is $\overline{\mathcal{E}} = \{(\overline{g}, \mu) \in \mathcal{G} \times \mathcal{X} \mid \mu = \mathcal{T}_{\overline{p}}(\mu)\}$, where $\overline{p}$ is derived from $\overline{g}$ as above. No additional conditions (including topological conditions) are imposed on the structure of $\mathcal{G}$ or on policies $g \in \mathcal{G}$. In particular, no policy in $\mathcal{G}$ is assumed to be upper or lower semicontinuous.

Example 7 (Stochastic systems)**.** The previous classes of models are formulated with a view to abstracting some of the unifying features in stochastic dynamical models that arise in many different contexts. In order to highlight their unifying analytical properties, we collect these different models into one. A ***stochastic system*** is one that is either a kernel system, or a stochastic dynamical system, or a stochastic dynamic economy, or a Markov decision process. A stochastic system is ***isotone (respectively, isotone infimum, isotone supremum)*** if the corresponding system is isotone (respectively, isotone infimum, isotone supremum).

Let's look at the models due to Hopenhayn and Prescott (1992) and Balbus, Dziewulski, Reffett, and Woźny (2019) to see how the ideas here take more concrete shape.

Example 8 (HP model)**.** Following Hopenhayn and Prescott (1992), consider the standard stochastic dynamic economy from Stokey and Lucas (1989). Let $X \subseteq \mathbb{R}^\ell$ be a convex Borel set, $Z \subseteq \mathbb{R}^k$ a compact set, each with the standard partial order ($\preceq_X$, $\preceq_Z$), q a kernel on $Z \times \mathcal{B}(Z)$ that satisfies the Feller property, $\Gamma : X \times Z \rightrightarrows X$ a feasibility correspondence that is nonempty valued, compact-valued, and continuous, A the graph of Γ, $F : A \to \mathbb{R}$ a bounded and continuous one-period return function, $\beta \in (0, 1)$ the constant discount rate, $v : X \times Z \to \mathbb{R}$ the unique value function associated with this problem, given by $v(x, z) = \sup_{x' \in \Gamma(x,z)}\{F(x', x, z) + \beta \int v(x', z')q(z, dz')\}$, and $\gamma(x, z) = \{x' \in \Gamma(x, z) \mid v(x, z) = F(x', x, z) + \beta \int v(x', z')q(z, dz')\}$ the policy correspondence. When convenient, we may use the Lebesgue completion of the Borel sigma-algebra on finite-dimensional Euclidean space. Suppose the economy satisfies the complementarity assumptions in Hopenhayn and Prescott (1992), that is, X is subcomplete, F is supermodular on $X \times X$ for each z and has increasing differences in $(x', x; z)$, Γ has strict complementarity, graph of $\Gamma(\cdot, \cdot, z)$ is a sublattice for each z, Γ is ascending, and q is an isotone kernel. With these assumptions, an ***HP model*** is given by $(X, Z, q, \Gamma, F, \beta)$. As shown in Hopenhayn and Prescott (1992), the HP model provides a unified model to study many central topics in dynamic economies such as investment theory, stochastic growth theory, and industry equilibrium. The *associated stochastic dynamic economy* is $((X, \preceq_X, \mathcal{B}(X)), (Z, \preceq_Z, \mathcal{B}(Z)), q, \mathcal{G})$, where $\mathcal{G}$ is the set of measurable selections from γ. The *associated measure theory model* is $(\mathcal{X}, \preceq_s, \Phi)$, derived as in Example 5.

Example 9 (BDRW model)**.** Following Balbus, Dziewulski, Reffett, and Woźny (2019), let I be a compact, perfect Hausdorff topological space of player characteristics and λ a regular probability measure on Borel sets of I, $\mathcal{B}(I)$, vanishing at each singleton. Let $A \subseteq \mathbb{R}^m$ be the action set, $\Gamma : I \rightrightarrows A$ the feasibility correspondence, and $\tilde{A}$ the graph of Γ. Let $\mathcal{M}$ be the set of regular probability measures on Borel sets of $I \times A$, let $\mathcal{R} = \{\mu \in \mathcal{M} \mid \mu|_I = \lambda\}$ be the subset of those measures with marginal on I equal to λ, and $\mathcal{D} = \{\mu \in \mathcal{R} \mid \mu(\tilde{A}) = 1\}$. Let $r : I \times A \times \mathcal{D} \to \mathbb{R}$ be the player payoff function. We assume the conditions in Assumption 1 (page 501) in Balbus, Dziewulski, Reffett, and Woźny (2019). The ***BDRW model*** is $((I, \mathcal{B}(I), \lambda), A, \tilde{A}, r)$. Let γ be the best response correspondence, $\gamma(i, \mu) = \arg\max_{a \in \Gamma(i)} r(i, a, \mu)$. A measure $\mu^* \in \mathcal{D}$ is a *distributional equilibrium* if $\mu^*(\{(i, a) \mid r(i, a, \mu^*) \geq r(i, a', \mu^*),\ \forall a' \in \Gamma(i)\} = 1$. Equivalently, $\mu^*(\{(i, a) \mid a \in \gamma(i, \mu^*)\} = 1$. As shown in Balbus, Dziewulski, Reffett, and Woźny (2019), the BDRW model can be used to understand many economic applications, including social distance model, linear non-atomic supermodular games, large stopping games, and keeping up with the Joneses.

The BDRW model does not fit directly into one of the classes of stochastic systems. Its equilibrium properties can be analyzed using its associated measure theory model as follows. For each measurable selection g from γ, define the mapping $T_g : \mathcal{D} \to \mathcal{D}$ by $T_g(\mu)(B) = \lambda[\tilde{g}^{-1}(B)]_\mu$, where $\tilde{g} : I \times \mathcal{D} \to I \times A$ is given by $\tilde{g}(i, \mu) = (i, g(i, \mu))$, and $[\tilde{g}^{-1}(B)]_\mu$ is the μ-section of $[\tilde{g}^{-1}(B)]$. Let $\mathcal{G}$ be the collection of all measurable selections g from γ. The *associated measure theory model* is $(\mathcal{D}, \preceq_s, \Phi)$, where $\preceq_s$ is stochastic order on $\mathcal{D}$ and $\Phi : \mathcal{D} \rightrightarrows \mathcal{D}$ is given by $\Phi(\mu) = \{T_g(\mu) \mid g \in \mathcal{G}\}$.

Acemoglu and Jensen (2015) propose a model of large dynamic economies with heterogeneous agents and stochastic monotonicity, extending the HP model. Balbus, Dziewulski, Reffett, and Woźny (2022) extend this to large games with no aggregate risk. Their models use some additional continuity assumptions and different methods to compute equilibrium, which are not assumed here.

All the models formulated above can be viewed either directly or through their associated measure theory model as special cases of universal models with complementarities.

Theorem 1. *Consider the class of poset models.*

1. *Every general model that is isotone (respectively, isotone infimum, isotone supremum) is a universal model that is isotone (respectively, isotone infimum, isotone supremum).*

2. *Every measure theory model that is isotone (respectively, isotone infimum, isotone supremum) is a universal model that is isotone (respectively, isotone infimum, isotone supremum).*

3. *For every stochastic system that is isotone (respectively, isotone infimum, isotone supremum), its associated measure theory model is isotone (respectively, isotone infimum, isotone supremum).*

4. *For every HP model and for every BDRW model, their associated measure theory model is isotone infimum and isotone supremum.*

Proof. Statements (1) and (2) follow immediately from the definitions. Consider statement (3) for the case of an isotone kernel system $(X, \preceq_X, \mathcal{B}(X), \mathcal{P})$ and consider the associated measure theory model $(\mathcal{X}, \preceq_s, \Phi)$, where $\mathcal{X}$ is the set of probability measures on $\mathcal{B}(X)$, $\preceq_s$ is stochastic order, and $\Phi : \mathcal{X} \rightrightarrows \mathcal{X}$ is given by $\Phi(\mu) = \{T_p(\mu) \mid p \in \mathcal{P}\}$. Notice that if $\underline{x} = \inf_X X \in X$ then the unit measure on $\underline{x}$, $\delta_{\underline{x}}$, satisfies $\inf_{\mathcal{X}} \mathcal{X} = \delta_{\underline{x}} \in \mathcal{X}$, and if $\overline{x} = \sup_X X \in X$ then $\sup_{\mathcal{X}} \mathcal{X} = \delta_{\overline{x}} \in \mathcal{X}$. Let $\hat{p} \in \mathcal{P}$ be isotone. Then for every increasing set A in X, the measurable function $x \mapsto \hat{p}(x, A)$ is isotone in x. Combined with $\mu \preceq_s \nu$, it follows that for every increasing set A in X, $T_{\hat{p}}(\mu)(A) = \int_X \hat{p}(x, A) d\mu(x) \leq \int_X \hat{p}(x, A) d\nu(x) = T_{\hat{p}}(\nu)(A)$, whence $T_{\hat{p}}(\mu) \preceq_s T_{\hat{p}}(\nu)$. Therefore, $\mu \mapsto T_{\hat{p}}(\mu)$ is an isotone selection from Φ. This shows that $(\mathcal{X}, \preceq_s, \Phi)$ is an isotone measure theory model. If $(X, \preceq_X, \mathcal{B}(X), \mathcal{P})$ is an isotone infimum kernel model with isotone $\underline{p} \in \mathcal{P}$, then the same argument shows that $\mu \mapsto T_{\underline{p}}(\mu)$ is an isotone selection from Φ. To see that it is the infimum selection, fix $p \in \mathcal{P}$ arbitrarily. Then $\underline{p} \preceq_k p$ implies that $\forall x \in X$ and $\forall A \subseteq X$ that is increasing, $\underline{p}(x, A) \leq p(x, A)$. Therefore, for every $\mu \in \mathcal{X}$ and for every increasing $A \subseteq X$, $T_{\underline{p}}(\mu)(A) = \int_X \underline{p}(x, A) d\mu(x) \leq \int_X p(x, A) d\mu(x) = T_p(\mu)(A)$, whence $T_{\underline{p}}(\mu) \preceq_s T_p(\mu)$. This shows that $(\mathcal{X}, \preceq_s, \Phi)$ is an isotone infimum measure theory model. A similar argument works if $\mathcal{P}$ is an isotone supremum kernel model.

Consider statement (3) for the case of a stochastic dynamical system $((S, \preceq_S, \mathcal{B}(S)), (Z, \preceq_Z, \mathcal{B}(Z)), q, \mathcal{G})$. Let its associated kernel system be $(S, \preceq_S, \mathcal{B}(S), \mathcal{P})$ with $\mathcal{P} = \{p \in ker(S \times \mathcal{B}(S)) \mid p(s, A) = q(s, [g^{-1}(A)]_s),\ g \in \mathcal{G}\}$, and its associated measure theory model be $(\mathcal{X}, \preceq_s, \Phi)$. If the stochastic dynamical system is isotone, then q is isotone and there is isotone $\hat{g} \in \mathcal{G}$. Consider the kernel $\hat{p}(s, A) = q(s, [\hat{g}^{-1}(A)]_s)$. Kernel $\hat{p}$ is isotone, because for every $\hat{s} \preceq_S \tilde{s}$ and for every increasing set $A \subseteq S$, $\hat{p}(\hat{s}, A) = q(\hat{s}, [\hat{g}^{-1}(A)]_{\hat{s}}) \leq q(\hat{s}, [\hat{g}^{-1}(A)]_{\tilde{s}}) \leq q(\tilde{s}, [\hat{g}^{-1}(A)]_{\tilde{s}}) = \hat{p}(\tilde{s}, A)$, where the first inequality follows from $[\hat{g}^{-1}(A)]_{\hat{s}} \subseteq [\hat{g}^{-1}(A)]_{\tilde{s}}$, using $\hat{g}$ is isotone and A is increasing, and the second inequality follows from q is isotone and $[\hat{g}^{-1}(A)]_{\tilde{s}}$ is an increasing set in Z. Moreover, either $\inf_S S \in S$ or $\sup_S S \in S$. This shows that the associated kernel system is isotone and therefore, the previous proof implies that the associated measure theory model is isotone. If the stochastic dynamical system is isotone infimum with isotone q and isotone $\underline{g} \in \mathcal{G}$, then the same argument shows that $\underline{p}(s, A) = q(s, [\underline{g}^{-1}(A)]_s)$ is an isotone kernel. To see that it is the lowest kernel in $\mathcal{P}$, fix $g \in \mathcal{G}$ arbitrarily and let $p(s, A) = q(s, [g^{-1}(A)]_s)$. Then $\underline{g} \preceq g$ implies that for every $s \in S$ and for every increasing A in S, $\underline{p}(s, A) = q(s, [\underline{g}^{-1}(A)]_s) \leq q(s, [g^{-1}(A)]_s) = p(s, A)$, where the inequality follows from $[\underline{g}^{-1}(A)]_s \subseteq [g^{-1}(A)]_s$, using $\underline{g} \preceq g$ and A is increasing. This shows that the associated kernel system is isotone infimum and therefore, its associated measure theory model is isotone infimum. A similar argument works if the stochastic dynamical system is isotone supremum.

Statement (3) for the case of stochastic dynamic economy and of Markov decision process is proved similarly.

Statement (4) for HP model follows from Theorem 9.6 (page 263) in Stokey and Lucas (1989), which shows that correspondence γ is nonempty valued, compact valued, and upper hemicontinuous, and from Proposition 2 (page 1395) in Hopenhayn and Prescott (1992), which shows that the functions $(x, z) \mapsto \underline{g}(x, z) := \inf \gamma(x, z)$ and $(x, z) \mapsto \overline{g}(x, z) := \sup \gamma(x, z)$ exist, are measurable, and are isotone.

Statement (4) for BDRW model follows from Lemma 1 (page 502) in Balbus, Dziewulski, Reffett, and Woźny (2019), which shows that the functions $(i, \mu) \mapsto \underline{g}(i, \mu) := \inf \gamma(i, \mu)$ and $(i, \mu) \mapsto \overline{g}(i, \mu) := \sup \gamma(i, \mu)$ exist, are measurable, and are isotone. Moreover, $\mathcal{D}$ is a chain complete poset in the stochastic order $\preceq_s$. It can be shown that the mappings $T_{\underline{g}}$ and $T_{\overline{g}}$ are isotone on $\mathcal{D}$ (for every $\mu, \nu \in \mathcal{D}$, $\mu \preceq_s \nu \Rightarrow T_{\underline{g}}(\mu) \preceq_s T_{\underline{g}}(\nu)$ and $T_{\overline{g}}(\mu) \preceq_s T_{\overline{g}}(\nu)$ in the stochastic order) and for every two measurable selections g, h from γ, if $g \preceq h$ (as measurable functions) then $T_g \preceq T_h$ (as mappings on $\mathcal{D}$, that is, for every $\mu \in \mathcal{D}$, $T_g(\mu) \preceq_s T_h(\mu)$ in the stochastic order). ∎

The next theorem shows equilibrium existence properties of universal models.

Theorem 2. *Consider the class of poset models.*

1. *Every universal model with complementarities has an equilibrium.*

2. *In every universal isotone infimum model $(X, \preceq, \Phi)$, $\mathcal{E}(\Phi)$ contains a chain sup-complete poset $\mathcal{E}(\underline{\Phi})$ such that $\inf_{\mathcal{E}(\underline{\Phi})} \mathcal{E}(\underline{\Phi}) = \inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$. In particular, every universal isotone infimum model has a smallest equilibrium.*

3. *In every universal isotone supremum model $(X, \preceq, \Phi)$, $\mathcal{E}(\Phi)$ contains a chain inf-complete poset $\mathcal{E}(\overline{\Phi})$ such that $\sup_{\mathcal{E}(\overline{\Phi})} \mathcal{E}(\overline{\Phi}) = \sup_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$. In particular, every universal isotone supremum model has a largest equilibrium.*

4. *Every universal model that is isotone infimum and isotone supremum has a smallest and a*

largest equilibrium.

Proof. Statement (1) follows immediately by applying Abian and Brown (1961) to the isotone selection f from Φ to conclude a fixed point exists, or by applying Markowsky (1976) to f to show that $\mathcal{E}(f)$ is a nonempty chain sup-complete (or inf-complete) poset and noting that $\mathcal{E}(f) \subseteq \mathcal{E}(\Phi)$. It is included here for cases (like Example 10) that are outside the scope of statements (2) and (3). For statement (2), let $(X, \preceq, \Phi)$ be a universal isotone infimum model. Statement (1) implies that $\mathcal{E}(\underline{\Phi})$ is a chain sup-complete poset contained in $\mathcal{E}(\Phi)$ and $\inf_{\mathcal{E}(\underline{\Phi})} \mathcal{E}(\underline{\Phi}) \in \mathcal{E}(\underline{\Phi})$. To see that $\inf_{\mathcal{E}(\underline{\Phi})} \mathcal{E}(\underline{\Phi}) = \inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$, let $A = \{x \in X \mid x \preceq \underline{\Phi}(x) \text{ and } \forall e \in \mathcal{E}(\Phi), x \preceq e\}$. The set A is nonempty as $\inf_X X \in A$. Let C be a chain in A. If C is empty, then $\sup_X C = \inf_X X \in A$. If C is not empty, let $y = \sup_X C$, which exists because X is chain sup-complete. Notice that $\forall x, x \in C$ implies $x \preceq \underline{\Phi}(x)$, and also, $x \preceq y$ implies $\underline{\Phi}(x) \preceq \underline{\Phi}(y)$. Thus, $\underline{\Phi}(y)$ is an upper bound for C, whence $y \preceq \underline{\Phi}(y)$. Moreover, $\forall e \in \mathcal{E}(\Phi), \forall x \in C, x \preceq e$ implies that $\forall e \in \mathcal{E}(\Phi)$, e is an upper bound for C, and therefore, $\forall e \in \mathcal{E}(\Phi), y \preceq e$. It follows that $y \in A$. In other words, every chain in A has an upper bound in A. By Zorn's lemma, let e^* be a maximal element in A. Then $e^* \preceq \underline{\Phi}(e^*)$ and $\forall e \in \mathcal{E}(\Phi), e^* \preceq e$. Therefore, $\underline{\Phi}(e^*) \preceq \underline{\Phi}(\underline{\Phi}(e^*))$ and $\forall e \in \mathcal{E}(\Phi), \underline{\Phi}(e^*) \preceq \underline{\Phi}(e) \preceq e$. Consequently, $\underline{\Phi}(e^*) \in A$. As e^* is maximal in A, it cannot be that $e^* \neq \underline{\Phi}(e^*)$, whence $e^* = \underline{\Phi}(e^*) \in \Phi(e^*)$. In other words, $e^* \in \mathcal{E}(\underline{\Phi}) \subseteq \mathcal{E}(\Phi)$ and $\forall e \in \mathcal{E}(\Phi), e^* \preceq e$, that is, $e^* = \inf_{\mathcal{E}(\underline{\Phi})} \mathcal{E}(\underline{\Phi}) = \inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi) \in \mathcal{E}(\Phi)$. This proves statement (2). Statement (3) is proved similarly. Statement (4) follows from statements (2) and (3). ∎

The statements in Theorem 2 are fairly weak. None of the statements requires any structure on the sets $\Phi(x)$ beyond isotone selections. This allows for situations not covered by existing general theorems on fixed points of correspondences on chain sup-complete posets, such as Smithson (1971) and its variants.

Example 10 (Not isotone in quasi weak set order)**.** Smithson (1971) proves existence of fixed points for a correspondence $\Phi : X \rightrightarrows X$ on a chain sup-complete poset X when the images satisfy one of the following conditions. Their condition I requires that Φ is isotone in the upper weak set order: For every $\hat{x} \preceq_X \tilde{x}$, $\forall \hat{y} \in \Phi(\hat{x}), \exists \tilde{y} \in \Phi(\tilde{x}), \hat{y} \preceq_X \tilde{y}$. Their condition II requires Φ is isotone in lower weak set order: For every $\hat{x} \preceq_X \tilde{x}$, $\forall \tilde{y} \in \Phi(\tilde{x}), \exists \hat{y} \in \Phi(\hat{x}), \hat{y} \preceq_X \tilde{y}$. Weak set order is defined using both conditions. Theorem 1.1 in Smithson (1971) (page 305) requires condition I (or condition II) to hold for Φ. Our Theorem 2 (statement (1)) does not require either condition. For example, let $X = \{1, 2, 3\}$ with the natural order and $\Phi : X \rightrightarrows X$ be given by $\Phi(1) = \{1, 3\}$,

$\Phi(2) = \{2\}$, and $\Phi(3) = \{1, 3\}$. Then Φ does not satisfy condition I in Smithson (1971), because $3 \in \Phi(1)$, but there is no $y \in \Phi(2)$ such that $3 \preceq y$, and Φ does not satisfy condition II in Smithson (1971), because $1 \in \Phi(3)$, but there is no $y \in \Phi(2)$ such that $y \preceq 1$. Nevertheless, Φ has an isotone selection and the example satisfies statement (1) in Theorem 2. Indeed, $\mathcal{E}(\Phi) = \{1, 2, 3\}$.

In addition to condition I or II, Smithson (1971) requires Φ to satisfy their condition III: For every chain $C \subseteq X$, $\exists$ isotone function $g : C \to X$ such that $\forall x \in C, g(x) \in \Phi(x)$ and if $x_0 = \sup_X C$ then $\exists y_0 \in \Phi(x_0)$ such that $\forall x \in C, g(x) \preceq_X y_0$. This is automatically satisfied if Φ has an isotone selection, in which case Abian and Brown (1961) applied to this selection prove existence of a fixed point, obviating the need for any additional conditions on the structure of images of Φ. This is the point that Example 10 highlights. As universal models with complementarities assume an isotone selection, conditions I and II in Smithson's theorem are not necessary for equilibrium existence.

Smithson's theorem applies when Φ has an isotone infimum or isotone supremum selection, because Φ has an isotone infimum (respectively, supremum) selection if, and only if, Φ satisfies condition II (respectively, condition I) in Smithson (1971) and $\forall x \in X$, $\inf_{\Phi(x)} \Phi(x)$ (respectively, $\sup_{\Phi(x)} \Phi(x)$) exists. This can be viewed as replacing Smithson's condition III with the simpler and more transparent one that $\Phi(x)$ has a smallest (or largest) point. In other words, in a universal isotone infimum model, in addition to invoking Abian and Brown (1961) to guarantee a fixed point, Smithson (1971) can also be invoked to prove existence of a fixed point. In either case, Markowsky (1976) implies further that $\inf_{\mathcal{E}(\underline{\Phi})} \mathcal{E}(\underline{\Phi})$ exists. Theorem 2 goes further by proving that $\inf_{\mathcal{E}(\underline{\Phi})} \mathcal{E}(\underline{\Phi}) = \inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$. A similar conclusion holds in every universal isotone supremum model, with Theorem 2 showing that $\sup_{\mathcal{E}(\overline{\Phi})} \mathcal{E}(\overline{\Phi}) = \sup_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$.

Li (1984) provides related results. Their Theorem 2.1 is a special case of Smithson (1971) and their Corollary 2.3 states that when Φ satisfies condition I in Smithson (1971) and $\forall x, \sup_{\Phi(x)} \Phi(x)$ exists, then $\mathcal{E}(\Phi)$ has a maximal element and is an inductive set. With the same conditions, our Theorem 2 strengthens their result to $\mathcal{E}(\Phi)$ has a largest element, whence $\mathcal{E}(\Phi)$ is an inductive set. (If necessary, their Assumption A3 can be translated into the subset $\hat{X}$ of X consisting of elements above the point y in their Assumption 3 for which there is $u \in \Phi(y)$ with $y \preceq_X u$, and considering the restricted correspondence $\hat{\Phi}(x) = \Phi(x) \cap \hat{X}$.)

Che, Kim, and Kojima (2021) use condition I (or II) from Smithson (1971) but replace condition III with a compactness condition: $\forall x, \Phi(x)$ is compact (closed subset of a compact metric space X). They make additional assumptions about a natural topology to go with the metric space and partial order. With their assumptions, their Theorem 6 concludes that $\mathcal{E}(\Phi)$ has a maximal (or minimal) element. We don't impose topological restrictions in the universal model: X is not assumed to be a metric space and the images $\Phi(x)$ are not assumed to be compact. If the images $\Phi(x)$ are compact lattices, or if we replace their compactness assumption with $\Phi(x)$ has a largest (or smallest) element and remove all metric space and topological assumptions, then their Theorem 6 is a special case of our Theorem 2, and we strengthen their conclusion to $\mathcal{E}(\Phi)$ has a largest (or smallest) element.

A motivation for working with isotone selections in universal models is that foundational models in economics (and elsewhere) may have discontinuities or non-convexities that limit the use of tools from topology and convex analysis. Isotone selections may provide a methodological advantage in these cases. Moreover, our condition that $\Phi(x)$ has a smallest (or largest) element is more transparent and intuitive as compared to condition III in Smithson (1971) or the related condition A2 in Li (1984): $\forall x, SF(x) = \{\xi \in X \mid \exists y \in \Phi(x), \xi \preceq_X y\}$ is inductively ordered. Manifestation of our conditions in large classes of models make them relevant to broad audiences studying models with complementarities within and outside economics. A hallmark of the analysis here is that we identify and prove the role such selections play in many aspects of the unified theory of equilibrium in universal models with complementarities developed with many new results throughout this paper.

Theorem 2 combined with Theorem 1 implies immediately that measure theory models that are isotone (respectively, isotone infimum, isotone supremum) have the properties in statement (1) (respectively, statement (2), statement (3)) of Theorem 2. The next theorem generalizes this to all stochastic systems.

Theorem 3. *Consider the class of stochastic systems.*

1. *In every isotone stochastic system, the set of isotone equilibria $\hat{\mathcal{E}}$ having the associated isotone kernel or policy is either a chain sup-complete set in which $\inf_{\hat{\mathcal{E}}} \hat{\mathcal{E}}$ exists or a chain inf-complete set in which $\sup_{\hat{\mathcal{E}}} \hat{\mathcal{E}}$ exists. In particular, every isotone stochastic system has an isotone equilibrium.*

2. *In every isotone infimum stochastic system, $\mathcal{E}$ contains a chain sup-complete subset of isotone equilibria $\underline{\mathcal{E}}$ such that $\inf_{\underline{\mathcal{E}}} \underline{\mathcal{E}}$ exists and $\inf_{\underline{\mathcal{E}}} \underline{\mathcal{E}} = \inf_{\mathcal{E}} \mathcal{E} = \inf_{\mathcal{E}^{iso}} \mathcal{E}^{iso}$. In particular, every isotone infimum stochastic system has a smallest equilibrium and it is isotone.*

3. *In every isotone supremum stochastic system, $\mathcal{E}$ contains a chain inf-complete subset of iso-*

tone equilibria $\overline{\mathcal{E}}$ *such that* $\sup_{\overline{\mathcal{E}}} \overline{\mathcal{E}}$ *exists and* $\sup_{\overline{\mathcal{E}}} \overline{\mathcal{E}} = \sup_{\mathcal{E}} \mathcal{E} = \sup_{\mathcal{E}^{iso}} \mathcal{E}^{iso}$. *In particular, every isotone supremum stochastic system has a largest equilibrium and it is isotone.*

4. *Every stochastic system that is isotone infimum and isotone supremum has a smallest and a largest equilibrium and both are isotone.*

Proof. Suppose the stochastic system being considered in statements (1)-(3) is a kernel system. For statement (1), consider an isotone kernel system $(X, \preceq_X, \mathcal{B}(X), \mathcal{P})$ with isotone $\hat{p} \in \mathcal{P}$, let $(\mathcal{X}, \preceq_s, \Phi)$ be the associated measure theory model, and let the set of isotone equilibria associated with $\hat{p}$ be $\hat{\mathcal{E}} = \{(\hat{p}, \mu) \mid \mu = \mathcal{T}_{\hat{p}}(\mu)\} \subseteq \mathcal{E}^{iso}$. Suppose $\inf_X X \in X$. Theorem 1 shows that $\mu \mapsto \mathcal{T}_{\hat{p}}(\mu)$ is an isotone selection from Φ and $\inf_{\mathcal{X}} \mathcal{X} \in \mathcal{X}$. By Markowsky (1976), the set $\mathcal{E}(\hat{p}) = \{\mu \in \mathcal{X} \mid \mu = \mathcal{T}_{\hat{p}}(\mu)\}$ is chain sup-complete with $\hat{\mu} := \inf_{\mathcal{E}(\hat{p})} \mathcal{E}(\hat{p}) \in \mathcal{E}(\hat{p})$. Using this, it can be shown that $\hat{\mathcal{E}}$ is chain sup-complete with $(\hat{p}, \hat{\mu}) = \inf_{\hat{\mathcal{E}}} \hat{\mathcal{E}}$. A similar proof shows that if $\sup_X X \in X$, then $\hat{\mathcal{E}}$ is chain inf-complete and $\sup_{\hat{\mathcal{E}}} \hat{\mathcal{E}}$ exists. For statement (2), suppose the kernel system has an isotone infimum kernel $\underline{p} \in \mathcal{P}$ (that is, $\underline{p}$ is isotone and $\forall p \in \mathcal{P}, \underline{p} \preceq_k p$). As $\inf_X X \in X$, the proof for statement (1) shows that $\underline{\mathcal{E}} = \{(\underline{p}, \mu) \mid \mu = \mathcal{T}_{\underline{p}}(\mu)\}$ is chain sup-complete, and both $\underline{\mu} = \inf_{\underline{\mathcal{E}}} \underline{\mathcal{E}}$ and $(\underline{p}, \underline{\mu}) = \inf_{\underline{\mathcal{E}}} \underline{\mathcal{E}}$ exist. Consider arbitrary $(p, \mu) \in \mathcal{E}$. We know that $\underline{p} \preceq_k p$, and by Theorem 1, $\underline{\mu} \preceq_s \mu$, whence $(\underline{p}, \underline{\mu})$ is a lower bound for $\mathcal{E}$. As $(\underline{p}, \underline{\mu}) \in \underline{\mathcal{E}} \subseteq \mathcal{E}$, it follows that $(\underline{p}, \underline{\mu}) = \inf_{\mathcal{E}} \mathcal{E}$. Combined with $(\underline{p}, \underline{\mu})$ is an isotone equilibrium, it follows that $(\underline{p}, \underline{\mu}) = \inf_{\mathcal{E}} \mathcal{E} = \inf_{\mathcal{E}^{iso}} \mathcal{E}^{iso}$. Statement (3) for an isotone supremum kernel system is proved similarly.

Suppose the stochastic system being considered in statements (1)-(3) is a stochastic dynamical system. For statement (1), consider an isotone stochastic dynamical system $((S, \preceq_S, \mathcal{B}(S)), (Z, \preceq_Z, \mathcal{B}(Z)), q, \mathcal{G})$ with isotone q and isotone $\hat{g} \in \mathcal{G}$, let $(\mathcal{X}, \preceq_s, \Phi)$ be the associated measure theory model, and let the set of isotone equilibria associated with $\hat{g}$ be $\hat{\mathcal{E}} = \{(\hat{g}, \mu) \mid \mu = \mathcal{T}_{\hat{p}}(\mu)\} \subseteq \mathcal{E}^{iso}$, where $\hat{p}$ is derived from $\hat{g}$ using $\hat{p}(s, A) = q(s, [\hat{g}^{-1}(A)]_s)$. Then q is isotone and $\hat{g}$ is isotone imply that $\hat{p}$ is an isotone kernel. Following the proof of statement (1) for isotone kernel systems, if $\inf_S S \in S$, then $\hat{\mathcal{E}}$ is chain sup-complete with smallest element, and if $\sup_S S \in S$, then $\hat{\mathcal{E}}$ is chain inf-complete with largest element. For statement (2), suppose the stochastic dynamical system has an isotone infimum $\underline{g} \in \mathcal{G}$ (that is, $\underline{g}$ is isotone and $\forall g \in \mathcal{G}, \underline{g} \preceq g$). In this case, the associated kernel $\underline{p}$ given by $\underline{p}(s, A) = q(s, [\underline{g}^{-1}(A)]_s)$ is an isotone infimum kernel in the associated kernel system. Following the proof for isotone infimum kernel systems, it follows that $\underline{\mathcal{E}} = \{(\underline{g}, \mu) \mid \mu = \mathcal{T}_{\underline{p}}(\mu)\}$ is chain sup-complete and $\exists (\underline{g}, \underline{\mu}) \in \underline{\mathcal{E}}$ such that $(\underline{g}, \underline{\mu}) = \inf_{\underline{\mathcal{E}}} \underline{\mathcal{E}} = \inf_{\mathcal{E}} \mathcal{E} = \inf_{\mathcal{E}^{iso}} \mathcal{E}^{iso}$. Statement (3) for an isotone supremum stochastic dynamical system is proved similarly.

Statements (1)-(3) for stochastic dynamic economies and Markov decision processes are proved in a manner similar to the proof for stochastic dynamical systems. Statement (4) follows from statements (2) and (3). ∎

Theorem 2 guarantees that the equilibrium set *contains* a nonempty chain sup-complete subset or a nonempty chain inf-complete subset. Theorem 4 strengthens this by presenting conditions that guarantee that *the equilibrium set is a nonempty chain sup-complete set or a nonempty chain inf-complete set or a nonempty chain complete set.* In a poset model $(X, \preceq, \Phi)$, for every $\hat{X} \subseteq X$, the ***model restricted to*** $\hat{X}$ is $(\hat{X}, \hat{\preceq}, \hat{\Phi})$, where $\hat{\preceq}$ is the restriction of the partial order $\preceq$ to $\hat{X}$ and $\hat{\Phi}$ is the restriction of Φ to $\hat{X}$ given by $\hat{\Phi}(x) = \Phi(x) \cap \hat{X}$. A ***universal isotone infimum model on upper intervals*** is a universal isotone infimum model $(X, \preceq, \Phi)$ in which for every $\hat{x} \in X$ such that $\hat{x} \preceq \overline{\Phi}(\hat{x})$, the model restricted to $\hat{X} = \{x \in X \mid \hat{x} \preceq x\}$ is a universal isotone infimum model. As earlier, this is equivalent to $\hat{\Phi}$ is isotone in lower weak set order and $\forall x \in \hat{X}, \underline{\hat{\Phi}}(x)$ exists. A ***universal isotone supremum model on lower intervals*** is a universal isotone supremum model $(X, \preceq, \Phi)$ in which for every $\hat{x} \in X$ such that $\underline{\Phi}(\hat{x}) \preceq \hat{x}$, the model restricted to $\hat{X} = \{x \in X \mid x \preceq \hat{x}\}$ is a universal isotone supremum model. This is equivalent to $\hat{\Phi}$ is isotone in upper weak set order and $\forall x \in \hat{X}, \overline{\hat{\Phi}}(x)$ exists.

Theorem 4. *Consider the class of poset models.*

1. *In every universal isotone supremum model* $(X, \preceq, \Phi)$ *that is isotone infimum on upper intervals,* $\mathcal{E}(\Phi)$ *is a nonempty, chain sup-complete set.*

2. *In every universal isotone infimum model* $(X, \preceq, \Phi)$ *that is isotone supremum on lower intervals,* $\mathcal{E}(\Phi)$ *is a nonempty, chain inf-complete set.*

3. *In every universal model* $(X, \preceq, \Phi)$ *that is isotone infimum on upper intervals and isotone supremum on lower intervals,* $\mathcal{E}(\Phi)$ *is a nonempty, chain complete set.*

4. *In each of (1)-(3) above,* $\inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi) = \inf_{\mathcal{E}(\underline{\Phi})} \mathcal{E}(\underline{\Phi})$ *and* $\sup_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi) = \sup_{\mathcal{E}(\overline{\Phi})} \mathcal{E}(\overline{\Phi})$.

Proof. To prove statement (1), let $(X, \preceq, \Phi)$ be a universal isotone supremum model that is isotone infimum on upper intervals. Then $(X, \preceq)$ is chain complete with $\inf_X X \in X$ and $\sup_X X \in X$, and Theorem 2 shows that $\mathcal{E}(\Phi)$ is nonempty with $\inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi) \in \mathcal{E}(\Phi)$ and $\sup_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi) \in \mathcal{E}(\Phi)$. To show that $\mathcal{E}(\Phi)$ is chain sup-complete, consider nonempty chain $C \subseteq \mathcal{E}(\Phi)$ and let $\overline{e} = \sup_X C \in X$, which exists because X is chain sup-complete. Then $e \in C$ implies $e \preceq \overline{\Phi}(e) \preceq \overline{\Phi}(\overline{e})$, whence $\overline{e} \preceq$

$\overline{\Phi}(\overline{e})$. Let $\hat{X} = \{x \in X \mid \overline{e} \preceq x\}$. By assumption, the model $(\hat{X}, \preceq, \hat{\Phi})$ is a universal isotone infimum model and therefore, has a smallest equilibrium, say, e^*. Then $e^* \in \hat{\Phi}(e^*) = \Phi(e^*) \cap \hat{X} \subseteq \Phi(e^*)$ and $\overline{e} \preceq e^*$ imply that e^* is an upper bound for C in $\mathcal{E}(\Phi)$. If $e \in \mathcal{E}(\Phi)$ is an arbitrary upper bound for C, then $\overline{e} \preceq e$ and therefore, $e \in \Phi(e) \cap \hat{X} = \hat{\Phi}(e)$, whence $e^* \preceq e$. Thus $e^* = \sup_{\mathcal{E}(\Phi)} C \in \mathcal{E}(\Phi)$. This proves statement (1). Statement (2) is proved similarly. Statement (3) follows from statements (1) and (2). Statement (4) follows from Theorem 2. ∎

Theorem 4 goes beyond existing results for structure of the fixed point set of correspondences on posets. It subsumes Markowsky (1976)'s result for isotone functions as a special case when Φ is singleton valued. More broadly, it extends the existence theorems in Abian and Brown (1961) and Smithson (1971) by providing additional structure to the fixed point set, goes beyond existence of maximal elements and inductive set structure of the fixed point set in chain sup-complete posets in Li (1984), and goes beyond existence of maximal or minimal fixed points in Che, Kim, and Kojima (2021). Theorem 4 goes beyond existing results by using natural and transparent assumptions in terms of isotone infimum (or supremum) selections, or equivalently, by using isotone in lower (or upper) weak set order and replacing condition III in Smithson (1971) or condition A2 in Li (1984) with the simpler assumption that infimum (or supremum) exists, which typically holds in models with complementarities.

Theorem 4 also generalizes the complete lattice structure theorem for fixed points of correspondences in lattice-based models proved in Sabarwal (2023b), which, in turn, generalized the well-known lattice-based theorems proved in Tarski (1955), Vives (1990), and Zhou (1994). Sabarwal (2023b) used similar assumptions about isotone infimum and supremum selections. Theorem 4 shows that the same assumptions over more general sets can be used to unify and generalize structure theorems from lattices to posets.

Example 11 (Stochastic systems, continued)**.** Consider a kernel system $(X, \preceq_X, \mathcal{B}(X), \mathcal{P})$. For each $p \in \mathcal{P}$ and $x \in X$, let $p(x)$ denote the probability measure $p(x, \cdot)$. The ***kernel system is strongly isotone (respectively, isotone) infimum on upper intervals*** if it is isotone infimum, the associated $\mathcal{P}$ has a largest kernel $\overline{p}$ (not necessarily isotone) in the kernel order, and for every function (respectively, isotone function) $\hat{p} : X \to \mathcal{X}$ lower than $\overline{p}$ (that is, $\forall x \in X, \hat{p}(x) \preceq_s \overline{p}(x)$), there is isotone kernel $\underline{\hat{p}} \in \mathcal{P}$ such that (1) $\forall x \in X, \hat{p}(x) \preceq_s \underline{\hat{p}}(x)$ and (2) $\forall p \in \mathcal{P}$ (respectively, $\forall p \in \mathcal{P}$ that is isotone), if $\forall x \in X, \hat{p}(x) \preceq_s p(x)$, then $\forall x \in X, \underline{\hat{p}}(x) \preceq_s p(x)$. A ***stochastic dynamical system is strongly isotone (respectively, isotone) infimum on***

upper intervals if it is isotone infimum, its state space S is chain sup-complete, the associated $\mathcal{G}$ has a largest function $\overline{g}$ (not necessarily isotone) in the pointwise partial order and for every function (respectively, isotone function) $\hat{g} : S \times Z \to S$ satisfying $\hat{g} \preceq \overline{g}$ (pointwise), there is isotone function $\underline{\hat{g}} \in \mathcal{G}$ such that (1) $\hat{g} \preceq \underline{\hat{g}}$ and (2) $\forall g \in \mathcal{G}$ (respectively, $\forall g \in \mathcal{G}$ that is isotone), $\hat{g} \preceq g \Rightarrow \underline{\hat{g}} \preceq g$. A ***stochastic dynamic economy or Markov decision process is strongly isotone (respectively, isotone) infimum on upper intervals*** is defined similarly. For every stochastic system, ***strongly isotone (respectively, isotone) supremum on lower intervals*** is defined similarly. It follows immediately that *if a stochastic system is strongly isotone infimum on upper intervals then it is isotone infimum on upper intervals*, and *if it is strongly isotone supremum on lower intervals then it is isotone supremum on lower intervals.*

Theorem 5. *Consider the class of stochastic systems.*

1. *In every isotone supremum stochastic system that is strongly isotone (respectively, isotone) infimum on upper intervals, $\mathcal{E}$ (respectively, $\mathcal{E}^{iso}$) is a nonempty, chain sup-complete set.*

2. *In every isotone infimum stochastic system that is strongly isotone (respectively, isotone) supremum on lower intervals, $\mathcal{E}$ (respectively, $\mathcal{E}^{iso}$) is a nonempty, chain inf-complete set.*

3. *In every stochastic system that is strongly isotone (respectively, isotone) infimum on upper intervals and strongly isotone (respectively, isotone) supremum on lower intervals, $\mathcal{E}$ (respectively, $\mathcal{E}^{iso}$) is a nonempty, chain complete set.*

4. *In every HP model, $\mathcal{E}^{iso}$ is a nonempty, chain complete set.*

5. *In each of (1)-(4) above, $\inf_{\mathcal{E}} \mathcal{E}$ and $\sup_{\mathcal{E}} \mathcal{E}$ exist, and $\inf_{\mathcal{E}} \mathcal{E} = \inf_{\mathcal{E}^{iso}} \mathcal{E}^{iso} = \inf_{\underline{\mathcal{E}}} \underline{\mathcal{E}}$, and $\sup_{\mathcal{E}} \mathcal{E} = \sup_{\mathcal{E}^{iso}} \mathcal{E}^{iso} = \sup_{\overline{\mathcal{E}}} \overline{\mathcal{E}}$.*

Proof. Consider statement (1) for an isotone supremum kernel system $(X, \preceq_X, \mathcal{B}(X), \mathcal{P})$ that is isotone infimum on upper intervals. (The case for strongly isotone infimum on upper intervals is similar). For each $p \in \mathcal{P}$ and $x \in X$, let $p(x)$ denote the probability measure $p(x, \cdot)$. We already know $\mathcal{E}^{iso}$ has a smallest and largest equilibrium. To show that $\mathcal{E}^{iso}$ is chain sup-complete, let C be a nonempty chain in $\mathcal{E}^{iso}$. Let $C_1 = \{p \in \mathcal{P} \mid \exists \mu \in \mathcal{X}, (p, \mu) \in C\}$ and for each $x \in X$, let $\hat{p}(x) = \sup_{\mathcal{X}}\{p(x) \mid p \in C_1\}$, which exists because $\{p(x) \mid p \in C_1\}$ is a chain in $\mathcal{X}$ and $\mathcal{X}$ is chain sup-complete. As $\forall p \in C_1$, p is isotone and $\forall x \in X, p(x) \preceq_s \overline{p}(x)$, it follows that $\hat{p}$ is isotone (as a function of x) and $\forall x \in X, \hat{p}(x) \preceq_s \overline{p}(x)$. Let $\underline{\hat{p}} \in \mathcal{P}$ be isotone such that (1) $\forall x \in X, \hat{p}(x) \preceq_s \underline{\hat{p}}(x)$ and (2) $\forall p \in \mathcal{P}$ that is isotone, if $\forall x \in X, \hat{p}(x) \preceq_s p(x)$, then $\forall x \in X, \underline{\hat{p}}(x) \preceq_s p(x)$. Let $C_2 = \{\mu \in \mathcal{X} \mid \exists p \in \mathcal{P}, (p, \mu) \in C\}$ and let $\hat{\mu} = \sup_{\mathcal{X}} C_2$, which exists because C_2 is a chain and $\mathcal{X}$ is chain sup-complete. Let $\hat{\mathcal{X}} = \{\mu \in \mathcal{X} \mid \hat{\mu} \preceq_s \mu\}$ and for $\mu \in \hat{\mathcal{X}}$, let $\hat{\Phi}(\mu) = \{\mathcal{T}_p(\mu) \mid p \in \mathcal{P}, \underline{\hat{p}} \preceq_k p\}$ where $\mathcal{T}_p$ is the adjoint operator defined by kernel p. Correspondence $\hat{\Phi} : \hat{\mathcal{X}} \rightrightarrows \hat{\mathcal{X}}$ is well defined, because $\hat{\mu} \preceq_s \mathcal{T}_{\underline{\hat{p}}}(\hat{\mu})$ and therefore, $\forall p \in \mathcal{P}$ satisfying $\underline{\hat{p}} \preceq_k p$, it follows that $\hat{\mu} \preceq_s \mu \Rightarrow \hat{\mu} \preceq_s \mathcal{T}_{\underline{\hat{p}}}(\hat{\mu}) \preceq_s \mathcal{T}_{\underline{\hat{p}}}(\mu) \preceq_s \mathcal{T}_p(\mu)$. Moreover, $\hat{\Phi}$ has an isotone infimum selection given by $\mu \mapsto \mathcal{T}_{\underline{\hat{p}}}(\mu)$. Let $\underline{\hat{\mu}}$ be the smallest fixed point of $\hat{\Phi}$. Then $(\underline{\hat{p}}, \underline{\hat{\mu}}) \in \mathcal{E}^{iso}$. Let $(p', \mu') \in \mathcal{E}^{iso}$ be an upper bound for C. Then $\forall x \in X, \hat{p}(x) \preceq_s p'(x)$ implies $\forall x \in X, \underline{\hat{p}}(x) \preceq_s p'(x)$, and moreover $\hat{\mu} \preceq_s \mu'$, and therefore, μ' is a fixed point of $\hat{\Phi}$, whence $\underline{\hat{\mu}} \preceq_s \mu'$. It follows that $(\underline{\hat{p}}, \underline{\hat{\mu}}) = \sup_{\mathcal{E}^{iso}} C$. A similar proof shows that $\mathcal{E}$ is chain sup-complete and has smallest and largest element.

Consider (1) for an isotone supremum stochastic dynamical system $((S, \preceq_S, \mathcal{B}(S)), (Z, \preceq_Z, \mathcal{B}(Z)), q, \mathcal{G})$ that is isotone infimum on upper intervals. Then $\mathcal{E}^{iso}$ has a smallest and largest equilibrium. Let C be a nonempty chain in $\mathcal{E}^{iso}$. Let $C_1 = \{g \in \mathcal{G} \mid \exists \mu \in \mathcal{X}, (g, \mu) \in C\}$. For each $(s, z) \in S \times Z$, let $\hat{g}(s, z) = \sup_S \{g(s, z) \mid g \in C_1\}$, which exists because $\{g(s, z) \mid g \in C_1\}$ is a chain and S is chain sup-complete. As $\forall g \in C_1$, g is isotone and $g \preceq \overline{g}$, it follows that $\hat{g}$ is isotone and $\hat{g} \preceq \overline{g}$. Let $\underline{\hat{g}} \in \mathcal{G}$ be isotone such that (1) $\hat{g} \preceq \underline{\hat{g}}$ and (2) for every $g \in \mathcal{G}$ that is isotone, $\hat{g} \preceq g \Rightarrow \underline{\hat{g}} \preceq g$. Let $C_2 = \{\mu \in \mathcal{X} \mid \exists g \in \mathcal{G}, (g, \mu) \in C\}$ and let $\hat{\mu} = \sup_{\mathcal{X}} C_2$, which exists because C_2 is a chain and $\mathcal{X}$ is chain sup-complete. Let $\hat{\mathcal{X}} = \{\mu \in \mathcal{X} \mid \hat{\mu} \preceq_s \mu\}$ and for $\mu \in \hat{\mathcal{X}}$, let $\hat{\Phi}(\mu) = \{\mathcal{T}_g(\mu) \mid g \in \mathcal{G}, \underline{\hat{g}} \preceq g\}$ where $\mathcal{T}_g$ is the adjoint operator defined by the kernel derived from g, as above. Correspondence $\hat{\Phi}$ is well defined, because $\hat{\mu} \preceq_s \mathcal{T}_{\underline{\hat{g}}}(\hat{\mu})$ and therefore, $\forall g \in \mathcal{G}$ with $\underline{\hat{g}} \preceq g$, $\hat{\mu} \preceq_s \mu \Rightarrow \hat{\mu} \preceq_s \mathcal{T}_{\underline{\hat{g}}}(\hat{\mu}) \preceq_s \mathcal{T}_{\underline{\hat{g}}}(\mu) \preceq_s \mathcal{T}_g(\mu)$. Moreover, $\hat{\Phi}$ has an isotone infimum selection given by $\mu \mapsto \mathcal{T}_{\underline{\hat{g}}}(\mu)$. Let $\underline{\hat{\mu}}$ be the smallest fixed point of $\hat{\Phi}$. Then $(\underline{\hat{g}}, \underline{\hat{\mu}}) \in \mathcal{E}^{iso}$. Let $(g', \mu') \in \mathcal{E}^{iso}$ be an upper bound for C. Then $\hat{g} \preceq g'$ implies $\underline{\hat{g}} \preceq g'$, and moreover $\hat{\mu} \preceq_s \mu'$, and therefore, μ' is a fixed point of $\hat{\Phi}$, whence $\underline{\hat{\mu}} \preceq_s \mu'$. It follows that $(\underline{\hat{g}}, \underline{\hat{\mu}}) = \sup_{\mathcal{E}^{iso}} C$. A similar proof shows that $\mathcal{E}$ is chain inf-complete and has smallest and largest element.

Statement (1) for the cases of stochastic dynamic economies and Markov decision processes are proved similarly. Statement (2) is proved similarly. Statement (3) follows from (1) and (2).

For statement (4), consider an HP model $(X, Z, q, \Gamma, F, \beta)$, with $v : X \times Z \to \mathbb{R}$ the unique value function given by $v(x, z) = \sup_{x' \in \Gamma(x,z)} \{F(x', x, z) + \beta \int v(x', z') q(z, dz')\}$ and $\gamma(x, z) = \{x' \in \Gamma(x, z) \mid v(x, z) = F(x', x, z) + \beta \int v(x', z') q(z, dz')\}$ the policy correspondence. The associated stochastic dynamic economy is $((X, \preceq_X, \mathcal{B}(X)), (Z, \preceq_Z, \mathcal{B}(Z)), q, \mathcal{G})$, where $\mathcal{G}$ is the set of measurable selections from γ with smallest selection $\underline{g}$ and largest selection $\overline{g}$. Theorem 3 implies that $\mathcal{E}^{iso}$ has a smallest and largest equilibrium. Let C be a nonempty chain in $\mathcal{E}^{iso}$. Let $C_1 = \{g \in \mathcal{G} \mid \exists \mu \in \mathcal{X}, (g, \mu) \in C\}$. For each $(x, z) \in X \times Z$, let $\hat{g}(x, z) = \sup_X \{g(x, z) \in X \mid g \in C_1\}$, which exists because $\{g(x, z) \mid g \in C_1\}$ is a chain and X is a complete lattice, hence chain sup-complete. As $\forall g \in C_1$, g is isotone and $g \preceq \overline{g}$, it follows that $\hat{g}$ is isotone and $\hat{g} \preceq \overline{g}$. For each $(x, z) \in X \times Z$, consider the restricted problem $\hat{\gamma}(x, z) = \arg\max_{x' \in \hat{\Gamma}(x,z)} F(x', x, z) + \beta \int v(x', z') q(z, dz')$, where $\hat{\Gamma}(x, z) = \Gamma(x, z) \cap [\hat{g}(x, z), \infty)$. The correspondence $\hat{\gamma}$ is nonempty valued because $\overline{g}$ is a selection from $\hat{\gamma}$. Moreover, $\hat{\Gamma}$ remains isotone in strong set order and the optimization problem continues to fit the framework of Topkis (1978) and Hopenhayn and Prescott (1992). Let $\underline{\hat{g}}$ be the isotone infimum selection from $\hat{\gamma}$, which is measurable (using Lebesgue completion of Borel sets if needed) because $X \times Z$ is in finite-dimensional Euclidean space. We show that $\forall (x, z)$, $\hat{\gamma}(x, z) = \gamma(x, z) \cap [\hat{g}(x, z), \infty)$, and therefore, $\underline{\hat{g}}$ is the isotone infimum measurable selection from $\gamma(x, z) \cap [\hat{g}(x, z), \infty)$. To see that $\hat{\gamma}(x, z) = \gamma(x, z) \cap [\hat{g}(x, z), \infty)$, suppose $\xi \in \hat{\gamma}(x, z)$. Then $\hat{g}(x, z) \preceq_X \xi$. Moreover, $\hat{g}(x, z) \preceq_X \overline{g}(x, z)$ and $\overline{g}(x, z) \in \gamma(x, z)$ imply $\overline{g}(x, z) \in \hat{\Gamma}(x, z)$, whence $F(\xi, x, z) + \beta \int v(\xi, z') q(z, dz') \geq F(\overline{g}(x, z), x, z) + \beta \int v(\overline{g}(x, z), z') q(z, dz') \geq F(x', x, z) + \beta \int v(x', z') q(z, dz'), \forall x' \in \Gamma(x, y)$, where the second inequality follows from $\overline{g}(x, z) \in \gamma(x, z)$. This shows that $\xi \in \gamma(x, z) \cap [\hat{g}(x, z), \infty)$. In the other direction, suppose $\xi \in \gamma(x, z) \cap [\hat{g}(x, z), \infty)$. Then $\hat{g}(x, z) \preceq \xi$ and $\forall x' \in \Gamma(x, z)$, $F(\xi, x, z) + \beta \int v(\xi, z') q(z, dz') \geq F(x', x, z) + \beta \int v(x', z') q(z, dz')$, whence $\xi \in \hat{\gamma}(x, z)$.

Let $C_2 = \{\mu \in \mathcal{X} \mid \exists g \in \mathcal{G}, (g, \mu) \in C\}$ and let $\hat{\mu} = \sup_{\mathcal{X}} C_2$, which exists because C_2 is a chain and $\mathcal{X}$ is chain sup-complete. Let $\hat{\mathcal{X}} = \{\mu \in \mathcal{X} \mid \hat{\mu} \preceq_s \mu\}$ and for $\mu \in \hat{\mathcal{X}}$, let $\hat{\Phi}(\mu) = \{\mathcal{T}_g(\mu) \mid g \in \mathcal{G}, \underline{\hat{g}} \preceq g\}$ where $\mathcal{T}_g$ is the adjoint operator defined by the kernel derived from g. Using $\hat{\gamma}(x, z) = \gamma(x, z) \cap [\hat{g}(x, z), \infty)$ and the same argument as in proof of statement (1) above, it can be shown that $\hat{\Phi}$ has an isotone infimum selection given by $\mu \mapsto \mathcal{T}_{\underline{\hat{g}}}(\mu)$. Let $\underline{\hat{\mu}}$ be the smallest fixed point of $\hat{\Phi}$. Then $(\underline{\hat{g}}, \underline{\hat{\mu}}) = \sup_{\mathcal{E}^{iso}} C$, showing that $\mathcal{E}^{iso}$ is chain sup-complete. Similarly $\mathcal{E}^{iso}$ is chain inf-complete.

Statement (5) follows from Theorem 3 and in the case of statement (4) using Theorem 1 as well. ∎

3 Equilibrium set comparisons

In order to compare equilibrium sets in poset models, we define the star chain complete set order as follows. For nonempty subsets A, B in a poset X, A is ***chain sup-complete in*** B, if for every nonempty chain $C \subseteq A$, $\sup_B C \in B$, and B is ***chain inf-complete in*** A, if for every nonempty chain $C \subseteq B$, $\inf_A C \in A$. Set A ***is lower than*** B ***in the star chain complete set order***, denoted $A \sqsubseteq^{*cc} B$, if A is chain sup-complete in B and B is chain inf-complete in A.

To define infimum and supremum for arbitrary subsets of a poset, we follow Sabarwal (2023b). For nonempty subsets E and A of poset X, the ***sup of*** E ***in*** A, denoted $\sup_A E$, is an element $\overline{e} \in A$ such that (1) $\overline{e}$ is an upper bound for E and (2) for every $a \in A$ that is an upper bound for E, $\overline{e} \preceq a$. The ***inf of*** E ***in*** A, denoted $\inf_A E$, is an element $\underline{e} \in A$ such that (1) $\underline{e}$ is a lower bound for E and (2) for every $a \in A$ that is a lower bound for E, $a \preceq \underline{e}$. Notice that $A = X$ gives the standard definition and $E \subseteq A \subseteq X$ gives the standard definition in the relative partial order. More generally, as E and A are arbitrary nonempty subsets of X, $\sup_A E$ and $\inf_A E$ might not exist in general even if X is a complete lattice. When they exist, they have some natural properties, proved in Sabarwal (2023b).

The star chain complete set order generalizes Sabarwal (2023b)'s star complete set order, defined as follows. For nonempty subsets A, B in a poset X, A is *sup-complete in* B, if for every nonempty $E \subseteq A$, $\sup_B E \in B$, and B is *inf-complete in* A, if for every nonempty $E \subseteq B$, $\inf_A E \in A$. Set A ***is lower than*** B ***in star complete set order***, denoted $A \sqsubseteq^{*c} B$, if A is sup-complete in B and B is inf-complete in A. For nonempty subsets A, B in a poset X, A *is join-complete in* B, if for every $x \in A$ and $y \in B$, $\sup_B\{x, y\} \in B$. Similarly, B *is meet-complete in* A, if for every $x \in A$ and $y \in B$, $\inf_A\{x, y\} \in A$, and A ***is lower than*** B ***in star lattice set order***, denoted $A \sqsubseteq^{*\ell} B$, if A is join-complete in B and B is meet-complete in A. Some properties of the star chain complete set order and its relation to other orders are as follows.

Theorem 6. *Let X be a poset and A, B, C be nonempty subsets of X.*

1. *Star chain complete set order*

   (a) $A \sqsubseteq^{*cc} A \iff A$ *is a chain complete poset (in the relative partial order from X)*

   (b) $A \sqsubseteq^{*cc} B \implies \inf_A A \preceq \inf_B B$ *and* $\sup_A A \preceq \sup_B B$*, whenever these exist*

   (c) $A \sqsubseteq^{*cc} B \implies A \sqsubseteq^{w} B$.

2. *Cross comparisons*

(a) $A \sqsubseteq^{*c} B \implies A \sqsubseteq^{*cc} B$

(b) *If B is chain inf-complete in A and A is a lattice, then B is meet-complete in A.*

(c) *If A is chain sup-complete in B and B is a lattice, then A is join-complete in B.*

(d) *If A and B are lattices, then $A \sqsubseteq^{*cc} B \implies A \sqsubseteq^{*\ell} B \implies A \sqsubseteq^{w} B$.*

(e) *If X is a lattice, then $A \sqsubseteq^{s} B \implies A \sqsubseteq^{*\ell} B \implies A \sqsubseteq^{w} B$.*

(f) *If A and B are subcomplete in X, then $A \sqsubseteq^{*c} B \Leftrightarrow A \sqsubseteq^{*cc} B \Leftrightarrow A \sqsubseteq^{*\ell} B \Leftrightarrow A \sqsubseteq^{w} B$*

Proof. For (1)(a), if $A \sqsubseteq^{*cc} A$, then for every nonempty chain $C \subseteq A$, $\inf_A C \in A$ and $\sup_A C \in A$, showing that A is chain complete. If A is chain complete, the reverse argument shows that $A \sqsubseteq^{*cc} A$. For (1)(b), suppose $A \sqsubseteq^{*cc} B$ and suppose $\underline{a} = \inf_A A$ and $\underline{b} = \inf_B B$ exist. Then $A \sqsubseteq^{*cc} B$ implies $a' = \inf_A\{\underline{b}\} \in A$, whence $\underline{a} \preceq a' \preceq \underline{b}$. Similarly, $\sup_A A \preceq \sup_B B$. For (1)(c), suppose $A \sqsubseteq^{*cc} B$ and consider arbitrary $a \in A$. Let $b' = \sup_B\{a\} \in B$. Then $a \preceq b'$. Similarly, for $b \in B$ there is $a' = \inf_A\{b\} \in A$ such that $a' \preceq b$.

Statement 2(a) follows immediately, because B is inf-complete in A implies B is chain inf-complete in A, and A is sup-complete in B implies A is chain sup-complete in B. For (2)(b), suppose B is chain inf-complete in A and A is a lattice. Let $x \in A, y \in B$. Let $E = \{y\} \subseteq B$ and let $\hat{y} = \inf_A E \in A$, which exists because B is chain inf-complete in A. Then $\hat{y}$ is a lower bound for E and for every $z \in A$ that is a lower bound for E, $z \preceq \hat{y}$. In other words, $\hat{y} \preceq y$ and for every $z \in A$ such that $z \preceq y$, it must be that $z \preceq \hat{y}$. As A is a lattice, let $\hat{a} = \inf_A\{x, \hat{y}\} \in A$. Then $\hat{y} \preceq y$ implies that $\hat{a}$ is a lower bound for $\{x, y\}$. Suppose $z \in A$ is a lower bound for $\{x, y\}$. Then $z \in A$ and $z \preceq y$ implies $z \preceq \hat{y}$ and therefore, z is a lower bound for $\{x, \hat{y}\}$, whence $z \preceq \hat{a}$. This shows that $\inf_A\{x, y\} = \hat{a} \in A$. Statement (2)(c) is proved similarly and (2)(d) follows from (2)(b) and (2)(c) and the last implication is due to Sabarwal (2023b). Statement 2(e) is proved in Sabarwal (2023b) To prove 2(f), it is sufficient to show that $A \sqsubseteq^{w} B \implies A \sqsubseteq^{*c} B$. Suppose $A \sqsubseteq^{w} B$. Let $E \subseteq A$ be nonempty. As A is subcomplete, let $\overline{a} = \sup_X E \in A$. Let $\hat{b} \in B$ such that $\overline{a} \preceq_X \hat{b}$, and let $U = \{b \in B \mid \overline{a} \preceq_X b\}$, which contains $\hat{b}$. As B is subcomplete, let $\overline{b} = \inf_X U \in B$. Then $\overline{a}$ is a lower bound of U implies $\overline{a} \preceq_X \overline{b}$, whence $\overline{b} = \sup_B E$, showing that A is sup-complete in B. Similarly, B is inf-complete in A. ∎

As shown in Theorem 6, the star chain complete set order and the star lattice set order are both stronger than the weak set order, and imply isotone infimum and supremum when these exist.

Additional comparisons are as follows.

For every nonempty subset A, B of poset X, $A \sqsubseteq^{*c} B \implies A \sqsubseteq^{*cc} B$. The converse is not true: Let $A = \{(1,0),(0,1)\}$ and $B = \{(2,0),(0,2)\}$ in $\mathbb{R}^2$ with the product order. Then $A \sqsubseteq^{*cc} B$ and $A \not\sqsubseteq^{*c} B$. When A and B are lattices, $A \sqsubseteq^{*cc} B \implies A \sqsubseteq^{*\ell} B$. The converse is not true: Let $A = \{0\}$ and $B = (0,1]$ in $\mathbb{R}$. Then $A \sqsubseteq^{*\ell} B$ and $A \not\sqsubseteq^{*cc} B$. On a lattice X, the star lattice set order is an intermediate notion between weak set order and strong set order. The converse of these statements is not true either.

When comparing subcomplete subsets of poset X, the star complete set order, star chain complete set order, star lattice set order, and weak set order are all equivalent. These equivalences do not hold for more general classes of subsets. The example above with $A = \{(1,0),(0,1)\}$ and $B = \{(2,0),(0,2)\}$ in $\mathbb{R}^2$ shows that even when A and B are chain complete and $A \sqsubseteq^{*cc} B$, it can be that $A \not\sqsubseteq^{*c} B$ and $A \not\sqsubseteq^{*\ell} B$. Strong set order is more restrictive. Even in the class of subcomplete subsets, these equivalences do not necessarily hold for the strong set order: Let $A = \{0,2\}$ and $B = \{1,3\}$, both are subcomplete (chains) in $\mathbb{N}$ (and in $\mathbb{R}$), $A \sqsubseteq^{w} B$ and $A \not\sqsubseteq^{s} B$. Additional examples can be constructed as well.

Recall that the strong set order is reflexive on the class of sublattices of a lattice, and is a partial order on this class. The weak set order is reflexive on the class of nonempty subsets of a poset, is not antisymmetric, but is transitive. Theorem 6 shows that the star chain complete set order is reflexive on the class of complete lattices in a poset. Like the weak set order, it is easy to see that it is not necessarily antisymmetric. Theorem 6 shows that on the class of subcomplete subsets of a poset, the star chain complete set order is transitive. This is not necessarily true more generally: Let $X = [0,3] \subseteq \mathbb{R}$, $A = [0,2) \cup \{3\}$, $B = \{1,3\}$, and $C = \{2,3\}$. In this case, $A \sqsubseteq^{*cc} B$ and $B \sqsubseteq^{*cc} C$, but $A \not\sqsubseteq^{*cc} C$, because $\inf_A\{2\}$ does not exist in A. In terms of comparison to strong set order, it is easy to check that $A \not\sqsubseteq^{s} B$, $B \not\sqsubseteq^{s} C$, and $A \not\sqsubseteq^{s} C$, and in terms of weak set order, $A \sqsubseteq^{w} C$. We want to compare different equilibrium sets or subsets of equilibrium sets and these are typically not subcomplete. Therefore, transitivity cannot be taken for granted and must be proved every time it appears in a theorem. We prove this in all the general results in this paper.

The extended S-model example in Sabarwal (2023b) is a universal model with complementarities. It provides additional distinctions between star complete set order, star chain complete set order, star lattice set order, and weak set order in a canonical model with complementarities.

Theorem 7. *Consider poset models in which X is chain complete.*

1. *In every universal isotone infimum model,* $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*cc} \mathcal{E}(\Phi)$.

2. *In every universal isotone supremum model,* $\mathcal{E}(\Phi) \sqsubseteq^{*cc} \mathcal{E}(\overline{\Phi})$.

3. *In every universal model that is isotone infimum and isotone supremum, in addition to (1) and (2),* $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*cc} \mathcal{E}(\overline{\Phi})$.

Proof. For statement (1), to show that $\mathcal{E}(\Phi)$ is inf-complete in $\mathcal{E}(\underline{\Phi})$, consider nonempty chain $C \subseteq \mathcal{E}(\Phi)$ and let $\underline{e} = \inf_X C \in X$, which exists because X is chain inf-complete. Let $\hat{X} = \{x \in X \mid x \preceq \underline{e}\}$ and $\Psi : \hat{X} \to \hat{X}$ be given $\Psi(x) = \underline{\Phi}(x)$. Then $e \in C \implies \underline{\Phi}(\underline{e}) \preceq \underline{\Phi}(e) \preceq e$, and therefore, $\underline{\Phi}(\underline{e})$ is a lower bound for C, whence $\underline{\Phi}(\underline{e}) \preceq \underline{e}$. Moreover, for every $x \in \hat{X}$, $\underline{\Phi}(x) \preceq \underline{\Phi}(\underline{e}) \preceq \underline{e}$. This shows that Ψ is well-defined and therefore, $(\hat{X}, \preceq_X, \Psi)$ is a universal isotone supremum model (a poset model in which $\hat{X}$ is chain inf-complete, $\sup_{\hat{X}} \hat{X} \in \hat{X}$, and Ψ is an isotone function). Let $\hat{e}$ be the largest fixed point of Ψ. Then $\hat{e} = \Psi(\hat{e}) = \underline{\Phi}(\hat{e})$ implies that $\hat{e} \in \mathcal{E}(\underline{\Phi})$, and $\hat{e} \preceq \underline{e}$ implies that $\hat{e}$ is a lower bound for C. If $e \in \mathcal{E}(\underline{\Phi})$ is an arbitrary lower bound for C, then $e \preceq \overline{e}$ and $e = \underline{\Phi}(e) = \Psi(e)$, showing that e is a fixed point of Ψ, whence $e \preceq \hat{e}$. Therefore, $\inf_{\mathcal{E}(\underline{\Phi})} C = \hat{e} \in \mathcal{E}(\underline{\Phi})$.

To show that $\mathcal{E}(\underline{\Phi})$ is sup-complete in $\mathcal{E}(\Phi)$, consider nonempty chain $C \subseteq \mathcal{E}(\underline{\Phi})$ and let $\bar{e} = \sup_X C \in X$, which exists as X is chain sup-complete. Let $\hat{X} = \{x \in X \mid \overline{e} \preceq x\}$ and $\Psi : \hat{X} \rightrightarrows \hat{X}$ be given by $\Psi(x) = \Phi(x) \cap \hat{X}$. Then $e \in C \implies e = \underline{\Phi}(e) \preceq \underline{\Phi}(\overline{e})$ and therefore, $\underline{\Phi}(\overline{e})$ is an upper bound for C, whence $\overline{e} \preceq \underline{\Phi}(\overline{e})$. Moreover, for every $x \in \hat{X}$, $\overline{e} \preceq \underline{\Phi}(\overline{e}) \preceq \underline{\Phi}(x)$, whence $\Phi(x) \subseteq \hat{X}$, and therefore, $\Psi(x) = \Phi(x)$. Consequently, $(\hat{X}, \preceq_X, \Psi)$ is a universal isotone infimum model ($\hat{X}$ is chain sup-complete, $\inf_{\hat{X}} \hat{X} \in \hat{X}$, and Ψ has an isotone infimum selection). Let $\hat{e}$ be the smallest fixed point of Ψ. Then $\hat{e} \in \Psi(\hat{e}) = \Phi(\hat{e}) \cap \hat{X} \implies \hat{e} \in \mathcal{E}(\Phi)$, and $\overline{e} \preceq \hat{e} \implies \hat{e}$ is an upper bound for C. If $e \in \mathcal{E}(\Phi)$ is an arbitrary upper bound for C, then $\overline{e} \preceq e$ and $e \in \Phi(e) \cap \hat{X} = \Psi(e)$, showing that e is a fixed point of Ψ, whence $\hat{e} \preceq e$. Therefore, $\sup_{\mathcal{E}(\Phi)} C \in \mathcal{E}(\Phi)$.

Statement (2) is proved similarly. Statement (3) is proved by following the proof of statement (1) with $\mathcal{E}(\overline{\Phi})$ instead of $\mathcal{E}(\Phi)$ and the proof of statement (2) with $\mathcal{E}(\underline{\Phi})$ instead of $\mathcal{E}(\Phi)$. ∎

In addition to comparative statics of the entire equilibrium set, Theorem 7 provides a formal theory of order approximation of equilibria. This extends to universal models the corresponding theory proposed and proved in Sabarwal (2023b) for general models with complementarities. Statement (1) shows that in models with an isotone infimum selection, every nonempty chain of equilibria has a largest lower bound among equilibria that arise using the infimum selection. Therefore, if $C \subseteq \mathcal{E}(\Phi)$ is a nonempty chain of equilibria of particular interest, it can be uniquely approximated from below in a formal order theoretic manner using an equilibrium from the infimum selection. In the special case that $C = \{e^*\}$ is a singleton, this proves that $\forall e^* \in \mathcal{E}(\Phi)$, $\exists$ unique $\hat{e} \in \mathcal{E}(\underline{\Phi})$ lower than e^* and closest to it among all equilibria associated with the infimum selection. In other words, *every equilibrium $e^* \in \mathcal{E}(\Phi)$ can be uniquely order approximated from below by an equilibrium in $\mathcal{E}(\underline{\Phi})$*. This may be particularly useful if the infimum selection is easier to work with or has some useful computational, dynamic, or theoretical properties (see, for example, Becker and Rincón-Zapatero (2021)). This result requires very little structure for the poset model (only chain complete X with a smallest point and isotone infimum selection). Notably, it does not require that X has a largest element.

Moreover, if $\mathcal{E}(\underline{\Phi})$ and $\mathcal{E}(\Phi)$ are lattices, then it follows that $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*\ell} \mathcal{E}(\Phi)$, and therefore, for every equilibrium $\hat{x} \in \mathcal{E}(\underline{\Phi})$ and $\tilde{x} \in \mathcal{E}(\Phi)$, if $\hat{x} \npreceq \tilde{x}$, then there is a different and unique equilibrium $\underline{\hat{x}} \in \mathcal{E}(\underline{\Phi})$ that is the largest equilibrium in $\mathcal{E}(\underline{\Phi})$ smaller than both of these equilibria.

Similarly, statement (2) shows that in universal isotone supremum models with chain complete X, every nonempty chain C of equilibria can be uniquely approximated from above as a smallest upper bound using equilibria from the supremum selection. In the special case that $C = \{e^*\}$ is a singleton, this proves that *every equilibrium $e^* \in \mathcal{E}(\Phi)$ can be uniquely order approximated from above by an equilibrium in $\mathcal{E}(\overline{\Phi})$*. This result also requires very little structure for the poset model (only chain complete X with a largest point and isotone supremum selection). For example, Rostek and Yoder (2020) show that in matching with complementarities, stable outcomes are characterized by the largest fixed point of a monotone operator. In particular, this result does not require that X has a smallest element.

In models with both isotone infimum and supremum selections, both sets of results hold. The next theorem proves that these results apply to every stochastic system.

Theorem 8. *Consider the class of stochastic systems.*

1. *In every isotone infimum stochastic system, $\underline{\mathcal{E}} \sqsubseteq^{*cc} \mathcal{E}$ and $\underline{\mathcal{E}} \sqsubseteq^{*cc} \mathcal{E}^{iso}$.*

2. *In every isotone supremum stochastic system, $\mathcal{E} \sqsubseteq^{*cc} \overline{\mathcal{E}}$ and $\mathcal{E}^{iso} \sqsubseteq^{*cc} \overline{\mathcal{E}}$.*

3. *In every stochastic system that is isotone infimum and isotone supremum, in addition to (1) and (2), $\underline{\mathcal{E}} \sqsubseteq^{*cc} \overline{\mathcal{E}}$.*

4. *In every isotone supremum stochastic system that is strongly isotone infimum on upper in-*

tervals, $\mathcal{E}$ is chain sup-complete in $\mathcal{E}^{iso}$.

5. *In every isotone infimum stochastic system that is strongly isotone supremum on lower intervals, $\mathcal{E}$ is chain inf-complete in $\mathcal{E}^{iso}$.*

6. *In every stochastic system that is strongly isotone infimum on upper intervals and strongly isotone supremum on lower intervals, $\mathcal{E}$ is chain complete in $\mathcal{E}^{iso}$.*

Proof. Suppose the stochastic system being considered in statements (1)-(3) is a kernel system. For statement (1), consider an isotone infimum kernel system $(X, \preceq_X, \mathcal{B}(X), \mathcal{P})$ with $\inf_X X \in X$ and isotone $\underline{p} \in \mathcal{P}$ such that $\forall p \in \mathcal{P}, \underline{p} \preceq_k p$ and consider the associated measure theory model $(\mathcal{X}, \preceq_s, \Phi)$. Let C be a nonempty chain in $\mathcal{E}$ and let $C' = \{\mu \in \mathcal{X} \mid (\exists p \in \mathcal{P}), (p, \mu) \in C\} \subseteq \mathcal{E}(\Phi)$. It follows that C' is a nonempty chain in $\mathcal{E}(\Phi)$ and using $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*cc} \mathcal{E}(\Phi)$, let $\hat{\mu} = \inf_{\mathcal{E}(\underline{\Phi})} C' \in \mathcal{E}(\underline{\Phi})$. Then $(\underline{p}, \hat{\mu}) \in \underline{\mathcal{E}}$ is a lower bound for C from $\underline{\mathcal{E}}$. If $(\underline{p}, \mu') \in \underline{\mathcal{E}}$ is an arbitrary lower bound for C from $\underline{\mathcal{E}}$, then μ' is a lower bound for C' from $\mathcal{E}(\underline{\Phi})$, whence $\mu' \preceq_s \hat{\mu}$. This shows that $(\underline{p}, \hat{\mu}) = \inf_{\underline{\mathcal{E}}} C \in \underline{\mathcal{E}}$. Now suppose C is a nonempty chain in $\underline{\mathcal{E}}$ and let $C' = \{\mu \in \mathcal{X} \mid (\underline{p}, \mu) \in C\} \subseteq \mathcal{E}(\Phi)$. As $\mathcal{X}$ is chain sup-complete, let $\overline{\mu} = \sup_{\mathcal{X}} C' \in \mathcal{X}$. Let $\hat{\mathcal{X}} = \{\mu \in \mathcal{X} \mid \overline{\mu} \preceq_s \mu\}$ and consider $\Psi : \hat{\mathcal{X}} \rightrightarrows \hat{\mathcal{X}}$ be given by $\Psi(\mu) = \Phi(\mu) \cap \hat{\mathcal{X}}$. Then $\mu \in C' \implies \mu = \underline{\Phi}(\mu) \preceq_s \underline{\Phi}(\overline{\mu})$ and therefore, $\overline{\mu} \preceq_s \underline{\Phi}(\overline{\mu})$. Moreover, for every $\mu \in \hat{X}$, $\overline{\mu} \preceq_s \underline{\Phi}(\overline{\mu}) \preceq_s \underline{\Phi}(\mu)$, whence $\Phi(\mu) \subseteq \hat{X}$, and therefore, $\Psi(\mu) = \Phi(\mu)$. Consequently, $(\hat{\mathcal{X}}, \preceq_s, \Psi)$ is an isotone infimum measure theory model. Let $\hat{\mu}$ be the smallest fixed point of Ψ. Then $\hat{\mu} = \underline{\Psi}(\hat{\mu}) = \underline{\Phi}(\hat{\mu})$ implies $(\underline{p}, \hat{\mu}) \in \underline{\mathcal{E}} \subseteq \mathcal{E}$ and $\overline{\mu} \preceq_s \hat{\mu}$ implies that $(\underline{p}, \hat{\mu})$ is an upper bound for C from $\mathcal{E}$. Let (p, μ) be an arbitrary upper bound for C from $\mathcal{E}$. Then $\underline{p} \preceq_k p$ and $\overline{\mu} \preceq_s \mu$ and $\mu \in \Phi(\mu)$ imply $\mu \in \Psi(\mu)$, whence $\hat{\mu} \preceq_s \mu$. This shows that $(\underline{p}, \hat{\mu}) = \sup_{\mathcal{E}} C \in \mathcal{E}$. Therefore, $\underline{\mathcal{E}} \sqsubseteq^{*cc} \mathcal{E}$. Similarly, $\underline{\mathcal{E}} \sqsubseteq^{*cc} \mathcal{E}^{iso}$. Statement (2) for an isotone supremum kernel system is proved similarly. Statement (3) is proved by following the proof of statement (1) with $\overline{\mathcal{E}}$ instead of $\mathcal{E}$ and the proof of statement (2) with $\underline{\mathcal{E}}$ instead of $\mathcal{E}$.

Suppose the stochastic system being considered in statements (1)-(3) is a stochastic dynamical system. For statement (1), consider an isotone infimum stochastic dynamical system $((S, \preceq_S, \mathcal{B}(S)), (Z, \preceq_Z, \mathcal{B}(Z)), q, \mathcal{G})$ with $\inf_S S \in S$, isotone q, and isotone $\underline{g} \in \mathcal{G}$ such that $(\forall g \in \mathcal{G}), \underline{g} \preceq g$ and consider the associated measure theory model $(\mathcal{X}, \preceq_s, \Phi)$. Let C be a nonempty chain in $\mathcal{E}$ and let $C' = \{\mu \in \mathcal{X} \mid (\exists g \in \mathcal{G}), (g, \mu) \in C\} \subseteq \mathcal{E}(\Phi)$. It follows that C' is a nonempty chain in $\mathcal{E}(\Phi)$ and using $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*cc} \mathcal{E}(\Phi)$, let $\hat{\mu} = \inf_{\mathcal{E}(\underline{\Phi})} C' \in \mathcal{E}(\underline{\Phi})$. Then $(\underline{g}, \hat{\mu}) \in \underline{\mathcal{E}}$ is a lower bound for C from $\underline{\mathcal{E}}$. If $(\underline{g}, \mu') \in \underline{\mathcal{E}}$ is an arbitrary lower bound for C from $\underline{\mathcal{E}}$, then μ' is a lower bound for C' from $\mathcal{E}(\underline{\Phi})$, whence $\mu' \preceq_s \hat{\mu}$. This shows that $(\underline{g}, \hat{\mu}) = \inf_{\underline{\mathcal{E}}} C \in \underline{\mathcal{E}}$. Now suppose C is a nonempty chain in $\underline{\mathcal{E}}$ and let $C' = \{\mu \in \mathcal{X} \mid (\underline{g}, \mu) \in C\} \subseteq \mathcal{E}(\Phi)$. As $\mathcal{X}$ is sup-complete, let $\overline{\mu} = \sup_{\mathcal{X}} C' \in \mathcal{X}$. Let $\hat{\mathcal{X}} = \{\mu \in \mathcal{X} \mid \overline{\mu} \preceq_s \mu\}$ and consider $\Psi : \hat{\mathcal{X}} \rightrightarrows \hat{\mathcal{X}}$ given by $\Psi(\mu) = \Phi(\mu) \cap \hat{\mathcal{X}}$. Then $\mu \in C' \implies \mu = \underline{\Phi}(\mu) \preceq_s \underline{\Phi}(\overline{\mu})$ and therefore, $\overline{\mu} \preceq_s \underline{\Phi}(\overline{\mu})$. Moreover, for every $\mu \in \hat{\mathcal{X}}$, $\overline{\mu} \preceq_s \underline{\Phi}(\overline{\mu}) \preceq_s \underline{\Phi}(\mu)$, whence $\Phi(\mu) \subseteq \hat{\mathcal{X}}$, and therefore, $\Psi(\mu) = \Phi(\mu)$. Consequently, $(\hat{\mathcal{X}}, \preceq_s, \Psi)$ is an isotone infimum measure theory model. Let $\hat{\mu}$ be the smallest fixed point of Ψ. Then $\hat{\mu} = \underline{\Psi}(\hat{\mu}) = \underline{\Phi}(\hat{\mu})$ implies $(\underline{g}, \hat{\mu}) \in \underline{\mathcal{E}} \subseteq \mathcal{E}$ and $\overline{\mu} \preceq_s \hat{\mu}$ implies that $(\underline{g}, \hat{\mu})$ is an upper bound for C from $\mathcal{E}$. Let (g, μ) be an arbitrary upper bound for C from $\mathcal{E}$. Then $\underline{g} \preceq g$ and $\overline{\mu} \preceq_s \mu$ and $\mu \in \Phi(\mu)$ imply $\mu \in \Psi(\mu)$, whence $\hat{\mu} \preceq_s \mu$. This shows that $(\underline{g}, \hat{\mu}) = \sup_{\mathcal{E}} C \in \mathcal{E}$. Therefore, $\underline{\mathcal{E}} \sqsubseteq^{*cc} \mathcal{E}$. Similarly, $\underline{\mathcal{E}} \sqsubseteq^{*cc} \mathcal{E}^{iso}$. Statement (2) for an isotone supremum stochastic dynamical system is proved similarly. Statement (3) is proved by following the proof of statement (1) with $\overline{\mathcal{E}}$ instead of $\mathcal{E}$ and the proof of statement (2) with $\underline{\mathcal{E}}$ instead of $\mathcal{E}$.

Statements (1)-(3) for stochastic dynamic economies and Markov decision processes are proved in a manner similar to the proof for stochastic dynamical systems.

For statement (4), consider an isotone supremum kernel system that is strongly isotone infimum on upper intervals. Let C be a nonempty chain in $\mathcal{E}$. Let $C_1 = \{p \in \mathcal{P} \mid \exists \mu \in \mathcal{X}, (p, \mu) \in C\}$ and $C_2 = \{\mu \in \mathcal{X} \mid \exists p \in \mathcal{P}, (p, \mu) \in C\}$. For each $x \in X$, let $\hat{p}(x) = \sup_{\mathcal{X}} \{p(x) \mid p \in C_1\}$, which exists because $\mathcal{X}$ is chain sup-complete. As the system is isotone supremum, $\exists \overline{p} \in \mathcal{P}$ such that for every $p \in \mathcal{P}$, $p \preceq_k \overline{p}$, whence $\forall x \in X$, $\hat{p}(x) \preceq_s \overline{p}(x)$. Let $\underline{\hat{p}} \in \mathcal{P}$ be an isotone kernel guaranteed by the assumption that the system is strongly isotone infimum on upper intervals. Let $\overline{\mu} = \sup_{\mathcal{X}} C_2$ and for $\overline{\mu} \preceq_S \mu$, let $\hat{\Phi}(\mu) = \{\mathcal{T}_p(\mu) \mid p \in \mathcal{P}, p \text{ is isotone}, \overline{\mu} \preceq_S \mathcal{T}_p(\mu)\}$. Then $\mu \mapsto \mathcal{T}_{\underline{\hat{p}}}(\mu)$ is an isotone infimum selection from $\hat{\Phi}$. Let $\hat{\mu}$ be the smallest fixed point. Then $(\underline{\hat{p}}, \hat{\mu})$ is an upper bound for C from $\mathcal{E}^{iso}$. If $(\tilde{p}, \tilde{\mu})$ is an upper bound for C from $\mathcal{E}^{iso}$, then $\tilde{p}$ is isotone and $\forall x \in X$, $\hat{p}(x) \preceq_s \tilde{p}(x)$, and therefore, $\forall x \in X$, $\underline{\hat{p}}(x) \preceq_k \tilde{p}(x)$, whence $\underline{\hat{p}} \preceq_k \tilde{p}$. Moreover, $\mu \in C_2 \Rightarrow \mu \preceq_s \tilde{\mu} \Rightarrow \overline{\mu} \preceq_s \tilde{\mu}$. As $\tilde{\mu}$ is a fixed point of $\hat{\Phi}$, $\hat{\mu} \preceq_s \tilde{\mu}$. Therefore, $(\underline{\hat{p}}, \hat{\mu}) = \sup_{\mathcal{E}^{iso}} C$. Statement (5) for an isotone infimum kernel system that is strongly isotone supremum on lower intervals is proved similarly. Statements (4) and (5) for stochastic dynamical systems, stochastic dynamic economies, and Markov decision processes are proved similarly. Statement (6) follows from statements (4) and (5). ∎

Statements (1)-(3) apply immediately to the HP model along with their interpretation about the role these play in the order approximation of arbitrary subsets of equilibria using only the isotone infimum or supremum selection.

In statement (4), if we replace *strongly isotone infimum on upper intervals* with *isotone infimum* *on upper intervals*, then $\mathcal{E}^{iso}$ *is chain sup-complete in* $\mathcal{E}$, following the same proof with the new definitions. Similarly, in statement (5), if we replace *strongly isotone supremum on lower intervals* with *isotone supremum on lower intervals*, then $\mathcal{E}^{iso}$ *is chain inf-complete in* $\mathcal{E}$. *With both these replacements*, it follows that $\mathcal{E}^{iso}$ *is chain complete in* $\mathcal{E}$.

In addition to a theory of order approximation of equilibria, the star chain complete set order is useful to prove MCS of the equilibrium set associated with different equilibrium selections in universal parametric models with complementarities, as formalized in the next section.

4 Universal parametric models with complementarities

Parametric models are used to study the effect of exogenous parameters on the working of a system and its equilibrium. We include these effects in a general manner by positing a partially ordered set T of parameters. A ***parametric poset model*** is a collection $((X, \preceq_X), (T, \preceq_T), \Phi)$, where $(X, \preceq_X)$ and $(T, \preceq_T)$ are posets and $\Phi : X \times T \rightrightarrows X$ is a correspondence. For each $t \in T$, the *poset model at t* is the triple $(X, \preceq_X, \Phi_t)$ where Φ_t is the t-section of Φ. An *equilibrium at t* is a fixed point of Φ_t, and the *equilibrium set at t* is $\mathcal{E}(\Phi_t) = \{x \in X \mid x \in \Phi(x, t)\}$. The ***equilibrium correspondence*** is $\mathcal{E} : T \rightrightarrows X$, $t \mapsto \mathcal{E}(\Phi_t)$. An *equilibrium selection* is a selection from the equilibrium correspondence. An *isotone equilibrium selection* is an equilibrium selection that is an isotone function. A parametric poset model $((X, \preceq_X), (T, \preceq_T), \Phi)$ has ***monotone comparative statics (MCS) of equilibrium*** if its equilibrium correspondence has an isotone selection.

A ***universal parametric model with complementarities*** (or ***universal parametric model***) is a parametric poset model $((X, \preceq_X), (T, \preceq_T), \Phi)$ in which $(X, \preceq_X)$ is either a chain sup-complete poset with $\inf_X X \in X$ or a chain inf-complete poset with $\sup_X X \in X$ and $\Phi : X \times T \rightrightarrows X$ has an isotone selection. A universal parametric model $((X, \preceq_X), (T, \preceq_T), \Phi)$ is ***isotone infimum*** if X is chain sup-complete with $\inf_X X \in X$ and Φ has an isotone infimum selection ($\forall (x, t) \in X \times T$, $\underline{\Phi}(x, t) \coloneqq \inf_{\Phi(x,t)} \Phi(x, t) \in \Phi(x, t)$, and the function $(x, t) \mapsto \underline{\Phi}(x, t)$ is isotone). The ***infimum equilibrium set at*** t is $\mathcal{E}(\underline{\Phi}_t) = \{x \in X \mid x = \underline{\Phi}(x, t)\}$. A universal parametric model $((X, \preceq_X), (T, \preceq_T), \Phi)$ is ***isotone supremum*** if X is chain inf-complete with $\sup_X X \in X$ and Φ has an isotone supremum selection ($\forall (x, t) \in X \times T$, $\overline{\Phi}(x, t) \coloneqq \sup_{\Phi(x,t)} \Phi(x, t) \in \Phi(x, t)$, and the function $(x, t) \mapsto \overline{\Phi}(x, t)$ is isotone). The ***supremum equilibrium set at*** t is $\mathcal{E}(\overline{\Phi}_t) = \{x \in X \mid x = \overline{\Phi}(x, t)\}$.

Example 12 (General parametric model with complementarities)**.** Following Sabarwal (2023b), a ***general parametric model with complementarities*** (or ***general parametric model***) is a parametric poset model $((X, \preceq_X), (T, \preceq_T), \Phi)$ in which $(X, \preceq_X)$ is a nonempty complete lattice and Φ has an isotone selection. A general parametric model is ***isotone infimum*** (respectively, ***isotone supremum***) if Φ has an isotone infimum (respectively, isotone supremum) selection. As shown in Sabarwal (2023b), the class of parametric general models is very large, including standard parametric models with complementarities used widely in economics. For example, every parametric Topkis model, every parametric Vives model, every parametric Zhou model, every parametric generalized MS model, and every parametric CKK model is a parametric general model that is isotone infimum and/or isotone supremum. Moreover, the class of general parametric models includes models not covered by any of these models.

Example 13 (Parametric measure theory model)**.** Let X be a Polish space with a closed partial order $\preceq_X$ and $\mathcal{M}(X)$ the set of finite measures on the Borel sets of X ($\mathcal{B}(X)$) with the stochastic order, $\preceq_s$, on $\mathcal{M}(X)$. Let $(T, \preceq_T)$ be a poset. A ***parametric measure theory model*** is $((\mathcal{X}, \preceq_s), (T, \preceq_T), \Phi)$, where $\mathcal{X} \subseteq \mathcal{M}(X)$, $\preceq_s$ is the stochastic order on $\mathcal{X}$, and $\Phi : \mathcal{X} \times T \rightrightarrows \mathcal{X}$ is a correspondence. For each t, the *measure theory model at* t is $(\mathcal{X}, \preceq_s, \Phi_t)$, where Φ_t is the t-section of Φ. It follows immediately that every parametric measure theory model is a parametric poset model. It retains the same definition of an equilibrium at t, the equilibrium set at t, and the equilibrium correspondence. A parametric measure theory model $((\mathcal{X}, \preceq_s), (T, \preceq_T), \Phi)$ is ***isotone*** (respectively, ***isotone infimum, isotone supremum***) if either $\mathcal{X}$ is chain sup-complete with $\inf_{\mathcal{X}} \mathcal{X} \in \mathcal{X}$ or $\mathcal{X}$ is chain inf-complete with $\sup_{\mathcal{X}} \mathcal{X} \in \mathcal{X}$ (respectively, $\mathcal{X}$ is chain sup-complete with $\inf_{\mathcal{X}} \mathcal{X} \in \mathcal{X}$, $\mathcal{X}$ is chain inf-complete with $\sup_{\mathcal{X}} \mathcal{X} \in \mathcal{X}$), and Φ has an isotone (respectively, isotone infimum, isotone supremum) selection.

Example 14 (Parametric kernel systems)**.** A ***parametric kernel system*** is a collection $((X, \preceq_X, \mathcal{B}(X)), (T, \preceq_T), \mathcal{P})$, where X is a Polish space, $\preceq_X$ is a closed partial order on X, $\mathcal{B}(X)$ are the Borel sets of X, $(T, \preceq_T)$ is a poset, and $\mathcal{P}$ is a correspondence $\mathcal{P} : T \rightrightarrows ker(X \times \mathcal{B}(X))$, $t \mapsto \mathcal{P}(t)$. For each $t \in T$, we view $\mathcal{P}(t)$ as the set of kernels governing the evolution of the system of Markov processes associated with the kernels in $\mathcal{P}(t)$. Each $t \in T$ is viewed as describing the exogenously specified environment in which the system $\mathcal{P}(t)$ is studied. The *kernel system at* t is $((X, \preceq_X, \mathcal{B}(X)), \mathcal{P}(t))$. The *associated parametric measure theory model* is $((\mathcal{X}, \preceq_s), (T, \preceq_T), \Phi)$, where $\mathcal{X}$ is the set of probability measures on $\mathcal{B}(X)$, $\preceq_s$ is stochastic order, and $\Phi : \mathcal{X} \times T \rightrightarrows \mathcal{X}$

is given by $\Phi(\mu,t) = \{\mathcal{T}_p(\mu) \mid p \in \mathcal{P}(t)\}$. An ***equilibrium at*** t is a pair (p,μ) such that $p \in \mathcal{P}(t)$, $\mu \in \mathcal{X}$, and $\mu = \mathcal{T}_p(\mu)$. The ***equilibrium set at*** t is $\mathcal{E}(t) = \{(p,\mu) \in \mathcal{P}(t) \times \mathcal{X} \mid \mu = \mathcal{T}_p(\mu)\}$ with the product order. The ***isotone equilibrium set at*** t is $\mathcal{E}^{iso}(t) = \{(p,\mu) \in \mathcal{E}(t) \mid p \text{ is isotone}\}$.

A parametric kernel system is ***isotone*** if either $\inf_X X \in X$ or $\sup_X X \in X$, and there is an isotone selection $t \mapsto p(t)$ from $\mathcal{P}$ such that $\forall t$, $p(t)$ is an isotone kernel. It is ***isotone infimum*** if $\inf_X X \in X$ and there is an isotone selection $\underline{p}$ from $\mathcal{P}$ such that $\forall t$, $\underline{p}(t)$ is an isotone kernel and $\forall p \in \mathcal{P}(t), \underline{p}(t) \preceq_k p$. The ***infimum equilibrium set at*** t is $\underline{\mathcal{E}}(t) = \{(\underline{p}(t),\mu) \in \mathcal{P}(t) \times \mathcal{X} \mid \mu = \mathcal{T}_{\underline{p}(t)}(\mu)\}$. It is ***isotone supremum*** if $\sup_X X \in X$ and there is an isotone selection $\overline{p}$ from $\mathcal{P}$ such that $\forall t$, $\overline{p}(t)$ is an isotone kernel and $\forall p \in \mathcal{P}(t), p \preceq_k \overline{p}(t)$. The ***supremum equilibrium set at*** t is $\overline{\mathcal{E}}(t) = \{(\overline{p}(t),\mu) \in \mathcal{P}(t) \times \mathcal{X} \mid \mu = \mathcal{T}_{\overline{p}(t)}(\mu)\}$.

Example 15 (Parametric stochastic dynamical systems)**.** A ***parametric stochastic dynamical system*** is a collection $((S,\preceq_S,\mathcal{B}(S)),(Z,\preceq_Z,\mathcal{B}(Z)),(T,\preceq_T),q,\mathcal{G})$, where $(S,\preceq_S,\mathcal{B}(S))$ and $(Z,\preceq_Z,\mathcal{B}(Z))$ are as in Example 4, $(T,\preceq_T)$ is a poset, q is a function $q : T \to ker(S \times \mathcal{B}(Z))$, and $\mathcal{G}$ is a correspondence $\mathcal{G} : T \rightrightarrows mbl(S\times Z, S)$. Each $t \in T$ describes the exogenously specified environment in which the stochastic dynamical system is studied. For each $t \in T$, we view $q(t)$ as the state dependent distribution of shocks when the exogenous parameter is t. Changing t to t' changes the state dependent distribution of shocks from $q(t)$ to $q(t')$. Similarly, $\mathcal{G}(t)$ is viewed as the set of policies being considered in environment t and these may change with t as well. Changes to $q(t)$ and $\mathcal{G}(t)$ affect the evolution of the stochastic dynamical system in an interdependent manner. The *stochastic dynamical system at* t is $((S,\preceq_S,\mathcal{B}(S)),(Z,\preceq_Z,\mathcal{B}(Z)),q(t),\mathcal{G}(t))$.

The *associated parametric kernel system* is $(S,\preceq_S,\mathcal{B}(S),(T,\preceq_T),\mathcal{P})$ with $\mathcal{P}(t) = \{p \in ker(S \times \mathcal{B}(S)) \mid p(s,A) = q(s,[g^{-1}(A)]_s),\ q = q(t),\ g \in \mathcal{G}(t)\}$, where $[g^{-1}(A)]_s$ is the s-section of $g^{-1}(A)$. The *associated parametric measure theory model* is $((\mathcal{X},\preceq_s),(T,\preceq_T),\Phi)$, where $\mathcal{X}$ is the set of probability measures on $\mathcal{B}(S)$, $\preceq_s$ is stochastic order, and $\Phi : \mathcal{X} \times T \rightrightarrows \mathcal{X}$ is given by $\Phi(\mu,t) = \{\mathcal{T}_p(\mu) \mid p \in \mathcal{P}(t)\}$. An ***equilibrium at*** t is a pair (g,μ) such that $g \in \mathcal{G}(t)$, $\mu \in \mathcal{X}$, and $\mu = \mathcal{T}_p(\mu)$, where p is derived from $q = q(t)$ and $g \in \mathcal{G}(t)$ as above, $p(s,A) = q(s,[g^{-1}(A)]_s)$. The ***equilibrium set at*** t is $\mathcal{E}(t) = \{(g,\mu) \in \mathcal{G}(t)\times\mathcal{X} \mid \mu = \mathcal{T}_p(\mu)\}$ with the product order. The ***isotone equilibrium set at*** t is $\mathcal{E}^{iso}(t) = \{(g,\mu) \in \mathcal{E}(t) \mid g \text{ is isotone}\}$.

Parametric complementarities are included as follows. A parametric stochastic dynamical system is ***isotone*** if either $\inf_S S \in S$ or $\sup_S S \in S$, the mapping $t \mapsto q(t)$ is isotone with $q(t)$ is isotone $\forall t$, and there is an isotone selection $t \mapsto g(t)$ from $\mathcal{G}$ such that $(\forall t)$, $g(t)$ is isotone. It is ***isotone infimum*** if $\inf_S S \in S$, the mapping $t \mapsto q(t)$ is isotone with $q(t)$ is isotone $\forall t$, and there is isotone selection $t \mapsto \underline{g}(t)$ from $\mathcal{G}$, such that $(\forall t)$, $\underline{g}(t)$ is isotone and $\forall g \in \mathcal{G}(t), \underline{g}(t) \preceq g$. The ***infimum equilibrium set at*** t is $\underline{\mathcal{E}}(t) = \{(\underline{g}(t), \mu) \in \mathcal{G}(t) \times \mathcal{X} \mid \mu = \mathcal{T}_{\underline{p}}(\mu)\}$, where $\underline{p}$ is derived from $q(t)$ and $\underline{g}(t)$ as above. It is ***isotone supremum*** if $\sup_S S \in S$, the mapping $t \mapsto q(t)$ is isotone with $q(t)$ is isotone $\forall t$, and there is isotone selection $t \mapsto \overline{g}(t)$ from $\mathcal{G}$, such that $(\forall t)$, $\overline{g}(t)$ is isotone and $\forall g \in \mathcal{G}(t), g \preceq \overline{g}(t)$. The ***supremum equilibrium set at*** t is $\overline{\mathcal{E}}(t) = \{(\overline{g}(t), \mu) \in \mathcal{G}(t) \times \mathcal{X} \mid \mu = \mathcal{T}_{\overline{p}}(\mu)\}$, where $\overline{p}$ is derived from $q(t)$ and $\overline{g}(t)$ as above.

Example 16 (Parametric stochastic dynamic economies)**.** A ***parametric stochastic dynamic economy*** is a collection $((X, \preceq_X, \mathcal{B}(X)), (Z, \preceq_Z, \mathcal{B}(Z)), (T, \preceq_T), q, \mathcal{G})$, where $(X, \preceq_X, \mathcal{B}(X))$ and $(Z, \preceq_Z, \mathcal{B}(Z))$ are as in Example 5, $(T, \preceq_T)$ is a poset, q is a function $q : T \to ker(Z \times \mathcal{B}(Z))$, and $\mathcal{G}$ is a correspondence $\mathcal{G} : T \rightrightarrows mbl(X \times Z, X)$. For each $g \in \mathcal{G}(t)$, let $\hat{g} : X \times Z \times Z \to X \times Z$ be given by $\hat{g}(x, z, z') = (g(x, z), z')$. The *associated parametric stochastic dynamical system* is $((X, \preceq_X, \mathcal{B}(X)), (Z, \preceq_Z, \mathcal{B}(Z)), (T, \preceq_T), q, \hat{\mathcal{G}})$. The *associated parametric kernel system* is $((S, \preceq_S, \mathcal{B}(S)), (T, \preceq_T), \mathcal{P})$, where $S = X \times Z$ with product partial order $\preceq_S$ and product sigma algebra $\mathcal{B}(S)$, and $\mathcal{P}(t) = \{p \in ker(S \times \mathcal{B}(S)) \mid p((x, z), A) = q(z, [\hat{g}^{-1}(A)]_{(x,z)}),\ q = q(t),\ g \in \mathcal{G}(t)\}$. The *associated parametric measure theory model* is $((\mathcal{X}, \preceq_s), (T, \preceq_T), \Phi)$, derived analogously. The *stochastic dynamic economy at* t is $((X, \preceq_X, \mathcal{B}(X)), (Z, \preceq_Z, \mathcal{B}(Z)), q(t), \mathcal{G}(t))$. An ***equilibrium at*** t is a pair (g, μ) such that $g \in \mathcal{G}(t)$, $\mu \in \mathcal{X}$, and $\mu = \mathcal{T}_p(\mu)$, where p is derived from $q = q(t)$ and $g \in \mathcal{G}(t)$ as above, $p((x, z), A) = q(z, [\hat{g}^{-1}(A)]_{(x,z)})$. The ***equilibrium set at*** t is $\mathcal{E}(t) = \{(g, \mu) \in \mathcal{G}(t) \times \mathcal{X} \mid \mu = \mathcal{T}_p(\mu)\}$ with the product order. The ***isotone equilibrium set at*** t is $\mathcal{E}^{iso}(t) = \{(g, \mu) \in \mathcal{E}(t) \mid g \text{ is isotone}\}$.

A parametric stochastic dynamic economy is ***isotone*** if either $\inf_X X \in X$ or $\sup_X X \in X$, the mapping $t \mapsto q(t)$ is isotone with $q(t)$ is isotone $\forall t$, and there is isotone selection $t \mapsto g(t)$ from $\mathcal{G}$, such that $(\forall t)$, $g(t)$ is isotone. It is ***isotone infimum*** if $\inf_X X \in X$, the mapping $t \mapsto q(t)$ is isotone with $q(t)$ is isotone $\forall t$, and there is isotone selection $t \mapsto \underline{g}(t)$ from $\mathcal{G}$, such that $(\forall t)$, $\underline{g}(t)$ is isotone, and $\forall g \in \mathcal{G}(t), \underline{g}(t) \preceq g$. The ***infimum equilibrium set at*** t is $\underline{\mathcal{E}}(t) = \{(\underline{g}(t), \mu) \in \mathcal{G}(t) \times \mathcal{X} \mid \mu = \mathcal{T}_{\underline{p}}(\mu)\}$, where $\underline{p}$ is derived from $q(t)$ and $\underline{g}(t)$ as above. It is ***isotone supremum*** if $\inf_X X \in X$, the mapping $t \mapsto q(t)$ is isotone with $q(t)$ is isotone $\forall t$, and there is isotone selection $t \mapsto \overline{g}(t)$ from $\mathcal{G}$, such that $(\forall t)$, $\overline{g}(t)$ is isotone, and $\forall g \in \mathcal{G}(t), g \preceq \overline{g}(t)$. The ***supremum equilibrium set at*** t is $\overline{\mathcal{E}}(t) = \{(\overline{g}(t), \mu) \in \mathcal{G}(t) \times \mathcal{X} \mid \mu = \mathcal{T}_{\overline{p}}(\mu)\}$, where $\overline{p}$ is derived from $q(t)$ and $\overline{g}(t)$ as above.

Example 17 (Parametric Markov decision processes)**.** A ***parametric Markov decision process*** is a collection $((X, \preceq_X, \mathcal{B}(X)), (Z, \preceq_Z, \mathcal{B}(Z)), (T, \preceq_T), q, \mathcal{G})$, where $(X, \preceq_X, \mathcal{B}(X))$ and $(Z, \preceq_Z, \mathcal{B}(Z))$ are as in Example 6, $(T, \preceq_T)$ is a poset, q is a function $q : T \to ker((X \times Z) \times \mathcal{B}(Z))$, and $\mathcal{G}$ is a correspondence $\mathcal{G} : T \rightrightarrows mbl(X \times Z, X)$. For each $g \in \mathcal{G}(t)$, let $\hat{g} : X \times Z \times Z \to X \times Z$ be given by $\hat{g}(x, z, z') = (g(x, z), z')$. $\mathcal{P}(t) = \{p \in ker(S \times \mathcal{B}(S)) \mid p((x, z), A) = q(z, [\hat{g}^{-1}(A)]_z),\ q = q(t),\ g \in \mathcal{G}(t)\}$. The *associated parametric stochastic dynamical system* is $((X, \preceq_X, \mathcal{B}(X)), (Z, \preceq_Z, \mathcal{B}(Z)), (T, \preceq_T), q, \hat{\mathcal{G}})$. The *associated parametric kernel system* and the *associated parametric measure theory model* are derived as in Example 16. The *Markov decision process at* t is $((X, \preceq_X, \mathcal{B}(X)), (Z, \preceq_Z, \mathcal{B}(Z)), q(t), \mathcal{G}(t))$. An ***equilibrium at*** t is a pair (g, μ) such that $g \in \mathcal{G}(t)$, $\mu \in \mathcal{X}$, and $\mu = \mathcal{T}_p(\mu)$, where p is derived from $q = q(t)$ and $g \in \mathcal{G}(t)$ as above, $p((x, z), A) = q(z, [\hat{g}^{-1}(A)]_z)$. The ***equilibrium set at*** t is $\mathcal{E}(t) = \{(g, \mu) \in \mathcal{G}(t) \times \mathcal{X} \mid \mu = \mathcal{T}_p(\mu)\}$ with the product order. The ***isotone equilibrium set at*** t is $\mathcal{E}^{iso}(t) = \{(g, \mu) \in \mathcal{E}(t) \mid g \text{ is isotone}\}$.

A parametric Markov decision process is ***isotone*** if either $(\inf_X X, \inf_Z Z) \in X \times Z$ or $(\sup_X X, \sup_Z Z) \in X \times Z$, the mapping $t \mapsto q(t)$ is isotone with $q(t)$ is isotone $\forall t$, and there is isotone selection $t \mapsto g(t)$ from $\mathcal{G}$, such that $(\forall t)$, $g(t)$ is isotone. It is ***isotone infimum*** if $(\inf_X X, \inf_Z Z) \in X \times Z$, the mapping $t \mapsto q(t)$ is isotone with $q(t)$ is isotone $\forall t$, and there is isotone selection $t \mapsto \underline{g}(t)$ from $\mathcal{G}$, such that $(\forall t)$, $\underline{g}(t)$ is isotone, and $\forall g \in \mathcal{G}(t), \underline{g}(t) \preceq g$. The ***infimum equilibrium set at*** t is $\underline{\mathcal{E}}(t) = \{(\underline{g}(t), \mu) \in \mathcal{G}(t) \times \mathcal{X} \mid \mu = \mathcal{T}_{\underline{p}}(\mu)\}$, where $\underline{p}$ is derived from $q(t)$ and $\underline{g}(t)$ as above. It is ***isotone supremum*** if $(\sup_X X, \sup_Z Z) \in X \times Z$, the mapping $t \mapsto q(t)$ is isotone with $q(t)$ is isotone $\forall t$, and there is isotone selection $t \mapsto \overline{g}(t)$ from $\mathcal{G}$, such that $(\forall t)$, $\overline{g}(t)$ is isotone, and $\forall g \in \mathcal{G}(t), g \preceq \overline{g}(t)$. The ***supremum equilibrium set at*** t is $\overline{\mathcal{E}}(t) = \{(\overline{g}(t), \mu) \in \mathcal{G}(t) \times \mathcal{X} \mid \mu = \mathcal{T}_{\overline{p}}(\mu)\}$, where $\overline{p}$ is derived from $q(t)$ and $\overline{g}(t)$ as above.

Example 18 (Parametric stochastic systems)**.** A ***parametric stochastic system*** is one that is either a parametric kernel system, or a parametric stochastic dynamical system, or a parametric stochastic dynamic economy, or a parametric Markov decision process. A parametric stochastic system is ***isotone (respectively, isotone infimum, isotone supremum)*** if the corresponding parametric system is isotone (respectively, isotone infimum, isotone supremum).

Example 19 (Parametric HP model)**.** A ***parametric HP model*** is given by $(X, Z, T, q, \Gamma, F, \beta)$, where X and Z are as in Example 8, $T \subseteq \mathbb{R}^m$ is a compact poset of parameters with standard partial order, $q : T \to ker(Z \times \mathcal{B}(Z))$ is an isotone function such that for every t, $q(t)$ is isotone and satisfies Feller property, $\Gamma : X \times Z \times T \rightrightarrows X$ is the feasibility correspondence that is nonempty valued, compact-valued, and continuous, A the graph of Γ, $F : A \to \mathbb{R}$ the (bounded) one-period return function, $\beta \in (0,1)$ the constant discount rate, $v : X \times Z \times T \to \mathbb{R}$ the unique value function associated with this problem, given by $v(x,z,t) = \sup_{x' \in \Gamma(x,z,t)} \{F(x',x,z,t) + \beta \int v(x',z',t) q(z,dz',t)\}$, and $\gamma(x,z,t) = \{x' \in \Gamma(x,z,t) \mid v(x,z,t) = F(x',x,z,t) + \beta \int v(x',z',t) q(z,dz',t)\}$ the policy correspondence. Under the natural parametric extension of the complementarity and continuity assumptions in Proposition 2 (page 1395) in Hopenhayn and Prescott (1992) and their compactness assumptions in Corollary 6 (page 1396), the functions $(x,z,t) \mapsto \underline{g}(x,z,t) := \inf \gamma(x,z,t)$ and $(x,z,t) \mapsto \overline{g}(x,z,t) := \sup \gamma(x,z,t)$ exist, are measurable, and are isotone. The *HP model at t* is $(X, Z, q(t), \Gamma_t, F_t, \beta)$ defined in terms of the appropriate sections. The *associated parametric stochastic dynamic economy* is $((X, \preceq_X, \mathcal{B}(X)), (Z, \preceq_Z, \mathcal{B}(Z)), (T, \preceq_T), q, \mathcal{G})$, where $\mathcal{G}(t)$ is the set of measurable selections from $\gamma(t)$. In particular, Corollary 7 (page 1396) in Hopenhayn and Prescott (1992) is the special case when $T = \{a, b\}$ with $a \prec b$.

Example 20 (Parametric BDRW model)**.** Consider the BDRW model in Example 9 and let $(T, \preceq_T)$ be a poset. Moreover, suppose Assumptions 2 (i), (ii), and (iii) (page 505) in Balbus, Dziewulski, Reffett, and Woźny (2019) are satisfied. (Assumption 2 (iv) is not used in this example.) Let $\Gamma : I \times T \rightrightarrows A$ be the feasibility correspondence, $\tilde{A}$ the graph of Γ, $r : I \times A \times \mathcal{D} \times T \to \mathbb{R}$ the player payoff function, and γ be the best response correspondence, $\gamma(i,\mu,t) = \arg\max_{a \in \Gamma(i,t)} r(i,a,\mu,t)$. The ***parametric BDRW model*** is $((I, \mathcal{B}(I), \lambda), T, A, \tilde{A}, r)$. The *BDRW model at t* is $((I, \mathcal{B}(I), \lambda), A_t, \tilde{A}_t, r_t)$ defined in terms of the appropriate sections. With these assumptions, results in Balbus, Dziewulski, Reffett, and Woźny (2019) show that for each $t \in T$, the functions $(i,\mu,t) \mapsto \underline{g}(i,\mu,t) := \inf \gamma(i,\mu,t)$ and $(i,\mu,t) \mapsto \overline{g}(i,\mu,t) := \sup \gamma(i,\mu,t)$ exist, are measurable, and are isotone. For each selection g from γ, let $\tilde{g} : I \times \mathcal{D} \times T \to I \times A$ be given by $\tilde{g}(i,\mu,t) = (i, g(i,\mu,t))$. For each $t \in T$, let $\gamma(t)$ be the t-section of γ and $\mathcal{G}(t)$ be the collection of all measurable selections from $\gamma(t)$. For each $t \in T$ and each $g \in \mathcal{G}(t)$, define the mapping $T_g : \mathcal{D} \to \mathcal{D}$ by $T_g(\mu)(B) = \lambda[\tilde{g}^{-1}(B)]_{(\mu,t)}$, where $\tilde{g}$ is as above and $[\tilde{g}^{-1}(B)]_{(\mu,t)}$ is the (μ,t)-section of $[\tilde{g}^{-1}(B)]$. The *associated parametric measure theory model* is $((\mathcal{D}, \preceq_s), (T, \preceq_T), \Phi)$, where $\Phi : \mathcal{D} \times T \rightrightarrows \mathcal{D}$ is given by $\Phi(\mu,t) = \{T_g(\mu) \mid g \in \mathcal{G}(t)\}$.

Theorem 9. *Consider the class of parametric poset models.*

1. *Every general parametric model that is isotone (respectively, isotone infimum, isotone supremum) is a universal parametric model that is isotone (respectively, isotone infimum, isotone supremum).*

2. *Every parametric measure theory model that is isotone (respectively, isotone infimum, isotone supremum) is a universal parametric model that is isotone (respectively, isotone infimum, isotone supremum).*

3. *For every parametric stochastic system that is isotone (respectively, isotone infimum, isotone supremum), its associated parametric measure theory model is isotone (respectively, isotone infimum, isotone supremum).*

4. *For every parametric HP and every parametric BDRW model, the associated parametric measure theory model is isotone infimum and isotone supremum.*

Proof. Statements (1) and (2) follow immediately from the definitions. To prove statement (3) for a parametric kernel system $((X, \preceq_X, \mathcal{B}(X)), (T, \preceq_T), \mathcal{P})$, consider its associated parametric measure theory model $((\mathcal{X}, \preceq_s), (T, \preceq_T), \Phi)$. If the parametric kernel system is isotone, let $t \mapsto p(t)$ be an isotone selection from $\mathcal{P}$ such that $\forall t$, $p(t)$ is isotone. Then the selection $(\mu, t) \mapsto T_{p(t)}(\mu)$ from Φ is isotone, because for every $\mu \preceq_s \nu$ and for every $\hat{t} \preceq_T \tilde{t}$, $T_{p(\hat{t})}(\mu) \preceq_s T_{p(\tilde{t})}(\mu) \preceq_s T_{p(\tilde{t})}(\nu)$, where the first inequality follows from $p(\hat{t}) \preceq_k p(\tilde{t})$ and the second from $p(\tilde{t})$ is isotone. Moreover, if $\underline{x} = \inf_X X \in X$ then the unit measure on $\underline{x}$, $\delta_{\underline{x}}$, satisfies $\inf_{\mathcal{X}} \mathcal{X} = \delta_{\underline{x}} \in \mathcal{X}$, and if $\overline{x} = \sup_X X \in X$ then $\sup_{\mathcal{X}} \mathcal{X} = \delta_{\overline{x}} \in \mathcal{X}$. Therefore, the associated parametric measure theory model is isotone. If the parametric kernel system is isotone infimum, let $t \mapsto \underline{p}(t)$ be an isotone selection from $\mathcal{P}$ such that $\forall t$, $\underline{p}(t)$ is isotone and $\forall p \in \mathcal{P}(t)$, $\underline{p}(t) \preceq_k p$. The reasoning above shows that the selection $(\mu, t) \mapsto T_{\underline{p}(t)}(\mu)$ from Φ is isotone. This selection is the infimum selection follows from the statement that $\forall t$ and $\forall p \in \mathcal{P}(t)$, $\underline{p}(t) \preceq_k p$ implies that for every $\mu \in \mathcal{X}$, $T_{\underline{p}(t)}(\mu) \preceq_s T_p(\mu)$. The statement for parametric kernel systems that are isotone supremum is proved similarly.

To prove statement (3) for a parametric stochastic dynamical system $((S, \preceq_S, \mathcal{B}(S)), (Z, \preceq_Z, \mathcal{B}(Z)), (T, \preceq_T), q, \mathcal{G})$, consider its associated parametric kernel system be $(S, \preceq_S, \mathcal{B}(S), (T, \preceq_T), \mathcal{P})$ with $\mathcal{P}(t) = \{p \in ker(S \times \mathcal{B}(S)) \mid p(s, A) = q(s, [g^{-1}(A)]_s),\ q = q(t),\ g \in \mathcal{G}(t)\}$, where $[g^{-1}(A)]_s$ is the s-section of $g^{-1}(A)$. If the parametric stochastic dynamical system is isotone, the mapping $t \mapsto q(t)$ is isotone with $q(t)$ is isotone $\forall t$, and there is an isotone selection $t \mapsto g(t)$ from $\mathcal{G}$ such that $(\forall t)$, $g(t)$ is isotone. Consider the selection $t \mapsto p(t)$ from $\mathcal{P}$ given by $p(s, A, t) = q(s, [g^{-1}(A)]_s)$, where $q = q(t)$ and $g = g(t)$. The proof in Theorem 1 (models) shows that $\forall t$, $p(t)$ is an isotone kernel. To see that the selection is isotone, fix $\hat{t} \preceq_T \tilde{t}$ arbitrarily, and consider arbitrary $s \in S$ and arbitrary increasing set $A \subseteq S$. Then $p_{\hat{t}}(s, A) = q_{\hat{t}}(s, [g_{\hat{t}}^{-1}(A)]_s) \leq q_{\hat{t}}(s, [g_{\tilde{t}}^{-1}(A)]_s) \leq q_{\tilde{t}}(s, [g_{\tilde{t}}^{-1}(A)]_s) = p_{\tilde{t}}(s, A)$, where $p_{\hat{t}} = p(\hat{t})$, $p_{\tilde{t}} = p(\tilde{t})$, $q_{\hat{t}} = q(\hat{t})$, $q_{\tilde{t}} = q(\tilde{t})$, $g_{\hat{t}} = g(\hat{t})$, $g_{\tilde{t}} = g(\tilde{t})$, the first inequality follows from $[g_{\hat{t}}^{-1}(A)]_s \subseteq [g_{\tilde{t}}^{-1}(A)]_s$ using A is increasing and $g(\tilde{t}) \preceq g(\hat{t})$, and the second inequality follows from $q(\hat{t}) \preceq_k q(\tilde{t})$ and $[g_{\tilde{t}}^{-1}(A)]_s$ is an increasing set in Z. Therefore, the associated parametric kernel system is isotone, and the proof for that case implies that the associated parametric measure theory model is isotone.

If the parametric stochastic dynamical system is isotone infimum, the mapping $t \mapsto q(t)$ is isotone with $q(t)$ is isotone $\forall t$, and there is isotone selection $t \mapsto \underline{g}(t)$ from $\mathcal{G}$, such that $(\forall t)$, $\underline{g}(t)$ is isotone and $\forall g \in \mathcal{G}(t), \underline{g}(t) \preceq g$. The same argument shows that $t \mapsto \underline{p}(t)$ given by $\underline{p}_t(s, A) = q_t(s, [\underline{g}_t^{-1}(A)]_s)$ is isotone, and for $\forall t$, $\underline{p}(t)$ is an isotone kernel. A similar argument shows that for every t, as each $p \in \mathcal{P}(t)$ is of the form $p(s, A) = q_t(s, [g^{-1}(A)]_s)$, for some $g \in \mathcal{G}(t)$, $\underline{g}(t) \preceq g$ implies $\underline{p}(t) \preceq_k p$. Therefore, the associated parametric kernel system is isotone infimum, and the proof for that case implies that the associated parametric measure theory model is isotone infimum. The statement for parametric stochastic dynamical systems that are isotone supremum is proved similarly.

Statements (3) for the cases of parametric stochastic dynamic economies and parametric Markov decision processes are proved similarly. Statement (4) for parametric HP model follows from (3) for the case of parametric stochastic dynamic economy, using the existence of infimum and supremum selections from γ as described in Example 19. Statement (4) for parametric BDRW model follows by showing that in every parametric BDRW model, the mapping $(\mu, t) \mapsto T_{\underline{g}(t)}(\mu)$ is the isotone infimum selection from Φ, where $\underline{g}(t)$ is the t-section of $\underline{g}(i, \mu, t) := \inf \gamma(i, \mu, t)$, and the mapping $(\mu, t) \mapsto T_{\overline{g}(t)}(\mu)$ is the isotone supremum selection, where $\overline{g}(t)$ is the t-section of $\overline{g}(i, \mu, t) := \sup \gamma(i, \mu, t)$. ∎

Isotone properties in universal parametric models imply MCS of equilibrium in these models.

Theorem 10. *Consider the class of parametric poset models.*

1. *In every universal parametric model, there is an isotone equilibrium selection. Equivalently, every universal parametric model has MCS of equilibrium.*

2. *In every universal parametric isotone infimum model, the infimum equilibrium selection $t \mapsto \inf_{\mathcal{E}(\underline{\Phi}_t)} \mathcal{E}(\underline{\Phi}_t)$ is isotone and selects the smallest equilibrium in $\mathcal{E}(\Phi_t)$. Equivalently, every universal parametric isotone infimum model has MCS of infimum equilibrium.*

3. *In every universal parametric isotone supremum model, the supremum equilibrium selection $t \mapsto \sup_{\mathcal{E}(\overline{\Phi}_t)} \mathcal{E}(\overline{\Phi}_t)$ is isotone and selects the largest equilibrium in $\mathcal{E}(\Phi_t)$. Equivalently, every*

universal parametric isotone supremum model has MCS of supremum equilibrium.

4. *Every universal parametric isotone infimum and isotone supremum model has MCS of extremal (that is, both infimum and supremum) equilibrium.*

Proof. To prove statement (1), let $((X, \preceq_X), (T, \preceq_T), \Phi)$ be a universal parametric model with complementarities. Suppose X is chain sup-complete with $\inf_X X \in X$ and $f : X \times T \to X$ is an isotone selection from Φ. Then $\forall t \in T$, $(X, \preceq, \Phi_t)$ has an isotone selection f_t given by the t-section of f. By Theorem 2, the equilibrium set $\mathcal{E}(f_t)$ is chain sup-complete and has a smallest element. Let $\underline{e}(t) = \inf_{\mathcal{E}(f_t)} \mathcal{E}(f_t) \in \mathcal{E}(f_t) \subseteq \mathcal{E}(\Phi_t)$. Then $t \mapsto \underline{e}(t)$ is an equilibrium selection. To see that it is isotone, fix $\hat{t} \preceq_T \tilde{t}$. Let $A = \{x \in X \mid x \preceq_X f(x, \hat{t}) \text{ and } x \preceq_X \underline{e}(\tilde{t})\}$. Set A is nonempty as $\inf_X X \in A$. Let C be a chain in A. If $C = \emptyset$ then $\sup_X C = \inf_X X \in A$. Otherwise, let $y = \sup_X C$ which exists as X is sup-complete. For every $x \in C$, $x \preceq f(x, \hat{t})$ and also, $x \preceq y \implies f(x, \hat{t}) \preceq f(y, \hat{t})$, showing that $f(y, \hat{t})$ is an upper bound for C, whence, $y \preceq f(y, \hat{t})$. Moreover, $\forall x \in C$, $x \preceq \underline{e}(\tilde{t}) \implies f(x, \hat{t}) \preceq f(\underline{e}(\tilde{t}), \hat{t}) \preceq f(\underline{e}(\tilde{t}), \tilde{t}) = \underline{e}(\tilde{t})$. This shows that $y \preceq \underline{e}(\tilde{t})$. It follows that $y \in A$ showing that every chain in A has an upper bound in A. Let e^* be a maximal element of A. Then $e^* \preceq f(e^*, \hat{t})$ and $e^* \preceq \underline{e}(\tilde{t})$. Therefore, $f(e^*, \hat{t}) \preceq f(f(e^*, \hat{t}), \hat{t})$ and also, $f(e^*, \hat{t}) \preceq f(\underline{e}(\tilde{t}), \hat{t}) \preceq f(\underline{e}(\tilde{t}), \tilde{t}) = \underline{e}(\tilde{t})$ showing that $f(e^*, \hat{t}) \in A$, whence $e^* = f(e^*, \hat{t})$. Consequently, $\underline{e}(\hat{t}) \preceq e^* \preceq \underline{e}(\tilde{t})$, as desired. A similar argument works if X is chain inf-complete with $\sup_X X \in X$.

Statement (2) is proved similarly. The mapping selects the smallest equilibrium in the corresponding poset model follows from Theorem 2. Statement 3 follows similarly. Statement (4) follows from statements (2) and (3). ∎

Theorem 10 combined with Theorem 9 yields MCS of equilibrium in every stochastic system.

Theorem 11. *Consider the class of parametric stochastic systems.*

1. *Every isotone parametric stochastic system has MCS of equilibrium.*

2. *Every isotone infimum parametric stochastic system has MCS of infimum equilibrium.*

3. *Every isotone supremum parametric stochastic system has MCS of supremum equilibrium.*

4. *Every parametric stochastic system that is isotone infimum and isotone supremum has MCS of extremal equilibrium.*

5. *In each of (1)-(4), every equilibrium that is selected is an isotone equilibrium.*

Proof. Suppose the stochastic system being considered in statements (1)-(4) is a kernel system. For statement (1), consider an isotone parametric kernel system $((X, \preceq_X, \mathcal{B}(X)), (T, \preceq_T), \mathcal{P})$ with isotone selection $t \mapsto p(t)$ such that $\forall t, p(t)$ is an isotone kernel, and the associated parametric measure theory model $((\mathcal{X}, \preceq_s), (T, \preceq_T), \Phi)$. Suppose $\inf_X X \in X$. Theorem 9 shows that $(\mu, t) \mapsto \mathcal{T}_{p(t)}(\mu)$ is an isotone selection from $\Phi : \mathcal{X} \times T \rightrightarrows \mathcal{X}$. For each t, let $\mathcal{E}(p(t)) = \{\mu \in \mathcal{X} \mid \mu = \mathcal{T}_{p(t)}(\mu)\}$. Then $\mathcal{E}(p(t)) \subseteq \mathcal{E}(\Phi_t)$ is nonempty, chain sup-complete, and has a smallest element. Let $\underline{\mu}(t) = \inf_{\mathcal{E}(p(t))} \mathcal{E}(p(t)) \in \mathcal{E}(p(t)) \subseteq \mathcal{E}(\Phi_t)$. Then $t \mapsto (p(t), \underline{\mu}(t))$ is a selection from the equilibrium correspondence $\mathcal{E}$. To see that it is isotone, consider arbitrary $\hat{t} \preceq_T \tilde{t}$. Then $p(\hat{t}) \preceq_k p(\tilde{t})$ as the kernel system is isotone, and $\underline{\mu}(\hat{t}) \preceq_s \underline{\mu}(\tilde{t})$ by Theorem 10 for the associated isotone parametric measure theory model. A similar proof works if $\sup_X X \in X$. For statement (2), consider a parametric kernel system $((X, \preceq_X, \mathcal{B}(X)), (T, \preceq_T), \mathcal{P})$ with isotone infimum selection $t \mapsto \underline{p}(t)$ such that $\forall t, p(t)$ is an isotone kernel and for all $p \in \mathcal{P}(t)$, $\underline{p}(t) \preceq_k p$, and the associated parametric measure theory model $((\mathcal{X}, \preceq_s), (T, \preceq_T), \Phi)$, and suppose $\inf_X X \in X$. The same proof as above shows that $t \mapsto (\underline{p}(t), \underline{\mu}(t))$ is an isotone selection from the equilibrium correspondence $\mathcal{E}$. To see that it is the infimum equilibrium selection, fix t and consider $p \in \mathcal{P}(t)$ and $\mu \in \mathcal{X}$ such that $\mu = \mathcal{T}_p(\mu)$. Then $\underline{p}(t) \preceq_k p$ as the kernel system is isotone infimum and $\underline{\mu}(t) \preceq_s \mu$ by Theorem 3. Statement (3) is proved similarly. Statement (4) follows from statements (2) and (3). Statement (5) follows because the kernel associated with each equilibrium that is selected is isotone.

Suppose the stochastic system being considered in statements (1)-(4) is a stochastic dynamical system. For statement (1), consider an isotone parametric stochastic dynamical system $((S, \preceq_S, \mathcal{B}(S)), (Z, \preceq_Z, \mathcal{B}(Z)), (T, \preceq_T), q, \mathcal{G})$, with isotone selection $t \mapsto g(t)$ such that $\forall t, g(t)$ is isotone, and the associated parametric measure theory model $((\mathcal{X}, \preceq_s), (T, \preceq_T), \Phi)$. Suppose $\inf_S S \in S$. Theorem 9 shows that $(\mu, t) \mapsto \mathcal{T}_{p(t)}(\mu)$, where $p(t)$ is derived from $q(t)$ and $g(t)$, is an isotone selection from $\Phi : \mathcal{X} \times T \rightrightarrows \mathcal{X}$. Following the same steps as above, it follows that for each t, if we let $\mathcal{E}(p(t)) = \{\mu \in \mathcal{X} \mid \mu = \mathcal{T}_{p(t)}(\mu)\}$ and let $\underline{\mu}(t) = \inf_{\mathcal{E}(p(t))} \mathcal{E}(p(t)) \in \mathcal{E}(p(t)) \subseteq \mathcal{E}(\Phi_t)$, then $t \mapsto (p(t), \underline{\mu}(t))$ is an isotone selection from the equilibrium correspondence $\mathcal{E}$. A similar proof works for an arbitrary isotone infimum parametric stochastic dynamical system and an arbitrary isotone supremum parametric stochastic dynamical system, showing statements (2), (3) and (4). Statements (1)-(4) for parametric stochastic dynamic economies and parametric Markov decision processes are proved similarly. Statement (5) follows similarly. ∎

Corollary 12. *Every parametric HP model and every parametric BDRW model has MCS of extremal equilibrium, and the selected equilibria are isotone.*

Proof. Follows from Theorem 11 and Theorem 9. ∎

Theorem 13 shows that the star chain complete set order can be used to generalize Theorem 10 to prove results for parametric comparisons of all the equilibria associated with particular selections from Φ.

Theorem 13. *Consider parametric poset models in which X is chain complete.*

1. *In every universal parametric isotone infimum model, for every $\hat{t} \preceq \tilde{t}$,*
   (a) $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*cc} \mathcal{E}(\underline{\Phi}_{\tilde{t}})$ and (b) $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*cc} \mathcal{E}(\Phi_{\tilde{t}})$.

2. *In every universal parametric isotone supremum model, for every $\hat{t} \preceq \tilde{t}$,*
   (a) $\mathcal{E}(\overline{\Phi}_{\hat{t}}) \sqsubseteq^{*cc} \mathcal{E}(\overline{\Phi}_{\tilde{t}})$ and (b) $\mathcal{E}(\Phi_{\hat{t}}) \sqsubseteq^{*cc} \mathcal{E}(\overline{\Phi}_{\tilde{t}})$.

3. *In every universal parametric isotone infimum and isotone supremum model, for every $\hat{t} \preceq \tilde{t}$, in addition to (1) and (2), $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*cc} \mathcal{E}(\overline{\Phi}_{\tilde{t}})$.*

4. *In every universal parametric isotone infimum and isotone supremum model, for every $\hat{t} \preceq \tilde{t}$, if $(X, \preceq_X, \Phi_{\hat{t}})$ is isotone supremum on lower intervals and $(X, \preceq_X, \Phi_{\tilde{t}})$ is isotone infimum on upper intervals, then in addition to (1), (2) and (3), $\mathcal{E}(\Phi_{\hat{t}}) \sqsubseteq^{*cc} \mathcal{E}(\Phi_{\tilde{t}})$.*

Proof. For statement (1), fix $\hat{t} \preceq \tilde{t}$. To show that $\mathcal{E}(\underline{\Phi}_{\hat{t}})$ is chain sup-complete in $\mathcal{E}(\underline{\Phi}_{\tilde{t}})$, consider nonempty chain $C \subseteq \mathcal{E}(\underline{\Phi}_{\hat{t}})$ and let $\bar{e} = \sup_X C \in X$, which exists because X is chain sup-complete. Let $\hat{X} = \{x \in X \mid \bar{e} \preceq x\}$ and $\Psi : \hat{X} \to \hat{X}$ be given by $\Psi(x) = \underline{\Phi}_{\tilde{t}}(x)$. Then $e \in C$ and $\underline{\Phi}(x, t)$ is isotone imply $e = \underline{\Phi}_{\hat{t}}(e) \preceq \underline{\Phi}_{\hat{t}}(\bar{e}) \preceq \underline{\Phi}_{\tilde{t}}(\bar{e})$, whence $\bar{e} \preceq \underline{\Phi}_{\tilde{t}}(\bar{e})$. Moreover, for every $x \in \hat{X}$, $\bar{e} \preceq \underline{\Phi}_{\tilde{t}}(\bar{e}) \preceq \underline{\Phi}_{\tilde{t}}(x)$. This shows that Ψ is well-defined and therefore, $(\hat{X}, \preceq_X, \Psi)$ is a universal isotone infimum model (a poset model in which $\hat{X}$ is chain sup-complete with $\inf_{\hat{X}} \hat{X} \in \hat{X}$ and Ψ is an isotone function). Let $\hat{e}$ be the smallest fixed point of Ψ. Then $\bar{e} \preceq \hat{e} \implies \hat{e}$ is an upper bound for C and $\hat{e} \in \Psi(\hat{e}) = \underline{\Phi}_{\tilde{t}}(\hat{e}) \implies \hat{e} \in \mathcal{E}(\underline{\Phi}_{\tilde{t}})$. Let $e \in \mathcal{E}(\underline{\Phi}_{\tilde{t}})$ be an arbitrary upper bound for C. Then $\bar{e} \preceq e$ and $e = \underline{\Phi}_{\tilde{t}}(e) = \Psi(e)$, showing that e is a fixed point of Ψ, whence $\hat{e} \preceq e$. This shows that $\sup_{\mathcal{E}(\underline{\Phi}_{\tilde{t}})} C = \hat{e} \in \mathcal{E}(\underline{\Phi}_{\tilde{t}})$.

To show that $\mathcal{E}(\underline{\Phi}_{\tilde{t}})$ is chain inf-complete in $\mathcal{E}(\underline{\Phi}_{\hat{t}})$, consider nonempty chain $C \subseteq \mathcal{E}(\underline{\Phi}_{\tilde{t}})$ and let $\underline{e} = \inf_X C \in X$, which exists because X is chain inf-complete. Let $\hat{X} = \{x \in X \mid x \preceq \underline{e}\}$ and

$\Psi : \hat{X} \to \hat{X}$ be given by $\Psi(x) = \underline{\Phi}_{\hat{t}}(x)$. Then $e \in C$ and $\underline{\Phi}(x,t)$ is isotone imply $e = \underline{\Phi}_{\tilde{t}}(e) \succeq \underline{\Phi}_{\tilde{t}}(\underline{e}) \succeq \underline{\Phi}_{\hat{t}}(\underline{e})$, whence $\underline{e} \succeq \underline{\Phi}_{\hat{t}}(\overline{e})$. Moreover, for every $x \in \hat{X}$, $\underline{e} \succeq \underline{\Phi}_{\hat{t}}(\overline{e}) \succeq \underline{\Phi}_{\hat{t}}(x)$. This shows that Ψ is well-defined and therefore, $(\hat{X}, \preceq_X, \Psi)$ is a universal isotone supremum model (a poset model in which $\hat{X}$ is chain inf-complete with a largest element, and Ψ is an isotone function). Let $\hat{e}$ be the greatest fixed point of Ψ. Then $\overline{e} \succeq \hat{e} \implies \hat{e}$ is a lower bound for C and $\hat{e} \in \Psi(\hat{e}) = \underline{\Phi}_{\hat{t}}(\hat{e}) \implies \hat{e} \in \mathcal{E}(\underline{\Phi}_{\hat{t}})$. Let $e \in \mathcal{E}(\underline{\Phi}_{\hat{t}})$ be an arbitrary lower bound for C. Then $\overline{e} \succeq e$ and $e = \underline{\Phi}_{\hat{t}}(e) = \Psi(e)$, showing that e is a fixed point of Ψ, whence $\hat{e} \succeq e$. This shows that $\inf_{\mathcal{E}(\underline{\Phi}_{\hat{t}})} C = \hat{e} \in \mathcal{E}(\underline{\Phi}_{\hat{t}})$. It follows that $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*cc} \mathcal{E}(\underline{\Phi}_{\tilde{t}})$.

To show that $\mathcal{E}(\underline{\Phi}_{\hat{t}})$ is chain sup-complete in $\mathcal{E}(\Phi_{\tilde{t}})$, consider nonempty chain $C \subseteq \mathcal{E}(\underline{\Phi}_{\hat{t}})$ and let $\bar{e} = \sup_X C \in X$. Let $\hat{X} = \{x \in X \mid \overline{e} \preceq x\}$ and $\Psi : \hat{X} \rightrightarrows \hat{X}$ be given by $\Psi(x) = \Phi_{\tilde{t}}(x) \cap \hat{X}$. Then $e \in C$ and $\underline{\Phi}(x,t)$ is isotone imply $e = \underline{\Phi}_{\hat{t}}(e) \preceq \underline{\Phi}_{\hat{t}}(\overline{e}) \preceq \underline{\Phi}_{\tilde{t}}(\overline{e})$, whence $\overline{e} \preceq \underline{\Phi}_{\tilde{t}}(\overline{e})$. Moreover, for every $x \in \hat{X}$, $\overline{e} \preceq \underline{\Phi}_{\tilde{t}}(\overline{e}) \preceq \underline{\Phi}_{\tilde{t}}(x)$, whence $\Phi_{\tilde{t}}(x) \subseteq \hat{X}$, and therefore, $\Psi(x) = \Phi_{\tilde{t}}(x)$. Consequently, $(\hat{X}, \preceq_X, \Psi)$ is a universal isotone infimum model ($\hat{X}$ is chain sup-complete, $\inf_{\hat{X}} \hat{X} \in \hat{X}$, and $\underline{\Psi}$ is isotone). Let $\hat{e}$ be the smallest fixed point of Ψ. Then $\hat{e} \in \Psi(\hat{e}) = \Phi_{\tilde{t}}(\hat{e}) \cap \hat{X}$ implies $\hat{e} \in \mathcal{E}(\Phi_{\tilde{t}})$ and $\overline{e} \preceq \hat{e}$ implies that $\hat{e}$ is an upper bound for C. Let $e \in \mathcal{E}(\Phi_{\tilde{t}})$ be an arbitrary upper bound for C. Then $\overline{e} \preceq e$ and $e \in \Phi_{\tilde{t}}(e) \cap \hat{X} = \Psi(e)$, showing that e is a fixed point of Ψ, and consequently, $\hat{e} \preceq e$. Therefore, $\sup_{\mathcal{E}(\Phi_{\tilde{t}})} C \in \mathcal{E}(\Phi_{\tilde{t}})$.

Finally, $\mathcal{E}(\Phi_{\tilde{t}})$ is chain inf-complete in $\mathcal{E}(\underline{\Phi}_{\hat{t}})$ can be proved in a manner very similar to the proof for $\mathcal{E}(\underline{\Phi}_{\tilde{t}})$ is chain inf-complete in $\mathcal{E}(\underline{\Phi}_{\hat{t}})$. It follows that $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*cc} \mathcal{E}(\Phi_{\tilde{t}})$.

Statement (2) is proved similarly. For statement (3), $\mathcal{E}(\underline{\Phi}_{\hat{t}})$ is chain sup-complete in $\mathcal{E}(\overline{\Phi}_{\tilde{t}})$ can be proved in a manner very similar to the proof for $\mathcal{E}(\underline{\Phi}_{\hat{t}})$ is chain sup-complete in $\mathcal{E}(\underline{\Phi}_{\tilde{t}})$, and $\mathcal{E}(\overline{\Phi}_{\tilde{t}})$ is chain inf-complete in $\mathcal{E}(\underline{\Phi}_{\hat{t}})$ can be proved in a manner very similar to the proof for $\mathcal{E}(\overline{\Phi}_{\tilde{t}})$ is chain inf-complete in $\mathcal{E}(\overline{\Phi}_{\hat{t}})$. This shows that $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*cc} \mathcal{E}(\overline{\Phi}_{\tilde{t}})$.

For statement (4), $\mathcal{E}(\Phi_{\hat{t}})$ is chain sup-complete in $\mathcal{E}(\Phi_{\tilde{t}})$ can be proved in a manner very similar to the proof for $\mathcal{E}(\underline{\Phi}_{\hat{t}})$ is chain sup-complete in $\mathcal{E}(\Phi_{\tilde{t}})$ and using $\overline{\Phi}(x,t)$ instead of $\underline{\Phi}(x,t)$. $\mathcal{E}(\Phi_{\tilde{t}})$ is chain inf-complete in $\mathcal{E}(\Phi_{\hat{t}})$ can be proved similarly. This shows that $\mathcal{E}(\Phi_{\hat{t}}) \sqsubseteq^{*cc} \mathcal{E}(\Phi_{\tilde{t}})$. ∎

Theorem 13 provides a framework for new theories of MCS of the full equilibrium set, the infimum equilibrium set, and the supremum equilibrium set using the star complete chain set order, unifying the results for lattice-based models in Sabarwal (2023b) and those for stochastic systems here. A parametric lattice model $((X, \preceq_X), (T, \preceq_T), \Phi)$ has ***MCS of the full equilibrium set in the star chain complete set order***, if the mapping $t \mapsto \mathcal{E}(\Phi_t)$ is isotone in the star chain complete set order; that is, for every $\hat{t} \preceq \tilde{t}$, $\mathcal{E}(\Phi_{\hat{t}}) \sqsubseteq^{*cc} \mathcal{E}(\Phi_{\tilde{t}})$. It has ***MCS of the infimum equilibrium set in the star chain complete set order***, if the mapping $t \mapsto \mathcal{E}(\underline{\Phi}_{\hat{t}})$ is isotone in the star chain complete set order, and it has ***MCS of the supremum equilibrium set in the star chain complete set order***, if the mapping $t \mapsto \mathcal{E}(\overline{\Phi}_{\hat{t}})$ is isotone in the star chain complete set order.

Corollary 14. *Consider parametric poset models in which X is chain complete.*

1. *Every universal parametric isotone infimum model has MCS of the infimum equilibrium set in the star chain complete set order.*

2. *Every universal parametric isotone supremum model has MCS of the supremum equilibrium set in the star chain complete set order.*

3. *Every universal parametric isotone infimum and supremum model in which for every t, $(X, \preceq_X, \Phi_t)$ is isotone supremum on lower intervals and isotone infimum on upper intervals has, in addition to (1) and (2), MCS of the full equilibrium set in the star chain complete set order.*

Proof. Follows from Theorem 13. ∎

Statements (1) and (2) in Corollary 14 require very little structure on the parametric model (just isotone infimum or isotone supremum). Statement (1) implies that $\forall e^* \in \mathcal{E}(\underline{\Phi}_{\hat{t}})$, $\exists$ unique $\tilde{e} \in \mathcal{E}(\underline{\Phi}_{\tilde{t}})$ higher than e^* and closest to it among all equilibria in $\mathcal{E}(\underline{\Phi}_{\tilde{t}})$, and $\forall e^* \in \mathcal{E}(\underline{\Phi}_{\tilde{t}})$, $\exists$ unique $\hat{e} \in \mathcal{E}(\underline{\Phi}_{\hat{t}})$ lower than e^* and closest to it among all equilibria in $\mathcal{E}(\underline{\Phi}_{\hat{t}})$, and similarly for statement (2). Statement (3) provides a universal order approximation result for every equilibrium, that is, $\forall e^* \in \mathcal{E}(\Phi_{\hat{t}})$, $\exists$ unique $\tilde{e} \in \mathcal{E}(\Phi_{\tilde{t}})$ higher than e^* and closest to it among all equilibria at $\tilde{t}$, and $\forall e^* \in \mathcal{E}(\Phi_{\tilde{t}})$, $\exists$ unique $\hat{e} \in \mathcal{E}(\Phi_{\hat{t}})$ lower than e^* and closest to it among all equilibria at $\hat{t}$.

Theorem 13 provides additional order approximation results of this form. Statement (1.b) implies that $\forall e^* \in \mathcal{E}(\Phi_{\tilde{t}})$, $\exists$ unique $\hat{e} \in \mathcal{E}(\underline{\Phi}_{\hat{t}})$ lower than e^* and closest to it among all equilibria in $\mathcal{E}(\underline{\Phi}_{\hat{t}})$, and (2.b) implies that $\forall e^* \in \mathcal{E}(\Phi_{\hat{t}})$, $\exists$ unique $\tilde{e} \in \mathcal{E}(\underline{\Phi}_{\tilde{t}})$ higher than e^* and closest to it among all equilibria in $\mathcal{E}(\underline{\Phi}_{\tilde{t}})$.

When X is chain complete and Φ is singleton valued, *every universal parametric model satisfies the conditions in every statement in Theorem 13 and Corollary 14*, leading to the following corollary.

Corollary 15. *Every universal parametric model in which X is chain complete and Φ is singleton valued has MCS of the full equilibrium set in the star chain complete set order.*

Proof. Follows from Theorem 13 and Corollary 14. ∎

Corresponding statements for parametric stochastic systems are proved similarly.

Theorem 16. *Consider the class of parametric stochastic systems.*

1. *In every isotone infimum parametric stochastic system, for every $\hat{t} \preceq \tilde{t}$,*
   (a) $\underline{\mathcal{E}}(\hat{t}) \sqsubseteq^{*cc} \underline{\mathcal{E}}(\tilde{t})$, (b) $\underline{\mathcal{E}}(\hat{t}) \sqsubseteq^{*cc} \mathcal{E}(\tilde{t})$, and (c) $\underline{\mathcal{E}}(\hat{t}) \sqsubseteq^{*cc} \mathcal{E}^{iso}(\tilde{t})$.

2. *In every isotone supremum parametric stochastic system, for every $\hat{t} \preceq \tilde{t}$,*
   (a) $\overline{\mathcal{E}}(\hat{t}) \sqsubseteq^{*cc} \overline{\mathcal{E}}(\tilde{t})$, (b) $\mathcal{E}(\Phi_{\hat{t}}) \sqsubseteq^{*cc} \overline{\mathcal{E}}(\tilde{t})$, and (c) $\mathcal{E}^{iso}(\Phi_{\hat{t}}) \sqsubseteq^{*cc} \overline{\mathcal{E}}(\tilde{t})$.

3. *In every parametric stochastic system that is isotone infimum and isotone supremum, for every $\hat{t} \preceq \tilde{t}$, in addition to (1) and (2), $\underline{\mathcal{E}}(\hat{t}) \sqsubseteq^{*cc} \overline{\mathcal{E}}(\tilde{t})$.*

4. *In every parametric stochastic system that is isotone infimum and supremum, for every $\hat{t} \preceq_T \tilde{t}$, if the $\hat{t}$ section of the system is strongly isotone (respectively, isotone) supremum on lower intervals and the $\tilde{t}$ section is strongly isotone (respectively, isotone) infimum on upper intervals, then in addition to (1), (2), and (3), $\mathcal{E}(\hat{t}) \sqsubseteq^{*cc} \mathcal{E}(\tilde{t})$ (respectively, $\mathcal{E}^{iso}(\hat{t}) \sqsubseteq^{*cc} \mathcal{E}^{iso}(\tilde{t})$).*

Proof. Suppose the stochastic system being considered in statements (1)-(3) is a parametric kernel system. For statement (1), consider an isotone infimum parametric kernel system $((X, \preceq_X, \mathcal{B}(X)), (T, \preceq_T), \mathcal{P})$ with $\inf_X X \in X$, isotone infimum selection $t \mapsto \underline{p}(t)$ such that $\forall t, \underline{p}(t)$ is an isotone kernel and for all $p \in \mathcal{P}(t)$, $\underline{p}(t) \preceq_k p$, and the associated parametric measure theory model $((\mathcal{X}, \preceq_s), (T, \preceq_T), \Phi)$. Fix $\hat{t} \preceq \tilde{t}$. To show that $\underline{\mathcal{E}}(\tilde{t})$ is inf-complete in $\underline{\mathcal{E}}(\hat{t})$, let C be a nonempty chain in $\underline{\mathcal{E}}(\tilde{t})$ and let $C' = \{\mu \in \mathcal{X} \mid (\exists p \in \mathcal{P}(\tilde{t})), (p, \mu) \in C\} \subseteq \mathcal{E}(\Phi_{\tilde{t}})$. It follows that C' is a nonempty chain in $\mathcal{E}(\Phi_{\tilde{t}})$ and using $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*cc} \mathcal{E}(\Phi_{\tilde{t}})$, let $\hat{\mu} = \inf_{\mathcal{E}(\underline{\Phi}_{\hat{t}})} C' \in \mathcal{E}(\underline{\Phi}_{\hat{t}})$. Then $(\underline{p}, \hat{\mu}) \in \underline{\mathcal{E}}(\hat{t})$ is a lower bound for C from $\underline{\mathcal{E}}(\hat{t})$. If $(\underline{p}, \mu') \in \underline{\mathcal{E}}(\hat{t})$ is an arbitrary lower bound for C from $\underline{\mathcal{E}}(\hat{t})$, then μ' is a lower bound for C' from $\mathcal{E}(\underline{\Phi}_{\hat{t}})$, whence $\mu' \preceq_s \hat{\mu}$. This shows that $(\underline{p}, \hat{\mu}) = \inf_{\underline{\mathcal{E}}(\hat{t})} C \in \underline{\mathcal{E}}(\hat{t})$. Similarly, $\underline{\mathcal{E}}(\hat{t})$ is sup-complete in $\underline{\mathcal{E}}(\tilde{t})$, whence $\underline{\mathcal{E}}(\hat{t}) \sqsubseteq^{*cc} \underline{\mathcal{E}}(\tilde{t})$. For (1)(b) and (1)(c), $\underline{\mathcal{E}}(\hat{t}) \sqsubseteq^{*cc} \mathcal{E}(\tilde{t})$ and $\underline{\mathcal{E}}(\hat{t}) \sqsubseteq^{*cc} \mathcal{E}^{iso}(\tilde{t})$ are shown by adapting the proof in Theorem 8 and using the connection to the associated parametric measure theory model in Theorem 13. Statements (2) and (3) are proved similarly. The statements for parametric stochastic dynamical systems, parametric stochastic dynamic economies, and parametric Markov decision processes are proved similarly. Statement (4) is proved by adapting the proof in Theorem 5 and using isotonicity of the corresponding stochastic system. ∎

Theorem 16 formalizes monotone comparative statics of the infimum equilibrium set and the supremum equilibrium set for parametric stochastic systems in an analogous manner. A parametric stochastic system has ***MCS of the full equilibrium set in the star chain complete set order***, if the mapping $t \mapsto \underline{\mathcal{E}}(t)$ is isotone in the star chain complete set order ($\hat{t} \preceq \tilde{t} \Rightarrow \mathcal{E}(\hat{t}) \sqsubseteq^{*cc} \mathcal{E}(\tilde{t})$), and it has ***MCS of the full isotone equilibrium set in the star chain complete set order***, if the mapping $t \mapsto \underline{\mathcal{E}}^{iso}(t)$ is isotone in the star chain complete set order ($\hat{t} \preceq \tilde{t} \Rightarrow \mathcal{E}^{iso}(\hat{t}) \sqsubseteq^{*cc} \mathcal{E}^{iso}(\tilde{t})$). A parametric stochastic system has ***MCS of the infimum equilibrium set in the star chain complete set order***, if the mapping $t \mapsto \underline{\mathcal{E}}(t)$ is isotone in the star chain complete set order ($\hat{t} \preceq \tilde{t} \Rightarrow \underline{\mathcal{E}}(\hat{t}) \sqsubseteq^{*cc} \underline{\mathcal{E}}(\tilde{t})$), and it has ***MCS of the supremum equilibrium set in the star chain complete set order***, if the mapping $t \mapsto \overline{\mathcal{E}}(t)$ is isotone in the star chain complete set order ($\hat{t} \preceq \tilde{t} \Rightarrow \overline{\mathcal{E}}(\hat{t}) \sqsubseteq^{*cc} \overline{\mathcal{E}}(\tilde{t})$).

Corollary 17. *Consider the class of parametric stochastic systems.*

1. *Every isotone infimum (respectively, supremum) parametric stochastic system has MCS of the infimum (respectively, supremum) equilibrium set in the star chain complete set order.*

2. *Every parametric isotone infimum and supremum stochastic system in which for every t, the t section of the system is strongly isotone (respectively, isotone) supremum on lower intervals and strongly isotone (respectively, isotone) infimum on upper intervals has, in addition to (1), MCS of the full equilibrium set (respectively, full isotone equilibrium set) in the star chain complete set order.*

Proof. Follows from Theorem 16. ∎

Statement (1) in Corollary 17 requires very little structure on the parametric model (just isotone infimum or isotone supremum). Indeed, in every isotone stochastic system, if the state space (X or S) has smallest and largest points and the kernel correspondence $\mathcal{P}$ or the policy correspondence $\mathcal{G}$ is singleton valued, *the system satisfies the conditions in every statement in Theorem 16 and Corollary 17*, leading to the following corollary.

Corollary 18. *Every isotone parametric stochastic system in which the state space has smallest and largest points and the associated correspondence $\mathcal{P}$ or $\mathcal{G}$ is singleton valued has MCS of the full equilibrium set in the star chain complete set order.*

Proof. Follows from Theorem 13 and Corollary 14. ∎

Corollary 19. *Every parametric HP model has MCS of infimum equilibrium set, supremum equilibrium set, and isotone equilibrium set in the star chain complete set order. If the policy correspondence is singleton valued, then the model has MCS of the full equilibrium set in the star chain complete set order as well.*

Proof. Follows by showing that every parametric HP model satisfies the corresponding statement in Corollary 17, using arguments similar to those in proof of Theorem 5. ∎

All the results in Corollary 19 are new features of equilibrium in the parametric HP model that are unknown in the previous literature. In particular, MCS of isotone equilibrium set implies the following: $\forall (g^*, \mu^*) \in \mathcal{E}(\hat{t})$ that is isotone, $\exists$ unique $(\tilde{g}, \tilde{\mu}) \in \mathcal{E}(\tilde{t})$ that is isotone, higher than (g^*, μ^*) and closest to it among all isotone equilibria at $\tilde{t}$, and $\forall (g^*, \mu^*) \in \mathcal{E}(\tilde{t})$ that is isotone, $\exists$ unique $(\hat{g}, \hat{\mu}) \in \mathcal{E}(\hat{t})$ that is isotone, lower than (g^*, μ^*) and closest to it among all isotone equilibria at $\hat{t}$. These results can help in policy analysis by guaranteeing order-nearest equilibria before or after a policy change, not only in terms of optimal actions but also for the entire steady state distribution in the economy.

5 Conclusion

We develop a universal theory of equilibrium in models with complementarities, unifying lattice-based theories used widely in economics and elsewhere and poset-based theories useful to study stochastic systems in many settings. We use deeper and more foundational order theoretic arguments to unify and generalize existing results. This provides a common language to study central features of equilibrium in different models with complementarities. Our formulation and proofs open the way to study additional classes of phenomena with complementarities, whether dynamic, deterministic, or stochastic.

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