General theory of equilibrium in models with complementarities

By

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Abstract

We unify and generalize the equilibrium theory of foundational models of complementarities used widely in economics and other disciplines. Widely used results for existence of extremal equilibrium, nonempty complete lattice structure of the equilibrium set, and monotone comparative statics (MCS) of extremal equilibria are unified and generalized, subsuming the results for standard and neostandard models as special cases and allowing for new situations. Structure theorems due to Tarski (1955) and Zhou (1994) are generalized without using the strong set order or subcompleteness. Defining new set orders, we formulate new theories for structural comparisons of equilibrium sets, and prove new theorems for MCS of the infimum equilibrium set, the supremum equilibrium set, and the full equilibrium set. Order comparability of equilibrium sets provides a new theory of order approximation of equilibria as well. Our off-the-shelf theorems apply regardless of the manner in which individual choices are made as long as they satisfy our weak conditions, which are proved to hold in standard and neostandard models.

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1 Introduction

Complementarities arise in many areas of socioeconomic interaction in economics and other disciplines. For example, complementarities arise in microeconomics (consumer theory, producer theory), macroeconomics (coordination failures, bank runs, macro policy), econometrics (peer effects, neighborhood effects), game theory (games with strategic complements, monotone equilibrium selections, Bayesian coordination games), equilibrium theory (existence, stability,

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comparative statics), industrial organization (competitive strategy, mergers, antitrust policy), education (peer effects, conventions), global games (contagion effects), political science (regime change), sociology (riots), urban and regional economics (agglomeration effects), networks (coordination, technology adoption, peer effects), computer science (algorithm design, auction implementation), development (growth traps, micro lending), market design, marketing, finance, banking, and more.

Theoretical foundations for socioeconomic situations with complementarities have been proposed in a series of lattice-based models, including Topkis (1978), Topkis (1979), Milgrom and Roberts (1990), Shannon (1990), Vives (1990), Milgrom and Shannon (1994), Zhou (1994), and others. Collectively, we term these *standard models with complementarities*. A series of newer models are proposed in Quah and Strulovici (2009), Prokopovych and Yannelis (2017), Che, Kim, and Kojima (2021), and others. We term these *neostandard models with complementarities*. Each model has some specialized features but they share the following common structural characteristics. Decentralized optimal behavior manifests in some form of an increasing correspondence for each individual. Systemic responses are expressed as a joint or aggregate of individual correspondences. Equilibrium predictions are formulated as fixed points of the joint correspondence. Parameters formalize the effect of environmental variables on individual, systemic, and equilibrium outcomes.

We isolate the common structural foundations of these models and study their equilibrium properties in a unified and general manner, as follows. Consider a finitely indexed collection $(X_i, \preceq_i, F_i)_{i=1}^I$, where for each individual i, $(X_i, \preceq_i)$ is a nonempty, complete lattice of possible actions or choices for individual i and $F_i : X_i \times X_{-i} \to \mathbb{R}$ is the payoff to i from choosing x_i when others choose x_{-i}. For each x_{-i}, let $\Phi_i(x_{-i}) = \arg\max_{\xi \in X_i} F_i(\xi, x_{-i})$. Let $\Phi : X \rightrightarrows X$ be given by $\Phi(x) = \times_{i=1}^I \Phi_i(x_{-i})$. Equilibrium of the collective system is given by the set of fixed points of Φ, denoted $\mathcal{E}(\Phi)$. We define the *associated lattice model* for this system as $(X, \preceq, \Phi)$, where $X = \times_{i=1}^I X_i$, $\preceq$ is product order, and $\Phi = \times_{i=1}^I \Phi_i$ is the joint best response correspondence, as above. Parameters are included in a natural manner as a poset $(T, \preceq_T)$ with appropriately generalized parametric joint behavior modeled as a correspondence $\Phi : X \times T \rightrightarrows X$, parametric equilibrium $\mathcal{E}(\Phi_t)$ defined as the fixed point set of the t-section of Φ, and the equilibrium correspondence given by $\mathcal{E} : T \rightrightarrows X$, $t \mapsto \mathcal{E}(\Phi_t)$.

As a first contribution, we show that the patterns of decentralized interdependent behavior in all the different standard and neostandard models are unified in terms of the same isotone properties on the joint correspondence Φ. This means that the theory of equilibrium in all the standard and neostandard models is unified by studying the equilibrium set of the associated lattice model in which Φ has these properties. Moreover, our framework allows for situations that cannot be subsumed in any of the standard models.

As a second contribution, we show that the main equilibrium benefits of different models with complementarities, such as existence of extremal equilibria and monotone comparative statics (MCS) of extremal equilibria hold in the general model using only isotone infimum and isotone supremum selections from Φ. Intuitively, a general model $(X, \preceq_X, \Phi)$ or a parametric general model $((X, \preceq_X), (T, \preceq_T), \Phi)$ derived from decentralized individual behavior has an isotone infimum selection when the set of maximizers of each individual has a smallest element that is isotone in parameters, and similarly for isotone supremum selection. This is true in all the standard and neostandard models in a natural manner, thereby including their equilibrium properties as special cases of the results for the general model. Moreover, our conditions are strictly weaker allowing for cases that cannot be subsumed in the standard models. No other conditions are imposed on Φ. It is not assumed to have any continuity properties, it is not assumed to be isotone in the strong set order, and the images $\Phi(x)$ are not assumed to be complete lattices (or even lattices).

As a third contribution, we provide weaker conditions on correspondences under which the equilibrium set is a nonempty, complete lattice, generalizing the well-known structure theorems of Zhou (1994), Vives (1990), and Tarski (1955). We provide two sets of conditions in isotone models: One based on isotone infimum selection on upper intervals and another based on isotone supremum selection on lower intervals (these are defined in the next section).

We prove that both conditions hold in all the standard and neostandard models, and both are strictly weaker, allowing for cases not included in those models. This is important because widely used results in the literature imply that the set of maximizers in a general model is a complete sublattice, but this does not necessarily imply that it is subcomplete, which is a stricter condition. Subcompleteness is a requirement in Zhou (1994) to prove that the equilibrium set is a complete lattice. This leaves a gap in the general theory between individual behavior and structure of the systemic equilibrium set. Our results plug this gap.

Our conditions don't use the strong set order and are strictly more general. (For nonempty subsets A, B of lattice X, A is lower than B in the *strong set order*, denoted $A \sqsubseteq^s B$, if $\forall x \in A$, $\forall y \in B$, $x \wedge y \in A$ and $x \vee y \in B$.) This is important because additional classes of situations are being identified where the strong set order may not necessarily hold (for example, Che, Kim, and Kojima (2021) and Prokopovych and Yannelis (2017)). Moreover, even though widely used results in the literature imply that the set of maximizers is isotone in the strong set order, this does not necessarily imply that the equilibrium set is isotone in the strong set order. Indeed, this may not hold even in canonical situations, as shown below. Our results include these situations in a natural manner and provide a unified and more general solution.

Our conditions don't use subcompleteness and are strictly more general. Both subcompleteness and isotone in strong set order are requirements in Zhou's theorem. Moreover, our conditions don't use the uniform set order as in Vives (1990). (For nonempty subsets A, B of poset $(X, \preceq)$, A is lower than B in the *uniform set order*, denoted $A \sqsubseteq^u B$, if $\forall x \in A$, $\forall y \in B$, $x \preceq y$.) Our results subsume these situations in a natural manner and provide a unified solution.

A long-standing problem in the theory of complementarities is lack of structural comparability of the equilibrium set at a lower parameter value with one at a higher parameter value. Comparability of the equilibrium set in the uniform set order or the strong set order does not obtain even in standard examples, as shown below. MCS of extremal equilibria implies that the equilibrium set is isotone in the weak set order. (For nonempty subsets A, B of poset $(X, \preceq)$, A is lower than B in the *weak set order*, denoted $A \sqsubseteq^w B$, if $\forall x \in A$, $\exists y \in B$, $x \preceq y$, and $\forall y \in B$, $\exists x \in A$, $x \preceq y$.) This does not necessarily provide tight bounds for an equilibrium (or subset of equilibria) at a lower parameter value using equilibria at the higher parameter value, or vice versa.

As a fourth contribution, we formulate two new relations to compare nonempty subsets of a partially ordered set: Star complete set order and star lattice set order. These relations help solve the problem of structural comparability of equilibrium sets. They use a natural process for order bounding one set using elements of a different set, as follows. For nonempty subsets E and A of poset X, $\sup_A E$ is defined to be an element of A that is an upper bound of E and is the smallest upper bound of E *among elements of* A. Similarly, $\inf_A E$ is an element of A that is a lower bound of E and is the largest lower bound of E *among elements of* A. When these exist, $\inf_A E$ and $\sup_A E$ provide natural and tight order bounds of E from the set A. We say that A is lower than B in the *star complete set order*, denoted $A \sqsubseteq^{*c} B$, if for every nonempty $E \subseteq A$, $\sup_B E$ exists (in B), and for every nonempty $E \subseteq B$, $\inf_A E$ exists (in A). We say that A is lower than B in the *star lattice set order*, denoted $A \sqsubseteq^{*\ell} B$, if $\forall x \in A$ and $\forall y \in B$, $\sup_B\{x, y\} \in B$ and $\forall x \in A$ and $\forall y \in B$, $\inf_A\{x, y\} \in A$. We prove that on every lattice X, the star lattice set order is an intermediate notion between strong set order and weak set order: $A \sqsubseteq^s B \Rightarrow A \sqsubseteq^{*\ell} B \Rightarrow A \sqsubseteq^w B$. We prove that when comparing lattices A and B, the star complete set order is a strengthening of star lattice set order: $A \sqsubseteq^{*c} B \Rightarrow A \sqsubseteq^{*\ell} B \Rightarrow A \sqsubseteq^w B$. Additional properties are proved as well.

As a fifth contribution, we show that these set orders are an appropriate modification of the strong set order to prove new theorems for MCS of the entire equilibrium set and identify previously unknown structural relationships among equilibrium sets in these models. We say that a general parametric model with complementarities has *MCS of the full equilibrium set in the star complete set order*, if the mapping $t \mapsto \mathcal{E}(\Phi_t)$ is isotone in the star complete set order ($\hat{t} \preceq_T \tilde{t} \Rightarrow \mathcal{E}(\Phi_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\Phi_{\tilde{t}})$). A similar statement defines *MCS of the full equilibrium set in the star lattice set order.*

We prove that in every general parametric model with complementarities, *if the correspondence Φ_t satisfies our conditions for $\mathcal{E}(\Phi_t)$ to be a nonempty complete lattice, then the model has MCS of the full equilibrium set in both the star complete set order and the star lattice set order.* Every standard and neostandard model satisfies these conditions, and therefore, *all the standard and neostandard models with complementarities necessarily have MCS of the full equilibrium set in both the star complete set order and the star lattice set order.* This is not true for the strong set order (or the uniform set order).

General results are available with fewer assumptions as well. If the model only has an isotone infimum selection, denoted $\underline{\Phi}$, we can still conclude that the model has *MCS of the* *infimum equilibrium set (denoted $\mathcal{E}(\underline{\Phi}_t)$) in both the star complete set order and the star lattice set order* ($\hat{t} \preceq_T \tilde{t} \Rightarrow \mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\underline{\Phi}_{\tilde{t}})$ and $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*\ell} \mathcal{E}(\underline{\Phi}_{\tilde{t}})$). Similarly, if the model has an isotone supremum selection, then it has *MCS of the supremum equilibrium set in both the star complete set order and the star lattice set order*. Every standard and neostandard model falls naturally under one of these situations. Moreover, under natural conditions, we also prove that for every t, $\mathcal{E}(\underline{\Phi}_t) \sqsubseteq^{*c} \mathcal{E}(\Phi_t) \sqsubseteq^{*c} \mathcal{E}(\overline{\Phi}_t)$ and $\mathcal{E}(\underline{\Phi}_t) \sqsubseteq^{*\ell} \mathcal{E}(\Phi_t) \sqsubseteq^{*\ell} \mathcal{E}(\overline{\Phi}_t)$.

As a sixth contribution, we show that order comparability of equilibrium sets provides a new and general theory of order approximation of equilibria in general models with complementarities. For example, we prove that in every general model $(X, \preceq_X, \Phi)$ with an isotone infimum selection $\underline{\Phi}$, it must be that $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*c} \mathcal{E}(\Phi)$. Therefore, for every nonempty $E \subseteq \mathcal{E}(\Phi)$, $\inf_{\mathcal{E}(\underline{\Phi})} E \in \mathcal{E}(\underline{\Phi})$. In other words, if a nonempty subset E of equilibria formalizes a specialized equilibrium notion of interest, it can be uniquely and tightly approximated from below in a formal order theoretic manner using equilibria from the infimum selection. In the special case that $E = \{e^*\}$ is a singleton, this proves that *every equilibrium $e^* \in \mathcal{E}(\Phi)$ can be uniquely order approximated from below by an equilibrium using only the infimum selection*. This is particularly useful if the infimum selection is easier to work with or has some useful computational, dynamic, or theoretical properties. Our result requires very little structure in the model (only isotone infimum selection). *We prove an analogous result for every general model with a supremum selection ($\mathcal{E}(\Phi) \sqsubseteq^{*c} \mathcal{E}(\overline{\Phi})$), and similar results are proved for parametric models as well.* Every standard and neostandard model falls under one of these situations.

As a seventh contribution, we focus on the equilibrium theory of models with complementarities. That is, we study systemic influences of interdependence among individual choices and their equilibrium impact, given individual choice behavior. We take individual choice behavior as given. It can be the solution to an optimization problem, but we do not require that in the general case. Anything that is an accurate description of the situation being studied is permissible as long as it satisfies our weak conditions. This has several benefits. First, it provides unified, off-the-shelf theorems that apply regardless of the manner in which individual choices are made as long as they satisfy weak conditions. Second, our conditions are naturally satisfied in the standard and neostandard models, they are intuitively easy to check, and they allow for new situations. Third, our results can guide new research lines to discover more general properties of individual behavior that do not fall under the purview of the standard models but satisfy complementarities in our weaker setting. Fourth, our study isolates salient properties of equilibrium that generalize to important and large classes of applications beyond the scope of lattice-based models.

In this paper, we focus on models with complementarities in which the underlying space X is a nonempty complete lattice. In several classes of models this is not necessarily true, for example, models based on probability spaces with partial orders on probability measures. Such models include kernel systems, stochastic dynamical systems, dynamic macroeconomic models, and Markov decision processes. Those models require different methods to study equilibrium and its properties. Some examples are available in Hopenhayn and Prescott (1992), Amir (1996), Acemoglu and Jensen (2013), Acemoglu and Jensen (2015), Balbus, Dziewulski, Reffett, and Woźny (2019), Balbus, Dziewulski, Reffett, and Woźny (2022), Schlee and Khan (2022b), Schlee and Khan (2022a), and others. Structural properties of equilibrium in these types of models are unified and generalized in Sabarwal (2023b). Structural properties for models with continuity properties are unified and generalized in Sabarwal (2023a).

The paper is organized as follows. Section 2 defines general models with complementarities and proves the main results for existence of equilibrium, existence of extremal equilibrium, and nonempty complete lattice structure of the equilibrium set. Section 3 formulates the new set orders and proves comparative statics of the infimum equilibrium set, the supremum equilibrium set, and the full equilibrium set. It also formalizes the theory of order approximation of equilibria. Section 4 defines general parametric models with complementarities, proves the analogous results for these models, and includes the additional results on parametric monotone comparative statics of the equilibrium set. Section 5 concludes.

2 General models with complementarities

A *partial order* on a set X is a binary relation $\preceq$ that is reflexive, antisymmetric, and transitive. A ***partially ordered set (or, poset)***, is a set X along with a partial order $\preceq$ on it, denoted $(X, \preceq)$. For a poset $(X, \preceq)$ and subset A of X, the *relative partial order* on A is the usual one: For every $x, x' \in A$, $x \preceq_A x' \Leftrightarrow x \preceq x'$. It follows that $(A, \preceq_A)$ is a poset in the relative partial order. For posets $(X, \preceq_X)$ and $(Y, \preceq_Y)$, the Cartesian product $X \times Y$ is a poset under the *product partial order* given by $(x,y) \preceq (x',y') \Leftrightarrow x \preceq_X x'$ and $y \preceq_Y y'$. For posets $(X, \preceq_X)$ and $(Y, \preceq_Y)$, a function $f : X \to Y$ is ***isotone*** if for every $\hat{x}$ and $\tilde{x}$ in X, $\hat{x} \preceq_X \tilde{x} \implies f(\hat{x}) \preceq_Y f(\tilde{x})$.

Two points x, y in a poset $(X, \preceq)$ are *comparable* (or *ordered*), if $x \preceq y$ or $y \preceq x$. In this case, we say that x is lower than y when $x \preceq y$, or x is higher than y when $y \preceq x$. Points x, y are *strictly comparable* (or *strictly ordered*), if they are comparable and $x \neq y$. In this case, we say x is strictly lower than y, denoted $x \prec y$, or x is strictly higher than y, denoted $y \prec x$, as the case may be. A partial order is *complete* if every pair of points is comparable. A poset with a complete order is a *chain*. In other words, a chain is a poset in which every pair of points is comparable. Two points x, y are *incomparable* (or *noncomparable*, or *unordered*), if they are not comparable, that is, $x \npreceq y$ and $y \npreceq x$.

Let X be a poset and E a nonempty subset of X. An ***upper bound for*** E is an element $x \in X$ such that for every $e \in E$, $e \preceq x$. The ***sup of*** E ***in*** X, denoted $\sup_X E$, is an element $\overline{e} \in X$ such that (1) $\overline{e}$ is an upper bound for E and (2) for every $x \in X$ that is an upper bound for E, $\overline{e} \preceq x$. A ***lower bound for*** E is an element $x \in X$ such that for every $e \in E$, $x \preceq e$. The ***inf of*** E ***in*** X, denoted $\inf_X E$, is an element $\underline{e} \in X$ such that (1) $\underline{e}$ is a lower bound for E and (2) for every $x \in X$ that is a lower bound for E, $x \preceq \underline{e}$. When convenient, we denote $\underline{x} = \inf_X X$ and $\overline{x} = \sup_X X$.

A ***lattice*** is a poset $(X, \preceq)$ in which for every $x, y \in X, x \wedge y := \inf_X\{x, y\} \in X$ and $x \vee y := \sup_X\{x, y\} \in X$. A lattice $(X, \preceq)$ is ***complete***, if for every nonempty subset E of X, $\inf_X E \in X$ and $\sup_X E \in X$. It follows that if X is a complete lattice, then $\inf_X X \in X$ and $\sup_X X \in X$. Subset A of lattice X is ***subcomplete***, if for every nonempty $B \subseteq A$, $\inf_X B \in A$ and $\sup_X B \in A$.

For subsets A and B of lattice X, A ***is lower than*** B ***in the strong set order (SSO),*** $A \sqsubseteq^s B$, if for every $a \in A$ and $b \in B$, $a \wedge b \in A$ and $a \vee b \in B$. Topkis (1978) attributes the strong set order to Veinott (1989). Milgrom and Shannon (1994) use the term strong set order. Other terms used are the induced set ordering in Topkis (1998) and the lattice set order in Sabarwal (2021). Two other set orders used in the literature are: A ***is lower than*** B ***in***

the weak set order, denoted $A \sqsubseteq^w B$, if for every $x \in A$ there is $y \in B$ such that $x \preceq y$, and for every $y \in B$ there is $x \in A$ such that $x \preceq y$, and A ***is lower than*** B ***in the uniform set order***, denoted $A \sqsubseteq^u B$, if for every $x \in A$ and for every $y \in B$, $x \preceq y$. It follows immediately that for nonempty $A, B \subseteq X$, $A \sqsubseteq^u B \Rightarrow A \sqsubseteq^s B \Rightarrow A \sqsubseteq^w B$.

For arbitrary sets X and Y, a ***correspondence from*** X ***to*** Y, denoted $\Phi : X \rightrightarrows Y$, is a function from X to the power set of Y, $\Phi : X \to \mathcal{P}(Y)$. It is nonempty valued, if for every $x \in X$, $\Phi(x) \neq \emptyset$. It is singleton valued, if for every x in X, $\Phi(x)$ is a singleton subset of Y. A function $f : X \to Y$ is viewed as a correspondence that is singleton valued, and conversely (and in this case, we'll use either notation without further mention). A ***selection from correspondence*** Φ is a function $f : X \to Y$ such that $f(x) \in \Phi(x)$ for every $x \in X$. For a correspondence $\Phi : X \rightrightarrows X$, a point $x \in X$ is ***a fixed point of*** Φ, if $x \in \Phi(x)$, and the ***fixed point set*** of Φ is $\mathcal{E}(\Phi) = \{x \in X \mid x \in \Phi(x)\}$.

A ***lattice model*** is a triple $(X, \preceq, \Phi)$, where $(X, \preceq)$ is a nonempty complete lattice and $\Phi : X \rightrightarrows X$ is a correspondence. An ***equilibrium*** in the model is a fixed point of Φ. The ***equilibrium set*** of the model is the fixed point set $\mathcal{E}(\Phi)$.

A ***general model with complementarities*** (or, ***general model***) is a lattice model $(X, \preceq, \Phi)$ with an isotone selection. A lattice model $(X, \preceq, \Phi)$ is ***isotone supremum*** if for every $x \in X$, $\overline{\Phi}(x) := \sup_{\Phi(x)} \Phi(x) \in \Phi(x)$, and $x \mapsto \overline{\Phi}(x)$ is isotone. It is ***isotone infimum*** if for every $x \in X$, $\underline{\Phi}(x) := \inf_{\Phi(x)} \Phi(x) \in \Phi(x)$, and $x \mapsto \underline{\Phi}(x)$ is isotone. These definitions only require the weaker conditions that infimum or supremum to be taken over $\Phi(x)$ not X (it is easy to check that $\inf_X \Phi(x) \in \Phi(x) \Rightarrow \inf_{\Phi(x)} \Phi(x) = \inf_X \Phi(x) \in \Phi(x)$, and similarly for supremum). No other conditions are imposed on Φ. It is not assumed to have any continuity properties, $\Phi(x)$ is not assumed to be subcomplete in X, or a complete lattice, or even a lattice, and Φ is not assumed to be isotone in strong set order.

A general model arises from decentralized individual behavior in the following standard manner. Consider a finitely indexed collection $(X_i, \preceq_i, \Phi_i)_{i=1}^I$, where $\forall i$, $(X_i, \preceq_i)$ is a nonempty, complete lattice and $\forall i$, $\Phi_i : X_{-i} \rightrightarrows X_i$ is a correspondence. Here, $X_{-i} = \times_{j=1, j \neq i}^I X_j$ with the product order. Each X_i is viewed as the choice (or action) space of individual i and $\Phi_i(x_{-i})$ are the decentralized choices (or actions) of individual i that can depend on what others

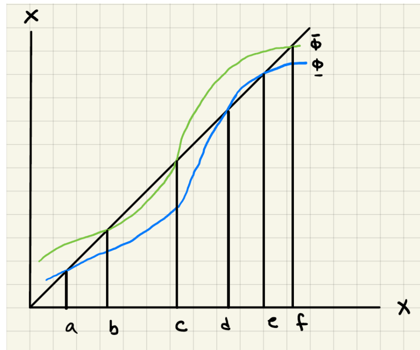

Figure 1: Standard S-model

are doing. As shown below, Φ_i is typically the solution to an optimization problem given individual-specific payoff functions, but we do not require that in the general case. Anything that is an accurate description of the situation being studied is permissible. Let $\Phi : X \rightrightarrows X$ be given by $\Phi(x) = \times_{i=1}^{I} \Phi_i(x_{-i})$. Equilibrium of the collective system is given by $\mathcal{E}(\Phi)$. The *associated lattice model* is $(X, \preceq, \Phi)$, where $X = \times_{i=1}^{I} X_i$, $\preceq$ is product order, and $\Phi = \times_{i=1}^{I} \Phi_i$ is the product correspondence. It is easy to check that if each Φ_i has an isotone infimum (respectively, supremum) selection then so does Φ, and the associated lattice model is a general model that is isotone infimum (respectively, supremum). All standard and neostandard models with complementarities are unified as special cases of this general framework.

Example 1 (Standard S-model)**.** Figure 1 shows a correspondence version of the standard S-model commonly used to motivate models with complementarities. Here, $(X, \preceq)$ is a chain, and $\forall x \in X$, $\Phi(x)$ is the interval given by $\Phi(x) = [\underline{\Phi}(x), \overline{\Phi}(x)]$. This model $(X, \preceq, \Phi)$ is an isotone infimum model and an isotone supremum model. If $\overline{\Phi}(x)$ is deleted from this example, the resulting model is isotone infimum but not isotone supremum, and if $\underline{\Phi}(x)$ is deleted from this example, the resulting model is isotone supremum but not isotone infimum. If both $\underline{\Phi}(x)$ and $\overline{\Phi}(x)$ are deleted from this example, the resulting model is a general model with complementarities that is neither isotone infimum nor isotone supremum.

Example 2 (Topkis model)**.** Following Topkis (1978) and Topkis (1979), the ***Topkis model*** is a finitely indexed collection $(X_i, \preceq_i, F_i)_{i=1}^{I}$, where for each i, X_i is a nonempty, compact sublattice of $\mathbb{R}^{m_i}$ in the natural product order (and using product order on products of X_i), for each i, $F_i : X_i \times X_{-i} \to \mathbb{R}$ has decreasing differences in (x_i, x_{-i}), and for each x_{-i}, $F_i(\cdot, x_{-i})$ is submodular and lower semicontinuous on X_i. The definitions of decreasing differences and submodular are the standard ones. On posets X, Y, a function $f : X \times Y \to \mathbb{R}$ satisfies decreasing differences if for every $\hat{x} \preceq \tilde{x}$, the difference $f(\tilde{x}, y) - f(\hat{x}, y)$ is (weakly) decreasing in y. On a lattice X, a function $f : X \to \mathbb{R}$ is submodular if for every $x, y \in X$, $f(x \wedge y) - f(x) \leq f(y) - f(x \vee y)$. Let $X = \times_{i=1}^{I} X_i$ with the product order, denoted $\preceq$. For each $x, y \in X$, let $G(x, y) = \sum_{i=1}^{I} F_i(y_i, x_{-i})$ and let $\Phi : X \rightrightarrows X$ be given by $\Phi(x) = \arg\min_{y \in X} G(x, y)$. Equilibrium of the Topkis model is given by $\mathcal{E}(\Phi)$. Therefore, equilibrium properties of the Topkis model can be studied equivalently using its ***associated lattice model*** $(X, \preceq, \Phi)$, where $X = \times_{i=1}^{I} X_i$, $\preceq$ is product order, and Φ is the correspondence as above. Topkis (1979) gives several concrete applications, including games with complementary products, minimum cut games, and competitive pricing with substitute products. Many additional applications are provided in Topkis (1998).

Example 3 (Vives model)**.** Following Vives (1990), the ***Vives model*** is a finitely indexed collection $(X_i, \preceq_i, F_i)_{i=1}^{I}$, where for each i, X_i is a nonempty, complete lattice (and using product order on products of X_i), for each i, $F_i : X_i \times X_{-i} \to \mathbb{R}$ has increasing differences in (x_i, x_{-i}), and for each x_{-i}, $F_i(\cdot, x_{-i})$ is supermodular and upper semicontinuous in Frink (1942)'s order interval topology on X_i. For each x_{-i}, let $\Phi_i(x_{-i}) = \arg\max_{\xi \in X_i} F_i(\xi, x_{-i})$. Let $\Phi : X \rightrightarrows X$ be given by $\Phi(x) = \times_{i=1}^{I} \Phi_i(x_{-i})$. Equilibrium properties of the Vives model can be studied equivalently using its ***associated lattice model*** $(X, \preceq, \Phi)$, where $X = \times_{i=1}^{I} X_i$, $\preceq$ is product order, and $\Phi = \times_{i=1}^{I} \Phi_i$ is the product correspondence as above. Vives (1990) gives several concrete applications of oligopoly games such as Bertrand, Cournot, and product selection with complementary products.

The Vives model may be viewed naturally as the model dual to the Topkis model. This can be formalized by invoking the duality between supermodular and submodular functions and between maximization and minimization, and by reformulating the correspondence in Topkis in the manner in Vives. With this identification, Vives model generalizes Topkis model by working with complete lattices rather than subcomplete lattices in finite-dimensional Euclidean spaces. In every Vives model, for every $x \in X$, $\Phi(x)$ is a nonempty, complete sublattice and Φ is isotone in the strong set order: $\hat{x} \preceq \tilde{x} \Rightarrow \Phi(\hat{x}) \sqsubseteq^s \Phi(\tilde{x})$.

Example 4 (MR model)**.** Following Milgrom and Roberts (1990), the ***MR model*** is a finitely indexed collection $(X_i, \preceq_i, F_i)_{i=1}^I$, where for each i, X_i is a nonempty, complete lattice (and using product order on products of X_i), for each i, $F_i : X_i \times X_{-i} \to \mathbb{R}$ has increasing differences in (x_i, x_{-i}), for each x_{-i}, $F_i(\cdot, x_{-i})$ is supermodular and upper semicontinuous in Frink (1942)'s order interval topology on X_i, and F_i is order continuous on X_{-i}. For each x_{-i}, let $\Phi_i(x_{-i}) = \arg\max_{\xi \in X_i} F_i(\xi, x_{-i})$. Let $\Phi : X \rightrightarrows X$ be given by $\Phi(x) = \times_{i=1}^I \Phi_i(x_{-i})$. The ***associated lattice model*** is $(X, \preceq, \Phi)$, where $X = \times_{i=1}^I X_i$, $\preceq$ is product order, and $\Phi = \times_{i=1}^I \Phi_i$ is the product correspondence as above. Milgrom and Roberts (1990) give several concrete applications including a Diamond-type search model, a Bertrand model, an arms race model, a Hendricks-Kovenock oil drilling model, and a Milgrom-Roberts model of modern manufacturing.

Example 5 (GMS model)**.** Following Shannon (1990) and Milgrom and Shannon (1994), the generalized Milgrom and Shannon model, or ***GMS model*** is a finitely indexed collection $(X_i, \preceq_i, F_i)_{i=1}^I$, where for each i, X_i is a nonempty, complete lattice (and using product order on products of X_i), for each i, $F_i : X_i \times X_{-i} \to \mathbb{R}$ has single crossing property in (x_i, x_{-i}), and for each x_{-i}, $F_i(\cdot, x_{-i})$ is quasisupermodular and upper semicontinuous in Frink (1942)'s order interval topology on X_i. For each x_{-i}, let $\Phi_i(x_{-i}) = \arg\max_{\xi \in X_i} F_i(\xi, x_{-i})$. Let $\Phi : X \rightrightarrows X$ be given by $\Phi(x) = \times_{i=1}^I \Phi_i(x_{-i})$. The definitions of quasisupermodular and single crossing property are the same as in Milgrom and Shannon (1994). The ***associated lattice model*** is $(X, \preceq, \Phi)$, where $X = \times_{i=1}^I X_i$, $\preceq$ is product order, and $\Phi = \times_{i=1}^I \Phi_i$ is the product correspondence as above.

As increasing differences implies single crossing property and supermodular implies quasisupermodular, the MR model is a special case of the GMS model. The GMS model here is more general than the corresponding one postulated in section 5 in Milgrom and Shannon (1994), because we do not impose the additional condition that for every i, F_i is continuous on X_{-i}. Milgrom and Shannon (1994) (and Milgrom and Roberts (1990)) use this continuity property to prove existence of extremal equilibria in their model. We do not use this property in this paper. Deleting this property has the additional benefit that it nests the Vives model as a special case and provides unified theorems for general models with complementarities. Results in Milgrom and Shannon (1994) show that in the GMS model, for every $x \in X$, $\Phi(x)$

is a nonempty, complete sublattice and Φ is isotone in the strong set order.

Example 6 (Zhou model)**.** Following Zhou (1994), the ***Zhou model*** is a lattice model $(X, \preceq, \Phi)$ in which Φ is isotone in the strong set order and for every x, $\Phi(x)$ is nonempty and subcomplete.

Example 7 (CKK model)**.** Che, Kim, and Kojima (2021) propose a model with weaker assumptions than Milgrom and Shannon (1994) using a weak dominance property formulated for pairs of objective functions: For $f, g : X \to \mathbb{R}$, $f \preceq_w g$, if for every $\tilde{x} \npreceq \hat{x}$ in X, $f(\tilde{x}) \geq (>) \max\{f(\hat{x} \wedge \tilde{x}), f(\hat{x})\} \Rightarrow \max\{g(\tilde{x}), g(\hat{x} \vee \tilde{x})\} \geq (>)\ g(\hat{x})$. With this property, their Theorem 2 shows that for every sublattice $S \subseteq X$, $\arg\max_S f \sqsubseteq^w \arg\max_S g$, whenever both sets are nonempty. For a parameterized collection of functions, we define the weak dominance property analogously: For a lattice X and poset T, a function $F : X \times T \to \mathbb{R}$ has *weak dominance property in* (x, t), if for every $\tilde{x} \npreceq_X \hat{x}$ and for every $\hat{t} \preceq_T \tilde{t}$, $F(\tilde{x}, \hat{t}) \geq (>) \max\{F(\hat{x} \wedge \tilde{x}, \hat{t}), F(\hat{x}, \hat{t})\} \Rightarrow \max\{F(\tilde{x}, \tilde{t}), F(\hat{x} \vee \tilde{x}, \tilde{t})\} \geq (>)\ F(\hat{x}, \tilde{t})$. It follows that for every $\hat{t} \preceq_T \tilde{t}$, $F_{\hat{t}} \preceq_w F_{\tilde{t}}$, where subscripts denote section of the function determined by the subscript.

A benefit of the formulation in Che, Kim, and Kojima (2021) is that it places no restriction on a given function, in the following sense: It is easy to check that for every function $f : X \to \mathbb{R}$ on a lattice X, $f \preceq_w f$ and $\arg\max_S f \sqsubseteq^w \arg\max_S f$ (whenever the set of maximizers is nonempty). The latter property follows because every nonempty subset A of a poset X satisfies $A \sqsubseteq^w A$, and therefore, no conditions on f (or S) are needed to obtain this property. As shown in section 5 in Che, Kim, and Kojima (2021), additional topological and order theoretic assumptions are needed to guarantee existence of fixed points and weak monotone comparative statics. Instead of those assumptions, we assume that the set of maximizers has either a smallest or a largest element (rather than just nonempty). This can follow from assumptions commonly used in the literature, for example, by assuming that the objective function is upper semicontinuous and quasisupermodular in the decision variable.

The above is translated into our framework as follows. The ***CKK-1 model*** is a finitely indexed collection $(X_i, \preceq_i, F_i)_{i=1}^I$, where for each i, X_i is a nonempty, complete lattice (and using product order on products of X_i) and for each i, $F_i : X_i \times X_{-i} \to \mathbb{R}$ has weak dominance property in (x_i, x_{-i}). For each i, fix a sublattice $S_i \subseteq X_i$, and for every x_{-i}, let $\Phi_i(x_{-i}) =$

$\arg\max_{\xi \in S_i} F_i(\xi, x_{-i})$, and suppose that $\underline{\Phi}_i(x_{-i}) \coloneqq \inf_{\Phi_i(x_{-i})} \Phi_i(x_{-i}) \in \Phi_i(x_{-i})$. Let $\Phi : X \rightrightarrows X$ be given by $\Phi(x) = \times_{i=1}^{I} \Phi_i(x_{-i})$. The ***associated lattice model*** is $(X, \preceq, \Phi)$, where $X = \times_{i=1}^{I} X_i$, $\preceq$ is product order, and $\Phi = \times_{i=1}^{I} \Phi_i$ is the product correspondence as above. As a second version, the ***CKK-2 model*** is a CKK-1 model in which the property that infimum exists is replaced with the property that supremum exists, that is, for each i and x_{-i}, $\overline{\Phi}_i(x_{-i}) \coloneqq \sup_{\Phi_i(x_{-i})} \Phi_i(x_{-i}) \in \Phi_i(x_{-i})$, and other aspects remain the same. A ***CKK model*** is one that is both CKK-1 and CKK-2. As mentioned above, if, instead, we add the assumption that for each i, S_i is subcomplete, and for each x_{-i}, $F_i(\cdot, x_{-i})$ is upper semicontinuous in Frink (1942)'s order interval topology and quasisupermodular, then the set of maximizers has both a smallest and a largest element. (As an aside, recall that quasisupermodularity is automatically satisfied if each X_i is a chain and as mentioned in Che, Kim, and Kojima (2021), on a chain, their model is the same as the GMS model and implies a correspondence isotone in the strong set order.) Che, Kim, and Kojima (2021) give several concrete applications including Pareto optimal choices, beauty contest game, stable many-to-one matchings, multidivisional organization, and matching with constraints.

Example 8 (ICKK model)**.** Che, Kim, and Kojima (2021) propose a further weakening for optimization on intervals using a weak interval dominance property: For $f, g : X \to \mathbb{R}$, $f \preceq_{wI} g$, if for every $\hat{x}, \tilde{x} \in X$, if $\tilde{x} \not\preceq \hat{x}$ and $\forall x \in J(\hat{x}, \tilde{x})$, $f(\tilde{x}) \geq (>)\ f(x)$ and $g(\hat{x}) \geq (>)\ g(x)$, then $f(\tilde{x}) \geq (>)\ \max_{\xi \in J(\hat{x} \wedge \tilde{x}, \hat{x})} f(\xi) \Rightarrow \max_{\xi \in J(\tilde{x}, \hat{x} \vee \tilde{x})} g(\xi) \geq (>)\ g(\hat{x})$. Here, $J(x, y) \coloneqq [x \wedge y, x \vee y]$ is the smallest interval containing x and y. With this property, their Theorem 3 implies that for every subinterval S in X, $\arg\max_S f \sqsubseteq^w \arg\max_S g$, whenever both sets are nonempty. For a parameterized collection of functions, we define the weak interval dominance property analogously: For a lattice X and poset T, a function $F : X \times T \to \mathbb{R}$ has *weak interval dominance property in* (x, t), if for every $\tilde{x} \not\preceq_X \hat{x}$ and for every $\hat{t} \preceq_T \tilde{t}$, if $\forall x \in J(\hat{x}, \tilde{x})$, $F(\tilde{x}, \hat{t}) \geq (>)\ F(x, \hat{t})$ and $F(\hat{x}, \tilde{t}) \geq (>)\ F(x, \tilde{t})$, then $F(\tilde{x}, \hat{t}) \geq (>)\ \max_{\xi \in J(\hat{x} \wedge \tilde{x}, \hat{x})} F(\xi, \hat{t}) \Rightarrow \max_{\xi \in J(\tilde{x}, \hat{x} \vee \tilde{x})} F(\xi, \tilde{t}) \geq (>)\ F(\hat{x}, \tilde{t})$.

As above, assuming that the argmax either has a smallest element or a largest element (rather than just nonempty), this is translated into our framework as follows. The ***ICKK-1 model*** is a finitely indexed collection $(X_i, \preceq_i, F_i)_{i=1}^{I}$, where for each i, X_i is a nonempty, complete lattice (and using product order on products of X_i) and for each i, $F_i : X_i \times X_{-i} \to \mathbb{R}$ has weak dominance property in (x_i, x_{-i}). For each i, fix a subinterval S_i in X_i, and for every x_{-i}, let $\Phi_i(x_{-i}) = \arg\max_{\xi \in S_i} F_i(\xi, x_{-i})$ and suppose that $\underline{\Phi}_i(x_{-i}) := \inf_{\Phi_i(x_{-i})} \Phi_i(x_{-i}) \in \Phi_i(x_{-i})$. Let $\Phi : X \rightrightarrows X$ be given by $\Phi(x) = \times_{i=1}^{I} \Phi_i(x_{-i})$. The ***associated lattice model*** is $(X, \preceq, \Phi)$, where $X = \times_{i=1}^{I} X_i$, $\preceq$ is product order, and $\Phi = \times_{i=1}^{I} \Phi_i$ is the product correspondence as above. As a second version, the ***ICKK-2 model*** is an ICKK-1 model in which the property that infimum exists is replaced with the property that supremum exists, that is, for each i and x_{-i}, $\overline{\Phi}_i(x_{-i}) := \sup_{\Phi_i(x_{-i})} \Phi_i(x_{-i}) \in \Phi_i(x_{-i})$, and other aspects remain the same. An ***ICKK model*** is one that is both ICKK-1 and ICKK-2. As earlier, if, instead, we add the assumption that for each i and x_{-i}, $F_i(\cdot, x_{-i})$ is upper semicontinuous in Frink (1942)'s order interval topology and quasisupermodular, then the set of maximizers has both a smallest and a largest element.

Example 9 (GQS model)**.** Quah and Strulovici (2009) propose an interval dominance order and show that with their property, maximizers on an interval are isotone in the strong set order. Che, Kim, and Kojima (2021) provide a generalization using the interval dominance property: For $f, g : X \to \mathbb{R}$, $f \preceq_I g$, if for every $\hat{x}, \tilde{x} \in X$, if $\tilde{x} \not\preceq \hat{x}$ and $\forall x \in J(\hat{x}, \tilde{x})$, $f(\tilde{x}) \geq (>)\ f(x)$ and $g(\hat{x}) \geq (>)\ g(x)$, then $f(\tilde{x}) \geq (>)\ f(\hat{x} \wedge \tilde{x}) \Rightarrow g(\hat{x} \vee \tilde{x}) \geq (>)\ g(\hat{x})$. With this property, their Theorem S1 implies that for every subinterval S in lattice X, $\arg\max_S f \sqsubseteq^s \arg\max_S g$, whenever both sets are nonempty. Moreover, $f \preceq_I g \Rightarrow f \preceq_{wI} g$. For a parameterized collection of functions, we define the interval dominance property analogously: For a lattice X and poset T, a function $F : X \times T \to \mathbb{R}$ has *interval dominance property in* (x, t), if for every $\tilde{x} \not\preceq_X \hat{x}$ and for every $\hat{t} \preceq_T \tilde{t}$, if $\forall x \in J(\hat{x}, \tilde{x})$, $F(\tilde{x}, \hat{t}) \geq (>)\ F(x, \hat{t})$ and $F(\hat{x}, \tilde{t}) \geq (>)\ F(x, \tilde{t})$, then $F(\tilde{x}, \hat{t}) \geq (>)\ F(\hat{x} \wedge \tilde{x}, \hat{t}) \Rightarrow F(\hat{x} \vee \tilde{x}, \tilde{t}) \geq (>)\ F(\hat{x}, \tilde{t})$.

The generalized Quah Strulovici infimum model, or ***GQS-1 model*** is a finitely indexed collection $(X_i, \preceq_i, F_i)_{i=1}^{I}$, where for each i, X_i is a nonempty, complete lattice (and using product order on products of X_i), and for each i, $F_i : X_i \times X_{-i} \to \mathbb{R}$ has interval dominance property in (x_i, x_{-i}). For each i, fix a subinterval S_i in X_i, and for every x_{-i}, let $\Phi_i(x_{-i}) = \arg\max_{\xi \in S_i} F_i(\xi, x_{-i})$ and suppose that $\underline{\Phi}_i(x_{-i}) := \inf_{\Phi_i(x_{-i})} \Phi_i(x_{-i}) \in \Phi_i(x_{-i})$. Let $\Phi : X \rightrightarrows X$ be given by $\Phi(x) = \times_{i=1}^{I} \Phi_i(x_{-i})$. The ***associated lattice model*** is $(X, \preceq, \Phi)$, where $X = \times_{i=1}^{I} X_i$, $\preceq$ is product order, and $\Phi = \times_{i=1}^{I} \Phi_i$ is the product correspondence as above. As a second version, the ***GQS-2 model*** is a GQS-1 model in which the property that infimum exists is replaced with the property that supremum exists, that is, for each i and x_{-i}, $\overline{\Phi}_i(x_{-i}) := \sup_{\Phi_i(x_{-i})} \Phi_i(x_{-i}) \in \Phi_i(x_{-i})$, and other aspects remain the same. A ***GQS model*** is one that is both GQS-1 and GQS-2. As above, if, instead, we add the assumption that for each i and x_{-i}, $F_i(\cdot, x_{-i})$ is upper semicontinuous in Frink (1942)'s order interval topology and quasisupermodular, then the set of maximizers has both a smallest and a largest element.

Example 10 (GCKK model)**.** For convenience, we summarize the models based on Che, Kim, and Kojima (2021) as follows. A generalized CKK-1 model, or ***GCKK-1 model***, is one that is either a CKK-1 model or a ICKK-1 model or a GQS-1 model. A generalized CKK-2 model, or ***GCKK-2 model***, is one that is either a CKK-2 model or a ICKK-2 model or a GQS-2 model. A generalized CKK model, or ***GCKK model***, is one that is either a CKK model or a ICKK model or a GQS model.

Example 11 (PY model)**.** Prokopovych and Yannelis (2017) propose a model with weaker conditions on payoff functions than those assumed in standard models in the case when decisions take values in compact chains. Following Prokopovych and Yannelis (2017), the ***PY-1 (respectively, PY-2) model*** is a finitely indexed collection $(X_i, \preceq_i, F_i)_{i=1}^I$, where for each i, $(X_i, \preceq_i)$ is a nonempty, complete chain (hence compact in the order interval topology), and using product order on products of X_i, for each i, $F_i : X_i \times X_{-i} \to \mathbb{R}$ is transfer weakly upper semicontinuous in x_i, and F_i satisfies the downward (respectively, upward) transfer single crossing property in (x_i, x_{-i}). We use the same definitions of these properties as Prokopovych and Yannelis (2017) and as they show, these conditions allow for new classes of games not covered by standard models or by Reny (1999). For each x_{-i}, let $\Phi_i(x_{-i}) = \arg\max_{\xi \in X_i} F_i(\xi, x_{-i})$. Let $\Phi : X \rightrightarrows X$ be given by $\Phi(x) = \times_{i=1}^I \Phi_i(x_{-i})$. The ***associated lattice model*** is $(X, \preceq, \Phi)$, where $X = \times_{i=1}^I X_i$, $\preceq$ is product order, and $\Phi = \times_{i=1}^I \Phi_i$ is the product correspondence as above. A ***PY model*** is one that is both PY-1 and PY-2. Prokopovych and Yannelis (2017) give several concrete applications including partnership game, Bertrand duopoly, and war of attrition.

Example 12 (Standard and neostandard models)**.** Finally, a ***standard model with complementarities***, or ***standard model***, is one that is either a Topkis model, or Vives model, or MR model, or GMS model. A ***neostandard model*** is on that is either GCKK or PY. A

neostandard-1 model is one that is either GCKK-1 or PY-1, and a ***neostandard-2 model*** is one that is either GCKK-2 or PY-2.

Theorem 1 shows that the patterns of decentralized interdependent behavior in all the different standard and neostandard models are unified in terms of the same isotone properties on the joint correspondence Φ.

Theorem 1. *Consider the class of standard and neostandard models with complementarities.*

1. *For every Topkis, Vives, MR, GMS, Zhou, GCKK, and PY model, the associated lattice model $(X, \preceq, \Phi)$ is isotone infimum and isotone supremum.*

2. *For every GCKK-1, for every PY-1 (respectively, GCKK-2, PY-2) model, the associated lattice model $(X, \preceq, \Phi)$ is isotone infimum (respectively, supremum).*

Proof. In statement (1), the result for Topkis model follows from Theorem 1.2 in Topkis (1979), for Vives model from Theorem 3.1 in Vives (1990), for MR and GMS model from Theorems 4 and A4 in Milgrom and Shannon (1994), for Zhou model, it follows immediately from the assumptions in the Zhou model, and for GCKK and for PY models, it follows from statement (2). To prove statement (2), consider a CKK-1 model $(X_i, \preceq_i, F_i)_{i=1}^I$ and its associated lattice model $(X, \preceq, \Phi)$. Fix i and sublattice S_i, suppose $\hat{x}_{-i} \preceq \tilde{x}_{-i}$, and let $a_i = \inf_{\Phi_i(\hat{x}_{-i})} \Phi_i(\hat{x}_{-i}) \in \Phi_i(\hat{x}_{-i})$ and $b_i = \inf_{\Phi_i(\tilde{x}_{-i})} \Phi_i(\tilde{x}_{-i}) \in \Phi_i(\tilde{x}_{-i})$. Theorem 2 in Che, Kim, and Kojima (2021) implies that $\Phi_i(\hat{x}_{-i}) \sqsubseteq^w \Phi_i(\tilde{x}_{-i})$, and therefore, there is $z \in \Phi_i(\hat{x}_{-i})$ with $z \preceq b_i$, from which it follows that $a_i \preceq z \preceq b_i$. As i is arbitrary and using the product partial order, it follows that $x \mapsto \underline{\Phi}(x)$ is isotone. The proof for ICKK-1 and GQS-1 model is similar, using Theorem 3 and Theorem S1 in Che, Kim, and Kojima (2021), respectively. The statements for CKK-2, ICKK-2 and GQS-2 models are proved similarly. For a PY-1 model $(X_i, \preceq_i, F_i)_{i=1}^I$, Theorem 1 in Prokopovych and Yannelis (2017) shows that $\forall i, \forall x_{-i}, \inf_{X_i} \Phi_i(x_{-i}) \in \Phi_i(x_{-i})$. Fix i and suppose $\hat{x}_{-i} \preceq \tilde{x}_{-i}$. Let $a_i = \inf_{X_i} \Phi_i(\hat{x}_{-i})$ and $b_i = \inf_{X_i} \Phi_i(\tilde{x}_{-i})$. Lemma 5 in Prokopovych and Yannelis (2017) implies that there is $z \in \Phi_i(\hat{x}_{-i})$ with $z \preceq b_i$, from which it follows that $a_i \preceq z \preceq b_i$. As i is arbitrary and using the product partial order, it follows that $x \mapsto \underline{\Phi}(x)$ is isotone. The proof for PY-2 model is similar. ∎

		$F2$			
		L	ML	MH	H
$F1$	L	$2,2$	$1,1$	$0,0$	$0,0$
	ML	$1,1$	$2,2$	$0,1$	$0,0$
	MH	$2,0$	$1,1$	$1,2$	$1,1$
	H	$1,0$	$2,0$	$2,1$	$2,2$

Table 1: Discrete Bertrand duopoly

An example shows that the general model includes isotone infimum and supremum models that cannot be realized from the standard models.

Example 13 (Discrete Bertrand duopoly)**.** Consider two restaurants competing as Bertrand duopolists. Each restaurant can choose to sell lower quality restaurant meal for lower prices or higher quality restaurant meal for higher prices. There are four quality-price categories ranked from low (L) to medium-low (ML) to medium-high (MH) to high (H), with $L \prec ML \prec MH \prec H$. Suppose market conditions are as follows. If firm 2 prices low, it is in firm 1's best interest to price either low or medium-high (that is, at the lower end of low quality/price segment or lower end of high quality/price segment). It can survive if it prices medium-low or high but at a lower profit. Similarly, if firm 2 prices medium-low, it is in firm 1's best interest to price either medium-low or high (that is, at the higher end of low quality/price segment or higher end of high quality/price segment). It can survive if it prices low or medium-high but at a lower profit. If firm 2 prices medium-high or high, it is in firm 1's best interest to price at the high end. It can survive at lower profit if it prices medium-high but cannot survive otherwise. For firm 2, the incentives are straightforward, that is, to copy firm 1 action. An assignment of payoffs with these features is given in the bimatrix in Table 1.

It is easy to see that this Example 13 does not fit any of the standard models, because each of these models would require best response of each firm to be isotone in the strong set order, but the best response of firm 1, denoted Φ_1, violates this property, because $\Phi_1(L) \not\sqsubseteq^s \Phi_1(ML)$. On the other hand, it is easy to check that the associated lattice model is both isotone infimum and isotone supremum. Similarly, Feng and Sabarwal (2020) show a limitation of applying the standard model to subgames in two stage dynamic games. Three additional examples that violate standard assumptions but fit in our framework are the following.

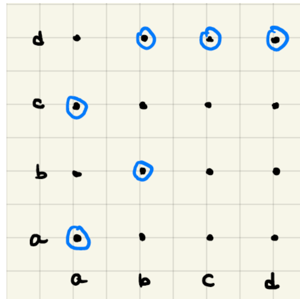

Figure 2: Not isotone in strong set order

Example 14 (Not isotone in strong set order)**.** Consider the correspondence Φ on a chain $X = \{a, b, c, d\}$ with $a \prec b \prec c \prec d$ given by $\Phi(a) = \{a, c\}, \Phi(b) = \{b, d\}, \Phi(c) = \Phi(d) = \{d\}$, as shown in Figure 2. In the corresponding lattice model $(X, \preceq, \Phi)$, Φ is not isotone in the strong set order because $\Phi(a) \not\sqsubseteq^s \Phi(b)$ as $b \wedge c \notin \Phi(a)$ and also $b \vee c \notin \Phi(b)$. It is easy to check that the model is isotone infimum and isotone supremum. This gives one of the simplest examples in which the correspondence looks very isotone but is not isotone in the strong set order.

Example 15 (Not isotone in weak set order)**.** Let $X = \{1, 2, 3, 4\}$ with the natural order and $\Phi : X \rightrightarrows X$ be given by $\Phi(1) = \{2, 4\}$, $\Phi(2) = \{1, 3\}$, $\Phi(3) = \{3\}$, and $\Phi(4) = \{1, 4\}$. Then Φ is not isotone in the weak set order, because $4 \in \Phi(1)$, but there is no $y \in \Phi(2)$ such that $4 \preceq y$, and also $1 \in \Phi(4)$, but there is no $y \in \Phi(3)$ such that $y \preceq 1$. Nevertheless, the selection $f(1) = 2, f(2) = 3, f(3) = 3, f(4) = 4$ is isotone and the model is a general model with complementarities. This example shows that our framework allows for cases not covered by Smithson (1971). See Sabarwal (2023b) for a more detailed discussion.

Example 16 (Not subcomplete valued)**.** Consider the constant correspondence: $\Phi(x) = [3, 4) \cup \{5\}$, for $x \in [0, 10]$, as shown in Figure 3. For every x, $\Phi(x)$ is a nonempty, complete sublattice but $\Phi(x)$ is not subcomplete. The corresponding lattice model $(X, \preceq, \Phi)$ with $X = [0, 10] \subset \mathbb{R}$ with the natural order model does not fit the Zhou model which requires subcompleteness. Moreover, it does not fit other standard models such as GMS, Vives, or Topkis, because those models imply that the correspondence is compact valued. On the other hand, it is easy to see that the model is isotone infimum and isotone supremum.

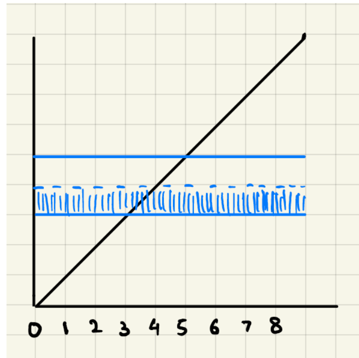

Figure 3: Model without subcompleteness

Theorem 2 shows that an equilibrium always exists in general models with complementarities, a smallest equilibrium always exists in isotone infimum models (and it is the same as the smallest equilibrium of the model with only the infimum selection from Φ), and a largest equilibrium always exists in isotone supremum models (and it is the same as the largest equilibrium of the model with only the supremum selection from Φ).

Theorem 2. *Consider the class of lattice models.*

1. *Every general model with complementarities has an equilibrium.*

2. *In every isotone supremum model* $(X, \preceq, \Phi)$*, the equilibrium set* $\mathcal{E}(\Phi)$ *contains a nonempty, complete lattice* $\mathcal{E}(\overline{\Phi})$ *such that* $\sup_{\mathcal{E}(\overline{\Phi})} \mathcal{E}(\overline{\Phi}) = \sup_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$*. In particular, every isotone supremum model has a largest equilibrium.*

3. *In every isotone infimum model* $(X, \preceq, \Phi)$*, the equilibrium set* $\mathcal{E}(\Phi)$ *contains a nonempty, complete lattice* $\mathcal{E}(\underline{\Phi})$ *such that* $\inf_{\mathcal{E}(\underline{\Phi})} \mathcal{E}(\underline{\Phi}) = \inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$*. In particular, every isotone infimum model has a smallest equilibrium.*

4. *Every general model that is isotone infimum and isotone supremum has a smallest and a largest equilibrium.*

Proof. Statement (1) follows immediately by applying Tarski (1955) to the isotone selection f from Φ to show that $\mathcal{E}(f)$ is a nonempty complete lattice, and noting that $\mathcal{E}(f) \subseteq \mathcal{E}(\Phi)$. It is included here for cases (like Example 15) that are outside the scope of statements (2) and (3). To prove statement (2), let $(X, \preceq, \Phi)$ be an isotone supremum model. Statement (1) implies that $\mathcal{E}(\overline{\Phi})$ is a nonempty, complete lattice in $\mathcal{E}(\Phi)$. Let $e^* = \sup_X A$, where $A = \{x \in X \mid x \preceq \overline{\Phi}(x)\}$. We know from Tarski's theorem that $e^* = \sup_{\mathcal{E}(\overline{\Phi})} \mathcal{E}(\overline{\Phi}) \in \mathcal{E}(\overline{\Phi})$. As $\mathcal{E}(\overline{\Phi}) \subseteq \mathcal{E}(\Phi)$, it follows that $e^* \in \mathcal{E}(\Phi)$. Let $e \in \mathcal{E}(\Phi)$ be an arbitrary equilibrium in the model $(X, \preceq, \Phi)$. Then $e \preceq \overline{\Phi}(e)$, and therefore, $e \in A$, whence $e \preceq e^*$. This shows that $e^* = \sup_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi) = \sup_{\mathcal{E}(\overline{\Phi})} \mathcal{E}(\overline{\Phi})$. In other words, e^* is also the largest equilibrium in the general model $(X, \preceq, \Phi)$. Statement (3) is proved similarly. Statement (4) follows from statements (2) and (3). ∎

Corollary 3. *Consider the class of standard and neostandard models.*

1. *Every Topkis, Vives, MR, GMS, Zhou, GCKK, and PY model has a smallest and a largest equilibrium.*

2. *Every GCKK-1, PY-1 (respectively, GCKK-2, PY-2) model has a smallest (respectively, largest) equilibrium.*

Proof. Follows from Theorem 1 and Theorem 2. ∎

The theorem and corollary provide unified results for existence of extremal equilibria in standard and neostandard models with complementarities. For reference, existence of extremal equilibria in the Topkis model is shown in Theorem 3.1 in Topkis (1979), in the Vives model in Theorem 4.2 in Vives (1990), and in the Zhou model it follows from Theorem 1 in Zhou (1994). Milgrom and Shannon (1994) prove existence of extremal equilibria using the additional assumption that for every i, F_i is order continuous on X_{-i}.

Example 17 (Standard S-model, continued)**.** Figure 1 shows that $\mathcal{E}(\Phi) = [a, b] \cup [c, d] \cup [e, f]$, $\mathcal{E}(\underline{\Phi}) = \{a, d, e\}$, and $\mathcal{E}(\overline{\Phi}) = \{b, c, f\}$. Consistent with Theorem 2, $\inf_{\mathcal{E}(\underline{\Phi})} \mathcal{E}(\underline{\Phi}) = a = \inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$ and $\sup_{\mathcal{E}(\overline{\Phi})} \mathcal{E}(\overline{\Phi}) = f = \sup_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$.

Example 18 (Not isotone in strong set order, continued)**.** Figure 2 shows that $\mathcal{E}(\Phi) = \{a, b, d\}$, $\mathcal{E}(\underline{\Phi}) = \{a, b, d\}$, and $\mathcal{E}(\overline{\Phi}) = \{d\}$. Consistent with Theorem 2, $\inf_{\mathcal{E}(\underline{\Phi})} \mathcal{E}(\underline{\Phi}) = a = \inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$ and $\sup_{\mathcal{E}(\overline{\Phi})} \mathcal{E}(\overline{\Phi}) = d = \sup_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$.

Example 19 (Not isotone in weak set order, continued)**.** The model in Example 15 has an isotone selection and is a general model with complementarities (but it is neither isotone infimum nor isotone supremum). The equilibrium set is $\mathcal{E}(\Phi) = \{3, 4\}$.

Example 20 (Not subcomplete valued, continued)**.** Figure 3 shows that $\mathcal{E}(\Phi) = [3, 4) \cup \{5\}$, $\mathcal{E}(\underline{\Phi}) = \{3\}$, and $\mathcal{E}(\overline{\Phi}) = \{5\}$. Therefore, $\inf_{\mathcal{E}(\underline{\Phi})} \mathcal{E}(\underline{\Phi}) = 3 = \inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$ and $\sup_{\mathcal{E}(\overline{\Phi})} \mathcal{E}(\overline{\Phi}) = 5 = \sup_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$, consistent with Theorem 2.

Theorem 2 guarantees that the equilibrium set *contains* a nonempty complete lattice. Theorem 4 strengthens this by presenting conditions that guarantee that *the equilibrium set is a nonempty complete lattice*. We provide two sets of conditions. Both sets hold in the standard models due to Vives (1990) and Zhou (1994), and both are strictly weaker, allowing for cases not included in those models.

In a lattice model $(X, \preceq, \Phi)$, for each nonempty $\hat{X} \subseteq X$, the ***model restricted to*** $\hat{X}$ is $(\hat{X}, \hat{\preceq}, \hat{\Phi})$, where $\hat{\preceq}$ is the restriction of the partial order $\preceq$ to $\hat{X}$ and $\hat{\Phi}$ is the restriction of Φ to $\hat{X}$ given by $\hat{\Phi}(x) = \Phi(x) \cap \hat{X}$. When convenient, the same notation $\preceq$ is used for the restricted partial order $\hat{\preceq}$. A lattice model $(X, \preceq, \Phi)$ is ***isotone supremum on lower intervals*** if it is an isotone supremum model in which $\forall \hat{x} \in X$ such that $\underline{\Phi}(\hat{x}) \preceq \hat{x}$, the model restricted to $\hat{X} = [\underline{x}, \hat{x}]$ is an isotone supremum model. It is ***isotone infimum on upper intervals*** if it is an isotone infimum model in which $\forall \hat{x} \in X$ such that $\hat{x} \preceq \overline{\Phi}(\hat{x})$, the model restricted to $\hat{X} = [\hat{x}, \overline{x}]$ is an isotone infimum model. As earlier, these definitions do not use any continuity properties, strong set order, subcompleteness, or uniform set order.

Theorem 4. *Consider the class of lattice models.*

1. *In every isotone supremum model* $(X, \preceq, \Phi)$ *that is isotone infimum on upper intervals,* $\mathcal{E}(\Phi)$ *is a nonempty complete lattice.*

2. *In every isotone infimum model* $(X, \preceq, \Phi)$ *that is isotone supremum on lower intervals,* $\mathcal{E}(\Phi)$ *is a nonempty complete lattice.*

Proof. To prove statement (1), let $(X, \preceq, \Phi)$ be an isotone supremum model that is isotone infimum on upper intervals. Statement (3) of Theorem 2 shows that $\mathcal{E}(\Phi)$ is nonempty and $\inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi) \in \mathcal{E}(\Phi)$. To show that $\mathcal{E}(\Phi)$ is sup-complete, let $E \subseteq \mathcal{E}(\Phi)$ be nonempty. Let

$\overline{e} = \sup_X E \in X$, which exists because X is complete. Then $e \in E$ implies $e \preceq \overline{\Phi}(e) \preceq \overline{\Phi}(\overline{e})$, where the second inequality follows because $\overline{\Phi}$ is isotone, and therefore, $\overline{\Phi}(\overline{e})$ is an upper bound for E, whence $\overline{e} \preceq \overline{\Phi}(\overline{e})$. By assumption, the restricted model $([\overline{e}, \overline{x}], \preceq, \hat{\Phi})$ is an isotone infimum model and therefore, by statement (3) of Theorem 2, the restricted model has a smallest equilibrium, say, e^*. As $e^* \in \hat{\Phi}(e^*) = \Phi(e^*) \cap [\overline{e}, \overline{x}] \subset \Phi(e^*)$, it follows that e^* is an equilibrium in $(X, \preceq, \Phi)$. As $\overline{e} \preceq e^*$, it follows that e^* is an upper bound for E in $\mathcal{E}(\Phi)$. Let $e \in \mathcal{E}(\Phi)$ be an arbitrary upper bound for E. Then $e \in [\overline{e}, \overline{x}]$ and therefore, $e \in \Phi(e) \cap [\overline{e}, \overline{x}] = \hat{\Phi}(e)$. As e^* is the smallest equilibrium for $\hat{\Phi}$, it follows that $e^* \preceq e$. Thus $e^* = \sup_{\mathcal{E}(\Phi)} E \in \mathcal{E}(\Phi)$.

To show that $\mathcal{E}(\Phi)$ is inf-complete, let $E \subseteq \mathcal{E}(\Phi)$ be nonempty. Let $A = \{x \in \mathcal{E}(\Phi) \mid \forall e \in E, x \preceq e\}$. The set A is nonempty, because $\inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi) \in A$. Let $e^* = \sup_{\mathcal{E}(\Phi)} A \in \mathcal{E}(\Phi)$, which exists because $\mathcal{E}(\Phi)$ is sup-complete. Then $\forall e \in E$ and $\forall x \in A$, $x \preceq e$, and therefore, $\forall e \in E$, e is an upper bound for A, whence $\forall e \in E$, $e^* \preceq e$, showing that e^* is a lower bound for E. Let $x \in \mathcal{E}(\Phi)$ be an arbitrary lower bound for E. Then $x \in A$ and consequently, $x \preceq e^*$. This shows that $e^* = \inf_{\mathcal{E}(\Phi)} E \in \mathcal{E}(\Phi)$. Statement (2) is proved similarly. ∎

Theorem 4 provides a unified result generalizing existing results that guarantee the equilibrium set is a nonempty complete lattice. The proof is more general than the different approaches in the proofs in Zhou (1994) and Vives (1990).

The sufficient conditions in Theorem 4 are weaker than those in Zhou (1994), which requires a correspondence that is isotone in the strong set order and subcomplete valued. It is easy to check that Example 14 (Not isotone in strong set order) depicted in Figure 2 is an isotone supremum model that is isotone infimum on upper intervals. The correspondence in that example is not isotone in the strong set order. Similarly, it is easy to check that Example 16 (Not subcomplete valued) depicted in Figure 3 is an isotone supremum model that is isotone infimum on upper intervals. The correspondence in that example is not subcomplete valued.

The conditions in Theorem 4 are weaker than those in Vives (1990), which requires a correspondence that is isotone in the uniform set order. The correspondences in Example 14 (Not isotone in strong set order) and Example 16 (Not subcomplete valued) are not isotone in uniform set order. In the special case of singleton valued correspondences, both conditions collapse to an isotone function, recovering the theorem due to Tarski (1955).

Theorem 5. *Consider the class of standard and neostandard models.*

1. *For every GMS, GCKK, and PY model, the associated lattice model is isotone infimum on upper intervals and isotone supremum on lower intervals.*
2. *Every Zhou model is isotone infimum on upper intervals and isotone supremum on lower intervals.*
3. *In every Topkis, Vives, MR, GMS, Zhou, GCKK, and PY model, the equilibrium set $\mathcal{E}(\Phi)$ is a nonempty complete lattice.*

Proof. For statement (1), consider a GMS model $(X_i, \preceq_i, F_i)_{i=1}^I$ and its associated lattice model $(X, \preceq, \Phi)$. Theorem 1 shows that $(X, \preceq, \Phi)$ is isotone infimum and isotone supremum. Suppose $\hat{x} \preceq \overline{\Phi}(\hat{x})$ and consider the restricted model $(\hat{X}, \preceq, \hat{\Phi})$. Then $\forall i$, $\hat{x}_i \preceq_i \overline{\Phi}_i(\hat{x}_{-i})$. Now, $\forall i$ and $\forall x_{-i} \in [\hat{x}_{-i}, \overline{x}_{-i}]$, let $\Psi_i(x_{-i}) = \arg\max_{x_i \in [\hat{x}_i, \overline{x}_i]} F_i(x_i, x_{-i})$, and $\forall x \in \hat{X}$, let $\Psi(x) = \times_{i=1}^I \Psi_i(x_{-i})$. From Theorem 1, it follows that $x \mapsto \Psi(x)$ has an isotone infimum selection and an isotone supremum selection. We show that $\forall x \in \hat{X}$, $\Psi(x) = \hat{\Phi}(x)$. Fix i and $x_{-i} \in [\hat{x}_{-i}, \overline{x}_{-i}]$ arbitrarily. Suppose $\xi \in \hat{\Phi}_i(x_{-i})$. Then $\xi \in [\hat{x}_i, \overline{x}_i]$ and $\forall x_i \in [\hat{x}_i, \overline{x}_i]$, $F_i(\xi, x_{-i}) \geq F_i(x_i, x_{-i})$, whence $\xi \in \Psi_i(x_{-i})$. Suppose $\xi \in \Psi_i(x_{-i})$. Then $\hat{x}_i \preceq_i \overline{\Phi}_i(\hat{x}_{-i}) \preceq_i \overline{\Phi}_i(x_{-i})$, where the second inequality follows from isotone supremum. Consequently, $\forall x_i \in X_i$, $F_i(\xi, x_{-i}) \geq F_i(\overline{\Phi}_i(x_{-i}), x_{-i}) \geq F_i(x_i, x_{-i})$, where the first inequality follows from ξ is a maximizer on $[\hat{x}_i, \overline{x}_i]$ and the second from $\overline{\Phi}_i(x_{-i})$ is a maximizer on X_i. Therefore, $\xi \in \hat{\Phi}_i(x_{-i})$. It follows that $\Psi_i(x_{-i}) = \hat{\Phi}_i(x_{-i})$, whence $\Psi(x) = \hat{\Phi}(x)$. It follows that $(X, \preceq, \Phi)$ is isotone infimum on upper intervals. Similarly, it is isotone supremum on lower intervals. The statement for every GCKK model and every PY model is proved similarly.

For statement (2), consider a Zhou model $(X, \preceq, \Phi)$. Theorem 1 shows that it is an isotone infimum model. To show that it is isotone infimum on upper intervals, consider $\hat{x} \in X$ such that $\hat{x} \preceq \overline{\Phi}(\hat{x})$ and consider the restricted model $(\hat{X}, \preceq, \hat{\Phi})$ where $\hat{X} = [\hat{x}, \overline{x}]$ and $\hat{\Phi}(x) = \Phi(x) \cap [\hat{x}, \overline{x}]$. Then $\hat{x} \preceq \overline{\Phi}(\hat{x})$ implies that for every $x \in \hat{X}$, $\hat{x} \preceq \overline{\Phi}(\hat{x}) \preceq \overline{\Phi}(x) \preceq \overline{x}$, where the second inequality follows from Φ is isotone in strong set order. This shows that $\hat{\Phi}$ is nonempty valued. Moreover, $\hat{\Phi}(x)$ is subcomplete, because $\Phi(x)$ and $[\hat{x}, \overline{x}]$ are both subcomplete, and therefore, for every $x \in \hat{X}$, $\underline{\hat{\Phi}}(x) := \inf_{\hat{\Phi}(x)} \hat{\Phi}(x) \in \hat{\Phi}(x)$. Furthermore, $\hat{\Phi}$ is isotone in strong set order, because

Φ is isotone in the strong set order and the correspondence constant at $[\hat{x}, \overline{x}]$ is also isotone in the strong set order. Therefore, $x \mapsto \underline{\hat{\Phi}}(x)$ is isotone. This shows that the restricted model $(\hat{X}, \preceq, \hat{\Phi})$ is an isotone infimum model. The proof that $(X, \preceq, \Phi)$ is an isotone supremum model on lower intervals is similar.

Statement (3) follows from Theorem 1, Theorem 4, and statements (1) and (2) here. ∎

Theorem 5 shows that the foundational models due to Topkis, Vives, Milgrom and Roberts, Shannon, Milgrom and Shannon, and Zhou are all subsumed as special cases of Theorem 4. This is important because widely used results in the literature (for example, see Milgrom and Shannon (1994), Appendix, page 179) imply that the set of maximizers is a complete sublattice, and therefore, has a greatest and least element. But this does not necessarily imply that it is subcomplete, which is a stricter condition, as can also be seen in Example 16 (Not subcomplete valued). Subcompleteness is a requirement in Zhou (1994) to prove that the equilibrium set is a complete lattice. This leaves a gap between individual maximization behavior and structure of the equilibrium set in standard models with complementarities. Our results plug this gap.

3 Equilibrium set comparisons

A long-standing problem in the theory of complementarities is lack of structural comparability of equilibrium sets. Comparability in the strong set order does not obtain even in standard examples (see below) and comparability in the weak set order does not provide tight bounds for subsets of equilibria.

Example 21 (Noncomparability of equilibrium sets in strong set order in standard S-model)**.** Consider the canonical model $(X, \preceq, \Phi)$ in Example 1 (Standard S-model) depicted in Figure 1. Here, $X = [0, \overline{x}] \subset \mathbb{R}$ and for every $x \in X$, $\Phi(x)$ is the interval given by $\Phi(x) = [\underline{\Phi}(x), \overline{\Phi}(x)]$. The domain X is a subcomplete chain in $\mathbb{R}$, the correspondence Φ is nonempty valued, subcomplete valued, and isotone in the strong set order. The equilibrium set is $\mathcal{E}(\Phi) = [a, b] \cup [c, d] \cup [e, f]$. This shows that the model has uncountably many equilibria, the equilibrium set is a chain, and it is subcomplete in X. The infimum selection $\underline{\Phi}(x)$ is isotone and the equilibrium set corresponding to the infimum selection is $\mathcal{E}(\underline{\Phi}) = \{a, d, e\}$. The supremum selection $\overline{\Phi}(x)$ is isotone and the equilibrium set corresponding to it is $\mathcal{E}(\overline{\Phi}) = \{b, c, f\}$. Each of $\mathcal{E}(\underline{\Phi})$ and $\mathcal{E}(\overline{\Phi})$ is a finite chain and is subcomplete in $\mathbb{R}$.

The three equilibrium sets $\mathcal{E}(\underline{\Phi})$, $\mathcal{E}(\Phi)$, and $\mathcal{E}(\overline{\Phi})$ have a lot of structure. Each is nonempty, a chain, and subcomplete in $\mathbb{R}$. Moreover, $\mathcal{E}(\underline{\Phi})$ and $\mathcal{E}(\overline{\Phi})$ are finite sets and $\mathcal{E}(\Phi)$ is a finite union of compact, convex intervals.

Given the order structure of the equilibrium sets, the structure and isotonicity of Φ, $\underline{\Phi}$, and $\overline{\Phi}$, and the natural ranking of their images in the strong set order, $\underline{\Phi}(x) \sqsubseteq^s \Phi(x) \sqsubseteq^s \overline{\Phi}(x)$, for every $x \in X$, it may be expected that the corresponding equilibrium sets $\mathcal{E}(\underline{\Phi})$, $\mathcal{E}(\Phi)$, and $\mathcal{E}(\overline{\Phi})$ are ranked similarly in the strong set order as well.

In fact, no two distinct equilibrium sets are comparable in the strong set order: $\mathcal{E}(\underline{\Phi}) \not\sqsubseteq^s \mathcal{E}(\Phi)$, because $\inf_X\{d, c\} \notin \mathcal{E}(\underline{\Phi})$; $\mathcal{E}(\Phi) \not\sqsubseteq^s \mathcal{E}(\overline{\Phi})$, because $\sup_X\{d, c\} \notin \mathcal{E}(\overline{\Phi})$; and $\mathcal{E}(\underline{\Phi}) \not\sqsubseteq^s \mathcal{E}(\overline{\Phi})$, because $\inf_X\{d, c\} \notin \mathcal{E}(\underline{\Phi})$ and also, $\sup_X\{d, c\} \notin \mathcal{E}(\overline{\Phi})$. The other pairings are less interesting but also incomparable in the strong set order: $\mathcal{E}(\Phi) \not\sqsubseteq^s \mathcal{E}(\underline{\Phi})$, because $\sup_X\{e, f\} \notin \mathcal{E}(\underline{\Phi})$; $\mathcal{E}(\overline{\Phi}) \not\sqsubseteq^s \mathcal{E}(\Phi)$, because $\inf_X\{a, b\} \notin \mathcal{E}(\overline{\Phi})$; and $\mathcal{E}(\overline{\Phi}) \not\sqsubseteq^s \mathcal{E}(\underline{\Phi})$, because $\inf_X\{a, b\} \notin \mathcal{E}(\overline{\Phi})$ and also, $\sup_X\{e, f\} \notin \mathcal{E}(\underline{\Phi})$.

An insight here is that the equilibrium sets are comparable in the following sense. Consider $d \in \mathcal{E}(\underline{\Phi})$ and $y \in [c, d) \subset \mathcal{E}(\Phi)$. Then $a \in \mathcal{E}(\underline{\Phi})$ is lower than both d and y, and *among elements of* $\mathcal{E}(\underline{\Phi})$, a is the largest of the lower bounds for both d and y. More generally, for every $x \in \mathcal{E}(\underline{\Phi})$ and $y \in \mathcal{E}(\Phi)$, there is $\underline{e} \in \mathcal{E}(\underline{\Phi})$ such that $\underline{e}$ is lower than x and y, and for every $e \in \mathcal{E}(\underline{\Phi})$ that is lower than x and y, $e \preceq \underline{e}$. Similarly, for $y \in (c, d] \subset \mathcal{E}(\Phi)$ and $c \in \mathcal{E}(\overline{\Phi})$, $f \in \mathcal{E}(\overline{\Phi})$ is higher than both y and c, and among elements of $\mathcal{E}(\overline{\Phi})$, f is the smallest of the upper bounds for both y and c. More generally, for every $x \in \mathcal{E}(\Phi)$ and $y \in \mathcal{E}(\overline{\Phi})$, there is $\overline{e} \in \mathcal{E}(\overline{\Phi})$ such that $\overline{e}$ is higher than x and y, and for every $e \in \mathcal{E}(\overline{\Phi})$ that is higher than x and y, $\overline{e} \preceq e$.

We show that this insight holds much more generally. It does not require that X is a subset of the reals, or an interval, or finite dimensional, or convex, or even a vector space. It does not require that any of the equilibrium sets is a chain or is subcomplete in X. It does not require that the correspondence satisfy continuity properties such as upper or lower hemicontinuity. We show that this insight holds in every model that is isotone infimum and/or isotone supremum.

In fact, we show that another strong result holds in such models. For every nonempty $E \subseteq \mathcal{E}(\Phi)$, $\inf_{\mathcal{E}(\underline{\Phi})} E \in \mathcal{E}(\underline{\Phi})$ and $\sup_{\mathcal{E}(\overline{\Phi})} E \in \mathcal{E}(\overline{\Phi})$, where $\inf_{\mathcal{E}(\underline{\Phi})} E$ and $\sup_{\mathcal{E}(\overline{\Phi})} E$ are defined below in a suitable manner generalizing the ideas above. This shows that every subset of equilibria in a model with complementarities has a largest lower bound among equilibria of the corresponding infimum model and smallest upper bound among equilibria of the corresponding supremum model.

In order to develop a formal language to compare equilibrium sets, we define the following new concepts to formalize order bounds of one set using a different set. The definitions are stated for arbitrary posets.

For nonempty subsets E and A of poset X, the ***sup of*** E ***in*** A, denoted $\sup_A E$, is an element $\overline{e} \in A$ such that (1) $\overline{e}$ is an upper bound for E and (2) for every $a \in A$ that is an upper bound for E, $\overline{e} \preceq a$. The ***inf of*** E ***in*** A, denoted $\inf_A E$, is an element $\underline{e} \in A$ such that (1) $\underline{e}$ is a lower bound for E and (2) for every $a \in A$ that is a lower bound for E, $a \preceq \underline{e}$. Notice that $A = X$ gives the standard definition, as stated above, and $E \subseteq A \subseteq X$ gives the standard definition in the relative partial order. More generally, as E and A are arbitrary nonempty subsets of X, $\sup_A E$ and $\inf_A E$ might not exist in general even if X is a complete lattice. When they exist, they have some natural properties, as follows.

Theorem 6. *Let X be a poset, $E \subseteq X$ be nonempty, and $A \subseteq B \subseteq X$ with A nonempty.*

1. $\inf_A E \preceq \inf_B E \preceq \sup_B E \preceq \sup_A E$, *whenever these exist.*
2. $\inf_A E = \inf_B E \iff \inf_B E \in A$
3. $\sup_B E = \sup_A E \iff \sup_B E \in A$

Proof. For statement (1), suppose $\inf_A E$ and $\inf_B E$ exist. Then $\inf_A E \in A \subseteq B$ implies that $\inf_A E \in B$ and $\inf_A E$ is a lower bound for E. As $\inf_B E$ is the largest of such bounds in B, it follows that $\inf_A E \preceq \inf_B E$. Now suppose $\inf_B E$ and $\sup_B E$ exist. Let $b \in B$. Then $\inf_B E \preceq b \preceq \sup_B E$. Finally, suppose $\sup_B E$ and $\sup_A E$ exist. Then $\sup_A E \in A \subseteq B$ implies that $\sup_A E \in B$ and $\sup_A E$ is an upper bound for E. As $\sup_B E$ is the smallest of such bounds, it follows that $\sup_B E \preceq \sup_A E$.

For statement (2), if $\inf_A E = \inf_B E$, then $\inf_A E \in A$ implies $\inf_B E \in A$. If $\inf_B E \in A$, then combined with $\inf_B E$ is a lower bound for E and $\inf_A E$ is the largest of such lower bounds, it follows that $\inf_B E \preceq \inf_A E$. Combined with statement (1), it follows that $\inf_A E = \inf_B E$. Statement (3) is proved similarly. ∎

These order bounds are used to define new set orders useful for equilibrium set comparisons, as follows.

For nonempty subsets A, B of poset X, A is ***sup-complete in*** B, if for every nonempty $E \subseteq A$, $\sup_B E \in B$, and B is ***inf-complete in*** A, if for every nonempty $E \subseteq B$, $\inf_A E \in A$. Set A ***is lower than*** B ***in the star complete set order***, denoted $A \sqsubseteq^{*c} B$, if A is sup-complete in B and B is inf-complete in A.

For nonempty subsets A, B in a poset X, A ***is join-complete in*** B, if for every $x \in A$ and $y \in B$, $\sup_B\{x, y\} \in B$. Similarly, B ***is meet-complete in*** A, if for every $x \in A$ and $y \in B$, $\inf_A\{x, y\} \in A$. Set A ***is lower than*** B ***in the star lattice set order***, denoted $A \sqsubseteq^{*\ell} B$, if A is join-complete in B and B is meet-complete in A.

Some properties of the star complete set order and star lattice set order and their relation to strong set order and weak set order are as follows.

Theorem 7. *Let X be a poset and A, B, C be nonempty subsets of X.*

1. *Star complete set order*
   (a) $A \sqsubseteq^{*c} A \iff A$ *is a complete lattice (in the relative partial order from X)*
   (b) $A \sqsubseteq^{*c} B \implies \inf_A A \preceq \inf_B B$ *and* $\sup_A A \preceq \sup_B B$*, whenever these exist*
   (c) $A \sqsubseteq^{*c} B \implies A \sqsubseteq^{w} B$.
2. *Star lattice set order*
   (a) $A \sqsubseteq^{*\ell} A \iff A$ *is a lattice (in the relative partial order from X)*
   (b) $A \sqsubseteq^{*\ell} B \implies \inf_A A \preceq \inf_B B$ *and* $\sup_A A \preceq \sup_B B$*, whenever these exist*
   (c) $A \sqsubseteq^{*\ell} B \implies A \sqsubseteq^{w} B$.
3. *Cross comparisons*

(a) *If B is inf-complete in A and A is a lattice, then B is meet-complete in A.*

(b) *If A is sup-complete in B and B is a lattice, then A is join-complete in B.*

(c) *If A and B are lattices, then $A \sqsubseteq^{*c} B \implies A \sqsubseteq^{*\ell} B \implies A \sqsubseteq^{w} B$.*

(d) *If X is a lattice, then $A \sqsubseteq^{s} B \implies A \sqsubseteq^{*\ell} B \implies A \sqsubseteq^{w} B$.*

Proof. For (1)(a), if $A \sqsubseteq^{*c} A$, then for every nonempty $E \subseteq A$, $\inf_A E \in A$ and $\sup_A E \in A$, showing that A is a complete lattice. If A is a complete lattice, the reverse argument shows that $A \sqsubseteq^{*c} A$. For (1)(b), suppose $A \sqsubseteq^{*c} B$ and suppose $\underline{a} = \inf_A A$ and $\underline{b} = \inf_B B$ exist. Then $A \sqsubseteq^{*c} B$ implies $a' = \inf_A\{\underline{b}\} \in A$, whence $\underline{a} \preceq a' \preceq \underline{b}$. Similarly, $\sup_A A \preceq \sup_B B$. For (1)(c), suppose $A \sqsubseteq^{*c} B$ and consider arbitrary $a \in A$. Let $b' = \sup_B\{a\} \in B$. Then $a \preceq b'$. Similarly, for $b \in B$ there is $a' = \inf_A\{b\} \in A$ such that $a' \preceq b$.

For (2)(a), if $A \sqsubseteq^{*\ell} A$, then for every $x, y \in A$, $\inf_A\{x, y\} \in A$ and $\sup_A\{x, y\} \in A$, showing that A is a lattice. If A is a lattice, the reverse argument shows that $A \sqsubseteq^{*\ell} A$. For (2)(b) suppose $A \sqsubseteq^{*\ell} B$ and suppose $\inf_A A$ and $\inf_B B$ exist. By definition, $\inf_A A \in A$ and $\inf_B B \in B$, and therefore, $z = \inf_A\{\inf_A A, \inf_B B\} \in A$. Combined with $\inf_A A \preceq z$, it follows that $\inf_A A = z \preceq \inf_B B$, where the inequality follows from z is a lower bound for $\{\inf_A A, \inf_B B\}$. Similarly, $\sup_A A \preceq \sup_B B$ whenever these exist. For (2)(c), suppose $A \sqsubseteq^{*\ell} B$ and consider $x \in A$. As B is nonempty, let $y \in B$, and by hypothesis, $\sup_B\{x, y\} \in B$. As $\sup_B\{x, y\}$ is an upper bound for $\{x, y\}$, it follows that $x \preceq \sup_B\{x, y\}$. Similarly, for $y \in B$, let x be any element of A. Then $\inf_A\{x, y\} \in A$ is such that $\inf_A\{x, y\} \preceq y$.

For (3)(a), suppose B is inf-complete in A and A is a lattice. Let $x \in A, y \in B$. Let $E = \{y\} \subset B$ and let $\hat{y} = \inf_A E \in A$, which exists because B is inf-complete in A. Then $\hat{y}$ is a lower bound for E and for every $z \in A$ that is a lower bound for E, $z \preceq \hat{y}$. In other words, $\hat{y} \preceq y$ and for every $z \in A$ such that $z \preceq y$, it must be that $z \preceq \hat{y}$. As A is a lattice, let $\hat{a} = \inf_A\{x, \hat{y}\} \in A$. Then $\hat{y} \preceq y$ implies that $\hat{a}$ is a lower bound for $\{x, y\}$. Suppose $z \in A$ is a lower bound for $\{x, y\}$. Then $z \in A$ and $z \preceq y$ implies $z \preceq \hat{y}$ and therefore, z is a lower bound for $\{x, \hat{y}\}$, whence $z \preceq \hat{a}$. This shows that $\inf_A\{x, y\} = \hat{a} \in A$. Statement (3)(b) is proved similarly and (3)(c) follows from (3)(a), (3)(b) and (2)(c).

For (3)(d), suppose X is a lattice and $A \sqsubseteq^{s} B$. Consider $x \in A$ and $y \in B$. By hypothesis, $\inf_X\{x, y\} \in A$ and therefore, by statement (2) of Theorem 6, $\inf_A\{x, y\} = \inf_X\{x, y\} \in A$.

Similarly, $\sup_B\{x,y\} = \sup_X\{x,y\} \in B$. This shows that $A \sqsubseteq^{*\ell} B$. The implication $A \sqsubseteq^w B$ is a special case of (2)(c). ▮

As shown in Theorem 7, the star complete set order and the star lattice set order are both stronger than the weak set order, and imply isotone infimum and supremum when these exist. When comparing lattices A and B, the star complete set order is stronger than the star lattice set order. On a lattice X, the star lattice set order is an intermediate notion between weak set order and strong set order.

Recall that the strong set order is reflexive on the class of sublattices of a lattice, and is a partial order on this class. The weak set order is reflexive on the class of nonempty subsets of a poset, is not antisymmetric, but is transitive. Theorem 7 shows that the star complete set order is reflexive on the class of complete lattices in a poset, and the star lattice set order is reflexive on lattices in a poset. Like the weak set order, it is easy to see that neither relation is necessarily antisymmetric. In fact, neither is necessarily transitive either, as shown by the following. Let $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid (0,0) \preceq (x_1, x_2) \preceq (3,1)\}$ with the product partial order. Let $A = \{(t,0) \in X \mid 0 \leq t < 2\} \cup \{(3,0)\}$, $B = \{(1,1),(3,1)\}$, and $C = \{(2,1),(3,1)\}$. It is easy to check that $A \sqsubseteq^{*\ell} B$ and $B \sqsubseteq^{*\ell} C$, but $A \not\sqsubseteq^{*\ell} C$, because $\inf_A\{(3,0),(2,1)\}$ does not exist in A. Similarly, it is easy to check that $A \sqsubseteq^{*c} B$ and $B \sqsubseteq^{*c} C$, but $A \not\sqsubseteq^{*c} C$, because $\inf_A\{(2,1)\}$ does not exist in A. In terms of strong set order, it is easy to check that $A \not\sqsubseteq^s B$, $B \not\sqsubseteq^s C$, and $A \not\sqsubseteq^s C$, and in terms of weak set order, $A \sqsubseteq^w C$. For this reason, transitivity cannot be taken for granted and must be proved every time it appears in a theorem. We prove this in all the general results in this paper.

Star complete set order and star lattice set order are useful to compare different versions of the equilibrium set in general models with complementarities. They help provide natural comparisons among the entire equilibrium set, the equilibrium set corresponding to the infimum selection, and the equilibrium set corresponding to the supremum selection, as follows.

Theorem 8. *Consider the class of lattice models.*

1. *In every isotone infimum model, $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*c} \mathcal{E}(\Phi)$. Moreover, if $\mathcal{E}(\Phi)$ is a lattice, then $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*\ell} \mathcal{E}(\Phi)$.*

2. *In every isotone supremum model,* $\mathcal{E}(\Phi) \sqsubseteq^{*c} \mathcal{E}(\overline{\Phi})$*. Moreover, if* $\mathcal{E}(\Phi)$ *is a lattice, then* $\mathcal{E}(\Phi) \sqsubseteq^{*\ell} \mathcal{E}(\overline{\Phi})$*.*

3. *In every lattice model that is isotone infimum and supremum, in addition to (1) and (2),* $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*c} \mathcal{E}(\overline{\Phi})$ *and* $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*\ell} \mathcal{E}(\overline{\Phi})$*.*

Proof. For statement (1), to show that $\mathcal{E}(\Phi)$ is inf-complete in $\mathcal{E}(\underline{\Phi})$, consider nonempty $E \subset \mathcal{E}(\Phi)$. Let $\underline{e} = \inf_X E \in X$, which exists because X is a complete lattice. Let $\Psi : [\underline{x}, \underline{e}] \to [\underline{x}, \underline{e}]$ be given $\Psi(x) = \underline{\Phi}(x)$. Then $e \in E$ implies $\underline{\Phi}(\underline{e}) \preceq \underline{\Phi}(e) \preceq e$, and therefore, $\underline{\Phi}(\underline{e})$ is a lower bound for E, whence $\underline{\Phi}(\underline{e}) \preceq \underline{e}$. Moreover, for every $x \in [\underline{x}, \underline{e}]$, $\underline{x} \preceq \underline{\Phi}(x) \preceq \underline{\Phi}(\underline{e}) \preceq \underline{e}$. This shows that Ψ is well-defined and therefore, $([\overline{e}, \overline{x}], \preceq_X, \Psi)$ is a lattice model in which Ψ is an isotone function. Let $\hat{e}$ be the largest fixed point of Ψ. Then $\hat{e} = \Psi(\hat{e}) = \underline{\Phi}(\hat{e})$ implies that $\hat{e} \in \mathcal{E}(\underline{\Phi})$, and $\hat{e} \preceq \underline{e}$ implies that $\hat{e}$ is a lower bound for E. If $e \in \mathcal{E}(\underline{\Phi})$ is an arbitrary lower bound for E, then $e \preceq \overline{e}$ and $e = \underline{\Phi}(e) = \Psi(e)$, showing that e is a fixed point of Ψ, whence $e \preceq \hat{e}$. Therefore, $\inf_{\mathcal{E}(\underline{\Phi})} E = \hat{e} \in \mathcal{E}(\underline{\Phi})$.

To show that $\mathcal{E}(\underline{\Phi})$ is sup-complete in $\mathcal{E}(\Phi)$, consider nonempty $E \subset \mathcal{E}(\underline{\Phi})$. Let $\bar{e} = \sup_X E \in X$, which exists as X is complete. Consider $\Psi : [\overline{e}, \overline{x}] \rightrightarrows [\overline{e}, \overline{x}]$ given by $\Psi(x) = \Phi(x) \cap [\overline{e}, \overline{x}]$. Then $e \in E \subset \mathcal{E}(\underline{\Phi})$ implies $e = \underline{\Phi}(e) \preceq \underline{\Phi}(\overline{e})$. Therefore, $\underline{\Phi}(\overline{e})$ is an upper bound for E, whence $\overline{e} \preceq \underline{\Phi}(\overline{e})$. Moreover, for every $x \in [\overline{e}, \overline{x}]$, $\overline{e} \preceq \underline{\Phi}(\overline{e}) \preceq \underline{\Phi}(x) \preceq \overline{x}$. This shows that $\underline{\Phi}(x) \in \Psi(x)$, and therefore, Ψ is nonempty valued and $([\overline{e}, \overline{x}], \preceq_X, \Psi)$ is an isotone infimum model. Let $\hat{e}$ be the smallest fixed point of Ψ. Then $\hat{e} \in \Psi(\hat{e}) = \Phi(\hat{e}) \cap [\overline{e}, \overline{x}] \implies \hat{e} \in \mathcal{E}(\Phi)$ and $\overline{e} \preceq \hat{e} \implies \hat{e}$ is an upper bound for E. If $e \in \mathcal{E}(\Phi)$ is an arbitrary upper bound for E, then $\overline{e} \preceq e$ and $e \in \Phi(e) \cap [\overline{e}, \overline{x}] = \Psi(e)$, showing that e is a fixed point of Ψ, whence $\hat{e} \preceq e$. Therefore, $\sup_{\mathcal{E}(\Phi)} E \in \mathcal{E}(\Phi)$.

The statement $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*\ell} \mathcal{E}(\Phi)$ follows from statement (3)(c) in Theorem 7. Statement (2) is proved similarly. Statement (3) is proved by following the proof of statement (1) with $\mathcal{E}(\overline{\Phi})$ instead of $\mathcal{E}(\Phi)$ and the proof of statement (2) with $\mathcal{E}(\underline{\Phi})$ instead of $\mathcal{E}(\Phi)$. ∎

Statement (3) in Theorem 8 does not follow automatically from statements (1) and (2), because transitivity cannot be assumed. That is, even when $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*c} \mathcal{E}(\Phi)$ and $\mathcal{E}(\Phi) \sqsubseteq^{*c} \mathcal{E}(\overline{\Phi})$, we must still prove that $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*c} \mathcal{E}(\overline{\Phi})$. The proof shows how to do this in a natural manner.

In addition to comparative statics of the entire equilibrium set, Theorem 8 provides a foundation for a formal theory of order approximation of equilibria. Statement (1) shows that in models with an isotone infimum selection, every subset of equilibria has a largest lower bound among equilibria that arise using the infimum selection. Therefore, if $E \subseteq \mathcal{E}(\Phi)$ formalizes a specialized equilibrium notion or an equilibrium refinement, it can be uniquely approximated from below in a formal order theoretic manner using equilibria from the infimum selection. In the special case that $E = \{e^*\}$ is a singleton, this proves that *every equilibrium* $e^* \in \mathcal{E}(\Phi)$ *can be uniquely order approximated from below by an equilibrium in* $\mathcal{E}(\underline{\Phi})$. This may be particularly useful if the infimum selection is easier to work with or have some useful computational, dynamic, or theoretical properties (see, for example, Becker and Rincón-Zapatero (2021)). This result requires very little structure for the lattice model (only isotone infimum selection).

Moreover, if $\mathcal{E}(\Phi)$ is a lattice, then $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*\ell} \mathcal{E}(\Phi)$ implies that for every equilibrium $\hat{x} \in \mathcal{E}(\underline{\Phi})$ and $\tilde{x} \in \mathcal{E}(\Phi)$, if $\hat{x} \not\preceq \tilde{x}$, then there is a different and unique equilibrium $\underline{\hat{x}} \in \mathcal{E}(\underline{\Phi})$ that is the largest equilibrium in $\mathcal{E}(\underline{\Phi})$ smaller than both of these equilibria.

Similarly, statement (2) formalizes the notion that in models with an isotone supremum selection, every nonempty subset E of equilibria can be uniquely approximated from above as a smallest upper bound using equilibria from the supremum selection. In the special case that $E = \{e^*\}$ is a singleton, this proves that *every equilibrium* $e^* \in \mathcal{E}(\Phi)$ *can be uniquely order approximated from above by an equilibrium in* $\mathcal{E}(\overline{\Phi})$. This result also requires very little structure for the lattice model (only isotone supremum selection). For example, Rostek and Yoder (2020) show that in matching with complementarities, stable outcomes are characterized by the largest fixed point of a monotone operator.

In models with both isotone infimum and supremum selections, both sets of results hold.

Theorem 8 implies corresponding comparisons among equilibrium sets in all standard models with complementarities and order approximation of equilibria in these models as well.

Corollary 9. *In every Topkis, Vives, MR, GMS, Zhou, GCKK, and PY model,*

1. *(a)* $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*c} \mathcal{E}(\Phi)$ *(b)* $\mathcal{E}(\Phi) \sqsubseteq^{*c} \mathcal{E}(\overline{\Phi})$ *(c)* $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*c} \mathcal{E}(\overline{\Phi})$
2. *(a)* $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*\ell} \mathcal{E}(\Phi)$ *(b)* $\mathcal{E}(\Phi) \sqsubseteq^{*\ell} \mathcal{E}(\overline{\Phi})$ *(c)* $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*\ell} \mathcal{E}(\overline{\Phi})$

Proof. Special cases of statements (1), (2), and (3) in Theorem 8. ∎

If only the infimum or supremum selection is available, Theorem 8 implies that in every CKK-1, ICKK-1, GQS-1, and PY-1 model, $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*c} \mathcal{E}(\Phi)$ and in every CKK-2, ICKK-2, GQS-2, and PY-2 model, $\mathcal{E}(\Phi) \sqsubseteq^{*c} \mathcal{E}(\overline{\Phi})$.

As shown in Example 21, comparability of equilibrium sets in the strong set order may fail even in the standard S-model used to motivate complementarities. On the other hand, our results show that the equilibrium sets in Example 21 are naturally comparable in star complete set order and in the star lattice set order. That is, $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*c} \mathcal{E}(\Phi)$; $\mathcal{E}(\Phi) \sqsubseteq^{*c} \mathcal{E}(\overline{\Phi})$; and $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*c} \mathcal{E}(\overline{\Phi})$; and also, $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*\ell} \mathcal{E}(\Phi)$; $\mathcal{E}(\Phi) \sqsubseteq^{*\ell} \mathcal{E}(\overline{\Phi})$; and $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*\ell} \mathcal{E}(\overline{\Phi})$.

Example 22 (Extended S-model)**.** Consider the extended S-model depicted in Figure 4 with correspondence $\Phi(x) = [\underline{\Phi}(x), \overline{\Phi}(x)]$ and the same interpretation as Figure 1. The full equilibrium set is $\mathcal{E}(\Phi) = [x_1, x_2] \cup [x_3, x_4] \cup [x_5, x_6] \cup [x_7, x_7] \cup [x_7, x_{10}]$, the equilibrium set corresponding to the infimum selection is $\mathcal{E}(\underline{\Phi}) = \{x_1, x_4, x_5, x_8, x_9\}$, and the one corresponding to to the supremum selection is $\mathcal{E}(\overline{\Phi}) = \{x_2, x_3, x_6, x_6, x_{10}\}$. In this case, $\sup_X\{x_4, x_3\} = x_4 \notin \mathcal{E}(\overline{\Phi})$, but $\sup_{\mathcal{E}(\overline{\Phi})}\{x_4, x_3\} = x_6 \in \mathcal{E}(\overline{\Phi})$ as guaranteed by the star lattice set order using Theorem 8, and similarly, $\inf_X\{x_8, x_6\} = x_6 \notin \mathcal{E}(\underline{\Phi})$, but $\inf_{\mathcal{E}(\underline{\Phi})}\{x_8, x_6\} = x_5 \in \mathcal{E}(\underline{\Phi})$. Moreover, this shows a distinction between using the star complete set order to compare equilibrium sets and a standard application of the weak set order using extremal equilibria, as follows. We know that $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{w} \mathcal{E}(\overline{\Phi})$, because $\forall b \in \mathcal{E}(\overline{\Phi})$, $x_1 \preceq b$ and $\forall a \in \mathcal{E}(\underline{\Phi})$, $a \preceq x_{10}$, but this does not give tight bounds for a given equilibrium. The star complete set order is more specific and gives the closest order approximation, for example, $\sup_{\mathcal{E}(\overline{\Phi})}\{x_4\} = x_6 \in \mathcal{E}(\overline{\Phi})$, and this is the "closest equilibrium" to x_4 (in terms of order) when considering equilibria in $\mathcal{E}(\overline{\Phi})$, and similarly, $\inf_{\mathcal{E}(\underline{\Phi})}\{x_7\} = x_5 \in \mathcal{E}(\underline{\Phi})$, and more generally, a similar computation for every subset E of equilibria. It is easy to modify this example to provide order approximations of equilibria in parametric models as well, even when the strong set order fails to apply.

These results extend to models with parametric complementarities, as shown next.

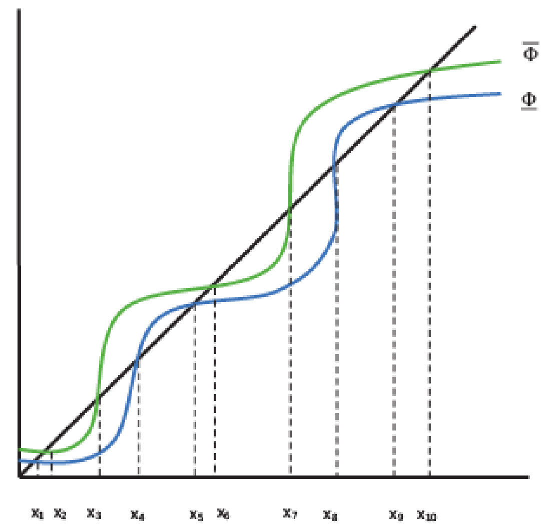

Figure 4: Extended S-model

4 General parametric models with complementarities

Parametric models are used to study the effect of exogenous parameters on the decision making environment and the corresponding equilibrium. We include these effects in a general manner by positing a partially ordered set T of parameters. A ***parametric lattice model*** is a collection $((X, \preceq_X), (T, \preceq_T), \Phi)$, where $(X, \preceq_X)$ is a nonempty, complete lattice, $(T, \preceq_T)$ is a nonempty poset, and $\Phi : X \times T \rightrightarrows X$ is a correspondence. For each $t \in T$, the ***lattice model at*** t is the triple $(X, \preceq_X, \Phi_t)$ where Φ_t is the t-section of Φ.

An ***equilibrium at*** t is a fixed point of Φ_t, that is, a point $x \in X$ such that $x \in \Phi(x,t)$. The ***equilibrium set at*** t is $\mathcal{E}(\Phi_t) = \{x \in X \mid x \in \Phi(x,t)\}$. The ***equilibrium correspondence*** is $\mathcal{E} : T \rightrightarrows X, t \mapsto \mathcal{E}(\Phi_t)$. An ***equilibrium selection*** is a selection from the equilibrium correspondence. An ***isotone equilibrium selection*** is an equilibrium selection that is an isotone function. A parametric lattice model $((X, \preceq_X), (T, \preceq_T), \Phi)$ has ***monotone comparative statics (MCS) of equilibrium*** if its equilibrium correspondence has an isotone selection.

A ***general parametric model with complementarities*** is a parametric lattice model $((X, \preceq_X), (T, \preceq_T), \Phi)$ with an isotone selection. A parametric lattice model $((X, \preceq_X), (T, \preceq_T), \Phi)$ is ***isotone supremum*** if $\forall (x,t) \in X \times T$, $\overline{\Phi}(x,t) \coloneqq \sup_{\Phi(x,t)} \Phi(x,t) \in \Phi(x,t)$, and the function $(x,t) \mapsto \overline{\Phi}(x,t)$ is isotone. It is ***isotone infimum*** if $\forall (x,t) \in X \times T$, $\underline{\Phi}(x,t) := \inf_{\Phi(x,t)} \Phi(x,t) \in \Phi(x,t)$, and the function $(x,t) \mapsto \underline{\Phi}(x,t)$ is isotone. As earlier, neither strong set order nor completeness nor subcompleteness nor continuity properties are used in these definitions.

Example 23 (Parametric Topkis model)**.** The ***parametric Topkis model*** is a finitely indexed collection $((X_i, \preceq_i, F_i)_{i=1}^I, (T, \preceq_T))$, where for each i, X_i is a nonempty, complete lattice, T is a poset (and using product order on Cartesian products), for each i, $F_i : X_i \times X_{-i} \times T \to \mathbb{R}$ has decreasing differences in (x_i, x_{-i}) $(\forall t)$, and decreasing differences in (x_i, t) $(\forall x_{-i})$, and for each (x_{-i}, t), $F_i(\cdot, x_{-i}, t)$ is submodular on X_i and upper semicontinuous in order interval topology. For each $x, y \in X$ and $t \in T$, let $G(x,y,t) = \sum_{i=1}^I F_i(y_i, x_{-i}, t)$ and let $\Phi : X \times T \rightrightarrows X$ be given by $\Phi(x,t) = \arg\min_{y \in X} G(x,y,t)$. The ***associated parametric lattice model*** is $((X, \preceq_X), (T, \preceq_T), \Phi)$, where $X = \times_{i=1}^I X_i$, $\preceq_X$ is product order, $(T, \preceq_T)$ is a poset, and Φ is the correspondence as above.

Example 24 (Parametric Vives model)**.** The ***parametric Vives model*** is a finitely indexed collection $((X_i, \preceq_i, F_i)_{i=1}^I, (T, \preceq_T))$, where for each i, X_i is a nonempty, complete lattice, T is a poset (and using product order on Cartesian products), for each i, $F_i : X_i \times X_{-i} \times T \to \mathbb{R}$ has increasing differences in (x_i, x_{-i}) $(\forall t)$, and increasing differences in (x_i, t) $(\forall x_{-i})$, and for each (x_{-i}, t), $F_i(\cdot, x_{-i}, t)$ is supermodular on X_i and upper semicontinuous in order interval topology. For each (x_{-i}, t), let $\Phi_i(x_{-i}, t) = \arg\max_{\xi \in X_i} F_i(\xi, x_{-i}, t)$. Let $\Phi : X \times T \rightrightarrows X$ be given by $\Phi(x,t) = \times_{i=1}^I \Phi_i(x_{-i}, t)$. The ***associated parametric lattice model*** is $((X, \preceq_X), (T, \preceq_T), \Phi)$, where $X = \times_{i=1}^I X_i$, $\preceq$ is product order, $(T, \preceq_T)$ is a poset, and $\Phi = \times_{i=1}^I \Phi_i$ is the product correspondence as above.

Example 25 (Parametric GMS model)**.** The ***parametric GMS model*** is a finitely indexed collection $((X_i, \preceq_i, F_i)_{i=1}^I, (T, \preceq_T))$, where for each i, X_i is a nonempty, complete lattice, T is a poset (and using product order on Cartesian products), for each i, $F_i : X_i \times X_{-i} \times T \to \mathbb{R}$ has single crossing property in (x_i, x_{-i}) $(\forall t)$, and single crossing property in (x_i, t) $(\forall x_{-i})$, and for each (x_{-i}, t), $F_i(\cdot, x_{-i}, t)$ is quasisupermodular on X_i and upper semicontinuous in order interval topology. For each (x_{-i}, t), let $\Phi_i(x_{-i}, t) = \arg\max_{\xi \in X_i} F_i(\xi, x_{-i}, t)$. Let $\Phi : X \times T \rightrightarrows X$ be given by $\Phi(x,t) = \times_{i=1}^I \Phi_i(x_{-i}, t)$. The ***associated parametric lattice***

model is $((X, \preceq_X), (T, \preceq_T), \Phi)$, where $X = \times_{i=1}^{I} X_i$, $\preceq_X$ is product order, $(T, \preceq_T)$ is a poset, and $\Phi = \times_{i=1}^{I} \Phi_i$ is the product correspondence as above. The ***parametric MR model*** is nested as a special case of the parametric GMS model.

Example 26 (Parametric Zhou model)**.** The ***parametric Zhou model*** is a parametric lattice model $((X, \preceq_X), (T, \preceq_T), \Phi)$ in which Φ is isotone in the strong set order and for every (x, t), $\Phi(x, t)$ is nonempty and subcomplete.

Example 27 (Parametric CKK model)**.** The ***parametric CKK-1 model*** is a finitely indexed collection $((X_i, \preceq_i, F_i)_{i=1}^{I}, (T, \preceq_T))$, where for each i, X_i is a nonempty, complete lattice, T is a poset (and using product order on Cartesian products), for each i, $F_i : X_i \times X_{-i} \times T \to \mathbb{R}$ has weak dominance property in (x_i, x_{-i}) $(\forall t)$, and weak dominance property in (x_i, t) $(\forall x_{-i})$. For each i, fix a sublattice $S_i \subseteq X_i$, and for every (x_{-i}, t), let $\Phi_i(x_{-i}, t) = \arg\max_{\xi \in S_i} F_i(\xi, x_{-i}, t)$, and suppose that $\underline{\Phi}_i(x_{-i}, t) \coloneqq \inf_{\Phi_i(x_{-i},t)} \Phi_i(x_{-i}, t) \in \Phi_i(x_{-i}, t)$. Let $\Phi : X \times T \rightrightarrows X$ be given by $\Phi(x, t) = \times_{i=1}^{I} \Phi_i(x_{-i}, t)$. The ***associated parametric lattice model*** is $((X, \preceq_X), (T, \preceq_T), \Phi)$, where $X = \times_{i=1}^{I} X_i$, $\preceq_X$ is product order, $(T, \preceq_T)$ is a poset, and $\Phi = \times_{i=1}^{I} \Phi_i$ is the product correspondence as above. As a second version, the ***parametric CKK-2 model*** is a parametric CKK-1 model in which the property that infimum exists is replaced with the property that supremum exists and other aspects remain the same. A ***parameteric CKK model*** is one that is both parametric CKK-1 and parametric CKK-2. As mentioned above, if, instead, we add the assumption that for each i and (x_{-i}, t), $F_i(\cdot, x_{-i}, t)$ is upper semicontinuous in order interval topology and quasisupermodular, then the set of maximizers has both a smallest and a largest element.

Example 28 (Parametric ICKK model)**.** The ***parametric ICKK-1 model*** is a finitely indexed collection $((X_i, \preceq_i, F_i)_{i=1}^{I}, (T, \preceq_T))$, where for each i, X_i is a nonempty, complete lattice, T is a poset (and using product order on Cartesian products), for each i, $F_i : X_i \times X_{-i} \times T \to \mathbb{R}$ has weak interval dominance property in (x_i, x_{-i}) $(\forall t)$, and weak interval dominance property in (x_i, t) $(\forall x_{-i})$. For each i, fix a subinterval $S_i \subseteq X_i$, and for every (x_{-i}, t), let $\Phi_i(x_{-i}, t) = \arg\max_{\xi \in S_i} F_i(\xi, x_{-i}, t)$ and suppose that $\underline{\Phi}_i(x_{-i}, t) \coloneqq \inf_{\Phi_i(x_{-i},t)} \Phi_i(x_{-i}, t) \in \Phi_i(x_{-i}, t)$. Let $\Phi : X \times T \rightrightarrows X$ be given by $\Phi(x, t) = \times_{i=1}^{I} \Phi_i(x_{-i}, t)$. The ***associated parametric lattice model*** is $((X, \preceq_X), (T, \preceq_T), \Phi)$, where $X = \times_{i=1}^{I} X_i$, $\preceq_X$ is product order,

$(T, \preceq_T)$ is a poset, and $\Phi = \times_{i=1}^{I} \Phi_i$ is the product correspondence as above. As a second version, the ***parametric ICKK-2 model*** is a parametric ICKK-1 model in which the property that infimum exists is replaced with the property that supremum exists and other aspects remain the same. A ***parametric ICKK model*** is one that is both parametric ICKK-1 and parametric ICKK-2. As mentioned above, if, instead, we add the assumption that for each i and (x_{-i}, t), $F_i(\cdot, x_{-i}, t)$ is upper semicontinuous in order interval topology and quasisupermodular, then the set of maximizers has both a smallest and a largest element.

Example 29 (Parametric GQS model)**.** The ***parametric GQS-1 model*** is a finitely indexed collection $((X_i, \preceq_i, F_i)_{i=1}^{I}, (T, \preceq_T))$, where for each i, X_i is a nonempty, complete lattice, T is a poset (and using product order on Cartesian products), for each i, $F_i : X_i \times X_{-i} \times T \to \mathbb{R}$ has interval dominance property in (x_i, x_{-i}) $(\forall t)$, and interval dominance property in (x_i, t) $(\forall x_{-i})$. For each i, fix a subinterval $S_i \subseteq X_i$, and for every (x_{-i}, t), let $\Phi_i(x_{-i}, t) = \arg\max_{\xi \in S_i} F_i(\xi, x_{-i}, t)$ and suppose that $\underline{\Phi}_i(x_{-i}, t) \coloneqq \inf_{\Phi_i(x_{-i},t)} \Phi_i(x_{-i}, t) \in \Phi_i(x_{-i}, t)$. Let $\Phi : X \times T \rightrightarrows X$ be given by $\Phi(x, t) = \times_{i=1}^{I} \Phi_i(x_{-i}, t)$. The ***associated parametric lattice model*** is $((X, \preceq_X), (T, \preceq_T), \Phi)$, where $X = \times_{i=1}^{I} X_i$, $\preceq_X$ is product order, $(T, \preceq_T)$ is a poset, and $\Phi = \times_{i=1}^{I} \Phi_i$ is the product correspondence as above. As a second version, the ***parametric GQS-2 model*** is a parametric GQS-1 model in which the property that infimum exists is replaced with the property that supremum exists and other aspects remain the same. A ***parametric GQS model*** is one that is both parametric GQS-1 and parametric GQS-2. As mentioned above, if, instead, we add the assumption that for each i and (x_{-i}, t), $F_i(\cdot, x_{-i}, t)$ is upper semicontinuous in order interval topology and quasisupermodular, then the set of maximizers has both a smallest and a largest element.

Example 30 (Parametric GCKK model)**.** The above models based on Che, Kim, and Kojima (2021) are summarized as follows. A generalized parametric CKK-1 model, or ***parametric GCKK-1 model*** is one that is either a parametric CKK-1 model or a parametric ICKK-1 model or a parametric GQS-1 model. A generalized parametric CKK-2 model, or ***parametric GCKK-2 model*** is one that is either a parametric CKK-2 model or a parametric ICKK-2 model or a parametric GQS-2 model. A generalized parametric CKK model, or ***parametric GCKK model*** is one that is either a parametric CKK model or a parametric ICKK model or a parametric GQS model.

Example 31 (Parametric PY model)**.** A ***parametric PY-1 (respectively, PY-2) model*** is a finitely indexed collection $((X_i, \preceq_i, F_i)_{i=1}^I, (T, \preceq_T))$, where for each i, $(X_i, \preceq_i)$ is a nonempty, complete chain (hence compact in the order interval topology), and using product order on products of X_i, for each i, $F_i : X_i \times X_{-i} \times T \to \mathbb{R}$ is transfer weakly upper semicontinuous in x_i, and F_i satisfies the downward (respectively, upward) transfer single crossing property in (x_i, x_{-i}) for each t, and in (x_i, t) for each x_{-i}. For each (x_{-i}, t), let $\Phi_i(x_{-i}, t) = \arg\max_{\xi \in X_i} F_i(\xi, x_{-i}, t)$. Let $\Phi : X \times T \rightrightarrows X$ be given by $\Phi(x, t) = \times_{i=1}^I \Phi_i(x_{-i}, t)$. The ***associated parametric lattice model*** is $((X, \preceq_X), (T, \preceq_T), \Phi)$, where $X = \times_{i=1}^I X_i$, $\preceq_X$ is product order, $(T, \preceq_T)$ is a poset, and $\Phi = \times_{i=1}^I \Phi_i$ is the product correspondence as above. A ***parametric PY model***, is one that is both parametric PY-1 and parametric PY-2.

Example 32 (Parametric standard and neostandard models)**.** Finally, a ***standard parametric model with complementarities***, or ***standard parametric model***, is one that is either a parametric Topkis model, or parametric Vives model, or parametric MR model, or parametric GMS model. A ***parametric neostandard model*** is on that is either parametric GCKK or parametric PY. A ***parametric neostandard-1 model*** is one that is either parametric GCKK-1 or parametric PY-1, and a ***parametric neostandard-2 model*** is one that is either parametric GCKK-2 or parametric PY-2.

Theorem 10 shows that the patterns of decentralized interdependent behavior in all the different standard and neostandard parametric models are unified in terms of the same isotone properties on the joint correspondence Φ.

Theorem 10. *Consider the class of standard and neostandard parametric models.*

1. *For every parametric Topkis, Vives, MR, GMS, Zhou, GCKK, and PY model, its associated parametric lattice model is isotone infimum and isotone supremum.*

2. *For every parametric GCKK-1, PY-1 (respectively, GCKK-2, PY-2) model, its associated parametric lattice model is isotone infimum (respectively, supremum).*

Proof. Similar to that of Theorem 1. ∎

Theorem 11. *Consider the class of parametric lattice models.*

1. *In every general parametric model with complementarities, there are two isotone equilibrium selections. In particular, every general parametric model with complementarities has MCS of equilibrium.*

2. (a) *Every parametric isotone supremum model has two isotone equilibrium selections: $t \mapsto \sup_{\mathcal{E}(\overline{\Phi}_t)} \mathcal{E}(\overline{\Phi}_t)$ and $t \mapsto \inf_{\mathcal{E}(\overline{\Phi}_t)} \mathcal{E}(\overline{\Phi}_t)$. The two selections are different if, and only if, there is even one t such that $\mathcal{E}(\overline{\Phi}_t)$ has multiple equilibria.*

   (b) *The supremum selection $t \mapsto \sup_{\mathcal{E}(\overline{\Phi}_t)} \mathcal{E}(\overline{\Phi}_t)$ selects the largest equilibrium in the general model at t, for every $t \in T$.*

   (c) *Every parametric isotone supremum model has MCS of supremum equilibrium.*

3. (a) *Every parametric isotone infimum model has two isotone equilibrium selections: $t \mapsto \inf_{\mathcal{E}(\underline{\Phi}_t)} \mathcal{E}(\underline{\Phi}_t)$ and $t \mapsto \sup_{\mathcal{E}(\underline{\Phi}_t)} \mathcal{E}(\underline{\Phi}_t)$. The two selections are different if, and only if, there is even one t such that $\mathcal{E}(\underline{\Phi}_t)$ has multiple equilibria.*

   (b) *The infimum selection $t \mapsto \inf_{\mathcal{E}(\underline{\Phi}_t)} \mathcal{E}(\underline{\Phi}_t)$ selects the smallest equilibrium in the general model at t, for every $t \in T$.*

   (c) *Every parametric isotone infimum model has MCS of infimum equilibrium.*

4. *Every parametric isotone infimum and supremum model has MCS of extremal equilibrium.*

Proof. To prove statement (1), let $((X, \preceq_X), (T, \preceq_T), \Phi)$ be a general parametric model with complementarities and $f : X \times T \to X$ be an isotone selection. Letting f_t denote the section of f determined by t, it follows that $\forall t \in T$, $(X, \preceq, f_t)$ is a general model with complementarities. As f_t is an isotone function, Tarski's theorem implies that the equilibrium set $\mathcal{E}(f_t)$ is a nonempty complete lattice. Let $\overline{e}(t) = \sup_{\mathcal{E}(f_t)} \mathcal{E}(f_t) \in \mathcal{E}(f_t) \subseteq \mathcal{E}(\Phi_t)$ Then $t \mapsto \overline{e}(t)$ is an equilibrium selection. To see that it is isotone, fix $\hat{t} \preceq_T \tilde{t}$. We know that $\overline{e}(\hat{t}) = \sup_X A$, where $A = \{x \in X \mid x \preceq_X f(x, \hat{t})\}$ and that $\overline{e}(\hat{t}) \in A$. Similarly, $\overline{e}(\tilde{t}) = \sup_X B$, where $B = \{x \in X \mid x \preceq_X f(x, \tilde{t})\}$. As f is isotone, $A \subseteq B$, and therefore, $\overline{e}(\hat{t}) \in B$, whence $\overline{e}(\hat{t}) \preceq \overline{e}(\tilde{t})$, as desired. Similarly, it can be shown that $t \mapsto \underline{e}(t) := \inf_{\mathcal{E}(f_t)} \mathcal{E}(f_t)$ is an isotone equilibrium selection.

Statement (2)(a) is proved similarly and the second part of (2)(a) follows from the fact that $\forall t, \inf_{\mathcal{E}(\overline{\Phi}_t)} \mathcal{E}(\overline{\Phi}_t) = \sup_{\mathcal{E}(\overline{\Phi}_t)} \mathcal{E}(\overline{\Phi}_t) \Leftrightarrow \mathcal{E}(\overline{\Phi}_t)$ is a singleton. Statement 2(b) follows from Theorem 2. Statement (2)(c) follows from (2)(b). Statement (3) is proved similarly. Statement (4) follows from statements (2)(c) and (3)(c). ∎

Statements 2(a) and 3(a) in Theorem 11 prove that there are other isotone equilibrium selections in general models with complementarities besides the extremal ones under weak conditions, that is, if there is even one t for which there are multiple equilibria at t for the corresponding equilibrium set. Thus, if we typically expect multiple equilibria in models with complementarities, we may also expect more isotone equilibrium selections than the extremal ones commonly identified in standard models. This shows another new structural feature of equilibrium in general models with complementarities.

Corollary 12. *Consider the class of standard and neostandard parametric models.*

1. *Every parametric Topkis, Vives, MR, GMS, Zhou, GCKK, and PY model has MCS of extremal equilibrium.*
2. *Every parametric GCKK-1, PY-1 (respectively, GCKK-2, PY-2) model has MCS of infimum (respectively, supremum) equilibrium.*

Proof. Follows from Theorem 10 and Theorem 11. ∎

Theorem 13 generalizes Theorem 11 to provide results for parameteric comparisons of entire equilibrium sets (as compared to particular equilibrium selections).

Theorem 13. *Consider the class of parametric lattice models.*

1. *In every parametric isotone infimum model, for every $\hat{t} \preceq \tilde{t}$,*
   (a) $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\underline{\Phi}_{\tilde{t}})$, (b) $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\Phi_{\tilde{t}})$ and (c) $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*\ell} \mathcal{E}(\underline{\Phi}_{\tilde{t}})$.
   Moreover, (d) if $\mathcal{E}(\Phi_{\tilde{t}})$ is a lattice, then $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*\ell} \mathcal{E}(\Phi_{\tilde{t}})$.
2. *In every parametric isotone supremum model, for every $\hat{t} \preceq \tilde{t}$,*
   (a) $\mathcal{E}(\overline{\Phi}_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\overline{\Phi}_{\tilde{t}})$, (b) $\mathcal{E}(\Phi_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\overline{\Phi}_{\tilde{t}})$ and (c) $\mathcal{E}(\overline{\Phi}_{\hat{t}}) \sqsubseteq^{*\ell} \mathcal{E}(\overline{\Phi}_{\tilde{t}})$.
   Moreover, (d) if $\mathcal{E}(\Phi_{\hat{t}})$ is a lattice, then $\mathcal{E}(\Phi_{\hat{t}}) \sqsubseteq^{*\ell} \mathcal{E}(\overline{\Phi}_{\tilde{t}})$.

3. *In every parametric lattice model that is isotone infimum and isotone supremum, for every* $\hat{t} \preceq \tilde{t}$*, in addition to (1) and (2),* $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\overline{\Phi}_{\tilde{t}})$ *and* $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*\ell} \mathcal{E}(\overline{\Phi}_{\tilde{t}})$.

4. *In every parametric lattice model that is isotone infimum and isotone supremum, if for every* t*,* $(X, \preceq_X, \Phi_t)$ *is either isotone supremum on lower intervals or isotone infimum on upper intervals, then for every* $\hat{t} \preceq \tilde{t}$*, in addition to (1), (2) and (3),* $\mathcal{E}(\Phi_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\Phi_{\tilde{t}})$ *and* $\mathcal{E}(\Phi_{\hat{t}}) \sqsubseteq^{*\ell} \mathcal{E}(\Phi_{\tilde{t}})$.

Proof. For statement (1), fix $\hat{t} \preceq \tilde{t}$. To show that $\mathcal{E}(\underline{\Phi}_{\hat{t}})$ is sup-complete in $\mathcal{E}(\underline{\Phi}_{\tilde{t}})$, consider nonempty $E \subseteq \mathcal{E}(\underline{\Phi}_{\hat{t}})$. Let $\bar{e} = \sup_X E \in X$, as X is complete. Consider $\Psi : [\overline{e}, \overline{x}] \to [\overline{e}, \overline{x}]$ given by $\Psi(x) = \underline{\Phi}_{\tilde{t}}(x)$. Then $e \in E \subseteq \mathcal{E}(\underline{\Phi}_{\hat{t}})$ and $\underline{\Phi}(x, t)$ is isotone imply $e = \underline{\Phi}_{\hat{t}}(e) \preceq \underline{\Phi}_{\hat{t}}(\overline{e}) \preceq \underline{\Phi}_{\tilde{t}}(\overline{e})$. Therefore, $\underline{\Phi}_{\tilde{t}}(\overline{e})$ is an upper bound for E, whence $\overline{e} \preceq \underline{\Phi}_{\tilde{t}}(\overline{e})$. Moreover, for every $x \in [\overline{e}, \overline{x}]$, $\overline{e} \preceq \underline{\Phi}_{\tilde{t}}(\overline{e}) \preceq \underline{\Phi}_{\tilde{t}}(x) \preceq \overline{x}$. This shows that Ψ is well-defined and therefore, $([\overline{e}, \overline{x}], \preceq_X, \Psi)$ is a lattice model in which Ψ is an isotone function. Let $\hat{e}$ be the smallest fixed point of Ψ. Then $\overline{e} \preceq \hat{e} \implies \hat{e}$ is an upper bound for E and $\hat{e} \in \Psi(\hat{e}) = \underline{\Phi}_{\tilde{t}}(\hat{e}) \implies \hat{e} \in \mathcal{E}(\underline{\Phi}_{\tilde{t}})$. Let $e \in \mathcal{E}(\underline{\Phi}_{\tilde{t}})$ be an arbitrary upper bound for E. Then $\overline{e} \preceq e$ and $e = \underline{\Phi}_{\tilde{t}}(e) = \Psi(e)$, showing that e is a fixed point of Ψ, whence $\hat{e} \preceq e$. This shows that $\sup_{\mathcal{E}(\underline{\Phi}_{\tilde{t}})} E = \hat{e} \in \mathcal{E}(\underline{\Phi}_{\tilde{t}})$.

To show that $\mathcal{E}(\underline{\Phi}_{\tilde{t}})$ is inf-complete in $\mathcal{E}(\underline{\Phi}_{\hat{t}})$, consider nonempty $E \subseteq \mathcal{E}(\underline{\Phi}_{\tilde{t}})$. Let $\underline{e} = \inf_X E \in X$. Consider $\Psi : [\underline{x}, \underline{e}] \to [\underline{x}, \underline{e}]$ given by $\Psi(x) = \underline{\Phi}_{\hat{t}}(x)$. Then $e \in E \subseteq \mathcal{E}(\underline{\Phi}_{\tilde{t}})$ and $\underline{\Phi}(x, t)$ is isotone imply $e = \underline{\Phi}_{\tilde{t}}(e) \succeq \underline{\Phi}_{\tilde{t}}(\overline{e}) \succeq \underline{\Phi}_{\hat{t}}(\overline{e})$. Therefore, $\underline{\Phi}_{\hat{t}}(\overline{e})$ is a lower bound for E, whence $\underline{e} \succeq \underline{\Phi}_{\hat{t}}(\overline{e})$. Moreover, for every $x \in [\underline{x}, \underline{e}]$, $\underline{e} \succeq \underline{\Phi}_{\hat{t}}(\overline{e}) \succeq \underline{\Phi}_{\hat{t}}(x) \succeq \underline{x}$. This shows that Ψ is well-defined and therefore, $([\overline{e}, \overline{x}], \preceq_X, \Psi)$ is a lattice model in which Ψ is an isotone function. Let $\hat{e}$ be the greatest fixed point of Ψ. Then $\overline{e} \succeq \hat{e} \implies \hat{e}$ is a lower bound for E and $\hat{e} \in \Psi(\hat{e}) = \underline{\Phi}_{\hat{t}}(\hat{e}) \implies \hat{e} \in \mathcal{E}(\underline{\Phi}_{\hat{t}})$. Let $e \in \mathcal{E}(\underline{\Phi}_{\hat{t}})$ be an arbitrary lower bound for E. Then $\overline{e} \succeq e$ and $e = \underline{\Phi}_{\hat{t}}(e) = \Psi(e)$, showing that e is a fixed point of Ψ, whence $\hat{e} \succeq e$. This shows that $\inf_{\mathcal{E}(\underline{\Phi}_{\hat{t}})} E = \hat{e} \in \mathcal{E}(\underline{\Phi}_{\hat{t}})$. It follows that $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\underline{\Phi}_{\tilde{t}})$.

To show that $\mathcal{E}(\underline{\Phi}_{\hat{t}})$ is sup-complete in $\mathcal{E}(\Phi_{\tilde{t}})$, consider nonempty $E \subseteq \mathcal{E}(\underline{\Phi}_{\hat{t}})$. Let $\bar{e} = \sup_X E \in X$. Consider $\Psi : [\overline{e}, \overline{x}] \rightrightarrows [\overline{e}, \overline{x}]$ given by $\Psi(x) = \Phi_{\tilde{t}}(x) \cap [\overline{e}, \overline{x}]$. Then $e \in E \subseteq \mathcal{E}(\underline{\Phi}_{\hat{t}})$ and $\underline{\Phi}(x, t)$ is isotone imply $e = \underline{\Phi}_{\hat{t}}(e) \preceq \underline{\Phi}_{\hat{t}}(\overline{e}) \preceq \underline{\Phi}_{\tilde{t}}(\overline{e})$. Therefore, $\underline{\Phi}_{\tilde{t}}(\overline{e})$ is an upper bound for E, whence $\overline{e} \preceq \underline{\Phi}_{\tilde{t}}(\overline{e})$. Moreover, for every $x \in [\overline{e}, \overline{x}]$, $\overline{e} \preceq \underline{\Phi}_{\tilde{t}}(\overline{e}) \preceq \underline{\Phi}_{\tilde{t}}(x) \preceq \overline{x}$. This shows that $\underline{\Phi}_{\tilde{t}}(x) \in \Psi(x)$, and therefore, Ψ is nonempty valued and $([\overline{e}, \overline{x}], \preceq_X, \Psi)$ contains an isotone infimum model. Let $\hat{e}$ be the smallest fixed point of Ψ. Then $\hat{e} \in \Psi(\hat{e}) = \Phi_{\tilde{t}}(\hat{e}) \cap [\overline{e}, \overline{x}]$ implies $\hat{e} \in \mathcal{E}(\Phi_{\tilde{t}})$ and $\overline{e} \preceq \hat{e}$ implies that $\hat{e}$ is an upper bound for E. Let $e \in \mathcal{E}(\Phi_{\tilde{t}})$ be an arbitrary upper bound for E. Then $\overline{e} \preceq e$ and $e \in \Phi_{\tilde{t}}(e) \cap [\overline{e}, \overline{x}] = \Psi(e)$, showing that e is a fixed point of Ψ, and consequently, $\hat{e} \preceq e$. Therefore, $\sup_{\mathcal{E}(\Phi_{\tilde{t}})} E \in \mathcal{E}(\Phi_{\tilde{t}})$.

Finally, $\mathcal{E}(\Phi_{\tilde{t}})$ is inf-complete in $\mathcal{E}(\underline{\Phi}_{\hat{t}})$ can be proved in a manner very similar to the proof for $\mathcal{E}(\underline{\Phi}_{\tilde{t}})$ is inf-complete in $\mathcal{E}(\underline{\Phi}_{\hat{t}})$. It follows that $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\Phi_{\tilde{t}})$. Statements (1)(c) and (1)(d) follow from Theorem 7, statement (3)(c).

Statement (2) is proved similarly. For statement (3), $\mathcal{E}(\underline{\Phi}_{\hat{t}})$ is sup-complete in $\mathcal{E}(\overline{\Phi}_{\tilde{t}})$ can be proved in a manner very similar to the proof for $\mathcal{E}(\underline{\Phi}_{\hat{t}})$ is sup-complete in $\mathcal{E}(\underline{\Phi}_{\tilde{t}})$, and $\mathcal{E}(\overline{\Phi}_{\tilde{t}})$ is inf-complete in $\mathcal{E}(\underline{\Phi}_{\hat{t}})$ can be proved in a manner very similar to the proof for $\mathcal{E}(\overline{\Phi}_{\tilde{t}})$ is inf-complete in $\mathcal{E}(\overline{\Phi}_{\hat{t}})$. This shows that $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\overline{\Phi}_{\tilde{t}})$. Theorem 7 now implies that $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*\ell} \mathcal{E}(\overline{\Phi}_{\tilde{t}})$.

For statement (4), to show that $\mathcal{E}(\Phi_{\hat{t}})$ is sup-complete in $\mathcal{E}(\Phi_{\tilde{t}})$, consider nonempty $E \subseteq \mathcal{E}(\Phi_{\hat{t}})$. The hypotheses imply that $\sup_{\mathcal{E}(\Phi_{\hat{t}})} \mathcal{E}(\Phi_{\tilde{t}})$ is an upper bound of E in $\mathcal{E}(\Phi_{\tilde{t}})$ and $\mathcal{E}(\Phi_{\tilde{t}})$ is complete. Therefore, $\hat{e} = \inf_{\mathcal{E}(\Phi_{\tilde{t}})}\{x \in \mathcal{E}(\Phi_{\tilde{t}}) \mid x \text{ is an upper bound of } E\} \in \mathcal{E}(\Phi_{\tilde{t}})$, whence $\sup_{\mathcal{E}(\Phi_{\tilde{t}})} E = \hat{e} \in \mathcal{E}(\Phi_{\tilde{t}})$. Similarly, $\mathcal{E}(\Phi_{\tilde{t}})$ is inf-complete in $\mathcal{E}(\Phi_{\hat{t}})$. This shows that $\mathcal{E}(\Phi_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\Phi_{\tilde{t}})$. Theorem 7 now implies that $\mathcal{E}(\Phi_{\hat{t}}) \sqsubseteq^{*\ell} \mathcal{E}(\Phi_{\tilde{t}})$. ∎

In Theorem 13, transitivity of any of the equilibrium correspondences in t cannot be assumed automatically, but it follows immediately from transitivity of the partial order on T, $\preceq_T$, as follows. For example, in statement (1) in Theorem 13, consider a parametric isotone infimum model and fix arbitrarily $t_1 \preceq_T t_2$ and $t_2 \preceq_T t_3$. Then $\mathcal{E}(\underline{\Phi}_{t_1}) \sqsubseteq^{*c} \mathcal{E}(\underline{\Phi}_{t_2})$ and $\mathcal{E}(\underline{\Phi}_{t_2}) \sqsubseteq^{*c} \mathcal{E}(\underline{\Phi}_{t_3})$. To conclude that $\mathcal{E}(\underline{\Phi}_{t_1}) \sqsubseteq^{*c} \mathcal{E}(\underline{\Phi}_{t_3})$, we use transitivity of $\preceq_T$ to conclude that $t_1 \preceq_T t_3$ and then apply statement (1) using $t_1 \preceq_T t_3$.

Theorem 13 provides a foundation for new theories of monotone comparative statics of the full equilibrium set, the infimum equilibrium set, and the supremum equilibrium set. A parametric lattice model $((X, \preceq_X), (T, \preceq_T), \Phi)$ has ***monotone comparative statics (MCS) of the full equilibrium set in the star complete set order*** if the mapping $t \mapsto \mathcal{E}(\Phi_t)$ is isotone in the star complete set order; that is, for every $\hat{t} \preceq \tilde{t}$, $\mathcal{E}(\Phi_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\Phi_{\tilde{t}})$. It has ***MCS of the infimum equilibrium set in the star complete set order*** if the mapping $t \mapsto \mathcal{E}(\underline{\Phi}_{\hat{t}})$

is isotone in the star complete set order, and it has ***MCS of the supremum equilibrium set in the star complete set order*** if the mapping $t \mapsto \mathcal{E}(\overline{\Phi}_{\hat{t}})$ is isotone in the star complete set order. Statements for MCS in the star lattice set order are defined analogously.

Corollary 14. *Consider the class of parametric lattice models.*

1. *Every parametric isotone infimum model has MCS of the infimum equilibrium set in the star complete set order and in the star lattice set order.*
2. *Every parametric isotone supremum model has MCS of the supremum equilibrium set in the star complete set order and in the star lattice set order.*
3. *Every parametric isotone infimum and supremum model in which for every t, $(X, \preceq_X, \Phi_t)$ is either isotone supremum on lower intervals or isotone infimum on upper intervals has, in addition to (1) and (2), MCS of the full equilibrium set in the star complete set order and in the star lattice set order.*

Proof. Follows from Theorem 13. ∎

Statements (1) and (2) in Corollary 14 require very little structure on the parametric model (just isotone infimum or isotone supremum). Statement (1) in Corollary 14 generalizes to the entire infimum equilibrium set the statement about MCS of the infimum equilibrium point in Theorem 11, statement (2) generalizes to the entire supremum equilibrium set the statement about MCS of the supremum equilibrium point, and statement (3) provides two generalizations of the statement about MCS of extremal equilibria, one to MCS of the extremal equilibrium sets and the other to MCS of the entire equilibrium set. These results complement theories of monotone selections of equilibrium, as in Echenique (2002), and theories based on the uniform set order, as in Shannon (1995) and Echenique and Sabarwal (2003).

Indeed, when Φ is singleton valued, *every general parametric model with complementarities satisfies the assumptions in every statement in Theorem 13 and Corollary 14*, leading to the following corollary.

Corollary 15. *For every general parametric model with complementarities $(X, \preceq, \Phi)$ in which Φ is singleton valued, the model necessarily has MCS of the full equilibrium set in both the star complete set order and the star lattice set order.*

Proof. Follows from Theorem 13 and Corollary 14. ∎

Correspondence Φ is singleton valued is a common situation in applications and is also the framework in Tarski (1955), and therefore, this corollary necessarily applies to both situations without additional assumptions. This is not true for the strong set order (or the uniform set order). Example 21 shows that comparability of parametric equilibrium sets may not hold in the strong set order even in the canonical S-model used to motivate complementarities. This can be seen by parameterizing the model in Example 21 by $T = \{0, 1, 2\}$, and defining Φ_0, Φ_1, and Φ_2 as follows: $\Phi_0(x) = \underline{\Phi}(x)$, $\Phi_1(x) = [\underline{\Phi}(x), \overline{\Phi}(x)]$, and $\Phi_2(x) = \overline{\Phi}(x)$. Similarly, Example 22 shows a distinction between standard applications of the weak set order using extremal equilibrium selections and the "closest approximation" (in terms of order) guaranteed here.

Theorem 13 implies the following corollaries about standard parametric models.

Corollary 16. *In every parametric Topkis, Vives, MR, GMS, Zhou, GCKK, and PY model,*

1. *For every* $\hat{t} \preceq \tilde{t}$,
   (a) $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\underline{\Phi}_{\tilde{t}})$ (b) $\mathcal{E}(\overline{\Phi}_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\overline{\Phi}_{\tilde{t}})$ (c) $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\overline{\Phi}_{\tilde{t}})$
   (d) $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\Phi_{\tilde{t}})$ (e) $\mathcal{E}(\Phi_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\Phi_{\tilde{t}})$ (f) $\mathcal{E}(\Phi_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\overline{\Phi}_{\tilde{t}})$
2. *For every* $\hat{t} \preceq \tilde{t}$,
   (a) $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*\ell} \mathcal{E}(\underline{\Phi}_{\tilde{t}})$ (b) $\mathcal{E}(\overline{\Phi}_{\hat{t}}) \sqsubseteq^{*\ell} \mathcal{E}(\overline{\Phi}_{\tilde{t}})$ (c) $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*\ell} \mathcal{E}(\overline{\Phi}_{\tilde{t}})$
   (d) $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*\ell} \mathcal{E}(\Phi_{\tilde{t}})$ (e) $\mathcal{E}(\Phi_{\hat{t}}) \sqsubseteq^{*\ell} \mathcal{E}(\Phi_{\tilde{t}})$ (f) $\mathcal{E}(\Phi_{\hat{t}}) \sqsubseteq^{*\ell} \mathcal{E}(\overline{\Phi}_{\tilde{t}})$

Proof. Follows from Theorem 13. ∎

Corollary 17. *Every parametric Topkis, Vives, MR, GMS, Zhou, GCKK, and PY model has MCS of the infimum equilibrium set, the supremum equilibrium set, and the full equilibrium set in star complete set order and star lattice set order.*

Proof. Follows from Corollary 16. ∎

Corollary 18.
1. *In every parametric GCKK-1, PY-1 model, for every* $\hat{t} \preceq \tilde{t}$, $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\underline{\Phi}_{\tilde{t}})$, $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\Phi_{\tilde{t}})$, *and* $\mathcal{E}(\underline{\Phi}_{\hat{t}}) \sqsubseteq^{*\ell} \mathcal{E}(\underline{\Phi}_{\tilde{t}})$. *Therefore, every parametric GCKK-1, PY-1 model has MCS of the infimum equilibrium in set star complete set order and star lattice set order.*
2. *In every parametric GCKK-2, PY-2 model, for every* $\hat{t} \preceq \tilde{t}$, $\mathcal{E}(\overline{\Phi}_{\hat{t}}) \sqsubseteq^{*c} \mathcal{E}(\overline{\Phi}_{\tilde{t}})$, $\mathcal{E}(\Phi_{\tilde{t}}) \sqsubseteq^{*c} \mathcal{E}(\overline{\Phi}_{\hat{t}}) \sqsubseteq^{*c}$, *and* $\mathcal{E}(\overline{\Phi}_{\hat{t}}) \sqsubseteq^{*\ell} \mathcal{E}(\overline{\Phi}_{\tilde{t}})$. *Therefore, every parametric GCKK-2, PY-2 model*

has MCS of the supremum equilibrium set in star complete set order and star lattice set order.

Proof. Follows from Theorem 13. ∎

5 Conclusion

We unify and generalize the theory of equilibrium in standard models with complementarities prevalent in the literature. We show that the patterns of decentralized interdependent behavior in all the different standard and neostandard models are unified in terms of the same isotone properties of their joint correspondence. We prove that the main benefits of different standard models with complementarities such as existence of extremal equilibria and MCS of extremal equilibria hold in the general model using only isotone infimum and isotone supremum selections. These results do not require continuity, strong set order, subcompleteness, or even lattice valued correspondences. We provide weaker conditions on correspondences under which the equilibrium set is a nonempty complete lattice, generalizing the well-known structure theorems of Zhou (1994) and Tarski (1955). This helps plug a gap in the literature between individual behavior and structure of systemic equilibrium outcomes.

We formulate two new set orders and show that these relations formalize equilibrium set comparisons in general models with complementarities. Using these set orders, we prove new theorems for MCS of the infimum equilibrium set, the supremum equilibrium set, and the full equilibrium set. These theorems hold in all the standard and neostandard models under natural conditions and allow for additional new cases. Moreover, our results provide a new theory of order approximation of equilibria using only the infimum selection or the supremum selection. Taken together, these results unify and expand the theory of equilibrium in models with complementarities, increase its scope of application, identify previously unknown structural relationships among equilibrium sets in such models, and formulate new theories of MCS of equilibrium sets.

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