Characterizing robust solutions in monotone games

By

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Abstract

In game theory, p-dominance and its set-valued generalizations serve as important robust solution concepts. We show that in monotone games, (which include the broad classes of supermodular games, submodular games, and their combinations,) these concepts can be characterized in terms of pure strategy Nash equilibria in an auxiliary game of complete information. The auxiliary game is constructed in a transparent manner that is easy to follow and retains a natural connection to the original game. Our results show explicitly how to map these concepts to a corresponding Nash equilibrium thereby identifying a new bijection between robust solutions in the original game and equilibrium notions in the auxiliary game. Moreover, our characterizations lead to new results about the structure of entire classes of such solution concepts. In games with strategic complements, these classes are complete lattices. More generally, they are totally unordered. We provide several examples to highlight these results.

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1 Introduction

The concepts of p-dominance and its generalizations are useful to understand equilibrium selection and equilibrium stability in games. The notion of p-dominance due to Morris, Rob, and Shin (1995) and Kajii and Morris (1997), and its set-valued generalizations of p-best response set (p-BR set) and minimal p-best response set (p-MBR set) due to Tercieux (2006a) and Tercieux (2006b) have proved to be useful in this regard. Results in the literature are typically of the form that if there is a p-dominant equilibrium or a p-MBR set with particular properties, then something useful about robustness can be concluded. The question of solving for such p-dominant equilibrium or p-MBR set is not addressed. Indeed, to the best of our knowledge, there exists no systematic tool to solve for these concepts beyond applying the definition, which involves examining each admissible belief that a player may have about her opponents' actions and all admissible supports for these beliefs. As there are uncountably many (a continuum of) such beliefs even in two player, two action games, this can be a very complex task beyond the simplest of cases.

We show that in every monotone game, solving for p-dominant equilibria, p-MBR sets, and p-EBR sets (a new concept here) is equivalent to finding particular pure strategy Nash equilibria in a corresponding auxiliary game of complete information. The class of monotone games includes the widely used class of games with strategic complements (or supermodular games), games with strategic substitutes (submodular games), and games in which both types of players are present simultaneously. We allow for finitely many players, and finitely or infinitely many actions (up to a compact subset of reals).

The auxiliary game we construct is defined in a transparent manner that is easy to follow and retains a natural connection to the original game. Actions for each player are unchanged from the original game and payoffs rely only on two weighted averages of payoffs from the original game (both using weights p and $1 - p$). One weighted average uses the lowest profile of opponent actions, the other uses the highest profile. There is no need to consider other beliefs, form expectations for each one of those beliefs, and maximize over uncountably many beliefs. The dual payoffs for each player are formalized using a "high" copy and a "low" copy for each player in the original game. Thus, if the original game has I players, the auxiliary game has $2 \times I$ players, but each player in the auxiliary game still has $I - 1$ opponents; the identities of these opponents depend on whether a player has strategic complements or strategic substitutes in the original game. Action profiles x, y in the original game correspond naturally to profile (x, y) in the auxiliary game and *vice versa*.

We prove that a profile of actions a^* is a p-dominant equilibrium in the original game, if, and only if, (a^*, a^*) is a Nash equilibrium in the auxiliary game at p, and a^* is a strict p-dominant equilibrium in the original game, if, and only if, (a^*, a^*) is a strict Nash equilibrium in the auxiliary game at p.

In order to characterize set-valued generalizations of p-dominance, we formalize the new notions of exact p-best response set (p-EBR set) and extremal response equilibrium. A p-EBR set generalizes the notion of p-MBR set and is a special case of p-BR set, while extremal response equilibrium generalizes strict Nash equilibrium to allow for particular non-strict Nash equilibria. We characterize p-EBR sets and p-MBR sets as follows: In a monotone game, if an order interval $[y, x]$ of profiles of actions is a nonempty p-EBR set, then (x, y) with $x \geq y$ is an extremal response equilibrium in the auxiliary game. If best responses are interval-valued (Assumption 1), the converse is true as well. This yields a characterization of all p-EBR sets in a monotone game. An appropriate specialization yields a characterization of all p-MBR sets. Several examples highlight these results.

The auxiliary game construction and characterization theorems are useful to prove new theorems about the structure of the entire class of p-EBR sets and p-MBR sets in monotone games. In games with strategic complements, the class of nonempty p-EBR sets and p-MBR sets are both complete lattices (in the standard lattice set order). This generalizes the result due to Zhou (1994) for the special case of Nash equilibria (which are p-dominant equilibria for $p = 1$). Minimal deviations from strategic complements destroy this structure completely. If only two players have strict strategic substitutes, or if one player has strict strategic substitutes and another has strict strategic complements, then the class of nonempty p-EBR sets and p-MBR sets are both totally unordered. This generalizes the result for Nash equilibria in Roy and Sabarwal (2008) and Monaco and Sabarwal (2016). To our knowledge, the auxiliary game construction is the only method available at present to prove the structure theorems in the paper.

As shown in a growing literature, the notion of p-dominance has been useful to understand robustness of Nash equilibrium to incomplete information and common knowledge. Morris, Rob, and Shin (1995) show that in two player, finite action games, for all sufficiently small p, a p-dominant equilibrium is robust in the sense that it survives iterated deletion of strictly dominated strategies in nearby incomplete information games, extending earlier results on global games due to Carlsson and van Damme (1993) and shedding more light on the equilibrium selection results due to Harsanyi and Selten (1988). Kajii and Morris (1997) show that in finite player, finite action games, for all sufficiently small p, a p-dominant equilibrium is robust to incomplete information arising from generalized perturbations. Frankel, Morris, and Pauzner (2003) generalize Carlsson and van Damme (1993) to games with strategic complements. Hoffmann and Sabarwal (2019) generalize this to games with strategic complements, strategic substitutes, and their combinations. They show that p-dominant equilibria for sufficiently small p emerge as the unique global games selection, extending the result in Carlsson and van Damme (1993) with fewer and more standard assumptions.

The notion of p-MBR set has been similarly useful. Tercieux (2006a) shows that p-MBR set provides a set valued concept of stability and Tercieux (2006b) generalizes the robustness results in Kajii and Morris (1997) to p-BR sets. Durieu, Solal, and Tercieux (2011) find that for p sufficiently small, strategies selected by either perturbed joint or independent fictitious play processes must be contained in a unique p-MBR set. Additionally, Maruta (1997) shows the relation of p-dominance to evolutionary stable stochastic equilibrium selection, Oyama (2002) provides a connection to equilibrium selection under perfect foresight dynamics, and Oyama and Takahashi (2020) show a connection to generalized beliefs and robustness in binary-action supermodular games.

In the literature, the results are of the form that if there is a p-dominant equilibrium or a p-MBR set with particular properties, then something useful about robustness can be concluded. This is important, because a well-known limitation of p-dominance is its non-existence in many games (especially for low values of p), negating the application of results for those cases. On the other hand, a p-MBR set exists for all values of p (and therefore, so does a p-EBR set). Our results provide a single, unified tool to solve for all three solution concepts (p-dominance, p-MBR set, and p-EBR set) simultaneously. The tool is transparent and easy to use, and yields results even when p-dominant equilibria do not exist.

Given ubiquitous use of Nash equilibrium, our results can help practitioners design games and solve for outcomes that are more likely to withstand parametric perturbations and more likely to be evolutionarily stable. Moreover, our insights about robustness concepts in a game by considering their characterization in an auxiliary game may open the door for theorists to explore similar links between other versions of higher order beliefs or weakening of common knowledge and their corresponding formulations in new types of auxiliary games. For example, Tercieux (2006a) and Oyama and Tercieux (2009) propose the notions of iterated p-dominant equilibrium and iterated p-best response set. In general it is hard to find p-dominant equilibria, p-MBR sets, and by extension, their iterated versions. Our results can help in this direction by showing that in monotone games, at each step of the iteration, our tools are available to compute p-EBR sets and p-MBR sets (both of which are p-BR sets) for that iteration, and therefore, can provide a computational aid to apply these iterated concepts. Indeed, for a decreasing collection of profiles of actions $(S^0, S^1, \ldots, S^N)$, payoff functions restricted to S^n continue to satisfy the assumed properties, and therefore, the game restricted to S^n remains a monotone game.

The next section presents a motivating example. Section 3 defines the model and presents some results. Section 4 characterizes p-dominant equilibrium. Section 5 characterizes p-EBR set and p-MBR set and gives examples to apply these results. Section 6 presents structure theorems for the class of p-EBR sets and p-MBR sets. Section 7 concludes.

2 Motivating Example

Example 1. Consider the following two player, three action game with strategic complements. It modifies Example 1 in Tercieux (2006a) so that payoff of each player has increasing differences. Each player has three actions $\{A, B, C\}$ with $A \prec B \prec C$. Payoffs are given in the bimatrix in Figure 1. The game has three Nash equilibria: (A, A), (B, B), and (C, C).

Player 2

Player 1		A	B	C
	A	6,6	0,5	0,0
	B	5,0	7,7	7,5
	C	0,0	5,7	8,8

Figure 1: Motivating example

Player 2_H

Player 1_H	A	B	C
A	3,3	0,6	0,4
B	6,0	7,7	7,6.5
C	4,0	6.5,7	8,8

(a) Payoffs for High copies

Player 2_L

Player 1_L	A	B	C
A	6, 6	3, 5	3, 0
B	5, 3	6, 6	6, 2.5
C	0,3	2.5, 6	4, 4

(b) Payoffs for Low copies

Figure 2: Auxiliary game at $p = \frac{1}{2}$

As a first step, suppose we want to find p-dominant equilibria for $p = \frac{1}{2}$. (Several results in the literature show that in two player games, p-dominant equilibria for $p \leq \frac{1}{2}$ are strongly robust.) The standard way to do this would be to check every belief that puts probability at least $\frac{1}{2}$ on each opponent action and to determine which profiles satisfy the definition. Instead, we consider only two belief-based computations for each player, formalized by their high copy and low copy in the auxiliary game at $p = \frac{1}{2}$.

In the auxiliary game, the payoff for the high copy of player 1 is given by sum of weight p on player 1's payoff from (a_1, a_2) in the original game and weight $(1 - p)$ on player 1's payoff from (a_1, C) in the original game, where C is the highest action of player 2. Player 2's payoff is computed similarly. These payoffs are given in Figure 2a. Nash equilibria for interaction among high players are (B, B) and (C, C). The payoff for the low copy of player 1 is given by weight p on player 1's payoff from (a_1, a_2) in the original game and weight $(1 - p)$ on player 1's payoff from (a_1, A) in the original game, where A is the lowest action of player 2. Player 2's payoff is computed similarly. These payoffs are given in Figure 2b. Nash equilibria for interaction among low players are (A, A) and (B, B). No other beliefs are needed.

A special case of our Theorem 2 below is that in a game with strategic complements, a profile a^* is a p-dominant equilibrium in the original game, if, and only if, a^* is an equilibrium in interactions among both high players and low players. Only (B, B) satisfies this condition, and therefore, (B, B) is the unique p-dominant equilibrium for $p = \frac{1}{2}$. This also shows that $\{B\} \times \{B\}$ is the unique p-MBR set for $p = \frac{1}{2}$. The other two Nash equilibria, (A, A) and (C, C), are not robust using the criterion of p-dominance at $p = \frac{1}{2}$.

Going further, our auxiliary game construction and characterizations provide a single, unified tool to solve for p-dominant equilibria, p-MBR sets, and p-EBR sets for every $p \in [0, 1]$ and deliver results even when p-dominant equilibrium does not exist. This is important because p-dominant equilibrium may not exist for low values of p, but p-MBR and p-EBR sets exist for all p. As an illustration, consider the same example for $p = \frac{1}{4}$. Payoffs in the auxiliary game are constructed in the same manner as above and are shown in Figure 3. Although each of the original Nash equilibria is an equilibrium in interactions among either high players or low players, none of them are in both. Therefore, without additional computation, we know there is no p-dominant equilibrium for $p = \frac{1}{4}$. Moreover, without additional computation, the same auxiliary game shows that (using Theorem 3 and Theorem 4 below) the original game has exactly two p-EBR sets $\{B, C\} \times \{B, C\}$ and $\{A, B, C\} \times \{A, B, C\}$ and a unique p-MBR set $\{B, C\} \times \{B, C\}$ (the smallest of the p-EBR sets). Indeed, a similar analysis can be carried out for every $p \in [0, 1]$, as shown in Example 2 below.

Player 2_H

Player 1_H	A	B	C
A	1.5, 1.5	0, 6.5	0, 6
B	6.5, 0	7, 7	7, 6.25
C	6, 0	6.25, 7	8, 8

(a) Payoffs for High copies

Player 2_L

Player 1_L	A	B	C
A	6, 6	4.5, 5	4.5, 0
B	5, 4.5	5.5, 5.5	5.5, 1.25
C	0, 4.5	1.25, 5.5	2, 2

(b) Payoffs for Low copies

Figure 3: Auxiliary game at $p = \frac{1}{4}$

We prove results for more general cases in Theorems 2, 3, and 4. We allow for finitely many players and finitely or infinitely many actions, up to a compact subset of the reals. Our results apply to games with strategic complements, games with strategic substitutes, and games in which arbitrary numbers of both types of players are present simultaneously. We solve several classes of examples as well. In particular, in the standard Cournot duopoly with a continuum of quantity choices, we show that for every $p < 1$, there is no p-dominant equilibrium but there is a unique p-MBR and p-EBR set.

3 Monotone game and corresponding auxiliary game

Consider finite dimensional Euclidean space, $\mathbb{R}^n$, with the standard topology and the standard pointwise order, denoted $\leq$. It is a lattice in this order, with lattice operations defined componentwise. For subsets S and S' of $\mathbb{R}^n$, S **is lower than** S' **in the lattice set order**, denoted $S \sqsubseteq S'$, if for every $x \in S$ and for every $y \in S'$, $x \wedge y \in S$ and $x \vee y \in S'$.[1] Infimum and supremum of a subset S of $\mathbb{R}^n$ are denoted $\wedge S$ and $\vee S$, respectively.

Definition 1. *A* ***monotone game*** *is a collection* $\mathcal{G} = (\mathcal{A}_i, \pi_i)_{i \in \mathcal{I}}$*, where*

1. *$\mathcal{I}$ is a finite set of players, $\mathcal{I} = \{1, 2, \ldots, I\}$.*

2. *Each player $i \in \mathcal{I}$ has an action space $\mathcal{A}_i \subset \mathbb{R}$ that is a compact sublattice in $\mathbb{R}$. The space of profiles of actions is $\mathcal{A} = \prod_{i \in \mathcal{I}} \mathcal{A}_i$. For notational convenience, we use the same symbol $\leq$ to denote the product order on subsets of $\mathcal{A}$.*

3. *The payoff of each player i is $\pi_i : \mathcal{A} \to \mathbb{R}$. It is continuous and either π_i has increasing differences in (a_i, a_{-i}), or π_i has decreasing differences in (a_i, a_{-i}), where, as usual,*

   - *π_i has* ***increasing differences in*** *(a_i, a_{-i}), if for every $a_i \leq a'_i$ and $a_{-i} \leq a'_{-i}$, $\pi_i(a'_i, a_{-i}) - \pi_i(a_i, a_{-i}) \leq \pi_i(a'_i, a'_{-i}) - \pi_i(a_i, a'_{-i})$.*
   - *π_i has* ***decreasing differences in*** *(a_i, a_{-i}), if for every $a_i \leq a'_i$ and $a_{-i} \leq a'_{-i}$, $\pi_i(a_i, a_{-i}) - \pi_i(a'_i, a_{-i}) \leq \pi_i(a_i, a'_{-i}) - \pi_i(a'_i, a'_{-i})$.*

We say that **player** i **is a strategic complements player** if π_i has increasing differences in (a_i, a_{-i}), and **player** i **is a strategic substitutes player** if π_i has decreasing differences in (a_i, a_{-i}). The definition of monotone game naturally subsumes games with strategic complements, games with strategic substitutes, and arbitrary combinations of the two. A **game with strategic complements (GSC)** is a monotone game in which payoff of every player i has increasing differences in (a_i, a_{-i}). A **game with strategic substitutes (GSS)** is a monotone game in which payoff of every player i has decreasing differences in (a_i, a_{-i}). Given the relation of increasing differences to supermodularity and decreasing differences to submodularity, GSC and GSS are sometimes termed supermodular and submodular games, respectively.

[1] The lattice set order is due to Veinott (1989), as mentioned in Topkis (1978). Other terms used in the literature are induced set ordering and strong set order.

Uncertainty about play of opponents is formalized by a probability measure on actions of opponents. Let $\Delta[\mathcal{A}_{-i}]$ denote the set of probability measures on the Borel sigma-algebra on $\mathcal{A}_{-i}$. A measure $\mu \in \Delta[\mathcal{A}_{-i}]$ is viewed as player i's belief about play of opponents. Player i's expected payoff from playing $a_i \in \mathcal{A}_i$ when belief about play of opponents is $\mu \in \Delta[\mathcal{A}_{-i}]$ is $\pi_i(a_i, \mu) = \int_{\mathcal{A}_{-i}} \pi_i(a_i, a_{-i}) d\mu$. For a profile of opponent actions a_{-i}, and $p \in [0,1]$, let $M^p[\{a_{-i}\}] = \{\mu \in \Delta[\mathcal{A}_{-i}] \mid \mu(\{a_{-i}\}) \geq p\}$ be the set of beliefs that put probability at least p on a_{-i}.

Following Morris, Rob, and Shin (1995) and Kajii and Morris (1997), p-dominant equilibrium is defined as follows. For each $p \in [0,1]$, a profile of actions $a^* \in \mathcal{A}$ is a p**-dominant equilibrium**, if for every player $i \in \mathcal{I}$,

$$a_i^* \in \{a_i \in \mathcal{A}_i \mid \forall \mu \in M^p[\{a_{-i}^*\}], \forall a_i' \in \mathcal{A}_i, \pi_i(a_i, \mu) \geq \pi_i(a_i', \mu)\}.$$

A profile of actions $a^* \in \mathcal{A}$ is a **strict** p**-dominant equilibrium**, if for every player $i \in \mathcal{I}$,

$$a_i^* \in \{a_i \in \mathcal{A}_i \mid \forall \mu \in M^p[\{a_{-i}^*\}], \forall a_i' \in \mathcal{A}_i \setminus \{a_i\}, \pi_i(a_i, \mu) > \pi_i(a_i', \mu)\}.$$

Following Tercieux (2006a), a p-best response set and minimal p-best response set are defined as follows. For each player i, consider measurable $S_i \subseteq \mathcal{A}_i$, let $S = \prod_{i \in \mathcal{I}} S_i$ and $S_{-i} = \prod_{j \neq i} S_j$. For each $p \in [0,1]$, a p-belief for player i that opponents play in S_{-i} is a probability measure $\mu \in \Delta[\mathcal{A}_{-i}]$ that assigns to S_{-i} probability at least p. Let $M^p[S_{-i}]$ denote the set of all such probability measures, that is,

$$M^p[S_{-i}] = \{\mu \in \Delta[\mathcal{A}_{-i}] \mid \mu(S_{-i}) \geq p\}.$$

When each $S_i \subseteq \mathcal{A}_i$ is nonempty and compact, let $\Lambda_i[S_{-i}, p]$ be the set of player i's best responses when player i believes that opponents will play in S_{-i} with probability at least p, that is,

$$\Lambda_i[S_{-i}, p] = \big\{a_i \in \mathcal{A}_i \mid \exists \mu \in M^p[S_{-i}], \forall a_i' \in \mathcal{A}_i, \pi_i(a_i, \mu) \geq \pi_i(a_i', \mu)\big\}.$$

Moreover, let $\Lambda[S, p] = \prod_{i \in \mathcal{I}} \Lambda_i[S_{-i}, p]$. A set $S = \prod_{i \in \mathcal{I}} S_i$ is a p**-best response set**, or p**-BR set**, if $\Lambda[S, p] \subseteq S$. A set S is a **minimal** p**-best response set**, or p**-MBR set**, if S is a p-BR set and S does not contain any proper subset that is a p-BR set. The following properties of p-MBR sets due to Tercieux (2006a) are useful for the analysis here.

Proposition 1. ***(Tercieux, 2006a)*** *For each fixed* $p \in [0,1]$*,*

1. *Every game has a p-MBR set.*
2. *Every p-BR set contains a p-MBR set.*
3. *Two distinct p-MBR sets are disjoint.*
4. $\{a^*\}$ *is a p-MBR set, if, and only if,* a^* *is a (strict) p-dominant equilibrium.*
5. *If* S *is a p-MBR set, then* $\Lambda[S,p] = S$*.*

In developing the results here, an important insight is that in a monotone game, for each $p \in [0,1]$, we can construct bounds on measures in $M^p[S_{-i}]$ and on best responses to those measures as follows. For each $p \in [0,1]$ and each $z_{-i} \in \mathcal{A}_{-i}$, let $\underline{\mu}_{z_{-i}}, \bar{\mu}_{z_{-i}} \in \Delta[\mathcal{A}_{-i}]$ be defined as

$$\underline{\mu}_{z_{-i}} = p\delta_{z_{-i}} + (1-p)\delta_{\wedge \mathcal{A}_{-i}} \quad \text{and} \quad \bar{\mu}_{z_{-i}} = p\delta_{z_{-i}} + (1-p)\delta_{\vee \mathcal{A}_{-i}},$$

where, as usual, δ_a is the degenerate measure that puts probability one on a, $\wedge \mathcal{A}_{-i} = \inf \mathcal{A}_{-i}$, and $\vee \mathcal{A}_{-i} = \sup \mathcal{A}_{-i}$. The measure $\underline{\mu}_{z_{-i}}$ puts probability p on z_{-i} and probability $1-p$ on the lowest profile of opponent actions, and measure $\bar{\mu}_{z_{-i}}$ puts probability p on z_{-i} and probability $1-p$ on the highest profile of opponent actions. Payoff to player i from playing x_i when opponents are playing $\underline{\mu}_{z_{-i}}$ or $\bar{\mu}_{z_{-i}}$ is

$$\begin{aligned}
\pi_i(x_i, \underline{\mu}_{z_{-i}}) &= p\pi_i(x_i, z_{-i}) + (1-p)\pi_i(x_i, \wedge \mathcal{A}_{-i}) \\
\pi_i(x_i, \bar{\mu}_{z_{-i}}) &= p\pi_i(x_i, z_{-i}) + (1-p)\pi_i(x_i, \vee \mathcal{A}_{-i}).
\end{aligned}$$

The intuition that $\underline{\mu}_{z_{-i}}$ is the smallest measure that puts probability p on z_{-i} and $\bar{\mu}_{z_{-i}}$ is the largest is formalized using first order stochastic dominance. Recall that for $\mu', \mu \in \Delta[\mathcal{A}_{-i}]$, μ' first order stochastically dominates μ, denoted $\mu \leq_F \mu'$, if for every increasing set $E \subseteq \mathcal{A}_{-i}$, $\mu(E) \leq \mu'(E)$. As usual, a set $E \subseteq \mathcal{A}_{-i}$ is increasing if for each $x \in E$ and each $y \in \mathcal{A}_{-i}$, $x \leq y \Rightarrow y \in E$. Measures in $M^p[S_{-i}]$ and best responses to such measures are bounded as follows. (The definition of best response is the usual one: For each $\mu \in \Delta[\mathcal{A}_{-i}]$, $BR_i(\mu) = \arg\max_{a_i \in \mathcal{A}_i} \pi_i(a_i, \mu)$.)

Lemma 1. *Let $i \in \mathcal{I}$, $S_{-i} = \prod_{j \neq i} S_j$, and $p \in [0,1]$. For every $\mu \in M^p[S_{-i}]$,*

1. $\underline{\mu}_{\wedge S_{-i}} \ \leq_F \ \mu \ \leq_F \ \bar{\mu}_{\vee S_{-i}}$
2. *If $i \in \mathcal{I}$ is a strategic complements player, then $BR_i(\underline{\mu}_{\wedge S_{-i}}) \sqsubseteq BR_i(\mu) \sqsubseteq BR_i(\bar{\mu}_{\vee S_{-i}})$.*
3. *If $i \in \mathcal{I}$ is a strategic substitutes player, then $BR_i(\bar{\mu}_{\vee S_{-i}}) \sqsubseteq BR_i(\mu) \sqsubseteq BR_i(\underline{\mu}_{\wedge S_{-i}})$.*

Proof. For statement (1), let $\mu \in M^p[S_{-i}]$ and $E \subseteq \mathcal{A}_{-i}$ be increasing. If $S_{-i} \cap E \neq \emptyset$, then for every $x \in S_{-i} \cap E$, $x \leq \vee S_{-i} \leq \vee \mathcal{A}_{-i}$, whence $\bar{\mu}_{\vee S_{-i}}(E) = p + (1-p) = 1 \geq \mu(E)$. If $S_{-i} \cap E = \emptyset$, then as E is increasing, $\bar{\mu}_{\vee S_{-i}}(E) \geq 1 - p$, and moreover, it must be that $\mu(E) \leq 1 - p$, because if $\mu(E) > 1 - p$, then $\mu(S_{-i}) + \mu(E) > p + (1-p) = 1$, a contradiction. Thus, $\mu(E) \leq 1 - p \leq \bar{\mu}_{\vee S_{-i}}(E)$, whence $\mu \leq_F \bar{\mu}_{\vee S_{-i}}$. Similarly, it can be shown that $\underline{\mu}_{\wedge S_{-i}} \leq_F \mu$.

For statement (2), notice that for $\hat{x}_i, \tilde{x}_i \in \mathcal{A}_i$,

$$\pi_i(\tilde{x}_i, \mu) - \pi_i(\hat{x}_i, \mu) = \int_{\mathcal{A}_{-i}} \pi_i(\tilde{x}_i, x_{-i}) - \pi_i(\hat{x}_i, x_{-i}) d\mu(x_{-i}).$$

If player i is a strategic complements player, then $\pi_i(x_i, x_{-i})$ has increasing differences in (x_i, x_{-i}), and therefore, $\pi_i(x_i, \mu)$ has increasing differences in (x_i, μ), where the partial order on distributions is given by first order stochastic dominance. Consequently, by Topkis (1978), $BR_i(\underline{\mu}_{\wedge S_{-i}}) \sqsubseteq BR_i(\mu) \sqsubseteq BR_i(\bar{\mu}_{\vee S_{-i}})$. Statement (3) follows similarly. □

We analyze p-dominance and p-MBR sets using an auxiliary game defined as follows.

Definition 2. *Let $\mathcal{G} = (\mathcal{A}_i, \pi_i)_{i \in \mathcal{I}}$ be a monotone game. The* ***auxiliary game*** *at $p \in [0,1]$ is $\widetilde{\mathcal{G}} = (\widetilde{\mathcal{A}}_i, \widetilde{\pi}_i)_{i \in \widetilde{\mathcal{I}}}$, where*

1. *$\widetilde{\mathcal{I}} = \mathcal{I}_H \cup \mathcal{I}_L$ is a set of $2I$ players, two copies (one high, one low) for each player in $\mathcal{G}$. The high players are denoted by $\mathcal{I}_H = \{1_H, \ldots, i_H, \ldots, I_H\}$ and the low players by $\mathcal{I}_L = \{1_L, \ldots, i_L, \ldots, I_L\}$.*
2. *The action space for each $i_H \in \mathcal{I}_H$ and $i_L \in \mathcal{I}_L$ is the same as that for the corresponding player i in $\mathcal{G}$. That is, $\widetilde{\mathcal{A}}_{i_H} = \widetilde{\mathcal{A}}_{i_L} = \mathcal{A}_i$, with the same Euclidean order. The space of profiles of actions is $\widetilde{\mathcal{A}} = \mathcal{A}_H \times \mathcal{A}_L$, where $\mathcal{A}_H = \prod_{i \in \mathcal{I}_H} \widetilde{\mathcal{A}}_{i_H}$ and $\mathcal{A}_L = \prod_{i \in \mathcal{I}_L} \widetilde{\mathcal{A}}_{i_L}$. When*

convenient, profiles $x, y \in \mathcal{A}$ are identified naturally with profile $(x, y) \in \mathcal{A}_H \times \mathcal{A}_L$, and conversely.

3. *Payoffs are defined as follows.*

   - *If $i \in \mathcal{I}$ has increasing differences in (a_i, a_{-i}), then for the corresponding i_H and i_L,* $\quad \widetilde{\pi}_{i_H}(x, y) = \pi_i(x_i, \bar{\mu}_{x_{-i}}) \quad$ *and* $\quad \widetilde{\pi}_{i_L}(x, y) = \pi_i(y_i, \underline{\mu}_{y_{-i}})$.
   - *If $i \in \mathcal{I}$ has decreasing differences in (a_i, a_{-i}), then for the corresponding i_H and i_L,* $\quad \widetilde{\pi}_{i_H}(x, y) = \pi_i(x_i, \underline{\mu}_{y_{-i}}) \quad$ *and* $\quad \widetilde{\pi}_{i_L}(x, y) = \pi_i(y_i, \bar{\mu}_{x_{-i}})$.

The auxiliary game has the following features. Although there are a total of $2I$ players in the auxiliary game, each player has $I - 1$ opponents. The identity of the opponents depends on whether a player has strategic complements in the original game or strategic substitutes.

If player $i \in \mathcal{I}$ is a strategic complements player, then the payoff for its high copy i_H in $\widetilde{\mathcal{G}}$ is affected only by high copies of other players: $\widetilde{\pi}_{i_H}(x, y) = \pi_i(x_i, \bar{\mu}_{x_{-i}}) = p\pi_i(x_i, x_{-i}) + (1 - p)\pi_i(x_i, \vee\mathcal{A}_{-i})$. The payoff for its low copy i_L in $\widetilde{\mathcal{G}}$ is affected only by low copies of other players: $\widetilde{\pi}_{i_L}(x, y) = \pi_i(x_i, \underline{\mu}_{x_{-i}}) = p\pi_i(x_i, x_{-i}) + (1 - p)\pi_i(x_i, \wedge\mathcal{A}_{-i})$. The payoff for each type relies only on a $(p, 1-p)$-weighted average of payoffs from the original game. The p-weight is the same for both types. The $(1 - p)$-weight for the high type uses the highest profile of opponent actions, while that for the low type uses the lowest profile of opponent actions. There is no need to consider additional beliefs.

If player $i \in \mathcal{I}$ is a strategic substitutes player, then the payoff for its high copy i_H is affected only by low copies of other players, and the payoff for its low copy i_L is affected only by high copies of other players. Similar statements about computation of payoffs apply in this case.

The construction of the auxiliary game preserves strategic complements and strategic substitutes for each player.

Theorem 1. *Let $\mathcal{G}$ be a monotone game and $\widetilde{\mathcal{G}}$ the auxiliary game at $p \in [0, 1]$.*

1. *If $i \in \mathcal{I}$ is a strategic complements player in $\mathcal{G}$, then both i_H and i_L are strategic complements players in $\widetilde{\mathcal{G}}$.*

2. *If $i \in \mathcal{I}$ is a strategic substitutes player in $\mathcal{G}$, then both i_H and i_L are strategic substitutes players in $\widetilde{\mathcal{G}}$.*

3. *$\widetilde{\mathcal{G}}$ is a monotone game.*

Proof. For statement (1), suppose player i is a strategic complements player in $\mathcal{G}$. Then payoff for its high copy in $\widetilde{\mathcal{G}}$ is given by $\widetilde{\pi}_{i_H}(x, y) = \pi_i(x_i, \bar{\mu}_{x_{-i}}) = p\pi_i(x_i, x_{-i}) + (1 - p)\pi_i(x_i, \vee\mathcal{A}_{-i})$. If $p = 0$, then $\widetilde{\pi}_{i_H}$ is constant with respect to opponent actions $(x, y)_{-i_H}$, and therefore, $\widetilde{\pi}_{i_H}$ satisfies increasing differences in $(x_{i_H}, (x, y)_{-i_H})$ trivially. For $p > 0$, $\widetilde{\pi}_{i_H}$ does not depend on y, and therefore, $\widetilde{\pi}_{i_H}$ satisfies increasing differences in $(x_{i_H}, (x, y)_{-i_H})$, if, and only if, π_i satisfies increasing differences in (x_i, x_{-i}). This shows that i_H is a strategic complements player in $\widetilde{\mathcal{G}}$. A similar argument shows that i_L is a strategic complements player in $\widetilde{\mathcal{G}}$. Statement (2) is proved similarly. Statement (3) follows from statements (1) and (2). □

Payoff functions in the auxiliary game are designed to produce the highest and lowest best response for each type of player, in the following sense. If player i is a strategic complements player, then the payoff for its high copy is given by $\widetilde{\pi}_{i_H}(x, y) = \pi_i(x_i, \bar{\mu}_{x_{-i}})$, and therefore, the best response of its high copy is given by $\widetilde{BR}_{i_H}((x, y)_{-i_H}) = BR_i(\bar{\mu}_{x_{-i}})$, where $BR_i(\bar{\mu}_{x_{-i}})$ is the best response of player i to belief $\bar{\mu}_{x_{-i}}$ about opponent actions in the original game $\mathcal{G}$. Lemma 1 shows that if $S_{-i} = \{x_{-i}\}$, then $\bar{\mu}_{x_{-i}}$ is the highest belief that opponents play x_{-i} with probability at least p. As player i is a strategic complements player, Lemma 1 shows further that $BR_i(\bar{\mu}_{x_{-i}})$ is the highest best response. Similarly, the payoff for its low copy is $\widetilde{\pi}_{i_L}(x, y) = \pi_i(y_i, \underline{\mu}_{y_{-i}})$, and therefore, the best response for its low copy is $\widetilde{BR}_{i_L}((x, y)_{-i_L}) = BR_i(\underline{\mu}_{y_{-i}})$, where $BR_i(\underline{\mu}_{y_{-i}})$ is the best response of player i to belief $\underline{\mu}_{y_{-i}}$ about opponent actions in the original game $\mathcal{G}$. Lemma 1 shows that $\underline{\mu}_{y_{-i}}$ is the lowest belief that opponents play y_{-i} with probability at least p. As player i is a strategic complements player, Lemma 1 shows further that $BR_i(\underline{\mu}_{y_{-i}})$ is the lowest best response. To summarize, if player i is a strategic complements player, then

$$\widetilde{BR}_{i_H}((x, y)_{-i_H}) = BR_i(\bar{\mu}_{x_{-i}}) \quad \text{and} \quad \widetilde{BR}_{i_L}((x, y)_{-i_L}) = BR_i(\underline{\mu}_{y_{-i}}).$$

A similar argument shows that if player i is a strategic substitutes player, then

$$\widetilde{BR}_{i_H}((x,y)_{-i_H}) = BR_i(\underline{\mu}_{y_{-i}}) \quad \text{and} \quad \widetilde{BR}_{i_L}((x,y)_{-i_L}) = BR_i(\bar{\mu}_{x_{-i}}).$$

The definition of (pure strategy) Nash equilibrium is unchanged, both in the original game and in the auxiliary game.

In the special case when all players are strategic complements players, that is, for a GSC, the auxiliary game decomposes into two parts: an *upper auxiliary game*, denoted $\widetilde{\mathcal{G}}_H$, composed only of interactions among high types of players, and a *lower auxiliary game*, denoted $\widetilde{\mathcal{G}}_L$, composed only of interactions among low types of players. In this case, a profile (x, y) is a Nash equilibrium in $\widetilde{\mathcal{G}}$, if, and only if, x is a Nash equilibrium in $\widetilde{\mathcal{G}}_H$ and y is a Nash equilibrium in $\widetilde{\mathcal{G}}_L$. This decomposition does not hold either for general GSS or for games in which both types of players are present.

4 Characterizing p-dominant equilibrium

The main result here characterizes p-dominant and strict p-dominant equilibrium in terms of Nash equilibrium in the corresponding auxiliary game. The strength of Theorem 2 below is that in order to locate every p-dominant equilibrium and every strict p-dominant equilibrium, it is necessary and sufficient to consider a subset of pure strategy Nash equilibria in the auxiliary game. In general, locating p-dominant equilibria requires checking every possible belief that puts probability p or more on different profiles of opponent actions. This involves uncountably many beliefs even in simple cases with few players and few actions. Construction of the auxiliary game makes the task much easier by considering high and low payoffs only (each of which is a simple $(p, 1-p)$-weighted average) and by looking for particular Nash equilibria. There is no need to consider all possible beliefs, form expectations for each one of these beliefs, and maximize over uncountably many beliefs. In other words, p-dominant equilibrium and strict p-dominant equilibrium in the original game may be found by computing pure strategy Nash equilibrium in the auxiliary game.

Theorem 2. *Let $\mathcal{G}$ be a monotone game and $\widetilde{\mathcal{G}}$ the auxiliary game at $p \in [0, 1]$.*

1. *A profile of actions a^* is a p-dominant equilibrium in $\mathcal{G}$, if, and only if, (a^*, a^*) is a*

Nash equilibrium in $\widetilde{\mathcal{G}}$.

2. *A profile of actions a^* is a strict p-dominant equilibrium in $\mathcal{G}$, if, and only if, (a^*, a^*) is a strict Nash equilibrium in $\widetilde{\mathcal{G}}$.*

Proof. Consider statement (1). Suppose a^* is a p-dominant equilibrium in $\mathcal{G}$. Then for every player i and for every $\mu \in M^p(\{a^*_{-i}\})$, $a^*_i \in BR_i(\mu)$. In particular, $a^*_i \in BR_i(\bar{\mu}_{a^*_{-i}})$ and $a^*_i \in BR_i(\underline{\mu}_{a^*_{-i}})$. Suppose player i is a strategic complements player and consider its two copies i_H and i_L in $\widetilde{\mathcal{G}}$. Then $a^*_i \in BR_i(\bar{\mu}_{a^*_{-i}}) = \widetilde{BR}_{i_H}((a^*, a^*)_{-i_H})$ and $a^*_i \in BR_i(\underline{\mu}_{a^*_{-i}}) = \widetilde{BR}_{i_L}((a^*, a^*)_{-i_L})$. A similar argument works if player i is a strategic substitutes player. Thus, (a^*, a^*) is a Nash equilibrium in $\widetilde{\mathcal{G}}$. In the other direction, suppose (a^*, a^*) is a Nash equilibrium in $\widetilde{\mathcal{G}}$. Fix player $i \in \mathcal{I}$, and let a^*_i be the common value of $a^*_{i_H}$ and $a^*_{i_L}$. If i is a strategic complements player, then for every $\mu \in M^p(\{a^*_{-i}\})$,

$$BR_i(\underline{\mu}_{a^*_{-i}}) \sqsubseteq BR_i(\mu) \sqsubseteq BR_i(\bar{\mu}_{a^*_{-i}}).$$

Moreover, (a^*, a^*) is a Nash equilibrium implies that $a^*_i \in \widetilde{BR}_{i_H}((a^*, a^*)_{-i_H}) = BR_i(\bar{\mu}_{a^*_{-i}})$ and $a^*_i \in \widetilde{BR}_{i_L}((a^*, a^*)_{-i_L}) = BR_i(\underline{\mu}_{a^*_{-i}})$. As $BR_i(\mu) \neq \emptyset$, pick $b \in BR_i(\mu)$, and then properties of lattice set order imply that $a^*_i = (a^*_i \vee b) \wedge a^*_i \in BR_i(\mu)$. A similar argument works if i is a strategic substitutes player. Thus, for every i and for every $\mu \in M^p(\{a^*_{-i}\})$, $a^*_i \in BR_i(\mu)$, whence a^* is a p-dominant equilibrium in $\mathcal{G}$.

Consider statement (2). Suppose a^* is a strict p-dominant equilibrium. Then for every player i and for every $\mu \in M^p(\{a^*_{-i}\})$, $a^*_i = BR_i(\mu)$. In particular, $a^*_i = BR_i(\bar{\mu}_{a^*_{-i}})$ and $a^*_i = BR_i(\underline{\mu}_{a^*_{-i}})$. Suppose player i is a strategic complements player and consider its two copies i_H and i_L in $\widetilde{\mathcal{G}}$. Then $a^*_i = BR_i(\bar{\mu}_{a^*_{-i}}) = \widetilde{BR}_{i_H}((a^*, a^*)_{-i_H})$ and $a^*_i = BR_i(\underline{\mu}_{a^*_{-i}}) = \widetilde{BR}_{i_L}((a^*, a^*)_{-i_L})$. A similar argument works if i is a strategic substitutes player. Consequently, (a^*, a^*) is a strict Nash equilibrium in $\widetilde{\mathcal{G}}$. In the other direction, suppose (a^*, a^*) is a strict Nash equilibrium in $\widetilde{\mathcal{G}}$. Fix player $i \in \mathcal{I}$, and let a^*_i be the common value of $a^*_{i_H}$ and $a^*_{i_L}$. If i is a strategic complements player, then for every $\mu \in M^p(\{a^*_{-i}\})$,

$$\{a^*_i\} \sqsubseteq BR_i(\mu) \sqsubseteq \{a^*_i\},$$

whence $BR_i(\mu) = a^*_i$. A similar argument works if player i is a strategic substitutes player. Thus, a^* is a strict p-dominant equilibrium in $\mathcal{G}$. □

Theorem 2 provides a natural bijection between p-dominant equilibria in $\mathcal{G}$ and particular Nash equilibria in $\widetilde{\mathcal{G}}$, and between strict p-dominant equilibria in $\mathcal{G}$ and particular strict Nash equilibria in $\widetilde{\mathcal{G}}$.

Unpacking the proof shows that a^* is a p-dominant equilibrium, if, and only if, for every player i, $a_i^* \in BR_i(\bar{\mu}_{a_{-i}^*}) \cap BR_i(\underline{\mu}_{a_{-i}^*})$, and a^* is a strict p-dominant equilibrium, if, and only if, for every player i, $a_i^* = BR_i(\bar{\mu}_{a_{-i}^*}) = BR_i(\underline{\mu}_{a_{-i}^*})$. In other words, an alternative characterization of p-dominance in monotone games can be formulated in terms of a common fixed point of two related best response correspondences (an "upper" correspondence and a "lower" correspondence) in the original game.

In general, finding common fixed points of two correspondences is harder than finding fixed points of a single correspondence. This is where the auxiliary game formulation is helpful. Theorem 2 provides an easy and transparent technique by connecting this common fixed point computation to the notion of Nash equilibrium in a "larger" auxiliary game. And of course, the notion of Nash equilibrium is accessible to a broad audience in economics (and other fields). Moreover, the same tool in Theorem 2 also *helps to determine when such a common fixed point does not exist.* Several examples given below show this explicitly.

Another benefit of the auxiliary game construction is that more general set-valued concepts such as p-MBR set and p-EBR sets do not satisfy a common fixed point argument but can still be solved using the same construction (as shown in the next section). Having a general tool is important, because a well-known limitation of p-dominance is its non-existence in many games (especially for low values of p). On the other hand, a p-MBR set and a p-EBR set exist for all values of p. The auxiliary game construction provides a single, transparent, and easy-to-follow technique that applies simultaneously to all three cases (p-dominance, p-MBR set, p-EBR set) and yields results even when p-dominant equilibria do not exist. Theorem 2 helps to characterize singleton p-MBR sets, as follows.

Corollary 1. *Let $\mathcal{G}$ be a monotone game and $\widetilde{\mathcal{G}}$ the auxiliary game at $p \in [0, 1]$.*
For a profile of actions $a^* \in \mathcal{A}$,
$\{a^*\}$ is a p-MBR set in $\mathcal{G}$, if, and only if, (a^*, a^*) is a strict Nash equilibrium in $\widetilde{\mathcal{G}}$.

Proof. Tercieux (2006a) points out that $\{a^*\}$ is a singleton p-MBR set, if, and only if, a^* is a strict p-dominant equilibrium, and Theorem 2 shows that this is equivalent to (a^*, a^*) is a strict Nash equilibrium in $\widetilde{\mathcal{G}}$. □

5 Characterizing p-EBR set and p-MBR set

In order to characterize p-MBR sets more generally, it is useful to consider a new class of sets. A set $S = \prod_{i\in\mathcal{I}} S_i$ is an **exact p-best response set**, denoted **p-EBR** set, if $\Lambda(S,p) = S$. In other words, for every player i, $\Lambda_i(S_{-i},p) = S_i$. As shown in Proposition 1, every p-MBR set has this property, and therefore, p-EBR set is a generalization of p-MBR set. Moreover, it is immediate that every p-EBR set is a p-BR set. The converse to both statements is not necessarily true. In other words, p-EBR set is an intermediate notion between p-MBR set and p-BR set.

We shall also find it useful to formalize a notion of extremal response equilibrium in the auxiliary game. Let $\mathcal{G}$ be a monotone game and $\widetilde{\mathcal{G}}$ the auxiliary game at $p \in [0,1]$. A Nash equilibrium (x,y) in $\widetilde{\mathcal{G}}$ is an **extremal response equilibrium**, if all high players are best responding with their highest best response, and all low players are best responding with their lowest best response. That is, for every $i_H \in \mathcal{I}_H$, $x_{i_H} = \vee\widetilde{BR}_{i_H}((x,y)_{-i_H})$, and for every $i_L \in \mathcal{I}_L$, $y_{i_L} = \wedge\widetilde{BR}_{i_L}((x,y)_{-i_L})$. Notice that every strict Nash equilibrium is trivially an extremal response equilibrium. The definition generalizes this to the case when a player may have multiple best responses in equilibrium, in which case the extremal response equilibrium is the one in which high players are playing their highest best response and low players are playing their lowest best response.

As shown in Theorem 3, extremal response equilibrium emerges naturally as a necessary condition to study p-EBR sets. For sufficiency, the following assumption is helpful.[2]

Assumption 1. *For each player $i \in \mathcal{I}$, each $p \in [0,1]$, and each $a_{-i}, a'_{-i} \in \mathcal{A}_{-i}$ such that $a'_{-i} \geq a_{-i}$, $BR_i(p\delta_{a_{-i}} + (1-p)\delta_{a'_{-i}})$ is interval-valued.*

When $\mathcal{A}_i$ is an interval in $\mathbb{R}$, Assumption 1 is equivalent to the statement that $BR_i(p\delta_{a_{-i}} + (1-p)\delta_{a'_{-i}})$ is convex valued, for which any one of the following conditions is sufficient. Consider the corresponding payoff function $p\pi_i(\cdot,a_{-i}) + (1-p)\pi_i(\cdot,a'_{-i})$ for which $BR_i(p\delta_{a_{-i}} + (1-p)\delta_{a'_{-i}})$ is the best response. One sufficient condition for Assumption 1 to be satisfied is that this payoff function has a unique maximizer, in which case it is trivially convex valued. A standard sufficient condition is that the payoff function is strictly quasiconcave.[3]

[2] We use the term interval in a lattice-theoretic manner; in a lattice X, an interval $[a,b] = \{x \in X \mid a \leq x \leq b\}$. The definition applies to the case when X is discrete.

[3] See Christensen (2017) for necessary and sufficient conditions for when a function has a unique maxi-

A second sufficient condition is that each $\pi_i(\cdot, a_{-i})$ is concave in a_i, in which case the payoff function is concave, and therefore, the best response is convex valued. For a third sufficient condition, Choi and Smith (2017) give conditions under which the sum of two quasiconcave functions remains quasiconcave, so that if each $\pi_i(\cdot, a_{-i})$ is quasiconcave in a_i and has an increasing portion that is more concave than each decreasing portion, their weighted sum is quasiconcave in a_i as well, and this implies that best response is convex valued.

When $\mathcal{A}_i \subseteq \mathbb{R}$ is not convex, another sufficient condition based on diminishing returns is available. Say that π_i has **diminishing returns in** a_i, if for every $a_{-i} \in \mathcal{A}_{-i}$, and every $a_i'', a_i', a_i \in \mathcal{A}_i$ with $a_i'' > a_i' > a_i$, $\pi(a_i', a_{-i}) - \pi(a_i, a_{-i}) \geq \pi(a_i'', a_{-i}) - \pi(a_i', a_{-i})$. In other words, for fixed actions of opponents a_{-i}, the marginal return to player i from playing a_i' over a lower a_i are reduced when player i plays the higher pair a_i'' versus a_i'. Morris and Ui (2005) use this assumption with strategic complements. (In games with finitely many actions, a slight weakening may be achieved by considering actions adjacent to a given action only.) This assumption is useful because concavity applies to situations when action spaces are convex intervals and our methods apply to finite actions spaces as well. The assumption of diminishing returns can be seen as a type of concavity assumption corresponding to the case of finite action spaces. Because it is not obvious that diminishing returns implies interval valued best responses in the same way that concavity does, we include a proof of this result as follows.

Proposition 2. *If for every player $i \in \mathcal{I}$, π_i has diminishing returns in a_i, then Assumption 1 is satisfied.*

Proof. See Appendix. □

Lemma 2. *Let $\mathcal{G}$ be a monotone game, $\widetilde{\mathcal{G}}$ the auxiliary game at $p \in [0,1]$, and suppose Assumption 1 holds. If (x, y) is an extremal response equilibrium in $\widetilde{\mathcal{G}}$ with $x \geq y$, then for every $i \in \mathcal{I}$, $[y_i, x_i] \subseteq \Lambda_i(\{y_{-i}, x_{-i}\}, p)$.*

Proof. See Appendix. □

The next theorem gives the main result in this section. It is significant because solving for all p-MBR sets directly from the definition in Tercieux (2006a) involves evaluating each

mizer.

player's best responses to all beliefs which put at least probability p over arbitrary subsets of opponents' actions. This is a daunting task even with few players and few actions. Our approach reduces this problem to checking for extremal response equilibria in the auxiliary game. Indeed, Theorem 3 shows that in order to find p-EBR sets (a class that includes p-MBR sets), it is necessary and sufficient to consider extremal response equilibria in the auxiliary game. We don't need to evaluate best responses to uncountably many beliefs each time over different subsets of opponent actions.

Theorem 3. *Let $\mathcal{G}$ be a monotone game and $\widetilde{\mathcal{G}}$ the auxiliary game at $p \in [0,1]$.*

1. *If S is a nonempty p-EBR set in $\mathcal{G}$, then $(\vee S, \wedge S)$ is an extremal response equilibrium in $\widetilde{\mathcal{G}}$.*

2. *If (x, y) is an extremal response equilibrium in $\widetilde{\mathcal{G}}$ with $x \geq y$, then the interval $[y, x]$ is a nonempty p-BR set in $\mathcal{G}$.*

3. *Suppose Assumption 1 is satisfied.*

   (a) *Interval $[y, x]$ is a nonempty p-EBR set in $\mathcal{G}$, if, and only if, profile (x, y) with $x \geq y$ is an extremal response equilibrium in $\widetilde{\mathcal{G}}$.*

   (b) *Every nonempty p-EBR set S in $\mathcal{G}$ is an interval of the form $S = [\wedge S, \vee S]$.*

Proof. For statement (1), suppose for every i, $S_i = \Lambda_i[S_{-i}, p]$. Consider player i and suppose player i is a strategic complements player. Then $BR_i(\bar{\mu}_{\vee S_{-i}}) \subseteq \Lambda_i(S_{-i}, p) = S_i$ implies $\vee BR_i(\bar{\mu}_{\vee S_{-i}}) \leq \vee S_i$. Moreover, $\Lambda_i(S_{-i}, p) = S_i$ implies that there is $\hat{\mu} \in M^p[S_{-i}]$ such that $\vee S_i \in BR_i(\hat{\mu})$, and therefore, using strategic complements, $\vee S_i \leq \vee BR_i(\bar{\mu}_{\vee S_{-i}})$. It follows that $\vee S_i = \vee BR_i(\bar{\mu}_{\vee S_{-i}}) \in BR_i(\bar{\mu}_{\vee S_{-i}}) = \widetilde{BR}_{i_H}((\vee S, \wedge S)_{-i_H})$. Similarly, it can be shown that $\wedge S_i = \wedge BR_i(\underline{\mu}_{\wedge S_{-i}}) \in BR_i(\underline{\mu}_{\wedge S_{-i}}) = \widetilde{BR}_{i_L}((\vee S, \wedge S)_{-i_L})$. A similar argument holds if player i is a strategic substitutes player.

For statement (2), suppose (x, y) is an extremal response equilibrium in $\widetilde{\mathcal{G}}$ with $x \geq y$. For player i, consider $a_i \in \Lambda_i([y_{-i}, x_{-i}], p)$, and let $\mu \in M^p[y_{-i}, x_{-i}]$ be such that $a_i \in BR_i(\hat{\mu})$. If player i is a strategic complements player, then $BR_i(\underline{\mu}_{y_{-i}}) \sqsubseteq BR_i(\mu) \sqsubseteq BR_i(\bar{\mu}_{x_{-i}})$, and therefore, $y_{i_L} = \wedge BR_i(\underline{\mu}_{y_{-i}}) \leq a_i \leq \vee BR_i(\bar{\mu}_{x_{-i}}) = x_{i_H}$. A similar argument holds if player i is a strategic substitutes player. Identifying $y_{i_L} = y_i$ and $x_{i_H} = x_i$ implies that for every player i, $\Lambda_i([y_{-i}, x_{-i}], p) \subseteq [y_i, x_i]$.

For statement $(3)(a)$, only necessity needs to be proved. Suppose (x, y) with $x \geq y$ is an extremal response equilibrium in $\widetilde{\mathcal{G}}$. By statement (2), $[y, x]$ is a p-BR set, so that for each player i, $\Lambda_i([y_{-i}, x_{-i}], p) \subseteq [y_i, x_i]$. Moreover, by Lemma 2, $[y_i, x_i] \subseteq \Lambda_i(\{y_{-i}, x_{-i}\}, p)$. Therefore, for every player i, $\Lambda_i([y_{-i}, x_{-i}], p) = [y_i, x_i]$, as desired.

For statement $(3)(b)$, suppose S is a p-EBR set. By statement 1, $(\vee S, \wedge S)$ is an extremal response equilibrium in $\widetilde{\mathcal{G}}$, and therefore, for each player i,

$$S_i \subseteq [\wedge S_i, \vee S_i] \subseteq \Lambda_i(\{\wedge S_{-i}, \vee S_i\}, p) \subseteq \Lambda_i(S_{-i}, p) = S_i,$$

where the second inclusion follows from Lemma 2 and the equality from the assumption that S is a p-EBR set. Consequently, for every player i, $S_i = [\wedge S_i, \vee S_i]$. □

Theorem 3 shows the close connection between p-EBR sets in $\mathcal{G}$ and extremal response equilibria in $\widetilde{\mathcal{G}}$. Statement (1) shows the necessity of considering extremal response equilibria in order to find p-EBR sets in $\mathcal{G}$. When combined with Assumption 1, statement (3)(a) shows the necessity and sufficiency of extremal response equilibria for interval p-EBR sets. Statement (3)(b) exhausts all other possibilities by necessitating p-EBR sets to be intervals.

Theorem 3 yields straightforward recipes to compute p-EBR sets in $\mathcal{G}$. It also provides a natural bijection between p-EBR sets in $\mathcal{G}$ and the subset of extremal response equilibria (x, y) in $\widetilde{\mathcal{G}}$ with $x \geq y$. These are formalized as follows.

Corollary 2. *Let $\mathcal{G}$ be a monotone game, $\widetilde{\mathcal{G}}$ the auxiliary game at $p \in [0, 1]$, and suppose Assumption 1 is satisfied.*

1. *Every strict Nash equilibrium (x, y) in $\widetilde{\mathcal{G}}$ with $x \geq y$ yields a nonempty p-EBR set $[y, x]$ in $\mathcal{G}$.*

2. *If best responses are singleton-valued, then every Nash equilibrium (x, y) in $\widetilde{\mathcal{G}}$ with $x \geq y$ yields a nonempty p-EBR set $[y, x]$ in $\mathcal{G}$.*

3. *There is a natural bijection between nonempty p-EBR sets in $\mathcal{G}$ and extremal response equilibria (x, y) in $\widetilde{\mathcal{G}}$ with $x \geq y$.*

Proof. Statement (1) follows immediately from statement (3) of Theorem 3, because a strict Nash equilibrium is an extremal response equilibrium. Statement (2) follows from statement

(1), because with singleton best responses, every Nash equilibrium is strict. Statement (3) follows from statement (3) in Theorem 3, using the natural mapping $[y, x] \mapsto (x, y)$ restricted to nonempty p-EBR sets in $\mathcal{G}$. $\square$

We characterize p-MBR sets using an appropriate specialization of this theorem.

Theorem 4. *Let $\mathcal{G}$ be a monotone game and $\widetilde{\mathcal{G}}$ the auxiliary game at $p \in [0, 1]$.*

1. *If interval $[y, x]$ is a nonempty p-MBR set in $\mathcal{G}$, then (x, y) is an extremal response equilibrium in $\widetilde{\mathcal{G}}$ with $x \geq y$, and there is no other extremal response equilibrium (x', y') in $\widetilde{\mathcal{G}}$ with $y \leq y' \leq x' \leq x$.*
2. *Suppose Assumption 1 is satisfied as well.*
   (a) *Interval $[y, x]$ is a nonempty p-MBR set in $\mathcal{G}$, if, and only if, (x, y) is an extremal response equilibrium in $\widetilde{\mathcal{G}}$ with $x \geq y$, and there is no other extremal response equilibrium (x', y') in $\widetilde{\mathcal{G}}$ with $y \leq y' \leq x' \leq x$.*
   (b) *Every p-MBR set S in $\mathcal{G}$ is of the form $S = [\wedge S, \vee S]$.*

Proof. For statement (1), notice that a p-MBR set is a p-EBR set and therefore, Theorem 3 implies that (x, y) is an extremal response equilibrium in $\widetilde{\mathcal{G}}$ with $x \geq y$. Moreover, if there is a different extremal response equilibrium (x', y') in $\widetilde{\mathcal{G}}$ with $y \leq y' \leq x' \leq x$, then Theorem 3 implies that $[y', x']$ is a strictly smaller p-BR set contained in $[y, x]$, a contradiction.

For statement (2)(a), only necessity needs to be proved. Suppose (x, y) is an extremal response equilibrium in $\widetilde{\mathcal{G}}$ with $x \geq y$, and there is no other extremal response equilibrium (x', y') in $\widetilde{\mathcal{G}}$ with $y \leq y' \leq x' \leq x$. In this case, Theorem 3 implies that $[y, x]$ is an p-BR set in $\mathcal{G}$. Suppose to the contrary that $[y, x]$ is not a p-MBR set. Then there is a p-BR set S in $\mathcal{G}$ that is a proper subset of this set, that is, $S \subsetneq [y, x]$. By Proposition 1, S contains a p-MBR set, say, $K \subseteq S$, which, by Proposition 1, is a p-EBR set. If follows from Theorem 3 that $K = [\wedge K, \vee K]$ and also that $(\vee K, \wedge K)$ is an extremal response equilibrium in $\widetilde{\mathcal{G}}$. Finally, $[\wedge K, \vee K] \subseteq S \subsetneq [y, x]$ implies that $(\vee K, \wedge K)$ is a different extremal response equilibrium than (y, x) with $y \leq \wedge K \leq \vee K \leq x$, a contradiction.

Statement (2)(b) follows, because Proposition 1 shows that a p-MBR set is a p-EBR set, which, by Theorem 3 has the desired form. $\square$

Player 2_H

		A	B	C
Player 1_H	A	6p, 6p	0, 5p+7(1-p)	0, 8(1-p)
	B	5p+7(1-p), 0	7, 7	7, 5p+8(1-p)
	C	8(1-p), 0	5p+8(1-p), 7	8, 8

Figure 4: Payoffs for High copies

Player 2_L

		A	B	C
Player 1_L	A	6, 6	6(1-p), 5	6(1-p), 0
	B	5, 6(1-p)	7p+5(1-p), 7p+5(1-p)	7p+5(1-p), 5p
	C	0, 6(1-p)	5p, 7p+5(1-p)	8p, 8p

Figure 5: Payoffs for Low copies

The following examples highlight these results.

Example 2 (Motivating example, general case)**.** Consider the example from Section 2 with payoffs given in Figure 1. The game has three Nash equilibria: (A, A), (B, B), and (C, C). For each $p \in [0, 1]$, payoffs for the high players in the auxiliary game at p are given in Figure 4 and payoffs for low players in the auxiliary game at p are given in Figure 5. Nash equilibria in the auxiliary games are listed in Figure 6. Using Theorems 2, 3, and 4, we can provide the exhaustive list of all p-dominant equilibria, all p-EBR sets, and all p-MBR sets using Nash equilibria in Figure 6. These are listed in Figure 7.

This example provides a good overview of how robust solutions may change with changes in p. For p close to 1, that is, for $p \in [\frac{7}{8}, 1]$, all three Nash equilibria are p-dominant. For $p \in [\frac{5}{6}, \frac{7}{8})$, (A, A) is no longer p-dominant; for $p \in [\frac{1}{3}, \frac{5}{6})$, (B, B) is the only p-dominant equilibrium; for $p < \frac{1}{3}$, there is no p-dominant equilibrium, but there continue to be p-MBR sets and p-EBR sets. Figure 7 provides a visual depiction of the increasing restrictiveness of robust solutions as p moves closer to 0.

Example 3 (Multi-player coordination game)**.** Following example 1.5 in Sabarwal (2021), consider a society of I players, indexed $i = 1, \ldots, I$, each of whom can choose action $x_i \in \{0, 1\}$. Action 0 is viewed as the baseline action and 1 is a new action. Playing the

p	**NE: Lower auxiliary game**	**NE: Upper auxiliary game**
$p < \frac{1}{8}$	(A, A)	(C, C)
$p \in [\frac{1}{8}, \frac{1}{3})$	(A, A), (B, B)	(C, C)
$p \in [\frac{1}{3}, \frac{5}{6})$	(A, A), (B, B)	(B, B), (C, C)
$p \in [\frac{5}{6}, \frac{7}{8})$	(A, A), (B, B), (C, C)	(B, B), (C, C)
$p \geq \frac{7}{8}$	(A, A), (B, B), (C, C)	(A, A), (B, B), (C, C)

Figure 6: Nash equilibria in upper and lower auxiliary games

p	p**-EBR sets**	p**-MBR sets**	p**-dominant equilibria**
$p < \frac{1}{8}$	$\{A, B, C\} \times \{A, B, C\}$	$\{A, B, C\} \times \{A, B, C\}$	None
$p \in [\frac{1}{8}, \frac{1}{3})$	$\{A, B, C\} \times \{A, B, C\}$, $\{B, C\} \times \{B, C\}$	$\{B, C\} \times \{B, C\}$	None
$p \in [\frac{1}{3}, \frac{5}{6})$	$\{A, B, C\} \times \{A, B, C\}$, $\{A, B\} \times \{A, B\}$, $\{B, C\} \times \{B, C\}$, $\{B\} \times \{B\}$	$\{B\} \times \{B\}$	(B, B)
$p \in [\frac{5}{6}, \frac{7}{8})$	$\{A, B, C\} \times \{A, B, C\}$, $\{A, B\} \times \{A, B\}$, $\{B, C\} \times \{B, C\}$, $\{B\} \times \{B\}$, $\{C\} \times \{C\}$	$\{B\} \times \{B\}$, $\{C\} \times \{C\}$	(B, B), (C, C)
$p \geq \frac{7}{8}$	$\{A, B, C\} \times \{A, B, C\}$, $\{A, B\} \times \{A, B\}$, $\{B, C\} \times \{B, C\}$, $\{A\} \times \{A\}$, $\{B\} \times \{B\}$, $\{C\} \times \{C\}$	$\{A\} \times \{A\}$, $\{B\} \times \{B\}$, $\{C\} \times \{C\}$	(A, A), (B, B), (C, C)

Figure 7: All p-EBR sets, p-MBR sets and p-dominant equilibria

baseline action gives a person a baseline payoff normalized to 0, regardless of the actions of others. Payoff from action 1 depends positively on the fraction of people in society who play 1 and negatively on the fraction of people who play 0, as follows. Player i's payoff from playing 0 is $\pi_i(0, x_{-i}) = 0$, and from playing 1 is

$$\pi_i(1, x_{-i}) = a\left(\frac{1}{I-1}(\sum_{j\neq i} x_j)\right) - b\left(1 - (\frac{1}{I-1}(\sum_{j\neq i} x_j))\right),$$

where $a > 0$ measures intensity of benefit from coordination, $b > 0$ measures intensity of cost of miscoordination, and $a > b$. This is a GSC and it is easy to check directly that best responses are interval-valued, so Assumption 1 is satisfied. The game has two Nash equilibria, $(0, \ldots, 0)$ and $(1, \ldots, 1)$.

Consider the auxiliary game at p. (For convenience, notation for non-relevant players is suppressed.) For the high type of player i, best responses are as follows: $\tilde{B}_{i_H}(x_{-i}) = 0$, if $\sum_{j\neq i} x_j < \frac{(I-1)((a+b)p-a)}{(a+b)p}$; $\tilde{B}_{i_H}(x_{-i}) = 1$, if $\sum_{j\neq i} x_j > \frac{(I-1)((a+b)p-a)}{(a+b)p}$; and there is indifference when $\sum_{j\neq i} x_j = \frac{(I-1)((a+b)p-a)}{(a+b)p}$. For the low type of player i, best responses are as follows: $\tilde{B}_{i_L}(y_{-i}) = 0$, if $\sum_{j\neq i} y_j < \frac{(I-1)b}{(a+b)p}$; $\tilde{B}_{i_L}(y_{-i}) = 1$, if $\sum_{j\neq i} y_j > \frac{(I-1)b}{(a+b)p}$; and there is indifference when $\sum_{j\neq i} y_j = \frac{(I-1)b}{(a+b)p}$.

Extremal response equilibria are as follows. For $p < \frac{b}{a+b}$, the fraction $\frac{(I-1)((a+b)p-a)}{(a+b)p}$ is negative. Therefore, the only option in equilibrium is for all high players to play 1 and for all low players to play 0. The unique extremal response equilibrium is $(x, y) = (1, \ldots, 1; 0, \ldots, 0)$, and consequently, the unique p-EBR set and unique p-MBR set is the whole space and there is no p-dominant equilibrium. For $\frac{b}{a+b} \leq p < \frac{a}{a+b}$, $(x, y) = (1, \ldots, 1; 0, \ldots, 0)$ and $(x, y) = (1, \ldots, 1; 1, \ldots, 1)$ are both extremal response equilibria. Therefore, there are two p-EBR sets, a unique p-MBR set and a unique p-dominant equilibrium (involving coordination on 1). For $\frac{a}{a+b} \leq p \leq 1$, the three extremal response equilibria are $(x, y) = (1, \ldots, 1; 0, \ldots, 0)$, $(x, y) = (1, \ldots, 1; 1, \ldots, 1)$, and $(x, y) = (0, \ldots, 0; 0, \ldots, 0)$. There are three p-EBR sets, two p-MBR sets (one involving coordination on 0, the other on 1), and two p-dominant equilibria (identified with p-MBR sets).

This example is a base model for binary action coordination games. The results show how robust solutions depend on the relative intensity of the benefits of coordination and costs of miscoordination. When the relative cost of miscoordination is low, say, b is small relative to a, the range $[\frac{b}{a+b}, \frac{a}{a+b})$ is large, and the only p-dominant equilibrium in this range is to coordinate on 1. Beyond this range, both Nash equilbria are p-dominant, below this range, neither equilibrium is p-dominant.

Example 4 (Multi-player externality game)**.** Following example 1.6 from Sabarwal (2021), consider a group of I people and suppose there is a good the production of which benefits everyone in this group. For example, once an open source software (operating system, text editing, web design, and so on) is produced, all users benefit from it. Or, once a social media product (meme, gif, video, tutorial, and so on) is produced and distributed, all recipients can use it.

For convenience, the benefit of the good to each person is normalized to 1 and the good can be produced by a single person by incurring a positive cost $c < 1$. Each person i can either produce the good ($x_i = 1$) or not ($x_i = 0$). Once produced, there is no significant additional benefit for anyone else to produce the same good. A profile of actions is $x = (x_1, \ldots, x_I)$, with each $x_i \in \{0, 1\}$. Given x_{-i}, if player i chooses to produce the good, payoff is $\pi_i(1, x_{-i}) = 1 - c$. If player i chooses not to produce the good, payoff is $\pi_i(0, x_{-i}) = 1$, if $x_{-i} > 0$, and $\pi_i(0, x_{-i}) = 0$, if $x_{-i} = 0$. This is a GSS and has I Nash equilibria, given by the set of basis vectors in $\mathbb{R}^I$. The equilibrium set is totally unordered.

Consider the auxiliary game at p. The payoff from choosing 1 for a high type of player i when the action profile of low opponents is y_{-i} is $\tilde{\pi}_{i_H}(1, y_{-i}) = 1 - c$, and from choosing 0 is $\tilde{\pi}_{i_H}(0, y_{-i}) = p$, if $y_{-i} > 0$, and is $\tilde{\pi}_{i_H}(0, y_{-i}) = 0$, if $y_{-i} = 0$. (For convenience, notation for non-relevant players is suppressed.) Best responses are as follows. For $y_{-i} = 0$, $\tilde{B}_{i_H}(y_{-i}) = 1$. For $y_{-i} > 0$, $\tilde{B}_{i_H}(y_{-i}) = 0$, if $1 - c < p$, and $\tilde{B}_{i_H}(y_{-i}) = 1$, if $1 - c > p$. There is indifference at $1 - c = p$.

The payoff for a low type of player i, when the action profile of high opponents is x_{-i} is $\tilde{\pi}_{i_L}(1, x_{-i}) = 1 - c$, and is $\tilde{\pi}_{i_L}(0, x_{-i}) = 1$, if $x_{-i} > 0$, and $\tilde{\pi}_{i_L}(0, x_{-i}) = 1 - p$, if $x_{-i} = 0$. Best responses are as follows. For $x_{-i} > 0$, $\tilde{B}_{i_L}(x_{-i}) = 0$. For $x_{-i} = 0$, $\tilde{B}_{i_L}(x_{-i}) = 0$, if $c > p$, and $\tilde{B}_{i_L}(x_{-i}) = 1$, if $c < p$. There is indifference at $c = p$.

Extremal response equilibria are computed as follows. When $y = 0$, it is easy to check that for every p, profile $(x, y) = (1, \ldots, 1, 0, \ldots, 0)$ is an extremal response equilibrium with $x \geq y$. For $y \neq 0$, (x, y) is an extremal response equilibrium with $x \geq y$, if, and only if, y is a basis vector in $\mathbb{R}^I$, $x = y$, and $p > \max\{c, 1 - c\}$. This is proved as follows. Suppose

(x, y) is an extremal response equilibrium with $x \geq y$ and $y \neq 0$. Then $y \neq 0$ implies that at least one component is 1. Suppose two components are 1, say, $y_1 = y_2 = 1$. Then best responses imply that $x_{-1} = x_{-2} = 0$, whence $x = 0$, and this contradicts $x \geq y$. Thus y is a basis vector. Suppose $y_i = 1$ and $y_{-i} = 0$. Then $y_i = 1$ implies $x_{-i} = 0$ and extremal response equilibrium implies $c < p$. Moreover, $x \geq y$ implies $x_i = 1$. Therefore, $x = y$. Furthermore, for every $j \neq i$, $x_j = 0$ implies $1 - c < p$. Thus, $p > \max\{c, 1 - c\}$. The other direction is easy to check directly.

Therefore, for $p > \max\{c, 1 - c\}$, the number of p-dominant equilibria is I (given by the basis vectors in $\mathbb{R}^I$), the number of p-MBR sets is I, and the number of p-EBR sets is $I + 1$. For $p \leq \max\{c, 1 - c\}$, there are no p-dominant equilibria, one p-MBR set (the whole space), and one p-EBR set (the whole space).

This example is a base model for binary action games with strategic substitutes. There are I Nash equilibria and the Nash equilibrium set is totally unordered. The example shows how all of these Nash equilibria are p-dominant above the threshold $\max\{c, 1 - c\}$, and all of them disappear below this threshold. This critical threshold depends on the relative cost of providing the good.

Example 5 (Online content provision)**.** Following example 5.1 in Sabarwal (2021), consider an interaction between a consumer (player 1) and an online content provider (player 2). The content provider chooses the quality of content to provide from four levels $\{A, B, C, D\}$ with $A \prec B \prec C \prec D$. The consumer controls the intensity of the ad blocking software using four levels $\{A, B, C, D\}$ with $A \prec B \prec C \prec D$. When ad blocking is low, content providers are better off with low quality content sponsored with ads. When ad blocking is high, it is better to provide high quality content that can be monetized in other ways. The consumer is better off using high intensity of ad blocking when the quality of content is low. Payoffs are summarized in Figure 8. Player 1 exhibits strategic substitutes, player 2 exhibits strategic complements, and the game has two (nonstrict) Nash equilibria: (B, B) and (B, C).

Consider the auxiliary game for $p \in [0, 1]$. Payoffs for each type of player 1 are presented in Figure 9 and payoffs for each type of player 2 are presented in Figure 10.

For player 1_L, we see that if player 2_H plays A, A is the best response if $p \leq \frac{1}{4}$, B is the best response for $\frac{1}{4} \leq p \leq \frac{2}{3}$ and D is the best response for $p \geq \frac{2}{3}$. If player 2_H plays B, player 1_L's best response is A if $p \leq \frac{1}{4}$ and B if $p \geq \frac{1}{4}$. If player 2_H plays C, A is player

Player 2

Player 1	A	B	C	D
A	0, 4	2, 4	3, 2	4, 1
B	3, 3	**5, 4**	**4, 4**	3, 3
C	3, 1	4, 2	3, 3	2, 4
D	4, 1	3, 2	2, 3	1, 4

Figure 8: Online content provision

Player 2_H

Player 1_L	A	B	C	D
A	4-4p	4-2p	4-p	4
B	3	3+2p	3+p	3
C	2+p	2+2p	2+p	2
D	1+3p	1+2p	1+p	1

(a) Payoffs for 1_L

Player 2_L

Player 1_H	A	B	C	D
A	0	2p	3p	4p
B	2	3+2p	3+p	3
C	3	3+p	3	3-p
D	4	4-p	4-2p	4-3p

(b) Payoffs for 1_H

Figure 9: Auxiliary game payoffs for player 1

Player 2_L

Player 1_L	A	B	C	D
A	4	4	2	1
B	4-p	4	2+2p	1+2p
C	4-3p	4-2p	2+p	1+3p
D	4-3p	4-2p	2+p	1+3p

(a) Payoffs for 2_L

Player 2_H

Player 1_H	A	B	C	D
A	1+3p	2+2p	3-p	4-3p
B	1+2p	2+2p	3+p	4-p
C	1	2	3	4
D	1	2	3	4

(b) Payoffs for 2_H

Figure 10: Auxiliary game payoffs for player 2

1_L's best response for $p \leq \frac{1}{2}$ and B if $p \geq \frac{1}{2}$. Lastly, if player 2_H plays D, A is player 1_L's best response for all $p \in [0,1]$.

For player 1_H, we see that if player 2_L plays A, D is player 1_L's best response for all $p \in [0,1]$. If player 2_L plays B, player 1_H's best response is D if $p \leq \frac{1}{3}$ and B if $p \geq \frac{1}{3}$. If player 2_L plays C, D is player 1_H's best response for $p \leq \frac{1}{3}$ and B if $p \geq \frac{1}{3}$. Lastly, if player 2_L plays D, D is the best response if $p \leq \frac{1}{3}$, B is the best response for $\frac{1}{3} \leq p \leq \frac{3}{4}$ and A is the best response for $p \geq \frac{3}{4}$.

For player 2_L, we see that if player 1_L plays A, A and B are player 2_L's best responses for all $p \in [0,1]$. If player 1_L plays B, player 2_L's best response is B if $p \in (0,1)$. C is also a best response if $p = 1$, while A is also a best response if $p = 0$. If player 1_L plays C or D, B is player 2_L's best response for $p < \frac{2}{3}$ and C if $p > \frac{2}{3}$, and indifference at $p = \frac{1}{2}$. D is also a best response if $p = 1$.

For player 2_H, we see that if player 1_H plays A, D is player 2_H's best response for $p < \frac{1}{2}$, while B is her best response if $p > \frac{1}{2}$, and indifference at $p = \frac{1}{2}$. If player 1_H plays B, player 2_H's best response is D if $p < \frac{1}{2}$ and C if $p > \frac{1}{2}$. If player 1_H plays C or D, D is player 2_H's best response for all $p \in [0,1]$.

It follows that for $p \leq \frac{1}{2}$, the unique extremal response equilibrium is $(x,y) = (D,D,A,A)$, and hence the whole space $\{A,B,C,D\} \times \{A,B,C,D\}$ is the unique p-EBR set. For $p > \frac{1}{2}$, there are two extremal Nash equilibria, (D,D,A,A) and (B,C,B,B). Therefore, there are two p-EBR sets, the entire action space $\{A,B,C,D\} \times \{A,B,C,D\}$ and $\{B\} \times \{B,C\}$. The latter one is the unique p-MBR set of the game. This game does not have a (strict) p-dominant equilibrium.

This example shows what happens when there are only non strict Nash equilibria in the original game (and therefore, no strict p-dominant equilibria). For p above the threshold $\frac{1}{2}$, our method correctly yields both equilibria as belonging to the same p-MBR set even though there are no strict p-dominant equilibria.

Example 6 (Cournot duopoly)**.** Consider a Cournot duopoly in which each firm $i \in \{1,2\}$ chooses quantity of output $q_i \in [0,50]$ and has a constant marginal cost of production $MC(q_i) = 10$. Suppose inverse market demand is given by $P(q_1,q_2) = 100 - q_1 - q_2$. Profit of each firm i can be written as $\pi_i(q_i, q_{-i}) = (90 - q_i - q_{-i})q_i$. This is a GSS and profit functions are concave in own action. Therefore, Assumption 1 is satisfied.

Let $p \in [0,1]$ be arbitrary. As both firms have strategic substitutes, payoffs in the auxiliary game are given by, for each $i \in \{1,2\}$ and each $(x,y) \in \mathcal{A}_H \times \mathcal{A}_L$,

$$\widetilde{\pi}_{i_H}(x,y) = \pi_i(x_i, p\delta_{y_{-i}} + (1-p)\delta_0) = (90 - x_i - py_{-i})x_i, \text{ and}$$
$$\widetilde{\pi}_{i_L}(x,y) = \pi_i(y_i, p\delta_{x_{-i}} + (1-p)\delta_{50}) = (40 + 50p - y_i - px_{-i})y_i.$$

The auxiliary game has a unique Nash equilibrium given by

$$(q^*_{1_H}, q^*_{2_H}, q^*_{1_L}, q^*_{2_L}) = \Big(\frac{180 - 40p - 50p^2}{4 - p^2}, \frac{180 - 40p - 50p^2}{4 - p^2}, \frac{80 + 10p}{4 - p^2}, \frac{80 + 10p}{4 - p^2}\Big).$$

Theorems 2, 3, and 4 imply that the interval $[(q^*_{1_L}, q^*_{2_L}), (q^*_{1_H}, q^*_{2_H})] \subseteq \mathcal{A}$ is the unique p-EBR set and unique p-MBR set in the original game, and therefore, for $p < 1$ there is no p-dominant equilibrium. For $p = 1$, the unique p-EBR set, unique p-MBR set, and unique p-dominant equilibrium are given by the unique Nash equilibrium $(q^*_1, q^*_2) = (30, 30)$.

Notice that for $p = 0$, this interval contains those actions that survive the first round of the process of iteratively deleting strictly dominated actions, and as p increases to 1, this interval shrinks toward the unique Nash equilibrium in the original game, $(q^*_1, q^*_2) = (30, 30)$. Figure 11 illustrates the boundaries of these intervals for $p = 0$, $p = 0.5$ and $p = 1$. This example generalizes to finitely many firms in a natural manner.

To the best of the authors' knowledge, there is no other solved example with continuous action spaces in the literature. Solving for p-EBR sets, p-MBR sets and p-dominant equilibria in this case requires keeping track of best response sets over a continuum of beliefs, each over a continuum of actions, and over subsets of continuum of actions. Our results continue to apply to such cases, simply by solving for pure strategy Nash equilibria in the auxiliary game.

6 Structure of class of p-EBR sets and p-MBR sets

Let $\mathcal{S}_E$ be the collection of nonempty p-EBR sets in $\mathcal{G}$, and $\mathcal{S}_M$ be the collection of nonempty p-MBR sets in $\mathcal{G}$. From Theorems 3 and 4, we know that every nonempty p-EBR set and every nonempty p-MBR set is a nonempty interval in $\mathcal{A}$. As the set of nonempty intervals is partially ordered by the lattice set order, it follows that both $\mathcal{S}_E$ and $\mathcal{S}_M$ are partially

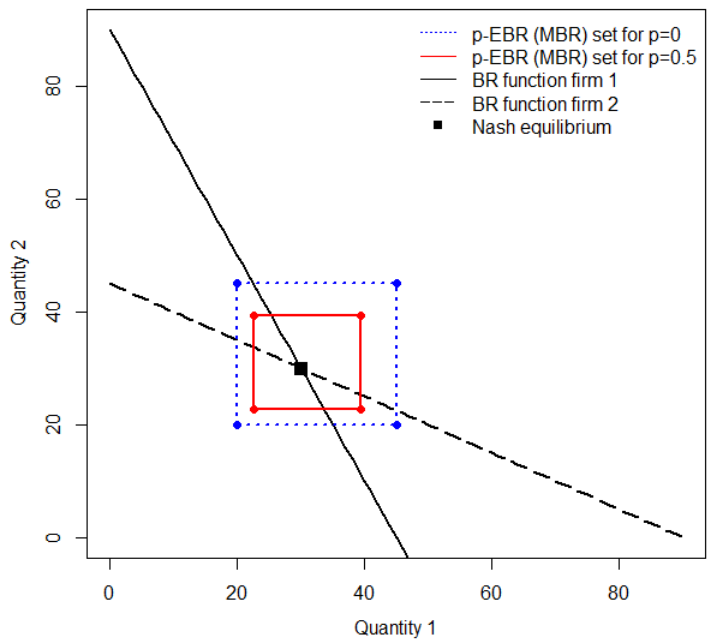

Figure 11: p-EBR (MBR) sets and Nash equilibrium in Cournot duopoly

ordered in the lattice set order.

In order to understand the structure of $\mathcal{S}_E$ and $\mathcal{S}_M$, it is helpful to have a notion of strict strategic complements and strict strategic substitutes, as follows. For subsets A, B of $\mathbb{R}^n$, A is **completely lower than** B, denoted $A \sqsubset_c B$, if $\forall a \in A$, $\forall b \in B$, $a < b$. (As usual, $a < b$ means $a \leq b$ and $a \neq b$.)

Definition 3. *Let $\mathcal{G}$ be a monotone game and $\widetilde{\mathcal{G}}$ be the auxiliary game at $p \in [0, 1]$.*

*Player $i \in \mathcal{I}$ has **strict strategic complements**, if for its high copy i_H, for every $x, x' \in \mathcal{A}_H$ and for every $y \in \mathcal{A}_L$, $x_{-i_H} < x'_{-i_H} \Rightarrow \widetilde{BR}_{i_H}((x, y)_{-i_H}) \sqsubset_c \widetilde{BR}_{i_H}((x', y)_{-i_H})$, and for its low copy i_L, for every $y, y' \in \mathcal{A}_L$ and for every $x \in \mathcal{A}_H$, $y_{-i_L} < y'_{-i_L} \Rightarrow \widetilde{BR}_{i_L}((x, y)_{-i_L}) \sqsubset_c \widetilde{BR}_{i_L}((x, y')_{-i_L})$.*

*Player $i \in \mathcal{I}$ has **strict strategic substitutes**, if for its high copy i_H, for every $y, y' \in \mathcal{A}_L$ and for every $x \in \mathcal{A}_H$, $y_{-i_L} < y'_{-i_L} \Rightarrow \widetilde{BR}_{i_H}((x, y')_{-i_H}) \sqsubset_c \widetilde{BR}_{i_H}((x, y)_{-i_H})$, and for its low copy i_L, for every $x, x' \in \mathcal{A}_H$ and for every $y \in \mathcal{A}_L$, $x_{-i_H} < x'_{-i_H} \Rightarrow \widetilde{BR}_{i_L}((x', y)_{-i_L}) \sqsubset_c \widetilde{BR}_{i_L}((x, y)_{-i_L})$.*

The next theorem shows that under Assumption 1, when $\mathcal{G}$ is a game with strategic complements, both $\mathcal{S}_E$ and $\mathcal{S}_M$ are a complete lattice. On the other hand, if $\mathcal{G}$ is a monotone game in which there are only two players with strict strategic substitutes or in which there is one player with strict strategic substitutes and one player with strict strategic complements, then both $\mathcal{S}_E$ and $\mathcal{S}_M$ are totally unordered. In other words, moving away from a GSC even in some minimal sense completely destroys the complete lattice structure of p-EBR sets and p-MBR sets. This is consistent with results for the set of Nash equilibria in these types of games, as shown in Roy and Sabarwal (2008) and in Monaco and Sabarwal (2016).

Theorem 5. *Let $\mathcal{G}$ be a monotone game, $p \in [0,1]$, $\widetilde{\mathcal{G}}$ be the auxiliary game at p, and suppose Assumption 1 holds.*

1. *If $\mathcal{G}$ is a GSC, then both $\mathcal{S}_E$ and $\mathcal{S}_M$ are complete lattices.*

2. *If $\mathcal{G}$ has either (1) two players with strict strategic substitutes, or (2) one player with strict strategic substitutes and one player with strict strategic complements, then both $\mathcal{S}_E$ and $\mathcal{S}_M$ are totally unordered.*

Proof. For statement (1), we show first that $\mathcal{S}_E$ is a complete lattice. Consider a collection of non-empty p-EBR sets indexed by t, denoted $\{[y^t, x^t] \in \mathcal{S}_E \mid t \in T\}$. Then for every $t \in T$, (x^t, y^t) is an extremal response equilibrium in $\widetilde{\mathcal{G}}$ with $x^t \geq y^t$. Thus, for every t, for every i_H, $x^t_{i_H} \in \vee\widetilde{BR}_{i_H}((x,y)_{-i_H})$, and for every i_L, $y^t_{i_L} \in \wedge\widetilde{BR}_{i_L}((x,y)_{-i_L})$.

As each player is a strategic complements player, the payoff of a high type is affected only by payoffs of other high types and the payoff of a low type is affected only by payoffs of other low types. That is, $\vee\widetilde{BR}_{i_H}((x,y)_{-i_H})$ does not depend on y and $\wedge\widetilde{BR}_{i_L}((x,y)_{-i_L})$ does not depend on x. Denote this as $\vee\widetilde{BR}_{i_H}(x_{-i_H})$ and $\wedge\widetilde{BR}_{i_L}(y_{-i_L})$ and notice that strategic complements implies that the corresponding joint best response functions denoted by $\vee\widetilde{BR}(x)$ and $\wedge\widetilde{BR}(y)$ are increasing, and therefore, the set of fixed points of each is a complete lattice.

As each x^t is a fixed point of $\vee\widetilde{BR}$ and each y^t is a fixed point of $\wedge\widetilde{BR}$, let $\underline{x}$ be the infimum of $\{x^t\}$ over the set of fixed points of $\vee\widetilde{BR}$, and $\overline{x}$ be the supremum, and similarly, let $\underline{y}$ be the infimum of $\{y^t\}$ over the set of fixed points of $\wedge\widetilde{BR}$ and $\overline{y}$ be the supremum. Then $\underline{x} \leq \wedge\{x^t\}$, $\overline{x} \geq \vee\{x^t\}$, $\underline{y} \leq \wedge\{y^t\}$, and $\overline{y} \geq \vee\{y^t\}$, and moreover, $(\underline{x}, \underline{y})$ and $(\overline{x}, \overline{y})$

are both extremal response equilibria in $\widetilde{\mathcal{G}}$ with $\underline{x} \geq \underline{y}$ and $\overline{x} \geq \overline{y}$. Consequently, both $[\underline{y}, \underline{x}]$ and $[\overline{y}, \overline{x}]$ are nonempty p-EBR sets in $\mathcal{G}$.

We show that $[\underline{y}, \underline{x}] = \inf_{\mathcal{S}_E}\{[y^t, x^t] \in \mathcal{S}_E \mid t \in T\}$ and $[\overline{y}, \overline{x}] = \sup_{\mathcal{S}_E}\{[y^t, x^t] \in \mathcal{S}_E \mid t \in T\}$. As for every t, $\underline{y} \leq y^t \leq \overline{y}$ and $\underline{x} \leq x^t \leq \overline{x}$, it follows that for every t,

$$[\underline{y}, \underline{x}] \sqsubseteq [y^t, x^t] \sqsubseteq [\overline{y}, \overline{x}].$$

Therefore, $[\underline{y}, \underline{x}]$ is a lower bound for $\{[y^t, x^t] \in \mathcal{S}_E \mid t \in T\}$ and $[\overline{y}, \overline{x}]$ an upper bound. To check that $[\underline{y}, \underline{x}]$ is the infimum, consider an abritrary nonempty p-EBR set $[\hat{y}, \hat{x}]$ that is also a lower bound. Then $(\hat{x}, \hat{y})$ is an extremal response equilibrium in $\widetilde{\mathcal{G}}$ with $\hat{x} \geq \hat{y}$, and for every t, $\hat{y} \leq y^t$ and $\hat{x} \leq x^t$. As $\underline{x}$ is the largest of the equilibria smaller than x^t, it follows that $\hat{x} \leq \underline{x}$ and similarly, $\hat{y} \leq \underline{y}$. That is, $[\hat{y}, \hat{x}] \sqsubseteq [\underline{y}, \underline{x}]$, and therefore, $[\underline{y}, \underline{x}]$ is the infimum. Similarly, $[\overline{y}, \overline{x}]$ is the supremum. This shows that $\mathcal{S}_E$ is a complete lattice.

To show that $\mathcal{S}_M$ is a complete lattice, consider a collection of nonempty p-MBR sets indexed by t, denoted $\{[y^t, x^t] \in \mathcal{S}_M \mid t \in T\}$. As $\mathcal{S}_M \subseteq \mathcal{S}_E$, each $[y^t, x^t]$ is a nonempty p-EBR set. For this collection of sets, let $\underline{x}$, $\overline{x}$, $\underline{y}$, and $\overline{y}$ be defined as above. Then $[\underline{y}, \underline{x}]$ is a nonempty p-EBR set and as $\mathcal{S}_E$ is complete, $[\underline{y}, \underline{x}] = \inf_{\mathcal{S}_E}\{[y^t, x^t] \in \mathcal{S}_M \mid t \in T\}$. We show that $[\underline{y}, \underline{x}]$ contains a unique p-MBR set of the form $[\underline{y}, \hat{x}]$ for some $\hat{x} \preceq \underline{x}$ and this is the desired infimum. A similar argument produces the desired supremum.

Notice first that every nonempty p-EBR set that is a subset of $[\underline{y}, \underline{x}]$ must be of the form $[\underline{y}, x']$ for some $x' \leq \underline{x}$. Consider a nonempty p-EBR set $[y, x] \subset [\underline{y}, \underline{x}]$. As $[\underline{y}, \underline{x}]$ is an infimum, it cannot be that $\underline{y} < y$, because if $\underline{y} < y$, then there is t' such that $\underline{y} \leq y^{t'} < y \leq x \leq \underline{x} \leq x^{t'}$, and this contradicts the fact that $[y^{t'}, x^{t'}]$ is a p-MBR set.

Notice next that as $[\underline{y}, \underline{x}]$ is a p-BR set, Proposition 1 implies that it contains at least one p-MBR set, (which must also be a p-EBR set,) and therefore, by the previous argument is of the form $[\underline{y}, x']$ for some $x' \leq \underline{x}$. Moreover, by Proposition 1, every p-MBR set is disjoint from every other p-MBR set, and therefore, there can be at most one p-MBR set that is a subset of $[\underline{y}, \underline{x}]$. It follows that there is a unique $\hat{x} \leq \underline{x}$ such that $[\underline{y}, \hat{x}]$ is the only p-MBR set contained in $[\underline{y}, \underline{x}]$. It is immediate that $[\underline{y}, \hat{x}] \sqsubseteq [\underline{y}, \underline{x}]$, and therefore, $[\underline{y}, \hat{x}]$ is a lower bound for $\{[y^t, x^t] \in \mathcal{S}_M \mid t \in T\}$. To check that this is the infimum, consider a p-MBR set

$[y', x']$ that is a larger lower bound, that is,

$$[\underline{y}, \hat{x}] \sqsubseteq [y', x'] \sqsubseteq [\underline{y}, \underline{x}].$$

This implies that $\underline{y} = y'$ and $\hat{x} \leq x' \leq \underline{x}$. As there is only one such p-MBR set, it follows that $\hat{x} = x'$, and therefore, $[\underline{y}, \hat{x}]$ is the infimum. A similar argument shows that there is unique $\hat{y} \geq \underline{y}$ such that $[\hat{y}, \overline{x}]$ is a p-MBR set and $[\hat{y}, \overline{x}] = \sup_{\mathcal{S}_M}\{[y^t, x^t] \in \mathcal{S}_M \mid t \in T\}$. Together, this shows that $\mathcal{S}_M$ is a complete lattice.

For statement (2), suppose $\mathcal{G}$ has two players with strict strategic substitutes, say, players 1 and 2, without loss of generality. Consider two distinct nonempty p-EBR sets $[y, x]$ and $[y', x']$, and suppose $[y, x] \sqsubseteq [y', x']$. Then $x \leq x'$, $y \leq y'$, and at least one inequality is strict. Suppose $x < x'$. As case 1, suppose $x_{-1} < x'_{-1}$. In this case, $x_{-1_H} < x'_{-1_H}$ and strict strategic substitutes implies $\widetilde{BR}_{1_L}((x', y)_{-1_L}) \sqsubset_c \widetilde{BR}_{1_L}((x, y)_{-1_L})$. Therefore, $\wedge\widetilde{BR}_{1_L}((x', y)_{-1_L}) < \wedge\widetilde{BR}_{1_L}((x, y)_{-1_L})$. As (x, y) and (x', y') are extremal response equilibria, $y'_{1_L} = \wedge\widetilde{BR}_{1_L}((x', y')_{-1_L}) = \wedge\widetilde{BR}_{1_L}((x', y)_{-1_L})$ and $y_{1_L} = \wedge\widetilde{BR}_{1_L}((x, y)_{-1_L})$, and this implies $y'_{1_L} < y_{1_L}$, a contradiction to $y \leq y'$. As case 2, suppose $x_{-1} = x'_{-1}$ and $x_1 < x'_1$. Then $x_{-2_H} < x'_{-2_H}$, and the same argument as in the previous case shows that $y'_{2_L} < y_{2_L}$, a contradiction to $y \leq y'$. A similar argument applies to the case $y < y'$. Therefore, $\mathcal{S}_E$ is totally unordered. As $\mathcal{S}_M$ is a subset of $\mathcal{S}_E$, it follows that $\mathcal{S}_M$ is totally unordered as well.

Now suppose $\mathcal{G}$ has one player with strict strategic substitutes (say, player 1) and one player with strict strategic complements (say, player 2). Consider two distinct nonempty p-EBR sets $[y, x]$ and $[y', x']$, and suppose $[y, x] \sqsubseteq [y', x']$. Then $x \leq x'$, $y \leq y'$, and at least one inequality is strict. Suppose $x < x'$. If $x_{-1} < x'_{-1}$, the same argument as in the case above yields a contradiction. If $x_{-1} = x'_{-1}$ and $x_1 < x'_1$, then $x_{-2} < x'_{-2}$, and therefore, $x_{-2_H} < x'_{-2_H}$. Player 2 has strict strategic complements implies $\widetilde{BR}_{2_H}((x, y)_{-2_H}) \sqsubset_c \widetilde{BR}_{2_H}((x', y)_{-2_H})$, and therefore, $\vee\widetilde{BR}_{2_H}((x, y)_{-2_H}) < \vee\widetilde{BR}_{2_H}((x', y)_{-2_H})$. As (x, y) and (x', y') are extremal response equilibria, $x_{2_H} = \vee\widetilde{BR}_{2_H}((x, y)_{-2_H})$ and $x'_{2_H} = \vee\widetilde{BR}_{2_H}((x', y')_{-2_H}) = \vee\widetilde{BR}_{2_H}((x', y)_{-2_H})$, and this implies $x_{2_H} < x'_{2_H}$, a contradiction to $x_{-1} = x'_{-1}$. A similar argument applies to the case $y < y'$. Therefore, $\mathcal{S}_E$ is totally unordered. As $\mathcal{S}_M$ is a subset of $\mathcal{S}_E$, it follows that $\mathcal{S}_M$ is totally unordered as well. □

Example 7. Consider the GSC in motivating example 2 and consider the case when $p \geq \frac{7}{8}$. In this case, as shown in Figure 7, the game has six p-EBR sets, $\{A, B, C\} \times \{A, B, C\}$,

$\{A, B\} \times \{A, B\}$, $\{B, C\} \times \{B, C\}$, $\{A\} \times \{A\}$, $\{B\} \times \{B\}$, and $\{C\} \times \{C\}$. The last three sets are also the three p-MBR sets for $p \geq \frac{7}{8}$. These sets are illustrated in Figure 12. By Theorem 5, the collection of p-EBR sets and the collection of p-MBR are complete lattices. In other words, for any two p-EBR (or p-MBR) sets, their infimum and supremum (in the lattice set order) are p-EBR (or p-MBR) sets as well. For example, consider the p-EBR sets $\{A, B, C\} \times \{A, B, C\}$ and $\{B\} \times \{B\}$. The infimum of these two sets is $\{A, B\} \times \{A, B\}$, the supremum is $\{B, C\} \times \{B, C\}$, and both are p-EBR sets.

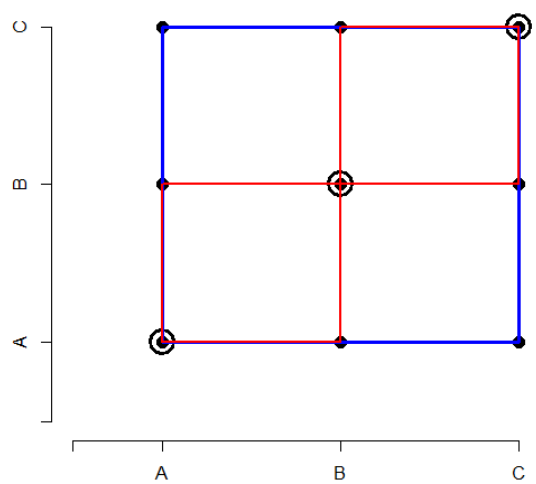

Figure 12: p-EBR and p-MBR sets for $p \geq \frac{7}{8}$

Example 8. In the multi-player externality game (Example 4), for $p > \max\{c, 1-c\}$, the number of p-MBR sets is I (each a singleton given by a basis vector), and the number of p-EBR sets is $I+1$ (all the p-MBR sets and the whole space). It is trivial to check that the collection of p-MBR sets is totally unordered. It is easy to see that the collection of p-EBR sets is totally unordered as well. Consider a p-EBR set that is a singleton, say, $S = \{e_i\}$, where e_i is the i-th basis vector, and the whole space $\mathcal{A}$. Then $S \not\sqsubseteq \mathcal{A}$, because $e_i \wedge (0, \ldots, 0) \notin S$ and $\mathcal{A} \not\sqsubseteq S$, because $(1, \ldots, 1) \vee e_i \notin S$.

The structure theorems in this section rely critically on the structure of Nash equilibria in the auxiliary game. Previous work has highlighted the structure of Nash equilibria in monotone games (see Sabarwal (2021) for a unified treatment). The bijections between robust solution concepts in the original game and particular Nash equilibria in the auxiliary game are important tools to import the structure of Nash equilibria in the auxiliary game to the class of p-EBR sets and p-MBR sets. To our knowledge, the auxiliary game construction is the only method available at present to prove these structure theorems.

7 Conclusion

We show that for monotone games, solving for p-dominant equilibrium, p-MBR set, and p-EBR set is equivalent to finding a corresponding Nash equilibrium in an auxiliary game of complete information. Our results provide a single, unified tool to solve for all three solution concepts simultaneously. The tool is transparent and easy to use, and yields results even when p-dominant equilibria do not exist. We also provide structure theorems for the classes of p-EBR sets and p-MBR sets. In a GSC, each class is a complete lattice. With minimal extensions beyond that, each class is totally unordered. Several examples highlight the results.

Appendix

Proof of Proposition 2

Proof. Let $i \in \mathcal{I}$, $a_{-i}, a'_{-i} \in \mathcal{A}_{-i}$ be such that $a'_{-i} \geq a_{-i}$, and consider $a''_i, a'_i, a_i \in \mathcal{A}_i$ such that $a''_i > a'_i > a_i$, and let $p \in [0,1]$. Notice that

$$
\begin{aligned}
& \pi_i(a'_i, p\delta_{a_{-i}} + (1-p)\delta_{a'_{-i}}) - \pi_i(a_i, p\delta_{a_{-i}} + (1-p)\delta_{a'_{-i}}) \\
= \; & p[\pi_i(a'_i, a_{-i}) - \pi_i(a_i, a_{-i})] + (1-p)[\pi_i(a'_i, a'_{-i}) - \pi_i(a_i, a'_{-i})] \\
\geq \; & p[\pi_i(a''_i, a_{-i}) - \pi_i(a'_i, a_{-i})] + (1-p)[\pi_i(a''_i, a'_{-i}) - \pi_i(a'_i, a'_{-i})] \\
= \; & \pi_i(a''_i, p\delta_{a_{-i}} + (1-p)\delta_{a'_{-i}}) - \pi_i(a'_i, p\delta_{a_{-i}} + (1-p)\delta_{a'_{-i}}),
\end{aligned}
$$

so that π_i satisfies decreasing returns in a_i against each belief $p\delta_{a_{-i}} + (1-p)\delta_{a'_{-i}}$. Suppose that $x, y \in BR_i(p\delta_{a_{-i}} + (1-p)\delta_{a'_{-i}})$ are such that $x > y$, and suppose $z \in \mathcal{A}_i$ is such that $x \geq z \geq y$. If $z \notin BR_i(p\delta_{a_{-i}} + (1-p)\delta_{a'_{-i}})$, then by decreasing returns in a_i,

$$
\begin{aligned}
0 \; > \; & \pi_i(z, p\delta_{a_{-i}} + (1-p)\delta_{a'_{-i}}) - \pi_i(y, p\delta_{a_{-i}} + (1-p)\delta_{a'_{-i}}) \\
\geq \; & \pi_i(x, p\delta_{a_{-i}} + (1-p)\delta_{a'_{-i}}) - \pi_i(z, p\delta_{a_{-i}} + (1-p)\delta_{a'_{-i}}),
\end{aligned}
$$

contradicting the optimality of x. Hence $BR_i(p\delta_{a_{-i}} + (1-p)\delta_{a'_{-i}})$ is interval-valued. $\square$

Proof of Lemma 2

Proof. Notice first that for each $a_{-i}, a'_{-i} \in \mathcal{A}_{-i}$, the best response correspondence $BR_i(p\delta_{a_{-i}} + (1-p)\delta_{a'_{-i}})$ is upper hemicontinuous in p. This follows from Berge's theorem, because the function $(a_i, p) \mapsto p\pi_i(a_i, a_{-i}) + (1-p)\pi_i(a_i, a'_{-i})$ is continuous, and therefore,

$$BR_i(p\delta_{a_{-i}} + (1-p)\delta_{a'_{-i}}) = \underset{a_i \in \mathcal{A}_i}{argmax}(p\pi_i(a_i, a_{-i}) + (1-p)\pi_i(a_i, a'_{-i}))$$

is upper hemicontinuous in p.

Suppose without loss of generality that player i is a strategic complements player, so that π_i satisfies increasing differences in (a_i, a_{-i}). The case when π_i satisfies decreasing differences in (a_i, a_{-i}) can be proven similarly.

We now show that $[y_i, x_i] \subseteq \Lambda_i(\{y_{-i}, x_{-i}\}, p)$. The remainder of the proof follows along the lines of the intermediate value theorem given for correspondences in Mutoh (2006), but adapted for our purposes. Because player $i \in \mathcal{I}$ is a complements player and (x, y) is an extremal response equilibrium, the definition of an auxiliary game implies

$$x_i = \vee \widetilde{BR}_{i_H}((x, y)_{-i}) = \vee BR_i(\bar{\mu}_{x_{-i}}) \text{ and } y_i = \wedge \widetilde{BR}_{i_L}((x, y)_{-i}) = \wedge BR_i(\underline{\mu}_{y_{-i}}),$$

so that $y_i, x_i \in \Lambda_i(\{y_{-i}, x_{-i}\}, p)$. Also, because $BR_i(y_{-i})$ and $BR_i(x_{-i})$ are best responses to beliefs $\delta_{y_{-i}}$ and $\delta_{x_{-i}}$, respectively, both of which put probability 1 on $\{y_{-i}, x_{-i}\}$, we have that $BR_i(y_{-i}), BR_i(x_{-i}) \subset \Lambda_i(\{y_{-i}, x_{-i}\}, p)$ as well. Using arguments similar to those in the proof to Theorem 1, it is readily verified that $T = \{\underline{\mu}_{y_{-i}}, \delta_{y_{-i}}\}$, $T' = \{\delta_{y_{-i}}, \delta_{x_{-i}}\}$, and $T'' = \{\delta_{x_{-i}}, \bar{\mu}_{x_{-i}}\}$ are partially ordered sets. By strategic complementarities, when $\geq_F$ is a partial order, $\mu \geq_F \nu$ implies $BR_i(\nu) \sqsubseteq BR_i(\mu)$, and hence we can write $[y_i, x_i]$ as

$$[y_i, x_i] = [y_i, \wedge BR_i(y_{-i})] \cup [\wedge BR_i(y_{-i}), \vee BR_i(x_{-i})] \cup [\vee BR_i(x_{-i}), x_i],$$

where the endpoints of all three intervals above are included in $\Lambda_i(\{y_{-i}, x_{-i}\}, p)$. Thus, the result follows by showing that all three intervals are included in $\Lambda_i(\{y_{-i}, x_{-i}\}, p)$.

We first show that $[y_i, \wedge BR_i(y_{-i})]$ is included in $\Lambda_i(\{y_{-i}, x_{-i}\}, p)$. Consider the beliefs

$\mu_\alpha \in \Delta(A_{-i})$ given by

$$\mu_\alpha = [\alpha p + (1-\alpha)]\delta_{y_{-i}} + [\alpha(1-p)]\delta_{\wedge \mathcal{A}_{-i}},$$

for each $\alpha \in [0,1]$. Note that for each such α, μ_α puts at least probability p on y_{-i}, and hence on $\{y_{-i}, x_{-i}\}$. Also, by the discussion above, $\wedge BR_i(y_{-i}) \in BR_i(\mu_0)$, and $y_i \in BR_i(\mu_1)$. Suppose there exists some $z \in \mathcal{A}_i$ such that $y_i < z < \wedge BR_i(y_{-i})$, but $z \notin \Lambda_i(\{y_{-i}, x_{-i}\}, p)$. Then $y_i < z < x_i$.

Consider the sets $K_z^- = \{\alpha \in [0,1] \mid \forall x \in BR_i(\mu_\alpha), x < z\}$ and $K_z^+ = \{\alpha \in [0,1] \mid \forall x \in BR_i(\mu_\alpha), z < x\}$. First, notice that K_z^- and K_z^+ are non-empty: From above, we know $y_i \in BR_i(\mu_1)$ and $y_i < z$. If some other $x \in BR_i(\mu_1)$ were such that $x \geq z$, then because $BR_i(\mu_1)$ is interval-valued, we would have $z \in BR_i(\mu_1)$ as well. Because the belief μ_1 puts probability at least p on $\{y_{-i}, x_{-i}\}$, we would have $z \in \Lambda_i(\{y_{-i}, x_{-i}\}, p)$, a contradiction. Hence, $1 \in K_z^-$. Likewise, $0 \in K_z^+$. Next, notice that $[0,1] = K_z^- \cup K_z^+$: Since each μ_α puts at least probability p on $\{y_{-i}, x_{-i}\}$, then for each $\alpha \in [0,1]$, $BR_i(\mu_\alpha) \subseteq \Lambda_i(\{y_{-i}, x_{-i}\}, p)$. Suppose $\alpha \in [0,1]$ is such that for some $x \in BR_i(\mu_\alpha)$, $x > z$. If there exists some other $y \in BR_i(\mu_\alpha)$ such that $z > y$, then because $BR_i(\mu_\alpha)$ is interval-valued, we have that $z \in BR_i(\mu_\alpha) \subseteq \Lambda_i(\{y_{-i}, x_{-i}\}, p)$, a contradiction. Hence, for each α such that for some $x \in BR_i(\mu_\alpha)$ we have $x > z$, it follows that $\alpha \in K_z^+$. Likewise, for each α such that for some $x \in BR_i(\mu_\alpha)$ we have $z > x$, it follows that $\alpha \in K_z^-$. Because each $\alpha \in [0,1]$ satisfies one of the two requirements, it follows that $[0,1] \subseteq K_z^- \cup K_z^+$, establishing equality.

Lastly, upper hemicontinuity of $BR_i(\mu_\alpha)$ in α implies that K_z^- is an open set, because it is the upper inverse of the open set $(-\infty, z)$ and K_z^+ is an open set, because it is the upper inverse of the open set (z, ∞). Taken together, this contradicts the connectedness of $[0,1]$. Therefore, $[y_i, \wedge BR_i(y_{-i})] \subseteq \Lambda_i(\{y_{-i}, x_{-i}\}, p)$.

A similar argument shows that $[\vee BR_i(x_{-i}), x_i] \subseteq \Lambda_i(\{y_{-i}, x_{-i}\}, p)$ by considering beliefs of the form $\mu_\alpha = [\alpha p + (1-\alpha)]\delta_{x_{-i}} + [\alpha(1-p)]\delta_{\vee A_{-i}}$. Likewise, it follows that $[\wedge BR_i(y_{-i}), \vee BR_i(x_{-i})] \subseteq \Lambda_i(\{y_{-i}, x_{-i}\}, p)$ by considering beliefs of the form $\mu_\alpha = (1-\alpha)\delta_{y_{-i}} + \alpha\delta_{x_{-i}}$. $\square$

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