Chaos in the UK New Keynesian Macroeconomy

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Abstract

We study the stability properties and conditions for the onset of Shilnikov chaos in the UK New Keynesian macroeconomy, as well as the shifts in the equilibrium dynamics under various policy regimes. We find that Shilnikov chaos emerges for a restricted part of the free parameters space in the baseline rational expectations UK model with no regime switching. When the UK's central bank showed a weak response to inflation in the high inflation regime, the chaos did not occur at all. But Shilnikov chaos appears easily in the case of the low-inflation regime, which is associated with the Bank of England's use of aggressive monetary policy in recent years. Tightening the monetary policy interest-rate-feedback rule via the Taylor coefficient is one of the policy alternatives proposed by the local analysis for restoring uniqueness. We find that doing so accelerates the emergence of unanticipated phenomena such as Shilnikov's chaotic dynamics.

Our results with UK data are thereby consistent with the results with US data by Barnett *et al.* (2020), who found that the adoption of an active interest rate feedback rule in recent years by the Federal Reserve produces Shilnikov chaos and unintentional downward drift in interest rates towards the lower bound. The source of the chaos and downward drift in interest rates is adoption of a myopic short-run interest-rate feedback rule without a terminal condition as long run anchor. A critical assumption of the results with US and UK data are existence of new Keynesian sticky prices. While the model's parameters were calibrated with pre-Brexit data, we expect that our results will be highly relevant post-Brexit, as the needed data become available. Changes in the geometry of the Shilnikov fractal attractor set can be expected to be revealing about changes in the level and nature of UK economic risk following Brexit.

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1. Introduction

The United Kingdom (UK) economy has undergone significant structural and policy

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changes in the past decades. The 1970s and the 1980s were characterized by volatile inflation and output growth (Benati (2004), Benati (2008)), whereas the period following the implementation of inflation targeting in 1992 saw low inflation and output volatility. Prior to the implementation of inflation targeting in 1992, the monetary authority used a passive monetary policy, by which the interest rate rose less than one-for-one with increase in inflation (Nelson and Nikolov (2004)). During this time, the impact of monetary policy shocks on inflation was considerable and positive (Castelnuovo and Surico (2005)). However, post inflation targeting period when the monetary authority used an active monetary policy, the inflation responses were smaller and negative. This paper attempts to capture the shifts in the stability dynamics of the model across the changing regimes. More specifically, the paper studies the possibility of onset of Shilnikov chaos under the different policy regimes for the UK economy. While the model's parameters were calibrated with pre-Brexit data, we expect the results to be highly relevant post-Brexit.

The standard New Keynesian (NK) model with an active monetary policy in accordance with the Taylor Principle was initially thought to be a sufficient criterion for determinacy in NK models. However, there is a large literature on the complex dynamics problems that NK models with aggressive interest rate policies produce. Sveen and Weinke (2005, 2007) show that inclusion of firm-specific capital in a standard NK model can lead to multiple equilibria with aggressive interest rate policies. Moreover, following the Taylor Principle is not sufficient in the presence of nominal capital income taxation (Røisland (2003) and Edge and Rudd (2007)), or in the presence of high government consumption (Natvik (2009), Galí et al. (2004)), or in the presence of trend inflation (Coibion and Gorodnichenko (2011) and Kiley (2007)). The policy implications of those papers are that the higher the effective capital income tax or government consumption, the more aggressively the interest rate should respond to inflation in order to attain a determinate equilibrium.

Another major obstacle to uniqueness of equilibrium in the NK economy is fiscal policy's inability or unwillingness to adjust primary surpluses to stabilize government debt, which may conflict with the central bank's inflation objective (Kumhof *et al.*, 2010). Regardless of the stance of fiscal policy, the role that the demand for money by agents plays in the monetary-transmission mechanism may also undermine the uniqueness of the equilibrium and encourage the onset of expectation-driven fluctuations (cf. Benhabib *et al.* 2001a,b). Further limits may result from the way preferences and technologies are introduced into the model.

In this paper, using the path-breaking work of Shilnikov (1965), we find that there may be further reasons to distrust the ability of Taylor rules to be conducive to stability.[6] We

[6] Consider, for example, the case in which the policy maker runs an active fiscal-monetary regime. Assume further that a change in the conduct of fiscal policy induces uniqueness of the equilibrium around the intended steady state. Then, the policy maker may be pressured to renounce discretion in fiscal policy by

show that this policy may induce a class of policy difficulties emerging from the onset of a chaotic attractor. If the economy becomes enmeshed in a chaotic attractor, the policy maker faces unwanted challenges. Within a chaotic attractor, there is sensitivity to initial conditions, even to infinitesimal changes in initial conditions. Because an initial condition can only be known to a finite degree of precision, long-term predictions become nearly impossible. It becomes impossible to predict dynamics far into the future. Small changes in the initial conditions have major effects on future temporal evolution. Furthermore, given the initial value of the predetermined variable, a continuum of initial values of the jump variables would result in admissible equilibria.

In this paper, we identify the subset of the parameter space for the UK model that supports stable solutions and the subset that supports chaotic dynamics. Based on the findings of Liu and Mumtaz (2011) spanning 1970Q1 to 2009Q1, we consider three different regimes: a model with fixed parameters, a model with high inflation volatility and lower monetary authority reaction to inflation, and a model with low inflation but high monetary authority reaction to inflation. In the model with constant parameters, the emergence of Shilnikov chaos is supported by a very narrow region. The model of high inflation and low reaction of monetary authorities is stable and does not support any form of Shilnikov chaos. Finally, the model with low inflation and a highly responsive monetary authority yields a large subspace of parameters that supports Shilnikov chaos. This case is consistent with the UK policy design in recent decades. Furthermore, the chaos subset expands with a higher intra-temporal elasticity of substitution between consumption and money, as well as a higher value of mark up over marginal cost, according to our research. As the price stickiness increases, the instability region expands. The central role of price stickiness is consistent with the findings of Barnett *et al.* (2020) with US data.

The liquidity trap observed in the UK during the post-inflation targeting period may be explained by the existence of this Shilnikov chaotic attractor in the low inflation and highly responsive monetary authority regime. The phenomenon has previously been linked mainly to the existence of a low-inflation steady state (see Benhabib *et al.* (2001a,b)). However, we offer a different explanation. The economy lingers in specific regions along the chaotic attractor, where model dynamics tend to evolve for a long time around inflation and nominal interest rates that are lower than expected. The resulting unintended downward drift of interest rates on the chaotic attractor set is consistent with the findings of Barnett et al (2020) with US data. Another consequence is the post-inflation targeting regime's lower and negative inflation responses to monetary policy shocks.

While the model's parameters were calibrated with pre-Brexit data, we expect that our results will be highly relevant post-Brexit. Changes in the geometry of the Shilnikov

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committing to a marginal tax rate above the real interest rate. As we show, a consequence could be Shilnikov chaos.

fractal attractor set can be expected to be revealing about changes in the level and nature of UK economic risk following Brexit.

We now present the plan of the paper. In Section 2, we present the model and the implied three-dimensional system of first-order differential equations. We also obtain stability results for the intended steady state, when monetary policy is active. In Section 3, we show that the three-dimensional dynamics, characterizing the solution of the model, can satisfy the requirements of the Shilnikov (1965) theorem under plausible calibration settings of the NK model for the UK. An example of chaotic dynamics is also discussed, along with its sensitivity to perturbations of the bifurcation parameter and the initial conditions. We finally consider the policy implications produced by the dynamics of the economy within the Shilnikov attractor set. The conclusion reassesses the main findings of the paper.

2. The model

Consider the optimization problem faced by household-firm *i* in the money-in-the utility-function, NK model in continuous time.[7] We shall call this problem Decision P.

Decision P:

$$\underset{c_i,m_i,l_i}{Max} \int_0^{\infty} \left[u(c_i, m_i) - f(l_i) - \frac{\eta}{2}(\pi_i - \pi^*)^2 \right] e^{-\rho t} dt$$

subject to

$$\begin{aligned} \dot{a}_i &= (R - \pi_i)a_i - Rm_i + \tfrac{p_i}{p} y(l_i) - c_i - \tau \\ \dot{p}_i &= \pi_i p_i \\ a_i(0) &= a_{i0} \\ p_i(0) &= p_{i0} \,. \end{aligned}$$

Decision P is produced from the framework developed originally in Benhabib *et al.* (2001a,b), which is still a standard reference point for ongoing theoretical research and policy advocacy (cf. *inter al.*, Benhabib *et al*. (2014), Tsuzuki, (2016), Le Riche *et al*., (2017)). For comparability, we adopt the same framework used with US data in Barnett *et al.* (2020), but with UK data we use the model generalized to include money in the utility

[7] The money in the utility function approach implicitly uses the derived utility function shown to exist by Arrow and Hahn (1971),if money has positive value in equilibrium. A long literature has repeatedly confirmed this existence from models having various explicit motives for holding money, such as transactions or liquidity constraints (e.g., Feenstra (1986), Poterba and Rotemberg (1987), and Wang and Yip (1992). Recently, in a dynamical framework, Benhabib *et al.* (2001a,b; 2002) have shown equivalence to a money in the production function model.
The mapping from explicit motives for holding money to the derived utility function does not have a unique inverse. Hence, money in the utility function models cannot reveal the explicit motive for holding money. But the ability to infer the explicit motive is not relevant to our research. Hence, for our purposes, we can assume that money has positive value in equilibrium, without conditioning upon an explicit motive.

function.

Prices are sticky in the sense of Rotemberg (1982). The objective of the household-firm optimizer is to maximize the discounted sum of a *net* utility stream, where $u(c_i, m_i)$ measures utility derived by household-firm i from consumption of the composite good, c_i, and from real money balances, m_i, under the time discount rate, ρ. It is assumed that $u(.,.)$ is twice continuously differentiable in all its arguments and that

$$u_c(c_i, m_i) > 0; \quad u_{cc}(c_i, m_i) < 0; \quad u_m(c_i, m_i) > 0; \quad u_{mm}(c_i, m_i) < 0, \tag{1}$$

where the function subscripts denote partial derivatives.

The function $f(l_i)$ measures the disutility of labor, where $f(l_i)$ is twice continuously differentiable, with $f_l > 0$ and $f_{ll} < 0$. The term $\frac{\eta}{2}(\pi_i - \pi^*)^2$ is standard to account for deviations of the percentage price change, $\pi_i = \frac{\dot{p}_i}{p_i}$, with regard to the intended rate π^*, where p_i is the price charged by individual i on the good it produces, and where the parameter η measures the degree to which household-firms dislike to deviate in their price-setting behavior from π^*. In the household-firm budget constraint, a_i denotes real financial wealth, consisting of interest-bearing government bonds, where R is the nominal interest rate and $y(l_i)$ is the amount of perishable goods, produced according to a production function using labor, l_i, as the only input. Real lump-sum taxes are denoted by τ.

Sales of good i are demand determined, so that

$$y(l_i) = \left(\frac{p_i}{p}\right)^{-\phi} y^d , \tag{2}$$

where $\phi > 1$ is the elasticity of substitution across varieties, and p is the aggregate price level. From the first order conditions solving the Hamiltonian for Decision (P), and assuming a symmetric equilibrium in which all household-firm units' behaviors are based on the same equations, and with $c = y(l)$ in equilibrium, we have that the following three-dimensional system of differential equations, which we shall call System *M*.[8]

System *M*:

$$\begin{aligned}
\dot{\mu}_1 &= (\rho - R + \pi)\mu_1 \\
\eta\dot{\pi} &= \rho(\pi - \pi^*)\eta - c(\mu_1, \pi)\left[(1 - \phi)\mu_1 + \phi c(\mu_1, \pi)^{\psi}\right] \\
\dot{a} &= (R - \pi)a - Rm(c(\mu_1, \pi), R) - \tau.
\end{aligned}$$

[8] Barnett et. al (2020), Benhabib *et al.* (2001a,b), and Tsuzuki (2016) for details. In system *M*, subscripts are dropped to simplify notation.

The first equation denotes the time evolution of the Lagrange multiplier associated with the continuous time budget constraint (or shadow price of the real value of aggregate per capita government liabilities, real balances, and bonds) at instant of time *t*. The second equation is the well-known New Keynesian Phillips Curve. The third equation is the budget constraint at time *t*. Solutions of system *M* are admissible equilibrium paths, if the Transversality Condition (TVC)

$$0 = \lim_{t\to\infty} e^{-\int_0^t [R(s)-\pi(s)]ds} a(t) \tag{3}$$

is satisfied.[9]

Following Benhabib *et al.* (2001a,b), we assume that the monetary authority adopts an inflation-targeting policy described by the interest rate feedback rule

$$R = R(\pi). \tag{4}$$

The function $R(\pi)$ is continuous, strictly convex, and satisfies the following properties.

Assumption 1. (Zero lower bound on nominal rates and Taylor principle). *Monetary authorities set the nominal interest rate as an increasing function of the inflation rate, so that*

$$R = R(\pi) > 0; \qquad R'(\pi) > 0; \qquad R''(\pi) > 0. \tag{5}$$

It is further assumed that there exists an inflation rate, π^*, at which the following steady-state Fisher equation is satisfied:

$$R(\pi^*) = \bar{R}. \tag{6}$$

Consider, moreover, the following definition (cf. Benhabib *et al.*, 2001a,b).

Definition 1. *Monetary policy is said to be active, if* $R'(\pi) > 1$ *and passive otherwise.*

Let us now turn our attention to fiscal policy. We assume that taxes are tuned according to fluctuations in total real government liabilities, *a*, so that

$$\tau = \tau(a). \tag{7}$$

It is further assumed that there exists a tax rate corresponding to the steady-state state level of real government liabilities

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[9] The TVC prevents households from engaging in Ponzi games.

$$\tau(a^*) = \bar{\tau}. \tag{8}$$

As in Leeper (1991), Woodford (2003), and Kumhof *et al.* (2010), we provide a definition of the fiscal policy stance. Let us consider the responses of a to its own variations

$$\frac{\partial \dot{a}}{\partial a} = R(\pi) - \pi - \tau'(a). \tag{9}$$

The dynamic path of total government liabilities is locally stable or unstable, according to the magnitude of the marginal tax rate, $\tau'(a)$. We can provide the following definition.

Definition 2. *Fiscal policy is passive, if* $\tau'(a^*) > R(\pi^*) - \pi^*$ *and active otherwise.*

Notice that adopting a passive fiscal policy is tantamount to committing to fiscal solvency under all circumstances.

2.1. Equilibrium properties and local stability analysis

The long-run properties of system *M* are well understood. As discussed by Benhabib *et al.* (2001a,b), system *M* presents two steady states, one where inflation reaches the so-called intended level $\pi = \pi^*$ and one where inflation is higher or lower than the intended rate according to whether monetary policy is passive or active. In the latter case, the unintended steady-state may be labeled as a liquidity trap, in which the interest rate is zero, or near-zero, and inflation is below the target level. For notational convenience, let us define $\mathbf{P}^* \equiv (\mu_1^*, \pi^*, a^*)$ to be triplet vector of $\mu_1, \pi,$ and a such that $\dot{\mu}_1 = \dot{\pi} = \dot{a} = 0$, and the inflation is at the intended rate π^*. Simple algebra shows that:

$$\mu_1^* = \frac{\phi}{\phi - 1} c(\mu_1^*, \pi^*)^{\psi}$$
$$\pi^* = \bar{R} - \rho$$
$$a^* = \frac{\bar{R} m(c(\mu_1^*, \pi^*), \bar{R}) + \bar{\tau}}{\rho}.$$

The local stability properties of $\mathbf{P}^*$ are fully described in the literature (Benhabib *et al.*, 2001a; Tsuzuki, 2016). We thereby have the following:

1. When monetary policy is passive, an active fiscal policy always induces uniqueness of the equilibrium. Conversely, a policy commitment to preserve fiscal solvency under all circumstances leads to an indeterminate equilibrium.

2. When monetary policy is active, the stability properties are more mixed. Specifically:

   2a. if money and consumption in the utility function are Edgeworth complements

($u^*_{cm} > 0$), a passive fiscal rule still induces uniqueness of the equilibrium. Conversely, no equilibria exist in the neighborhood of the steady state in the case of an active fiscal policy;

2b. if, conversely, money and consumption in the utility function are Edgeworth substitutes ($u^*_{cm} < 0$), there exists a critical threshold,

$$u^*_{cm} = \hat{u}^*_{cm}, \tag{10}$$

such that if $|u^*_{cm}| < |\hat{u}^*_{cm}|$, then the same stability properties as in (2a) arise. Conversely, when $|u^*_{cm}| > |\hat{u}^*_{cm}|$, full stability of the intended steady state is established when fiscal rule is passive, while indeterminacy of the equilibrium prevails when fiscal policy is active.

The development of the main points of the paper requires the intended steady state of system *M* to be a saddle-focus equilibrium. Both cases (2.a) and (2.b) are therefore possible candidates. However, to limit the number of cases, we focus on the (2.b) case. We therefore assume the following.

Assumption 2. *Money and consumption are Edgeworth substitutes, i.e.* $u^*_{cm} < 0$.[10]

Therefore, by limiting the discussion to the case of an active monetary policy under Assumption 2, the following statement applies (see Appendix 1).

Proposition 1: *Recall Assumption* 2 *and consider the case of an active monetary policy. Then two stability cases can occur according to the magnitude of* $|u^*_{cm}|$. *Consider, first, the case,* $|u^*_{cm}| < |\hat{u}^*_{cm}|$. *If fiscal policy is also active,* $\mathbf{P}^*$ *is a repellor, and there are no equilibrium paths converging to the steady state. If fiscal policy is passive,* $\mathbf{P}^*$ *is a saddle of index* 2, *and the equilibrium is locally unique. Consider now the case,* $|u^*_{cm}| > |\hat{u}^*_{cm}|$. *If fiscal policy is passive,* $\mathbf{P}^*$ *is an attractor, whereas if fiscal policy is active, there is a continuum of equilibria that converge to the steady-state* (*local indeterminacy*).

[10] The economic implications of a negative u_{cm} are well represented in Walsh (2010) for the general case of the utility function with non-zero interdependences between leisure, money, and consumption. Specifically, if $u_{cm} < 0$, a monetary injection that raises expected inflation will increase consumption, labor supply, and output, a situation described as an "asset substitution model" by Wang and Yip (1992). Since Edgeworth substitutability is a cardinal property, it is not econometrically testable. But closely related Morishima substitutability is ordinal and has been tested by Serletis and Xu (2019). They found (see their figure 11, p. 21) that consumer goods have consistently been net Morishima substitutes for monetary services throughout their sample period, beginning in 1967, but gross complements because of positive income effects. Since income effects are not relevant to Edgeworth substitutability, the finding of net Morishima substitutability is more relevant to our assumption. Consumer goods might be both net and gross substitutes for monetary services, if monetary services are augmented to include credit card services, as available with the Divisia monetary aggregates supplied by the Center for Financial Stability. Increased consumption is associated with increased use of credit card services.

3. Shilnikov chaos

Let us now focus on the case, $|u_{cm}^*| < |\hat{u}_{cm}^*|$. Consider a scenario where the policymaker runs an active fiscal-monetary regime. Then, by Proposition 1, the policy maker may be pressured to increase the marginal tax rate above the real interest rate. In this Section, we show that following this policy prescription may induce another class of difficulties.

3.1. An explicit variant of the model

Before proceeding with our analysis, we need to provide specific forms for the implicit functions presented in system *M*. Following the standard literature, we first assume that the utility function has constant relative risk aversion in a composite good, which in turn is produced with consumption goods and real balances via a CES aggregator as follows:

$$u(c,m) = \frac{\left[\kappa c^{1-\beta} + (1-\kappa) m^{1-\beta}\right]^{\frac{1-\Phi}{1-\beta}}}{1-\Phi}, \tag{11}$$

where $0 < \kappa < 1$ is a share parameter, β measures the intra-temporal elasticity of substitution between the two arguments, c and m, and $\Phi > 0$ is the inverse of the intertemporal elasticity of substitution. Since we have, for now, assumed that consumption and real money balances are Edgeworth substitutes, the following parametric restriction is implied.

Remark 1. $Sign(u_{cm}^*) = Sign(\beta - \Phi)$. Therefore, Assumption 2 requires $\beta < \Phi$.

Moreover, it is standard to assume that the disutility of labor is described by:

$$f(l) = \frac{l^{1+\psi}}{1+\psi}, \tag{12}$$

where $\psi > 0$ measures the preference weight of leisure in utility. Furthermore, following Carlstrom and Fuerst (2003), we also assume a production function linear in labor,

$$y(l) = Al, \tag{13}$$

where A denotes the productivity level in the composite goods production. Without loss of generality, we set $A = 1$.

Additionally, we use the specification of the Taylor principle in Benhabib *et al.* (2001a,b), and assume that monetary authorities observe the inflation rate and conduct market operations to ensure that

$$R(\pi) = \bar{R} e^{(C/\bar{R})(\pi - \pi^*)}, \tag{14}$$

where C is a positive constant. Notice that, from the above specifications given in (6), our chosen functional form in (16) implies that

$$R(\pi^*) = \bar{R}; \qquad R'(\pi^*) = C. \tag{15}$$

so that C can be interpreted as a Taylor coefficient measuring the intensity of the reaction to inflation targeting.

Finally, to satisfy the Transversality Condition defined in (3), we need to assume that the economy follows a Ricardian regime, according to the fiscal rule

$$\tau(a) = \alpha a - Rm, \tag{16}$$

where the marginal tax rate $\alpha \equiv \tau'(a) \in (0,1)$.

3.2. Shilnikov chaos in system *M*

In this section, we sketch the whole set of conditions that are necessary to prove the existence of a chaotic regime in the equilibrium dynamics described by system *M*. Consider the following generalized version of the Shilnikov (1965) theorem (Chen and Zhou (2011)).

Theorem 1. *Consider a system of ordinary differential equations*

$$\frac{dY}{dt} = f(Y, \alpha), \quad Y \in \mathbb{R}^3, \quad \alpha \in \mathbb{R}^1,$$

with f sufficiently smooth. Assume f has a hyperbolic saddle-focus equilibrium point, $Y_0 = 0$, at $\alpha = 0$, implying that eigenvalues of the Jacobian, $J = Df$, are of the form γ and $\chi \pm \xi i$, where γ, χ, and ξ are real constants with $\gamma\chi < 0$. Assume that the following conditions also hold:

$(H.1)$ *The saddle quantity, $\sigma \equiv |\gamma| - |\chi| > 0$;*
$(H.2)$ *There exists a homoclinic orbit, Γ_0, based at Y_0.*

Then the following results hold:

(1) *The Shilnikov map, defined in the neighborhood of Γ_0 possesses an infinite number of Smale horseshoes in its discrete dynamics;*
(2) *For any sufficiently small C^1-perturbation, g, of f, the perturbed system has at least a finite number of Smale horseshoes in the discrete dynamics of the Shilnikov map, defined in the neighborhood of Γ_0;*
(3) *Both the original and the perturbed system exhibit horseshoes chaos.*

The application of Theorem 1 to system *M* requires a set of restrictions on the parameter space of the model, such that the system possesses a hyperbolic saddle-focus equilibrium point, with a positive saddle quantity, and there exists a homoclinic orbit connecting the saddle-focus to itself.

3.3. Shilnikov attractor with a UK calibration of the model

We now show that, under specific circumstances, the calibration of the economy with UK data can well determine the Shilnikov scenario. We base the calibration of our model on the highly regarded Liu and Mumtaz (2011) estimation of a New Keynesian DSGE model for the UK, 1970Q1 to 2009Q1, with Markov switching. The paper uses Bayesian methods to estimate a baseline rational expectations model with no regime switching (full sample constant parameters), and four further versions of the model, in which the British economy is assumed to undergo some kind of structural changes across two regimes. The most interesting case for us is the model where deep and policy parameters are estimated independently in each regime (model M5). This type of estimation proxies a "sample split" or "breakpoint" type approach to modeling structural change (see also Lubik and Schorfheide (2007)) which allows us to study the onset of the Shilnikov phenomenon under different parameterizations and policy formulations. Barnett at al. (2020) similarly found Shilnikov chaos in basically the same standard NK model, but with US data and the version of the model without money. Their calibration of the US model is based on Lubik and Schorfheide (2007).

In this regard, it is interesting to realize with UK data that posterior estimates of the Taylor coefficient *C* for the case of no regime switching lead to a point value of 1.56, which is very close to the slope suggested by Taylor (1993). However, as soon as we allow for a policy rule switch, the Taylor coefficient *C* statistically bifurcates across the two regimes. One regime (prevailing in the mid-1970s) is associated with episodes of high inflation and has a posterior for the Taylor coefficient equal to 1.36. The other regime, which is characterized by a move towards a more active monetary policy, has a *C* coefficient equal to 2.71. In connection with these data, we anticipate that the Shilnikov phenomenon occurs for a narrow region of the parameter space in the case of the baseline rational expectations model with no regime switching, but does not occur at all in the high inflation regime. On the contrary, it comes up easily in the case of the low-inflation regime. It seems therefore that the tightening of the policy rule via the Taylor coefficient speeds up the emergence of the unexpected phenomena of chaotic dynamics.

Before starting the discussion of the examples, consider that Liu and Mumtaz' paper only provides direct evidence for the subset (ψ, ρ, Φ) of the deep parameters. However, considering the connection formula between the reduced form implications for the linearized Phillips curve of Rotemberg and Calvo models of costly and sticky price adjustments (cfr. Benhabib *et al*., 2014), we have

$$\eta = (\phi - 1)\frac{\delta^h}{(1-\delta^h)(1-F\delta^h)}, \tag{17}$$

where δ^h is the fraction of firms not able to change their price during the quarter and F is the discount factor. We can also proceed by calibrating η, provided that some prior information on $\phi = \frac{mark\_up}{mark\_up-1}$ is known.

Unfortunately, the Liu and Mumtaz (2011) paper is of no help in calibrating the (β, κ) parameters of the utility function. Airaudo and Zanna (2013), in an attempt to simulate cyclical and/or chaotic solutions in a DSGE NK model for the UK economy, set $\kappa \cong 0.97$. Since τ cancels out in the computation of the eigenvalues, we are only left with the β and ϕ elasticities as free parameters, along with the fiscal policy coefficient τ'.

The heavy parameterization of the model and the frequent numerical anomalies prevent us from deriving the parameter space three-dimensional (β, Φ, τ') critical surface, beyond which the Shilnikov pre-conditions are satisfied. Therefore, we resort to some parametric examples.

Example 1. (Constant parameters with no regime switching). *Let us first fill the vector of the observables* $(\bar{R}, \pi^*)$ *for the full sample* (1970*Q*1-2009*Q*1). *Drawing from World Bank databases, we use the averages of the short-run interest rate and the GDP deflator to obtain quarterly* $(\bar{R}, \pi^*) \cong (0.021, 0.017)$. *Therefore,* $r = \bar{R} - \pi^* \cong 0.004$ *is the real interest rate. Following Liu and Mumtaz* (2011)*, we set* $(\psi, \rho, \Phi) = (4.4, 0.01, 1.83)$. *Moreover, using a baseline estimation of a* 5% *mark-up as in Benhabib et al.,* (2001) and *in Benhabib et al.* (2014)*, we obtain* $\phi = 21$. *Now, recalling the point estimate of* $\delta^h = 0.55$*, we can also set* $\eta \cong 53.67$ (*see equation* 17) *after fixing F at the standard value of* 0.99. *In order to comply with Edgeworth substitutability* (*cf. Assumption* 2 *and Remark* 1)*, the parameter* β *is initially fixed to* 1.825*, a value below* (*but very close to*) *the value of* Φ. *Solving for the characteristic equation* (*A*.2 *in Appendix* 1) *gives*:

$$\lambda_1 = 0.004 - \tau',$$
$$\lambda_{2,3} \cong 0.005 - 0.00254C \pm 0.00254\sqrt{(C - 1.0000058)(C - 160770)}.$$

Therefore, in accordance with Proposition 1*, provided that* $C < \frac{0.005}{0.00254} = 1.968504$*, an active monetary-fiscal regime implies one negative eigenvalue and two eigenvalues with positive real parts.*[11] *Then, the saddle quantity* σ *is equal to*:

$$\sigma \equiv \tau' - 0.009 + 0.00254C.$$

[11] This numerical requirement on C corresponds to the condition $|u^*_{cm}| < |\hat{u}^*_{cm}|$ in Proposition 1.

Recall now that C *is estimated equal to* 1.56 *in this sample. Thus, if we set* $\tau' \gtrsim 0.00646$*, the saddle quantity is positive and pre-conditions for the existence of Shilnikov chaos are satisfied. To test the robustness of this result, we have checked the permanence of the Shilnikov pre-conditions for values of* β *and* ϕ *progressively drifting away from the chosen baseline. Specifically, after setting* $\tau' = 0.02$*, we have studied the characteristics of the eigenvalues of the linearized matrix and the sign of* σ *for a grid of values for the mark-up and* β *parameter. Results are in Table* 1*. What appears clear is that, given the assumption of Edgeworth substitutability and our baseline set of parameters, the pre-conditions of the Shilnikov phenomenon are only satisfied within a very narrow region of the domain of the* β *parameter. This region widens for higher degrees of price stickiness, although remaining small. Lowering values of* β *leads to full stability, initially through damped oscillations, and then along qualitatively monotone paths.*

Table 1. *Stability properties of the steady state and Shilnikov pre-conditions for varying mark-up and β elasticity. No switching regime.*

Beta \ Mark-up	*1.01*	*1.05*	*1.1*	*1.15*	*1.2*	*1.3*
1.825	*Shilnikov σ =0.0162*	*Shilnikov σ =0.0170*	*Shilnikov σ =0.0178*	*Shilnikov σ =0.0184*	*Shilnikov σ =0.0189*	*Shilnikov σ =0.0195*
1.82	*Stability + CE*	*Stability + CE*	*Shilnikov σ =0.0146*	*Shilnikov σ =0.0158*	*Shilnikov σ =0.0167*	*Shilnikov σ =0.0179*
1.80	*Stability + CE*	*Stability + CE*	*Stability + CE*	*Stability + CE*	*Shilnikov σ =0.0065*	*Shilnikov σ =0.0117*
1.78	*Stability + CE*	*Stability + CE*	*Stability + CE*	*Stability + CE*	*Stability + CE*	*Stability + CE*
1.50	*Stability + CE*	*Stability + CE*	*Stability + CE*	*Stability + CE*	*Stability + CE*	*Stability + CE*
1.10	*Stability + CE*	*Stability + CE*	*Stability + RE*	*Stability + RE*	*Stability + RE*	*Stability + RE*
0.99	*Stability + RE*	*Stability + RE*	*Stability + RE*	*Stability + RE*	*Stability + RE*	*Stability + RE*
0.50	*Stability + RE*	*Stability + RE*	*Stability + RE*	*Stability + RE*	*Stability + RE*	*Stability + RE*

Legend:

- *Region with green characters: combinations of the (β, ϕ) parameters such that Shilnikov pre-conditions in Theorem 1 are satisfied. σ is the saddle quantity.*
- *Region with blue characters: combinations of the (β, ϕ) parameters such that the Jacobian associated with system M has three eigenvalues with negative real parts. CE stands for Complex Eigenvalues.*
- *Region with red characters: combinations of the (β, ϕ) parameters such that the Jacobian associated with system M has three eigenvalues with negative real parts. RE stands for Real Eigenvalues.*

Example 2. (Low-inflation regime with high Taylor coefficient). *To fill the vector of observables,* $(\bar{R}, \pi^*)$, *we now exclude, for the computations from the World Bank databases, the episodes of two-digit high inflation from the middle of the 70's and beginning of the 80's. We obtain* $(\bar{R}, \pi^*) \cong (0.019, 0.011)$. *Therefore,* $r = \bar{R} - \pi^* \cong 0.008$ *is the average quarterly real interest rate. Following Liu and Mumtaz's* (2011) *estimations relative to Regime* 1 *in model M*5, *we set* $(\psi, \rho, \Phi) = (1.6, 0.01, 1.76)$. *Moreover, using a baseline estimation of a* 5% *mark-up as in Example* 1, *we obtain* $\phi = 21$. *Now, since* $\delta^h = 0.45$ *in this sample*, *we can also set* $\eta \cong 29.51$ *after fixing F to the standard value of* 0.99. *In order to comply with Edgeworth substitutability* (*cf. Assumption* 2 *and Remark* 1), *the parameter* β *is initially fixed to* 1.7, *which is close, but smaller than* Φ, *the inverse of the temporal elasticity of substitution. Solving for the characteristic equation* (*A*.2 *in Appendix* 1) *gives*:

$$\lambda_1 = 0.00756 - \tau',$$
$$\lambda_{2,3} \cong 0.005 - 1.9 \times 10^{-8} C \pm 1.9 \times 10^{-8} \sqrt{(C - 1.009247)(C - 7.429411.9 \times 10^{10})}.$$

Therefore, in accordance with Proposition 1, *an active monetary-fiscal regime implies three eigenvalues with positive real parts for* $C < \frac{0.005}{1.9 \times 10^{-8}}$ =263157.8947. *Conversely, a fiscal policy switch to a passive rule implies one negative eigenvalue and two eigenvalues with positive real parts. Consider now the case of a passive fiscal policy. Then, the saddle quantity* σ *is equal to*:

$$\sigma \equiv \tau' - 0.00256 + 1.9 \times 10^{-8} C.$$

Recall now that C *is estimated equal to* 2.27 *for the low inflation sample. Thus, if we set* $\tau' \gtrsim 0.0025$, *the saddle quantity is positive and pre-conditions for the existence of Shilnikov chaos are satisfied. We now check the permanence of the Shilnikov pre-conditions for values of* β *and* ϕ *progressively drifting away from the chosen baseline, just as we did in Example 1. After setting* $\tau' = 0.02$, *and solving the characteristic equation, we obtain the results in Table* 2. *Given the assumption of Edgeworth substitutability, the pre-conditions of the Shilnikov phenomenon are now satisfied in a very large region of the domain of the two control parameters, with the saddle quantity nearly flat with regard to both control parameters. Only one type of structural change of the eigenvalues is observed: a sign-preserving transition of the eigenvalues from complex to real, for a high enough mark-up index.*

Table 2. *Stability properties of the steady state and Shilnikov pre-conditions for varying mark-up and β elasticity. Low-inflation regime.*

Mark-up / Beta	*1.01*	*1.05*	*1.1*	*1.15*	*1.2*	*1.3*
1.75	*Shilnikov* $\sigma \cong 0.017435$	*Shilnikov* $\sigma \cong 0.017439$	*Shilnikov* $\sigma \cong 0.017441$	*Shilnikov* $\sigma \cong 0.017442$	*Shilnikov* $\sigma \cong 0.017443$	*Uniq. + RE*
1.70	*Shilnikov* $\sigma \cong 0.017416$	*Shilnikov* $\sigma \cong 0.017439$	*Shilnikov* $\sigma \cong 0.017440$	*Shilnikov* $\sigma \cong 0.017441$	*Uniq. + RE*	*Uniq. + RE*
1.65	*Shilnikov* $\sigma \cong 0.017396$	*Shilnikov* $\sigma \cong 0.017439$	*Shilnikov* $\sigma \cong 0.017439$	*Uniq. + RE*	*Uniq. + RE*	*Uniq. + RE*
1.60	*Shilnikov* $\sigma \cong 0.017376$	*Shilnikov* $\sigma \cong 0.017439$	*Shilnikov* $\sigma \cong 0.017439$	*Uniq. + RE*	*Uniq. + RE*	*Uniq. + RE*
1.50	*Shilnikov* $\sigma \cong 0.017341$	*Shilnikov* $\sigma \cong 0.017439$	*Shilnikov* $\sigma \cong 0.017439$	*Uniq. + RE*	*Uniq. + RE*	*Uniq. + RE*
1.10	*Shilnikov* $\sigma \cong 0.017254$	*Shilnikov* $\sigma \cong 0.017439$	*Shilnikov* $\sigma \cong 0.017438$	*Uniq. + RE*	*Uniq. + RE*	*Uniq. + RE*
0.50	*Shilnikov* $\sigma \cong 0.017226$	*Shilnikov* $\sigma \cong 0.017439$	*Uniq. + RE*	*Uniq. + RE*	*Uniq. + RE*	*Uniq. + RE*

Legend:

- *Region with green characters: combinations of the* (β, ϕ) *parameters such that the Jacobian associated with system M: i) has one negative and two complex conjugate eigenvalues with positive real parts; ii) Shilnikov pre-conditions in Theorem* 1 *are satisfied.* σ *is the saddle quantity.*
- *Region with dark blue characters: combinations of the* (β, ϕ) *parameters such that the Jacobian associated with system M has one eigenvalue with negative real part and two eigenvalues with positive real parts. RE stands for Real Eigenvalues.*

Example 3. (High-inflation regime, low Taylor coefficient). *To fill the vector of the observables,* $(\bar{R}, \pi^*)$*, we only consider the episodes of double-digits high inflation for the UK. We obtain* $(\bar{R}, \pi^*) \cong (0.0311, 0.0395)$. *Therefore,* $r = \bar{R} - \pi^* \cong -0.009$ *is the average quarterly real interest rate. Following the Liu and Mumtaz's* (2011) *estimations relative to their Regime* 2 *in model M*5*, we set* $(\psi, \rho, \Phi) = (1.11, 0.01, 2.23)$. *Moreover, as in Example* 1*, using a baseline estimation of a* 5% *mark-up, we obtain* $\phi = 21$. *Now, since* $\delta^h = 0.45$, *we can also set* $\eta \cong 29.51$ *after setting F to the standard value of* 0.99. *To comply with Edgeworth substitutability* (*cf. Assumption* 2 *and Remark* 1)*, the parameter* β *is initially fixed to* 2.2, *which is close, but smaller than* Φ*, the value of the inverse of the temporal elasticity of substitution estimated for this regime. Solving for the characteristic equation* (*A*.2 *in Appendix* 1) *gives*

$$\lambda_1 = -0.00905 - \tau',$$
$$\lambda_{2,3} \cong 0.005 - 0.00556C \pm 0.00556\sqrt{(C - 1.000000227)(C - 44912.09494)}$$

Since in this case $|u_{cm}^*| \cong 0.0111 > |\hat{u}_{cm}^*| = 0.001$, *in accordance with Proposition* 1*, an active monetary-fiscal regime*[12] *implies three eigenvalues with negative real parts for* $C > 1.0000002273$. *Conversely, a fiscal policy switch to a passive rule implies a continuum of equilibria that converge to the steady-state* (*local indeterminacy*)*. Consider now the case of a passive fiscal policy. Then, the saddle quantity* σ *is equal to*

$$\sigma \equiv \tau' - 0.00405 + 0.0556C$$

Recall now that C *is estimated equal to* 1.35 *for the high inflation sample. Thus, if we set* $\tau' \gtrsim 0.0071$, *the saddle quantity is positive, and pre-conditions for the existence of Shilnikov chaos are satisfied. Now, to verify that Shilnikov pre-conditions can never be detected in this sample, we allow* β *and* ϕ *to progressively drift away from the chosen baseline. After setting* $\tau' = 0.02$ *and solving the characteristic equation, we obtain the results in Table* 3. *For the high inflation sample, it is now clear that the pre-conditions for the onset of the Shilnikov phenomenon are never satisfied. Only one type of topological change of the eigenvalues is observed, namely a transition towards the case of three negative real eigenvalues, for a low enough* β *elasticity.*

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[12] Notice that, for this sample, the real interest rate is negative. Thus, an active fiscal policy might consider a negative marginal tax rate.

Table 3. *Stability properties of the steady state and Shilnikov pre-conditions for varying mark-up and β elasticity. High-inflation regime*

Beta \ Mark-up	*1.01*	*1.05*	*1.10*	*1.15*	*1.20*	*1.30*
2.20	*Stability* + *CE*	*Stability* + *CE*	*Stability* + *CE*	*Stability* + *CE*	*Stability* + *CE*	*Stability* + *CE*
2.10	*Stability* + *CE*	*Stability* + *CE*	*Stability* + *CE*	*Stability* + *CE*	*Stability* + *CE*	*Stability* + *CE*
2.00	*Stability* + *CE*	*Stability* + *CE*	*Stability* + *CE*	*Stability* + *CE*	*Stability* + *CE*	*Stability* + *RE*
1.50	*Stability* + *CE*	*Stability* + *CE*	*Stability* + *CE*	*Stability* + *CE*	*Stability* + *RE*	*Stability* + *RE*
1.10	*Stability* + *CE*	*Stability* + *CE*	*Stability* + *CE*	*Stability* + *RE*	*Stability* + *RE*	*Stability* + *RE*
0.50	*Stability* + *RE*	*Stability* + *RE*	*Stability* + *RE*	*Stability* + *RE*	*Stability* + *RE*	*Stability* + *RE*

Legenda:

- *Region with blue characters: combinations of the (β, ϕ) parameters such that the Jacobian associated with system M has three eigenvalues with negative real parts. CE stands for Complex Eigenvalues.*
- *Region with red characters: combinations of the (β, ϕ) parameters such that the Jacobian associated with system M has three eigenvalues with negative real parts. RE stands for Real Eigenvalues.*

The examples discussed above prove the following statement.

Lemma 1. (Fulfillment of pre-condition H.1 in Theorem 1). *There are regions of the parameter space at which* $\mathbf{P}^*$ *is a saddle-focus with* $\sigma > 0$.

We draw some interesting lessons from these examples. Using the UK data, it appears clear that the width of the region of the parameter space at which monetary activism may induce the birth of the chaotic attractor depends on the intensity of the Taylor coefficient. Low C coefficients (typically associated with high inflation periods in our sample) do not seem to introduce the perils of chaotic solutions to Decision P. Conversely, when the fight against inflation becomes tougher, then the risk of un-intended dynamics becomes substantial. Notice that the phenomenon is twice insidious, since no alert comes from a local analysis perspective, which continues to detect the desirable property of uniqueness of the equilibrium.

We now show that system M supports the existence of a family of homoclinic orbits doubly asymptotic to a saddle-focus in $\mathbb{R}^3$. (pre-condition $H.2$ in Theorem 1). Bella, Mattana, and Venturi (2017) describe in detail all necessary steps. We must first put system M into normal form by using the eigen-basis. We thereby obtain the following (truncated) system

$$\begin{pmatrix}\dot{w}_1\\ \dot{w}_2\\ \dot{w}_3\end{pmatrix}=\begin{bmatrix}\chi & -\xi & 0\\ \xi & \chi & 0\\ 0 & 0 & \gamma\end{bmatrix}\begin{pmatrix}w_1\\ w_2\\ w_3\end{pmatrix}+$$
$$+\begin{pmatrix}F_{1a}w_1w_2 + F_{1b}w_1w_3 + F_{1c}w_2w_3 + F_{1d}w_1^2 + F_{1e}w_2^2 + F_{1f}\, w_3^2\\ F_{2a}w_1w_2 + F_{2b}w_1w_3 + F_{2c}w_2w_3 + F_{2d}w_1^2 + F_{2e}w_2^2 + F_{2f}\, w_3^2\\ F_{3a}w_1w_2 + F_{3b}w_1w_3 + F_{3c}w_2w_3 + F_{3d}w_1^2 + F_{3e}w_2^2 + F_{3f}\, w_3^2\end{pmatrix}, \qquad (18)$$

where $(w_1, w_2, w_3)^T$ is the vector of transformed coordinates, and where the $F_{i,j}$ coefficients, with $i = 1,2,3$ and $j = a, b... f$, are combinations of the original parameters of the model, also depending on the values of three free constants, $\varphi_i, i = 1,2,3$, arising in the computation of the eigen-basis.

Once the normal form (18) is obtained, the method of undetermined coefficients (Shang and Han (2005)) can be applied to derive a polynomial approximation of the analytical expressions of both the two-dimensional unstable manifolds associated with λ_2 and λ_3, and of the one-dimensional stable manifold associated with λ_1. Then, with given parameters, conditions for the existence of the homoclinic loop, doubly asymptotic to the saddle-focus equilibrium point, rely on the existence of a triplet of free constants $(\Xi, \Psi, \Omega) \in (0,1)^3$ satisfying the split function, as in Barnett *et al.* (2020).

$$\Sigma = \Xi + \frac{F_{3f}\Xi^2}{\gamma} + (2\chi - \gamma)\frac{F_{3a}\Psi\Omega + F_{3d}\Psi^2 + F_{3e}\Omega^2}{(2\chi-\gamma)^2 + 4\xi^2} = 0, \qquad (19)$$

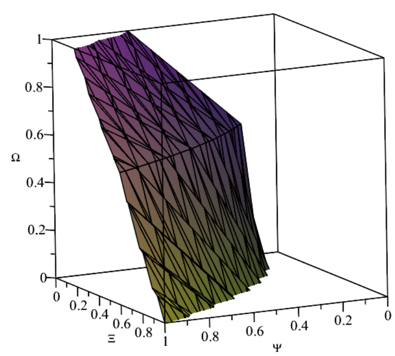
where $\gamma = \lambda_1$, $\chi = Re(\lambda_{2,3})$, and $\xi = Im(\lambda_{2,3})$. The reason why the three constants (Ξ, Ψ, Ω) are bound to belong to the cube $(0,1)^3$ is strictly related to the geometry of the stable and unstable manifolds, which intersect near the origin (in the transformed eigenspace) and form the homoclinic loop.[13]

To verify whether the split function (19) can be satisfied within the NK model, we retake the parameters in the two Examples 1 and 2 above and compute all necessary coefficients. We have the following.

[13] Cf. Kuznetsov (1998, p. 198) for the geometrical interpretation of the split function.

Example 4. *Set* $(\eta,\kappa,\psi,\rho,\Phi)\cong(53.67,0.97,4.4,0.01,1.83)$ *and* $(\bar{R},\pi^{*})\cong(0.021,0.017)$ *as in Example* 1. *From Table* 1, *we also choose* $\beta=1.825$ *and a mark-up level of* 1.05, *implying* $\phi=21$. *Then, recalling that* $C=1.56$, *we have that if* $\tau'>\tilde{\tau}'\equiv 0.005038$, $\mathbf{P}^{*}$ *is a saddle-focus with positive saddle quantity. Set therefore* $\tau'=0.02$ *as in Example* 1. *Then, there exists a surface of admissible combinations of* (Ξ,Ψ,Ω) *belonging to the cube* $(0,1)^3$ *such that the split function is satisfied* (*cf. Figure* 1).

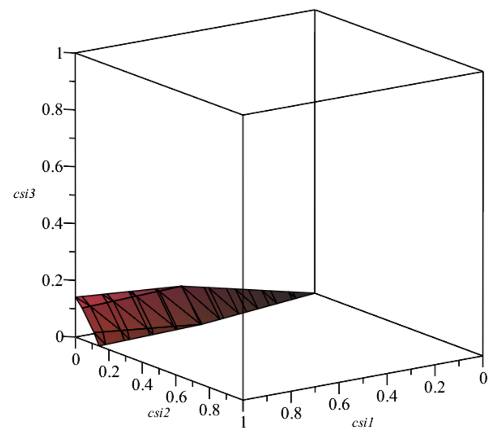
Figure 1. *Coordinates of the* (Ξ,Ψ,Ω) *constants satisfying the split function.*



We now drift away from the baseline $\tau'=0.02$ *and determine, given all other parameters, the interval of the marginal tax rate for which there emerges a solution of the split function for* $(\Xi,\Psi,\Omega)\in(0,1)^3$. *More precisely, we iteratively increase* τ' *above the critical value* $\tilde{\tau}'\equiv 0.005038$ *with a grid of* 0.001 *and keep track of the values such that there exists a solution for* $\Sigma=0$ *with* $(\Xi,\Psi,\Omega)\in(0,1)^3$. *The procedure reveals that there is a wide interval* $I_{\tau'}\cong(0.005038,0.301)$ *such that, for all* $\tau'\in I_{\tau'}$, *a homoclinic loop doubly asymptotic to the saddle-focus equilibrium point is established.*

Example 5. *Set* $(\eta,\kappa,\psi,\rho,\Phi)\cong(29.51,0.97,1.11,0.01,2.23)$ *and* $(\bar{R},\pi^{*})\cong(0.019,0.011)$ *as in Example* 2. *From Table* 2, *we also choose* $\beta=1.75$ *and a mark-up level of* 1.05, *implying* $\phi=21$. *Then, recalling that in this sample* $C=2.27$, *we have that if* $\tau'>\tilde{\tau}'\equiv0.00255$, $\mathbf{P}^{*}$ *is a saddle-focus with positive saddle quantity. Set now* $\tau'=0.02$ *as in Example* 2. *Then, there exists a surface of combinations of* (Ξ,Ψ,Ω) *belonging to the cube* $(0,1)^3$ *such that the split function is satisfied* (*cf. Figure* 2).

Figure 2. *Coordinates of the* (Ξ,Ψ,Ω) *constants satisfying the split function.*



We now drift away from the baseline $\tau'=0.02$ *and determine, given all other parameters, the interval of the marginal tax rate for which there emerges a solution of the split function for* $(\Xi,\Psi,\Omega)\in(0,1)^3$. *More precisely, we iteratively increase* τ' *above* $\tilde{\tau}'\equiv0.0025$ *with a grid of* 0.001 *and keep track of the values such that there exists a solution for* $\Sigma=0$ *with* $(\Xi,\Psi,\Omega)\in(0,1)^3$. *The procedure reveals that for all values of* $\tau'>0.07$, *there exists a homoclinic loop doubly asymptotic to the saddle-focus equilibrium point.*

The following statement is therefore implied.

Lemma 2. (Fulfillment of pre-condition H.2 in Theorem 1). *There exists a sub-region of the parameter space such that H.*1 *and H.*2 *in Theorem* 1 *are simultaneously satisfied.*

3.4. Existence and properties of the chaotic attractor

We can now go to the main result in this section. Let us take τ' as a bifurcation parameter. Define the difference $\nu=\tau'-\bar{\tau}'$, where $\bar{\tau}'$ belongs to the set of critical values such that an admissible solution of the split function exists for $(\Xi,\Psi,\Omega)\in(0,1)^3$. Let $V\subset\mathbb{R}$ be a sufficiently small open neighborhood of $\mathbf{0}$. We have the following result.

Proposition 2. (Existence of a Shilnikov chaotic attractor) *Assume that the parametric conditions in Lemmas* 1 *and* 2 *are satisfied. Let* $\nu \in V$. *Then, given a triplet of initial conditions* $(\tilde{w}_1(0), \tilde{w}_2(0), \tilde{w}_3(0))$, *sufficiently close to the origin, system* (18) *admits chaotic, perfect-foresight equilibrium solution. By topological equivalence, the result also applies to system M.*

Proof. Let Lemmas 1 and 2 apply. Then, Theorem 1 guarantees that there is a non-empty set of initial conditions giving rise to orbits of system (18) that will stay forever in the neighborhood of the origin. Since, by construction, these trajectories are also valid solution trajectories for the original system *M*, the statement in Proposition 2 is confirmed. ■

Consider now the following Example 6. To ease the numerical computations, we follow Freire *et al.* (2002), and construct the hypernormal (truncated) form of the vector field *M*. Then we derive the following versal deformation

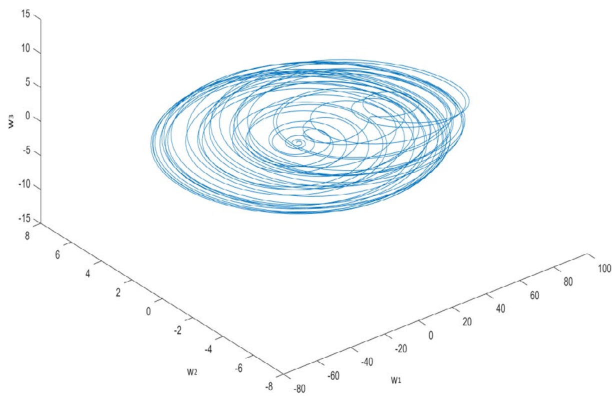
$$\begin{pmatrix} \dot{\tilde{w}}_1 \\ \dot{\tilde{w}}_2 \\ \dot{\tilde{w}}_3 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{bmatrix} \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{w}_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ d\tilde{w}_1^2 + k\tilde{w}_1^3 \end{pmatrix}, \qquad (20)$$

where $(\tilde{w}_1, \tilde{w}_2, \tilde{w}_3)$ is the new set of coordinates arising from the near-identity transformation, $\varepsilon_1 = \mathrm{Det}(\mathbf{J})$, $\varepsilon_2 = -\mathrm{B}(\mathbf{J})$, $\varepsilon_3 = \mathrm{Tr}(\mathbf{J})$, and where *d* and *k* are combinations of the coefficients of the non-linear terms.[14]

Example 6. *Set* $(\beta, \eta, \kappa, \psi, \phi, \rho, \Phi) \cong (1.75, 29.51, 0.97, 1.11, 21, 0.01, 2.23)$, $(\bar{R}, \pi^*) \cong (0.019, 0.011)$ *and* $(C, \tau') \cong (2.27, 0.02)$ *as in Example* 2. *Then, we know that there exists a family of homoclinic loops doubly asymptotic to a saddle-focus equilibrium point with positive saddle quantity. Consider initial conditions* $(\tilde{w}_1(0), \tilde{w}_2(0), \tilde{w}_3(0)) = (0.01, 0.01, 0.01)$. *Then, the perfect-foresight solution trajectory evolves chaotically along the attractor represented in Figure* 3.

Figure 3. *The chaotic attractor in the* $(\tilde{w}_1, \tilde{w}_2, \tilde{w}_3)$ *space.*

[14] The versal deformation in (20) is a homeomorphism that takes orbits of the flow generated by system *M* onto orbits of (18), preserving orientation and parametrization by time. Hence, system *M* and system (18) are $\mathbf{C}^0$ topologically conjugate (cf. Wiggins (1991), pp. 297-298).



The object in Figure 3 has the distinct shape of the Shilnikov attractor. The dynamics of the economy along the spiral attractor have periods of relative quiescence, when the phase point approaches the saddle-focus point. Conversely, when the phase point starts to spiral away from the saddle-focus point, there is the onset of irregular episodes of oscillatory activity. More details on the characteristics of chaotic time profiles of the variables are provided in the next sections.

3.5. Economic implications

Economic implications of Proposition 2 are manifold for the dynamics implied by NK models. First of all, the existence of a chaotic attractor implies that small changes in initial conditions can produce large changes in dynamics over time. Two economies, starting contiguously in the space of initial conditions, can follow completely different patterns over time. Since an initial condition is known only to a finite degree of precision, it is impossible to predict dynamics deterministically over extended periods of time.

Moreover, as discussed in Bella, Mattana and Venturi (2017) for the Lucas' growth model and in Barnett et al. (2020) for the NK model, within the Shilnikov attractor, given the initial value of the predetermined variable, there exists a *continuum* of initial values of the jump variables giving rise to admissible equilibria living inside a tubular neighborhood of the (perturbed) homoclinic orbit. This is a global indeterminacy phenomenon.

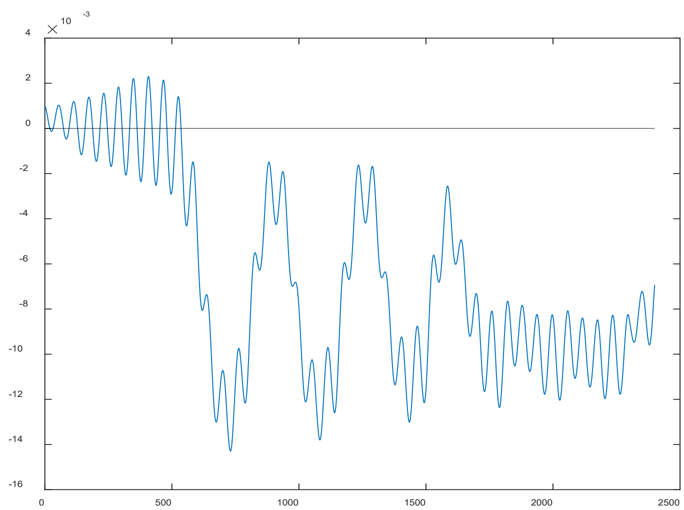
However, what is of particular interest for this paper are the qualitative "dimensions" of the chaotic attractor generated by the NK model.[15] Assume that the relative frequency at which an orbit visits different region of the attractor is largely heterogeneous. Then, across the volume of all possible coordinates contained in the attractor, the economy lingers on particular regions with higher "density." In the numerical simulations developed in the paper, it is evident that the emerging dynamics tend to evolve for a long

[15] Cf. Farmer *et al.* (1983) for a classical discussion on the relevant dimensions of a chaotic attractor.

time around lower-than-targeted inflation and nominal interest rates. This can be interpreted as a liquidity trap phenomenon, which now depends on the existence of a chaotic attractor and not on the influence of an unintended steady state as in much of related literature.

We have now reconstructed the inflation rate time profile from the chaotic attractor in Figure 4. To ease discussion, we have reported its moving average in Figure 4 (window = 100 iterates), which we have demeaned using the steady-state value of 0.011. First, it is central to observe that the dynamics of the inflation rate, in this numerical experiment, presents several irregular sudden drops (corresponding to the time when the phase point approaches the saddle-focus), before collapsing into an aperiodic cycle, centered on lower-than-expected coordinates. This phenomenon has a number of important predictions for the behavior of inflation in an economy enmeshed in a chaotic attractor, the most important of which being the possibility of long periods during which inflation is stubbornly low (akin to a deflationary equilibrium). In this example, average long-run inflation is approximately 1% lower than the quarterly steady-state value of 0.011. [16]

Figure 4. *The moving average of chaotic inflation rate* (*window* = 100 *iterations*).



The result of a persistent low inflation for economies evolving within the attractor appears pervasive in our data. We have conducted extensive numerical simulations, changing initial conditions and trying different combinations of the mark-up and β elasticity in the utility function for which the Shilnikov phenomenon is confirmed (cf. Table 2). Qualitative results are invariably confirmed. As a consequence of these empirical results, the following statement is implied.

Corollary 1. *Assume that the dynamics of the system evolve along the attractor set. Then, persistently low inflation rates with regard to the* (*unique*) *steady-state value can emerge.*

[16] Notice that the eigenvalues can be chosen such that inflation can also be persistently negative.

Corollary 1 has important implications for the debate regarding liquidity traps. As discussed in our introduction, this phenomenon has previously been linked mainly to the existence of a low-inflation steady state (cf., in particular, Benhabib *et al.*, 2001a,b) and to its basin of attraction. We offer an alternative explanation, based on the long-run peculiarities of a chaotic attractor and the evolution of the dynamics within that attractor set, such that the economy drifts into the liquidity trap without any policy intent.

The differences in the qualitative dynamics arising because of an unintended steady state or because of the existence of a chaotic attractor are striking. The time profile for inflation featured in Benhabib *et al.*, (2001a,b) for the case of $u_{cm} < 0$, so that consumption and real balances are substitutes, shows higher and higher amplitude oscillations around the active steady. When the saddle connection is established, inflation suddenly drops to the passive (lower) steady state value. This kind of predictable/regular behavior of the economy could be traced out by an econometric exercise.

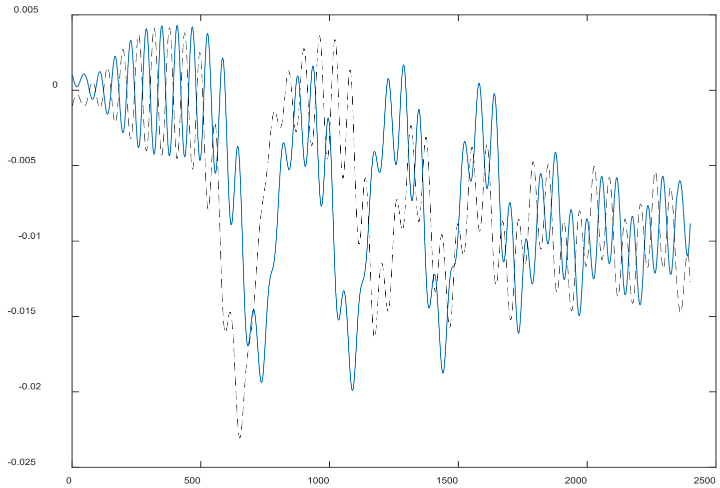
In our case, inflation, along the spiral attractor, has long quiescence periods, possibly characterized by a persistent and steep monotonic behavior, followed by bursts of irregular oscillatory activity. This kind of pattern is largely unpredictable and cannot be inferred by conventional econometric tools, since such behavior violates the regularity conditions for available statistical inference methodologies, such as the usually assumed properties of the likelihood function and polyspectra (see, e.g., Barnett *et al.* (1997) and Geweke (1992)).

We now proceed to derive more implications for monetary policy from a different point of view. Specifically, we explore the effects on the time profile of inflation caused by changes in the Taylor coefficient *C*. First results, although still provisional, are interesting and somewhat unexpected.

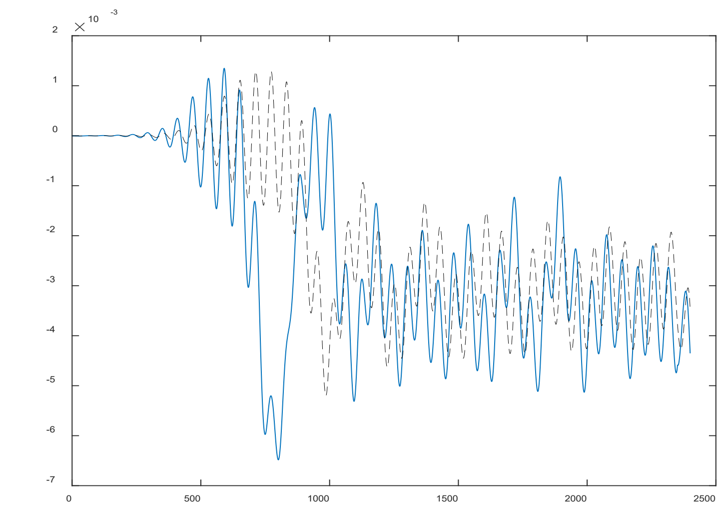
We find that for *C* below a certain threshold, inflation time profiles appear to have comparable statistical moments across different *C* and initial conditions. Figure 5, panel *a*, contrasts the moving averages (window = 100) obtained for *C* = 2.27, our baseline, and for *C* = 1.5. Both time profiles are initialized at $(\tilde{w}_1(0), \tilde{w}_2(0), \tilde{w}_3(0)) = (0.01, 0.01, 0.01)$. Panel *b* presents the moving averages for the same pair of *C*, but initialized at $(\tilde{w}_1(0), \tilde{w}_2(0), \tilde{w}_3(0)) = (0.1, 0, 0)$.

Figure 5. *Paths of chaotic inflation with lower-than-threshold C*

Panel a). $(\tilde{w}_1(0), \tilde{w}_2(0), \tilde{w}_3(0)) = (0.01, 0.01, 0.01)$
Dashed line: *C* = 2.27; Solid line *C* = 1.5



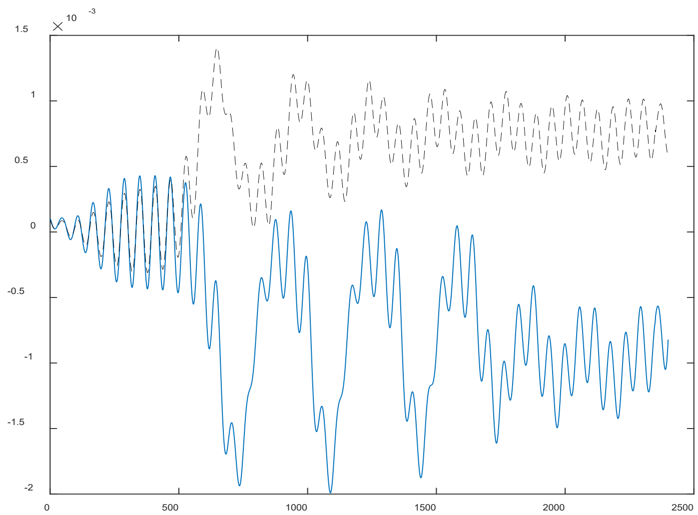
Panel b). $(\widetilde{w}_1(0), \widetilde{w}_2(0), \widetilde{w}_3(0)) = (0.1,0,0)$
Dashed line: *C* = 2.27; Solid line *C* = 1.5



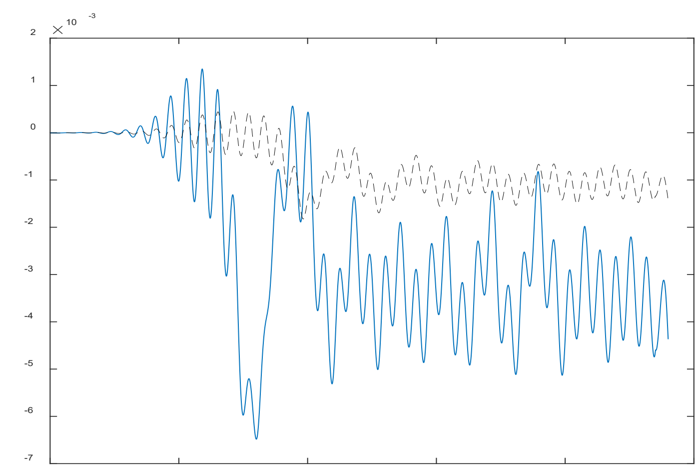
We see a qualitative change in the statistical moments of the simulated chaotic inflation when *C* is raised above a certain threshold. Preliminary results seem to point that when *C* > 2.9, the mean and volatility of simulated inflation after a given shock to the economy are considerably reduced. This implies that, although in the special situation of an economy evolving within a chaotic attractor, monetary policy is able to curb inflation volatility and keep inflation mean closer to the steady state intended value by using extreme values of the Taylor coefficient. However, we also find that extreme values of *C*, for specific initial conditions, may imply the paradoxical result of *raising* long-run inflation mean. Consider Figure 6, where we contrast the baseline simulated inflation (solid line) to time profiles of inflation generated for very high Taylor coefficients.

Figure 6. *Paths of chaotic inflation. Higher-than-threshold C*

Panel a). $(\widetilde{w}_1(0), \widetilde{w}_2(0), \widetilde{w}_3(0)) = (0.01, 0.01, 0.01)$
Solid line: $C = 2.27$; Dashed line: $C = 3$



Panel b). $(\widetilde{w}_1(0), \widetilde{w}_2(0), \widetilde{w}_3(0)) = (0.1, 0, 0)$
Solid line: $C = 2.27$; Dashed line: $C = 3$



The choice of the magnitude of the Taylor coefficient becomes therefore central in running monetary policy, when the economy is suspected to be entrapped in a chaotic attractor. We can think of C as being chosen such that the intended steady-state is targeted in the long-run.

4. Conclusions

By using the Shilnikov criterion, we discover a "route to chaos" in the money-in-utility NK model with standard policy design in the United Kingdom. Although this is not the first paper to find chaos in these kinds of models (cf. *inter al*. Benhabib et al. (2002)), our method shares with our paper using US data (Barnett *et al.* (2020)), the distinct advantage of allowing for a complete characterization of the fractal attractor and thereby of the intrinsic stochasticity implied by the induced probability measure on the fractal attractor set. But in our results with US data, we used a model without money.

The Shilnikov phenomenon occurs for a narrow region of the parameters space in the case of no regime switching (full sample), and it does not occur at all for the high inflation regime according to Liu and Mumtaz (2011) estimation of a NK DSGE model with Markov switching for the British economy. But it comes up easily in the case of the low-inflation regime. It seems therefore that the tightening of the policy rule through the Taylor coefficient speeds up the way towards the unexpected phenomenon of chaotic dynamics. Paradoxically, the adoption of an active Taylor feedback rule in the UK in recent years, while successfully lowering and stabilizing inflation, has increased the risk of Shilnikov chaos with unexpected downward drift in interest rates to the lower bound, at which the usefulness of the Taylor rule is compromised.

Furthermore, given the estimated values of the Taylor coefficient, we find a downward bias in the interest rate and inflation orbits, producing a phenomenon similar to a liquidity trap. The problems associated with the zero-lower bound on nominal interest rates, would thereby not be an intentional objective of central bank policy but of the dynamics of the system within the attractor set. The existence of this downward bias could explain the recent conundrum of very low real rates of interest that are well below the marginal product of capital.

Finally, in order to derive more implications of unintended effects of conventional monetary policy, we have used the Taylor coefficient as a bifurcation parameter and explored the effects on the simulated time profile of chaotic inflation. Although preliminary, the initial findings are interesting and somewhat unexpected. Specifically, given the baseline parameter values, there is a threshold of the Taylor coefficient (C) below which simulated inflation time profiles have comparable statistical moments across different C and initial conditions. Conversely, when the Taylor coefficient is chosen above the threshold, a qualitative change in the statistical moments of the simulated chaotic inflation is introduced, with a drastic reduction of the volatility and a mean that is much closer to the target. Paradoxically, extreme values of C may even imply an increase in the long-run inflation mean.

There has been much discussion of the potential consequences of Brexit for economic risk within the United Kingdom. Our results use only pre-Brexit data consistent with our calibration of parameters based on the Liu and Mumtaz (2011) well regarded UK model.

But the ability to produce the geometry of the fractal attractor set is likely to prove very valuable in exploring subsequent results with post-Brexit data. Comparison of the fractal attractor set's geometry before and after Brexit should provide detailed information about the nature of the changes in economic risk caused by Brexit, as the needed data become available.

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APPENDICES

Appendix 1: Proof of Proposition 1.

To establish local stability properties when Edgeworth substitutability prevails, we compute the Jacobian matrix of system *M*, evaluated at $\mathbf{P}^*$. We obtain:

$$\mathbf{J} = \begin{bmatrix} 0 & (1-R'(\pi^*))\mu_1^* & 0 \\ j_{21}^* & j_{22}^* & 0 \\ -\frac{\bar{R}}{u_{mm}^*}\left[\bar{R} - u_{cm}^* \frac{u_{mm}^* - u_{cm}^*\bar{R}}{u_{cc}^* u_{mm}^* - u_{cm}^{*2}}\right] & 0 & \bar{R} - \pi^* - \tau'(a^*) \end{bmatrix}, \quad \text{(A.1)}$$

where $j_{21}^* = -\frac{\psi\phi}{\eta} c^{*\psi} \frac{u_{mm}^* - u_{cm}^*\bar{R}}{u_{cc}^* u_{mm}^* - u_{cm}^{*2}} + \frac{(\phi-1)}{\eta} c^*$ and $j_{22}^* = \rho - \frac{\psi\phi c^{*\psi} u_{cm}^* \mu_1^*}{\eta(u_{cc}^* u_{mm}^* - u_{cm}^{*2})} R'(\pi^*)$ are both positive. The application of the Routh-Hurwitz stability criterion requires the computation of:

$$Tr(\mathbf{J}) = j_{22}^* + \bar{R} - \pi^* - \tau'(a^*), \quad \text{(A.2)}$$

$$Det(\mathbf{J}) = [\bar{R} - \pi^* - \tau'(a^*)][R'(\pi^*) - 1]\mu_1^* j_{21}^* \quad \text{(A.3)}$$

$$B(\mathbf{J}) = [R'(\pi^*) - 1]\mu_1^* j_{21}^* + [\bar{R} - \pi^* - \tau'(a^*)] j_{22}^* \quad \text{(A.4)}$$

which represent Trace, Determinant, and Sum of principal minors of **J**, respectively. We also need to define

$$G(\mathbf{J}) = -B(\mathbf{J}) + \frac{Det(\mathbf{J})}{Tr(\mathbf{J})} \quad \text{(A.5).}$$

We now study the local stability properties of system *M* in the neighborhood of $\mathbf{P}^*$, when monetary policy is active ($R'(\pi^*) > 1$). As discussed in the main text, we need now to use the Edgeworth elasticity $|u_{cm}^*|$. There exists a threshold such that, for any $|u_{cm}^*| < |\hat{u}_{cm}^*|$, $Tr(\mathbf{J}) > 0$ and, for any $|u_{cm}^*| > |\hat{u}_{cm}^*|$, $Tr(\mathbf{J}) < 0$. Therefore, the effects of changing fiscal policy diverge according to whether the observed Edgeworth elasticity is above or below the threshold. Consider first the case of $|u_{cm}^*| < |\hat{u}_{cm}^*|$. Then, if the fiscal policy is active ($\bar{R} - \pi^* - \tau'(a^*) > 0$), both $Det(\mathbf{J})$ and $Tr(\mathbf{J})$ are positive, whereas $G(\mathbf{J}) < 0$. There are therefore three sign variations in the Routh-Hurwitz criterion: $\mathbf{P}^*$ is a repellor. Consider now the case of a passive fiscal policy. Now, while $Tr(\mathbf{J})$ keeps its positive sign, $Det(\mathbf{J})$ becomes negative. Then, irrespectively of the sign of $G(\mathbf{J})$, there are two variations and one permanence in the Routh-Hurwitz scheme. $\mathbf{P}^*$ is a saddle of index 2. The opposite case of $|u_{cm}^*| > |\hat{u}_{cm}^*|$ uses similar arguments.