A Nonparametric Test for Testing Heterogeneity in Conditional Quantile Treatment Effects*†

Zongwu Cai[a], Ying Fang[b,c], Ming Lin[b,c,†], Shengfang Tang[b]

[a]Department of Economics, University of Kansas, Lawrence, KS 66045, USA.
[b]Wang Yanan Institute for Studies in Economics and Fujian Key Laboratory of Statistical Sciences, Xiamen University, Xiamen, Fujian 361005, China.
[c]Department of Statistics and Data Science, School of Economics, Xiamen University, Xiamen, Fujian 361005, China.

November 30, 2021

Abstract: This paper proposes a nonparametric test to assess whether there exists heterogeneously quantile treatment effect for an intervention on outcome of interest across different sub-populations defined by covariates of interest. Specifically, a consistent test statistic based on the Cramér-von Mises type criterion is developed to test if the treatment has a constant quantile effect for all subpopulations defined by covariates of interest. Under some regular conditions, the asymptotic behaviors of the proposed test statistic are investigated under both the null and alternative hypotheses. Furthermore, a nonparametric Bootstrap procedure is suggested to approximate the finite-sample null distribution of the proposed test and the asymptotic validity of the proposed Bootstrap test is theoretically justified. Through Monte Carlo simulations, we demonstrate the power properties of the test in finite samples. Finally, the proposed testing approach is applied to investigating whether there exists heterogeneity for the quantile treatment effect of maternal smoking during pregnancy on infant birth weight across different age groups of mothers.

Keywords: Bootstrap; Cramér-von Mises type test; Heterogeneity test; Nonparametric quantile regression; Quantile treatment effect.

*The authors acknowledge the financial supports from the National Science Fund of China (NSFC) for Distinguished Scholars (71625001) and the NSFC key projects with grant numbers 71631004 and 72033008.
†Corresponding author: Ming Lin (linming50@xmu.edu.cn).

1 Introduction

In program evaluation studies, it is important to learn about the heterogeneous impacts of policy variables on different points of the outcome distribution of interest. Examples include, but not limited to, evaluating the effects of government training programs on lower quantiles of earning distributions studied by LaLonde (1995) and Abadie, Angrist and Imbens (2002), the effects of the government-subsidized saving program on lower tails of savings distributions, and among many others. To characterize the heterogeneous effects along with the outcome distribution, quantile treatment effect (QTE), originally suggested by Doksum (1974) and Lehmann (1975) and defined as the difference between the quantiles of the marginal potential distributions of the treatment and control responses, provides a powerful tool to document such heterogeneity. In the last few decades, QTE has gained increasing popularity in economics, political science, and many other social, behavioral, and statistical sciences. Recent studies on QTE include, but not limited to, the papers by Abadie et al. (2002), Chernozhukov and Hansen (2005), Donald and Hsu (2014), Firpo (2007), Frölich and Melly (2013), and the references therein. Moreover, Tang (2020) provided a comprehensive survey on recent developments in modeling methods for QTE.

It is importantly noted that the aforementioned papers mainly focus on identification and estimation of the quantile treatment effect for the overall population or the treated group under various assumptions. It is generally believed in program evaluation literature that the effect of a treatment can be heterogeneous across different individuals as in Heckman and Robb (1985) and Heckman et al. (1997). Consequently, in many cases, researchers may be more interested in studying the effects of programs across different individuals instead of the effects for the overall population or the subpopulation of treated individuals. For example, it may be of substantive interest to investigate the heterogeneous effect of maternal smoking during pregnancy on infant birth weight across mothers with different ages. How to characterize the heterogeneity of treatment effects across different individuals is a challenge in the treatment effect literature and it has been extensively considered in the recent literature. Recently, to characterize the heterogeneous effect across different sub-populations defined by some covariates of interest, Abrevaya, Hsu and Lieli (2015) and Lee, Okui and Whang (2017) considered the partially conditional average treatment effect

(ATE). Different from Abrevaya et al. (2015) and Lee et al. (2017), to capture heterogeneities for both across-distribution and across-individuals simultaneously, Cai, Fang, Lin and Tang (2021)[1] and Zhou, Guo and Zhu (2021) proposed a partially conditional quantile treatment effect (PCQTE) model, whereas Tang, Cai, Fang and Lin (2021) considered a parametric model.

Our motivation in this paper comes actually from the empirical estimation results in Cai et al. (2021) and Tang et al. (2021) by investigating the QTE of maternal smoking during pregnancy on infant birth weight across different age groups of mothers. The main findings in Cai et al. (2021) are that there is a significant negative effect of smoking on infant birth weight across all mothers' ages and quantiles considered for both whites and blacks and there is substantial heterogeneity across different mothers' ages for whites but not for blacks. Motivated by these estimation results, from statistical and empirical perspectives, it is interesting to test whether or not the conditional QTEs conditional on mothers' ages, for both whites and blacks, change over mothers' ages, in other words, whether there exists heterogeneity for the QTEs of maternal smoking on infant birth weight across different age groups of mothers for both whites and blacks. To this end, we propose in this paper a novel test to assess whether there exists heterogeneously distributional effect for an intervention on outcome of interest across different sub-populations defined by covariates of interest. Specifically, a nonparametric test is developed for testing the null hypothesis that the treatment has a constant quantile effect for all subpopulations defined by covariates of interest. In other words, there is no heterogeneity in QTEs by covariates of interest. To this end, a consistent test statistic is constructed based on the Cramér-von Mises type criterion.

Under some regular conditions, we establish the asymptotic distribution of the proposed test statistic under both the null and alternative hypotheses and investigate the power of our test against a sequence of local alternatives. However, note that to calculate the critical value of the proposed test statistic under null hypothesis, one needs to consistently estimate the conditional density of the potential outcomes conditional on covariates of interest involved in the asymptotic bias and asymptotic variance, which is not an easy

[1] Please note that the working paper version of this paper in English with a modification can be downloaded at https://econpapers.repec.org/paper/kanwpaper/202005.htm.

task recognized in the literature. To overcome this problem, a nonparametric Bootstrap procedure is proposed to approximate the finite-sample null distribution of the proposed test. Furthermore, the asymptotic validity of the proposed Bootstrap test is justified. Through Monte Carlo simulations, we demonstrate the power properties of the test in finite samples. As an empirical illustration, the proposed testing approach is applied to investigating whether there exists heterogeneity for the QTE of maternal smoking during pregnancy on infant birth weight across different age groups of mothers. The testing results show that the QTE of maternal smoking on infant birth weight do change over mother's age for all quantile levels considered for whites but not for blacks, which support the findings obtained in Cai et al. (2021).

Interestingly, this paper is related to some works in literature. For example, first, Crump, Hotz, Imbens and Mitnik (2008) developed two nonparametric tests based on series approach, in which the first is to test whether a treatment has a zero average effect for all sub-populations defined by covariates, and the second is to test whether the ATE conditional on the covariates is identical for all sub-populations, in other words, whether heterogeneity exists in ATE by covariates. Second, Lee and Whang (2009) proposed to test whether the conditional QTE is significant, conditional on the whole set of covariates. By contrast, our focus is on testing if the partially conditional QTE is a constant, in which the constant needs to be estimated. More importantly, one may be interested in studying the heterogeneous effect on some particular covariates instead of the whole set of covariates, for example, the effect of maternal smoking during pregnancy on infant birth weight across mothers with different ages. Moreover, Escanciano and Goh (2014) considered a nonparametric test for testing the specification of a linear conditional quantile function over a continuum of quantile levels and they showed that the use of an orthogonal projection on the tangent space of nuisance parameters at each quantile index can improve power and facilitate the simulation of critical values via the application of a simple multiplier Bootstrap procedure. Finally, Dong, Li and Feng (2019) introduced a new approach to assess the lack of fit for quantile regression models. They first transformed the lack-of-fit tests for parametric quantile regression models into checking the equality of two conditional distributions of covariates. Then, by applying some successful two-sample test statistics in the literature, two tests are constructed to check the lack of fit for low and high dimensional quantile regression models. Finally, to calculate the p-values or critical values, they suggested adopting the wild Bootstrap procedure.

The remainder of this paper is organized as follows. Section 2 introduces the proposed test statistic and presents its asymptotic properties under the null hypothesis. Also, a Bootstrap procedure is suggested to approximate the finite-sample null distribution of the proposed test and the asymptotic validity of the proposed Bootstrap test is theoretically justified. In Sections 3 and 4, the finite sample properties of our test through Monte Carlo simulations are investigated and an empirical application is considered, respectively. Section 5 concludes the paper. Finally, the key steps for proving the theorems can be found in Appendix, together with some auxiliary lemmas with their detailed proofs given in Supplement.

2 Testing Heterogeneity for Conditional QTE

2.1 Test Statistic

Let us first introduce the model framework considered in this paper. To this end, let D_i be the binary treatment variable of individual i in population, where $D_i = 1$ if individual i receives the treatment of interest and otherwise, $D_i = 0$. Using the potential outcome framework initialized by Rubin (1974), define $Y_i(0)$ and $Y_i(1)$ to be the potential outcomes of individual i if it is in the control group or in the treated group, respectively. Also, assume that D_i and Y_i are observed, where Y_i is the realized outcome as $Y_i = (1 - D_i) \cdot Y_i(0) + D_i \cdot Y_i(1)$. In addition, suppose that X_i, a p-dimensional vector of pre-treatment variables for individual i, is observed too. Therefore, throughout the paper, it is assumed that $\{Y_i(0), Y_i(1), X_i, D_i\}$, $i = 1, \cdots, n$, are independent and identically distributed (iid).

Let Z_i be a d-dimensional sub-vector of X_i, where $1 \leq d \leq p$ and in particular, d is small and much smaller than p in many applications. To capture heterogeneities for both across-distribution and across-individuals simultaneously, Cai et al. (2021) and Zhou et al. (2021) considered a partially conditional quantile treatment effect model, which is defined as

$$\Delta_\tau(z) = q_{1,\tau}(z) - q_{0,\tau}(z), \tag{1}$$

where $\tau \in (0,1)$ is the quantile level and for $j = 0$ and $q_{j,\tau}(z)$ is the τ-th conditional quantile function of $Y_i(j)$ conditional on $Z_i = z$. It is important to note that for each individual in the population, only one of $Y_i(0)$ and $Y_i(1)$ is observable, so that due to the missing variable, the PCQTE parameter $\Delta_\tau(z)$ in (1) can not be identified without further restrictions on the data-generating distribution. To identify the functionals in (1), it is common in the treatment effect literature to assume that assignment to treatment is unconfounded and that the probability of assignment is bounded away from 0 and 1. Formally, the following assumption is imposed throughout the paper.

Assumption 1. *Assume that*
(i) (unconfoundedness) conditional on pre-treatment variables X_i, the potential outcomes are jointly independent from the treatment variable D_i, namely,

$$\big(Y_i(0), Y_i(1)\big) \perp\!\!\!\perp D_i \mid X_i,$$

where $\perp\!\!\!\perp$ indicates statistical independence, and
(ii) (overlap) for almost every $x \in \mathcal{X}$, where $\mathcal{X}$ is the support of X_i, there exists some small $\varepsilon > 0$ so that $\varepsilon < p(x) = P(D_i = 1|X_i = x) < 1 - \varepsilon$, where $p(x)$ is called the propensity score function.

Part (i) of Assumption 1 is often referred to as the (strongly) *ignorable treatment assignment*, *conditional independence assumption* or *selection on observables* in the econometrics and/or statistics literature and it requires that conditional on the observed individual characteristics X_i, the treatment assignment D_i is independent of the potential outcomes $Y_i(0)$ and $Y_i(1)$. Although it is a strong assumption, it has been extensively employed in many applied fields to study on the effect of treatments or programs, among others, see, for example, the papers by Abadie and Imbens (2006, 2016), Hirano, Imbens and Ridder (2003), Heckman et al. (1998), Dehejia and Wahba (1999), and Firpo (2007). Part (ii) of Assumption 1 states that in the population for almost all values of X_i, both treatment assignment levels have a positive probability of occurrence. In practice, however, there are often concerns about possible lack of common support. A common approach to address this problem is to drop observations with the propensity score close to zero or one, and focus on the treatment effect in the subpopulation with propensity score bounded away from zero and one; see Crump et al. (2009) for more details.

Under Assumption 1, Cai et al. (2021) showed that the PCQTE function $\Delta_\tau(z)$ is nonparametrically identified and further proposed a semiparametric estimation procedure to estimate $\Delta_\tau(z)$. Specifically, the proposed estimate for $\Delta_\tau(z)$ in Cai et al. (2021) is given by

$$\widehat{\Delta}_\tau(z) = \widehat{q}_{1,\tau}(z) - \widehat{q}_{0,\tau}(z), \tag{2}$$

where for $j = 0$ and 1,

$$\widehat{q}_{j,\tau} = \arg\min_q \frac{1}{n}\sum_{i=1}^{n} K_h(Z_i - z)\widehat{W}_{n,j}(X_i, D_i)\rho_\tau(Y_i - q)$$

with $\widehat{W}_{n,0}(X_i, D_i) = (1 - D_i)/[1 - \widehat{p}_n(X_i)]$, $\widehat{W}_{n,1}(X_i, D_i) = D_i/\widehat{p}_n(X_i)$, and $K_h(u) = K(u/h)/h$. Here, $K(\cdot)$ is a kernel function, h is the bandwidth parameter, $\rho_\tau(u) = u \cdot (\tau - I\{u \leq 0\})$ is the check function as in Koenker and Bassett (1978) and Koenker (2005), and $\widehat{p}_n(x)$ is the parametric estimate of $p(x)$. Furthermore, the consistency and asymptotic normality of the proposed semiparametric estimator $\widehat{\Delta}_\tau(z)$ are also established in Cai et al. (2021). The reader is referred to the paper by Cai et al. (2021) for more details.

As discussed in the introduction, in this paper, our interest is to investigate whether there exists heterogeneity in QTEs across different sub-populations defined by covariates of interest. To this end, the following hypothesis testing problem is investigated

$$H_0 : \Delta_\tau(z) = \delta_\tau \quad \text{for all } z \in \mathcal{Z} \quad \text{versus} \quad H_1 : \Delta_\tau(z) \neq \delta_\tau \text{ for some } \; z \in \mathcal{Z} \tag{3}$$

for some constant δ_τ, where $\mathcal{Z}$ is the support of Z_i. Under the null hypothesis, the partially conditional quantile effect of the treatment is a constant and under the alternative, the PCQTE varies across different sub-populations defined by Z_i. Furthermore, the above testing setting in (3) can be generalized to the following testing problem

$$H_0 : \Delta_\tau(z) = \Delta_{\tau,0}(z, \theta_\tau) \quad \text{versus} \quad H_1 : \Delta_\tau(z) \neq \Delta_{\tau,0}(z, \theta_\tau), \tag{4}$$

where $\Delta_{\tau,0}(z, \theta_\tau)$ is a known function with unknown parameter θ_τ. The purpose of the test in (4) is to see whether $\Delta_\tau(z)$ has a particular parametric form, say, a linear function as in

Tang et al. (2021).

In order to test whether the hypothesis testing problem formulated in (3) holds or not, the test statistic is constructed based on Cramér-von Mises criterion as follows. To this end, let

$$J = \int \Big(\Delta_\tau(z) - \delta_\tau\Big)^2 dz \geq 0,$$

where the integral is taken over $\mathcal{Z}$. Note that $J = 0$ if and only if the null hypothesis in (3) is true. Hence, a test statistic using the sample analogue of J is defined by

$$J_n = \int \Big(\widehat{\Delta}_\tau(z) - \hat{\delta}_\tau\Big)^2 dz,$$

where $\widehat{\Delta}_\tau(z)$ is the semiparametric estimator (2) of $\Delta_\tau(z)$, and $\hat{\delta}_\tau$ is a $\sqrt{n}$-consistent estimator for δ_τ, such as the estimator

$$\hat{\delta}_\tau = \frac{1}{n}\sum_{i=1}^{n} \widehat{\Delta}_\tau(Z_i) \tag{5}$$

proposed in Cai and Xiao (2012). Note that by following the same idea in Cai and Xiao (2012), one can show easily that $\hat{\delta}_\tau$ in (5) a is $\sqrt{n}$-consistent estimator of δ_τ.

Remark 1. If Z is taken to be X in (3), then, the hypothesis testing problem in (3) collapses into testing whether the conditional QTE is a constant for all values of the covariates. Different from our setting, Crump et al. (2008) considered to testing whether the conditional ATE is a constant or zero for all values of the covariates. However, note that even though the conditional ATE is equal to a constant, the conditional QTE may not be a constant. Consequently, our paper complements and extends the paper by Crump et al. (2008) on testing whether there exists treatment effect heterogeneity across different sub-populations defined by covariates.

Remark 2. Besides the testing issues displayed in (3) and (4), one may be interested in testing

$$H_0 : \Delta_\tau(z) \leq 0 \ \ (\text{or } \ \geq 0) \ \ \text{for all} \ \ z \in \mathcal{Z}, \tag{6}$$

which leads to studying the stochastic dominance between $Y(0)$ and $Y(1)$ for all Z. Recently, Lee, Song and Whang (2015, 2018) developed a general method for testing inequality restrictions on nonparametric functions using a Bootstrap procedure. Hence, the procedure as in Lee et al. (2015, 2018) may be used here to test the null hypothesis formulated in (6) being true or not. These extensions are beyond the scope of this paper but certainly worth pursuing in future research.

2.2 Limiting Distribution of Test Statistic J_n

This subsection is devoted to investigating the asymptotic properties of the proposed test statistic J_n. Although the asymptotic theory for J_n can be obtained for any d-dimensional Z_i with $d \ll p$, the result is presented only for $d = 1$ to save notation throughout the rest of this paper. As pointed out by Abrevaya et al. (2015), the case for $d = 1$ is the most relevant case in practice. Before studying the asymptotic properties of the proposed test statistic J_n, the following technical assumptions are needed, list below.

Assumption 2. *(Distributions of X_i and Z_i) X_i has a compact support $\mathcal{X}$ and the density function of X_i, $f_X(x)$, satisfies $\inf_{x\in\mathcal{X}} f_X(x) \geq c$ for some $c > 0$. Furthermore, the density function of Z_i, $f_Z(z)$ is twice continuously differentiable in $\mathcal{Z}$.*

Assumption 3. *(i) The conditional density function $f_{Y(j)|X}(y|x)$ is continuous and bounded on the support of $Y_i(j)$ and X_i for $j = 0$ and 1. (ii) The conditional density function $f_{Y(j)|Z}(y|z)$ is continuous and uniformly bounded away from zero in a neighborhood of $q_{j,\tau}(z)$ for $j = 0$ and 1. It is twice differentiable with respect to z, and its first derivative with respect to y is continuous and bounded on the support of $Y_i(j)$ and Z_i.*

Assumption 4. *For $j = 0$ and 1, the conditional quantile function $q_{j,\tau}(z)$ is continuously differentiable on the support of Z with bounded second order derivatives.*

Assumption 5. *(Kernel and bandwidth) (i) The kernel function $K(u)$ is a symmetric, continuously differentiable probability density function with compact support, $[-1, 1]$, say. (ii) $h \to 0$, $nh^2 \to \infty$ and $nh^4 \to 0$ as $n \to \infty$.*

Assumption 6. *(Parametric propensity score function) Suppose the propensity score function has a parametric form $p(x) = p(x;\theta_0)$ with a fixed dimensional parameter θ_0. Also, assume that the estimated propensity score function $\hat{p}_n(x) = p(x;\hat{\theta}_n)$ satisfies $\sup_{x\in\mathcal{X}} \big|p(x;\hat{\theta}_n) - p(x;\theta_0)\big| = O_p(n^{-1/2})$.*

The restriction imposed on the distribution of X_i in Assumption 2 is commonly used in the literature on treatment effect evaluation; see Hirano et al. (2003), Abadie and Imbens (2006, 2016), Firpo (2007), Abrevaya et al. (2015), and among others. Assumption 3 guarantees the conditional quantile function $q_{j,\tau}(z)$ for $j = 0$ and 1 is unique and well defined and the smoothness conditions imposed are easily satisfied in practice. The smoothness conditions on the conditional quantile function $q_{j,\tau}(z)$ for $j = 0$ and 1 imposed in Assumption 4 are also easily satisfied in practice. Assumption 5 on kernel function and bandwidth is frequently assumed in the literature on nonparametric estimation. Many commonly used kernel functions, such as the Epanechnikov kernel, satisfy the requirements. Assumption 6 typically holds for standard parametric estimation methods under reasonably mild regularity conditions.

Under the assumptions listed above, we now can state our main result on the asymptotic properties of the proposed test statistic J_n and its proof can be found in Appendix. For easy presentation, first, define some notations as follows. Let $\mu_0(z;u) = E\Big\{ \big[I\{Y_i(0) \le q_{0,\tau}(u)\} - \tau\big]^2/[1-p(X_i)] \,\big|\, Z_i = z \Big\}$ and $\mu_1(z;u) = E\Big\{ \big[I\{Y_i(1) \le q_{1,\tau}(u)\} - \tau\big]^2/p(X_i) \,\big|\, Z_i = z \Big\}$. Then, we have the following asymptotic results.

Theorem 1. *Suppose that Assumptions 1-6 are satisfied. Then, under the null hypothesis H_0 in (3), one has*

$$n\sqrt{h}\big(J_n - \mu_J\big) \xrightarrow{D} \mathcal{N}(0, \sigma_J^2),$$

where

$$\mu_J = \frac{\nu_0(K)}{nh} \int \left\{ \frac{\mu_1(z;z)}{f^2_{Y(1)|Z}(q_{1,\tau}(z)|z)} + \frac{\mu_0(z;z)}{f^2_{Y(0)|Z}(q_{0,\tau}(z)|z)} \right\} \frac{1}{f_Z(z)} dz,$$

and with $\nu_0(K) = \int K^2(u)du$,

$$\sigma_J^2 = 2 \int \Big(\int K(t)K(t+s)dt \Big)^2 ds \int \left\{ \frac{\mu_1(u;u)}{f^2_{Y(1)|Z}(q_{1,\tau}(u)|u)} + \frac{\mu_0(u;u)}{f^2_{Y(0)|Z}(q_{0,\tau}(u)|u)} \right\}^2 \frac{1}{f^2_Z(u)} du,$$

and under the alternative hypothesis H_1,

$$n\sqrt{h}\big(J_n - \mu_J\big) \xrightarrow{p} +\infty. \tag{7}$$

Following Theorem 1, an asymptotic significance level α_0 test is to reject H_0 if $n\sqrt{h}\big(J_n - \mu_J\big)/\sigma_J > C_{\alpha_0}$, where C_{α_0} is the α_0 upper-quantile of the standard normal distribution. Clearly, (7) implies that the proposed test is consistent. Note that to the best of our knowledge, the above asymptotic result for testing nonparametric QTE is new in the literature.

In addition to testing the null hypothesis against fixed alternatives, it is of interest to consider testing power for local departures from the null. Suppose that δ_τ is estimated using (5), we focus on a set of Pitman alternatives represented by

$$H_{1n} : \Delta_\tau(z) = \delta_\tau + \rho_n \cdot \zeta(z), \tag{8}$$

where $\rho_n = n^{-1/2}h^{-1/4} \to 0$ as $n \to \infty$ and the function $\zeta(z)$ satisfies

$$\int \zeta(z) f_Z(z)\, dz = 0 \quad \text{and} \quad 0 < \int \zeta^2(z) dz < \infty.$$

The following theorem shows that our test can distinguish alternatives H_{1n} that get closer to H_0 at rate $n^{-1/2}h^{-1/4}$ while maintaining a constant power level.

Theorem 2. *Under Assumptions 1-6, suppose that the local alternative (8) converges to the null in the sense that $\rho_n = n^{-1/2}h^{-1/4}$. Then,*

$$\frac{n\sqrt{h}}{\sigma_J}\big(J_n - \mu_J\big) \xrightarrow{D} \mathcal{N}\Big(\sigma_J^{-1} \int \zeta^2(z) dz, 1\Big).$$

Clearly, it follows from Theorem 2 that under the local alternative (8),

$$P\Big(n\sqrt{h}\big(J_n - \mu_J\big)/\sigma_J > C_{\alpha_0}\Big) \to 1 - \Phi\Big(C_{\alpha_0} - \sigma_J^{-1} \int \zeta^2(z) dz\Big),$$

where Φ is the cumulative distribution function of the standard normal distribution. This indicates that the testing power for local alternative (8) converges to a constant greater than the significance level α_0.

2.3 A Nonparametric Bootstrap Test

Theorem 1 provides the asymptotic null distribution of the test J_n. Consequently, one can perform tests for H_0 by comparing the value of J_n with its asymptotic critical value. However, as expected, it can not be used directly for an accurate calculation of critical values. This is because the test based on the asymptotic distribution might be sensitive to the choice of bandwidth h and the consistent estimation of μ_J and σ_J^2 in small samples. In particular, it is well known in the quantile regression literature that the consistent estimation of the conditional density of $Y_i(j)$ given Z_i involved in μ_J and σ_J^2 is not an easy task; see, for example, Koenker and Xiao (2004) and Cai and Xu (2008). To overcome this difficulty, following Chen et al. (2003) and Firpo et al. (2017), although other types of Bootstrap methods such as the multiplier Bootstrap in Escanciano and Goh (2014) and the wild Bootstrap in Dong et al. (2019) can be used, for simplicity, here a nonparametric Bootstrap procedure is proposed to determine the p-value for J_n. It involves the following steps.

(1) Generate the ith Bootstrap sample by drawing samples from the original sample $\big\{(Y_i, X_i,\, D_i)\big\}_{i=1}^n$ with replacement, denoted by $\big\{(Y_i^*, X_i^*, D_i^*)\big\}_{i=1}^n$.

(2) Compute the Bootstrap test statistic

$$J_n^* = \int \Big(\big(\widehat{\Delta}_\tau^*(z) - \widehat{\delta}_\tau^*\big) - \big(\widehat{\Delta}_\tau(z) - \widehat{\delta}_\tau\big)\Big)^2 dz,$$

where $\widehat{\Delta}_\tau^*(z)$ and $\widehat{\delta}_\tau^* = \sum_{i=1}^n \widehat{\Delta}_\tau^*(Z_i^*)/n$ are estimated using the Bootstrapping sample $\big\{(Y_i^*, X_i^*, D_i^*)\big\}_{i=1}^n$, and $\widehat{\Delta}_\tau(z)$ and $\widehat{\delta}_\tau = \sum_{i=1}^n \widehat{\Delta}_\tau(Z_i)/n$ are computed based on the original data.

(3) Repeat steps (1) and (2) a large number of times, say, B times, to obtain $\{J_n^{*(j)}\}_{j=1}^B$.

(4) Reject H_0 at significance level α_0 if J_n exceeds the $(1-\alpha_0)$-th sample quantile of $\{J_n^{*(j)}\}_{j=1}^B$.

Define $\widetilde{J}_n^* = n\sqrt{h}(J_n^* - \mu_J)/\sigma_J$. The following theorem justifies the asymptotic validity of the Bootstrap test with its proof given in Appendix.

Theorem 3. *Suppose the same conditions as in Theorem 1 are satisfied. Then under H_0 or H_1, we have*

$$\sup_{y\in\mathbb{R}}\left|P\big(\widetilde{J}_n^* \le y \,\big|\, \{Y_i, X_i, D_i\}_{i=1}^n\big) - \Phi(y)\right| = o_p(1),$$

where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal distribution.

Theorem 3 states that the Bootstrap statistic $\widetilde{J}_n^* = n\sqrt{h}(J_n^* - \mu_J)/\sigma_J$ converges to $\mathcal{N}(0,1)$ in distribution in probability. It is important to note that Theorem 3 holds true regardless of whether the null hypothesis is true or not. Therefore, when the null hypothesis is true, the Bootstrap test procedure leads to asymptotically correct size, because conditional on the data, the Bootstrap statistic J_n^* has the same asymptotic distribution as J_n. When the null hypothesis is false, because the test statistic $n\sqrt{h}\big(J_n - \mu_J\big)/\sigma_J$ diverges to $+\infty$ as the sample size n goes to infinity as shown in Theorem 1, whereas the Bootstrap critical value is still finite, the Bootstrap procedure leads to a consistent test.

3 Monte Carlo Studies

In this section, we investigate the finite sample performance of the proposed test J_n by means of simulation studies. The goal is to assess the accuracy and power of the proposed test for moderate sample sizes in various scenarios.

Let the data generating process (DGP) be:

$$Y(0) = \gamma_0\sqrt{U_0}X_2 \quad \text{and} \quad Y(1) = \lambda\cdot\rho_n\cdot X_1 + \gamma_1\sqrt{U_1}X_2,$$

where $\rho_n = n^{-1/2}h^{-1/4}$, $\gamma_0 = 1.0$, $\gamma_1 = 1.5$, U_0 and U_1 independently follow the $U[0,1]$ distribution, X_1 and X_2 are independently generated from $U[-1,1]$ and Beta$(3,1)$, respectively, and the propensity score function is

$$P(D=1|X_1,X_2) = \frac{\exp\{-0.5 + X_1 + X_2\}}{1+\exp\{-0.5 + X_1 + X_2\}}.$$

Finally, the conditional variable Z is taken to be X_1. Under this setting, by straightforward calculations, the conditional quantile function for $Y(j)$ for $j = 0$ and 1, conditional on $Z = z$, is given by $q_{0,\tau}(z) = \gamma_0 a_\tau$ and $q_{1,\tau}(z) = \lambda\,\rho_n\,z + \gamma_1 a_\tau$, respectively, where a_τ is the unique solution of equation $-2a^3 + 3a^2 - \tau = 0$ within the interval $(0, 1)$. Therefore, the PCQTE is

$$\Delta_\tau(z) = \lambda \, \rho_n \, z + (\gamma_1 - \gamma_0) a_\tau,$$

where λ in the above equation takes different values in the experiment so that we can investigate empirical sizes and local power curves of the test statistic J_n indexed by λ. It is easy to see that $\Delta_\tau(z)$ is equal to a constant only when $\lambda = 0$, which corresponds to the null hypothesis. The Bootstrap procedure outlined in Section 2.3 is used to determine the critical value. The number of Bootstrap replications is set as $B = 599$. To examine the size and local power performance of the test statistic J_n, three different sample sizes $n = 400$, $n = 800$ and $n = 1600$ are considered. To check the sensitivity of the test with respect to different values of the bandwidth h, motivated by the conditions in Theorem 1, $h = c\, n^{-1/3}$ is used with $c = 0.5$, 1.0 and 2.0. Finally, three quantiles levels, namely, $\tau = 0.25$, 0.5 and $\tau = 0.75$, are considered. The empirical sizes and local powers of the test J_n are computed using $1,000$ simulations under the nominal size $\alpha = 5\%$, respectively.

The empirical sizes of the test J_n based on Bootstrap critical value are reported in Table 1. It can be seen that the empirical sizes converge to their nominal sizes as the sample size n increases. Particularly, when the sample size increases to 1600, the test J_n performs well in all cases considered. Also, one can observe that the choice of the bandwidth h seems to have little influence on empirical sizes.

Next, Figures 1-3 display the estimated local power curves with nominal size $\alpha = 5\%$ of the test J_n for different quantile levels and different choices of the bandwidth. In general, the test J_n performs reasonably powerful in detecting the deviation from the null hypothesis in all cases considered. Specifically, it can be seen from these figures that the test J_n has power against local alternatives converging to the null at the rate of $\rho_n = n^{-1/2} h^{-1/4}$. Moreover, it is not surprising that the powers increase quickly with the value of λ increasing. It is also noticed from these figures that the bandwidth h in a certain range seems to have little impact on the power of the test.

Table 1: Empirical sizes of J_n (nominal size $\alpha = 5\%$)

		$h = 0.5n^{-1/3}$		
$\lambda = 0$	n	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$
	400	0.030	0.034	0.031
	800	0.040	0.037	0.045
	1600	0.047	0.053	0.044
		$h = 1.0n^{-1/3}$		
$\lambda = 0$	n	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$
	400	0.043	0.043	0.045
	800	0.058	0.052	0.055
	1600	0.044	0.053	0.053
		$h = 2.0n^{-1/3}$		
$\lambda = 0$	n	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$
	400	0.034	0.036	0.038
	800	0.056	0.056	0.054
	1600	0.053	0.051	0.053

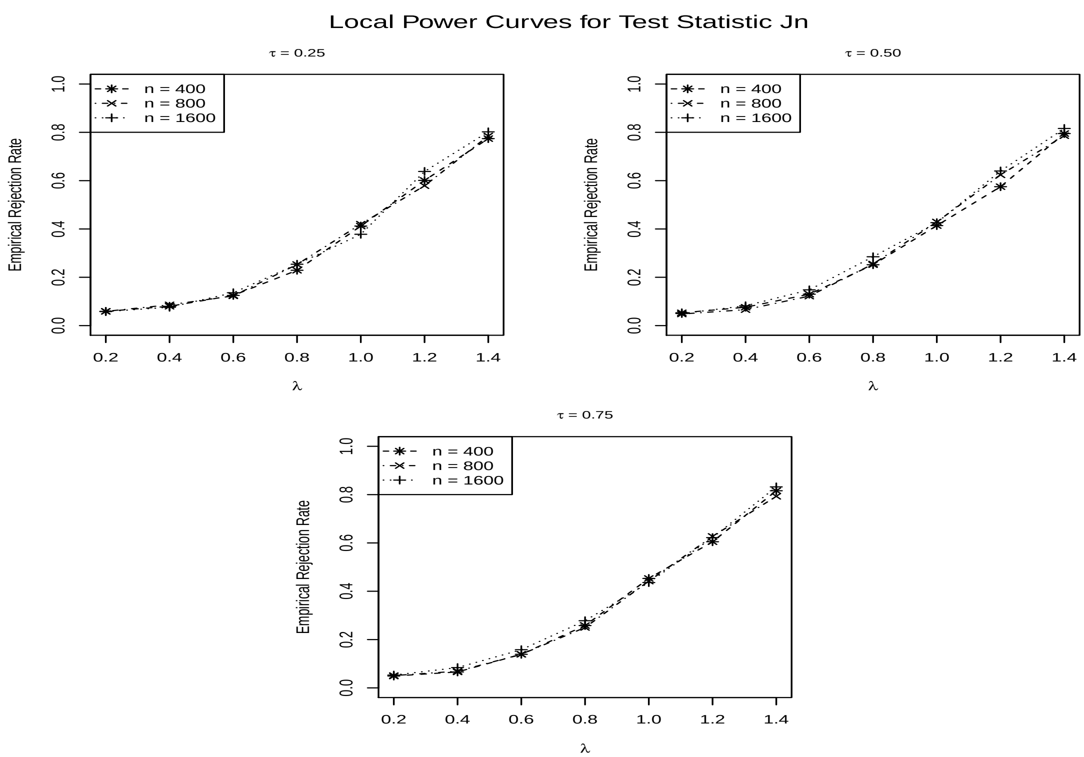

Figure 1: Local power curves for test statistic J_n with $\rho_n = n^{-1/2}h^{-1/4}$ and bandwidth $h = 0.5n^{-1/3}$ and nominal size $\alpha = 5\%$.

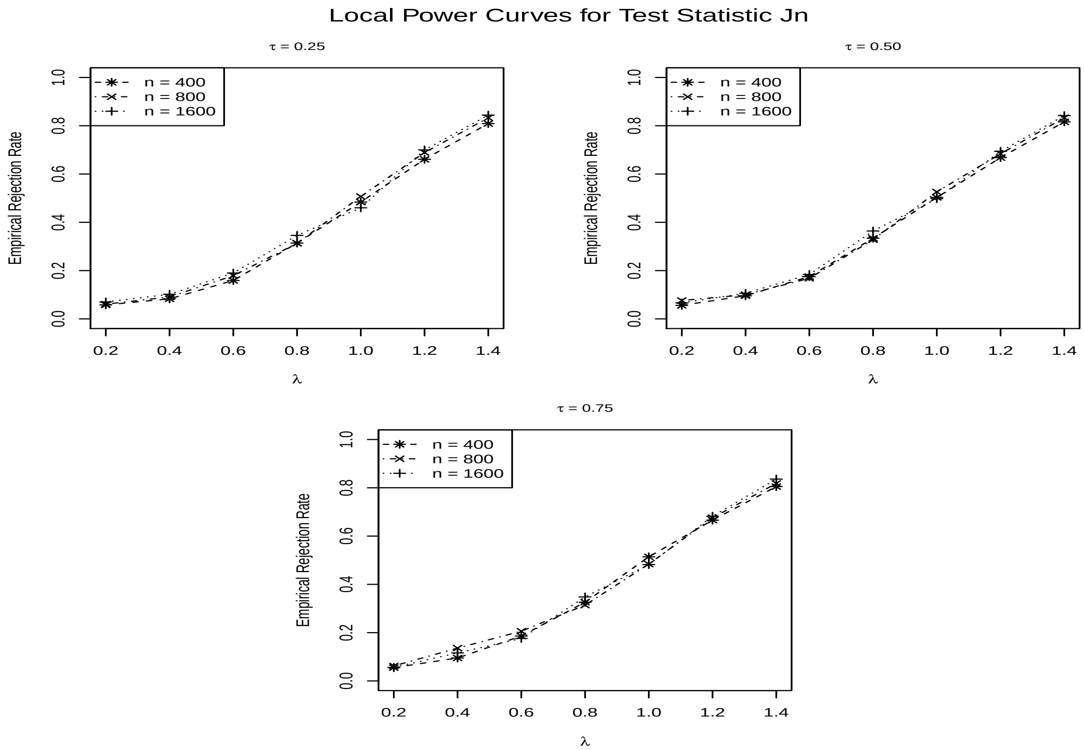

Figure 2: Local power curves for test statistic J_n with $\rho_n = n^{-1/2}h^{-1/4}$ and bandwidth $h = 1.0n^{-1/3}$ and nominal size $\alpha = 5\%$.

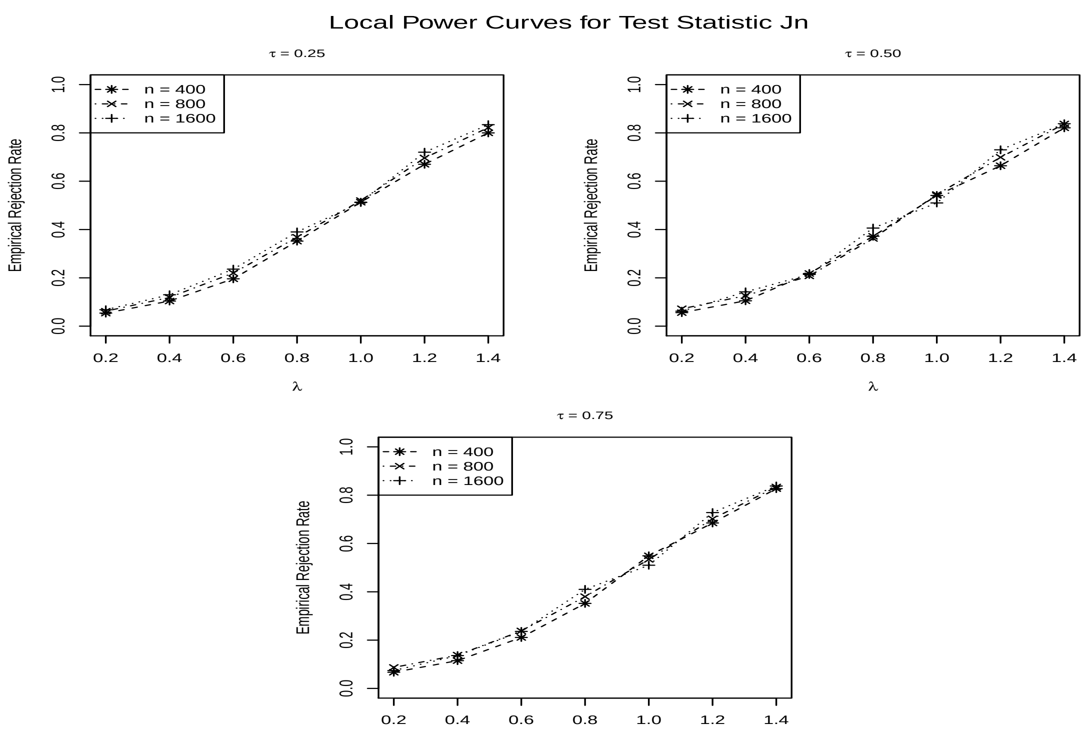

Figure 3: Local power curves for test statistic J_n with $\rho_n = n^{-1/2}h^{-1/4}$ and bandwidth $h = 2.0n^{-1/3}$ and nominal size $\alpha = 5\%$.

4 A Real Example

In this section, the proposed testing approach is applied to investigating whether there exists heterogeneity for the QTE of maternal smoking during pregnancy on infant birth weight across different age groups of mothers. To this end, we use the same dataset as in Abrevaya et al. (2015), composed of vital statistics collected by the North Carolina State Center Health Services between 1988 and 2002, accessible through the Odum Institute at the University of North Carolina. As in Abrevaya et al. (2015), our sample is limited to first-time mothers and as routine in the literature, we also treat blacks and whites as separate populations throughout. The number of observations is $157,989$ for the blacks group and $433,558$ for the whites group.

It is generally recognized that low infant birth weight is associated with health and human capital development throughout life as argued by Black et al. (2007) and Almond and Currie (2011), and maternal smoking during pregnancy is considered to be the most important preventable negative cause of low birth weight; see Kramer (1987) for more discussions. Recently, there have been several studies in the literature to explore how the effect of maternal smoking during pregnancy on infant birth weight varies across different values of the mother's ages by using program evaluation approach. In particular, Abrevaya et al. (2015) and Lee et al. (2017) considered the ATE of maternal smoking on infant birth weight conditional on different mothers' ages, and found different degrees of heterogeneity by age. The main qualitative finding in Abrevaya et al. (2015) and Lee et al. (2017) is that smoking has a more severe impact at higher ages. Different from the studies by Abrevaya et al. (2015) and Lee et al. (2017), Cai et al. (2021) considered the QTE of mothers' smoking status during pregnancy on infant birth weight conditional on different mothers' ages, and found that the QTEs for the quantile levels considered seem to change significantly over ages only for whites but not for blacks. Motivated by the estimation results in Cai et al. (2021), it is further interesting to test statistically whether or not the partially conditional QTEs, for whites and blacks, change over mothers' ages. Therefore, our interest in this section is to test whether the QTE of maternal smoking on infant birth weight varies across different age groups of mothers using the proposed testing approach in Section 2.

Since our interest is in exploring whether the QTE of maternal smoking during pregnancy on infant birth weight changes across different age groups of mothers, hence the conditional variable Z is the mother's age. In addition, the treatment variable D is a binary variable which takes value 1 if the mother smokes and 0 otherwise. The outcome variable of interest Y is the infant birth weight measured in grams. Also, in this example, $Y(0)$ denotes the infant birth weight for the untreated (no-smoking) group and $Y(1)$ stands for the infant birth weight for the treated (smoking) group.

To explore the treatment effect heterogeneity of mothers' smoking on infant birth weight using the proposed testing approach in Section 2, one needs to find certain baseline covariates such that the unconfoundedness assumption holds true, that is, the potential infant birth weight outcomes are independent of the smoking decision conditional on the baseline covariates. In this paper, we use the same set of covariates X as in Abrevaya et al. (2015), which includes the mother's age, education, month of first prenatal visit, number of prenatal visits, and indicators for the baby's gender, the mother's marital status, whether the father's age is missing, gestational diabetes, hypertension, amniocentesis, taking ultra sound exams, previous (terminated) pregnancies, and alcohol use; see Abrevaya et al. (2015) for the detailed discussion.

To use the proposed testing approach, another problem is how to estimate the unknown propensity score function $p(x)$. Following Abrevaya et al. (2015) and Cai et al. (2021), here a logist model is used to estimate the propensity score function $p(x)$ and the explanatory variables used in the logist model consist of all the elements of X, the square of the mother's age, and the interaction terms between the mother's age and all other elements of X. Finally, the partially conditional QTE is estimated for mothers aged between 20 and 30 for both whites and blacks.

Table 2 displays the testing results for testing whether the partially conditional QTE changes over mother's age. It can be seen clearly from Table 2 that one should reject the null hypothesis for whites for all quantiles considered at 5%. This means that PCQTEs do change over mother's age for all quantile levels considered at the significance level $\alpha = 5\%$ for whites. But, for blacks, there is a strong evidence to support the homogeneity of PCQTE over mother's age for all quantile levels considered. These testing results support strongly the empirical findings obtained in Cai et al. (2021).

Table 2: Test results for testing if PCQTE function changes over mother's age.

Quantile level	Test statistic J_n (Bootstrap p-value)	
τ	Whites	Blacks
0.10	0.022	0.573
0.25	0.002	0.307
0.50	0.033	0.151
0.75	0.035	0.474
0.90	0.042	0.326

5 Conclusion

Motivated by investigating whether or not the conditional QTEs conditional on mothers' ages, for both whites and blacks, change over mothers' ages, in other words, whether there exists heterogeneity for the QTEs of maternal smoking on infant birth weight across different age groups of mothers for whites and blacks, we propose a nonparametric versus constant test under quantile regression setting, which is applied to assessing whether there exists heterogeneously distributional effect for an intervention on outcome of interest across different sub-populations defined by covariates of interest. To test whether the null hypotheses of interest holds true or not, a consistent test statistic is proposed based on the Cramér-von Mises type criterion. To the best of our knowledge, it is believed that this test is novel in the QTE literature. Under some regular conditions, we establish the asymptotic distribution of the proposed test statistic under the null hypothesis and its consistency against fixed alternatives. We also study the power of our test against a sequence of local alternatives. Also, we propose a Bootstrap procedure to approximate the finite-sample null distribution of the proposed test. Furthermore, the asymptotic validity of the proposed Bootstrap test is also established.

Finally, some extensions of our paper can be considered. For example, the first is to consider the hypothesis formulated in (4) which is a test for nonparametric versus parametric. Second, one might be interesting in extending our results to time series cases, which has a potential in a wide range of applications. Such extensions can be warranted as a future research.

References

Abadie, A., J. Angrist, and G. Imbens (2002). Instrumental variables estimates of the effect of subsidized training on the quantiles of trainee earnings. *Econometrica 70*(1), 91–117.

Abadie, A. and G. W. Imbens (2006). Large sample properties of matching estimators for average treatment effects. *Econometrica 74*(1), 235–267.

Abadie, A. and G. W. Imbens (2016). Matching on the estimated propensity score. *Econometrica 84*(2), 781–807.

Abrevaya, J., Y.-C. Hsu, and R. P. Lieli (2015). Estimating conditional average treatment effects. *Journal of Business & Economic Statistics 33*(4), 485–505.

Almond, D. and J. Currie (2011). Killing me softly: The fetal origins hypothesis. *Journal of Economic Perspectives 25*(3), 153–72.

Bhattacharya, R. N. and R. R. Rao (1986). *Normal Approximation and Asymptotic Expansions*. Wiley, New York.

Black, S. E., P. J. Devereux, and K. G. Salvanes (2007). From the cradle to the labor market? the effect of birth weight on adult outcomes. *Quarterly Journal of Economics 122*(1), 409–439.

Cai, Z., Y. Fang, M. Lin, and S. Tang (2021). Estimation of partially conditional quantile treatment effect model. *China Journal of Econometrics 1*(4), 741–762.

Cai, Z. and Z. Xiao (2012). Semiparametric quantile regression estimation in dynamic models with partially varying coefficients. *Journal of Econometrics 167*(2), 413–425.

Cai, Z. and X. Xu (2008). Nonparametric quantile estimations for dynamic smooth coefficient models. *Journal of the American Statistical Association 101*(485), 1595–1608.

Chen, X., O. Linton, and I. Van Keilegom (2003). Estimation of semiparametric models when the criterion function is not smooth. *Econometrica 71*(5), 1591–1608.

Chernozhukov, V. and C. Hansen (2005). An IV model of quantile treatment effects. *Econometrica 73*(1), 245–261.

Crump, R. K., V. J. Hotz, G. W. Imbens, and O. A. Mitnik (2008). Nonparametric tests for treatment effect heterogeneity. *Review of Economics and Statistics 90*(3), 389–405.

Crump, R. K., V. J. Hotz, G. W. Imbens, and O. A. Mitnik (2009). Dealing with limited overlap in estimation of average treatment effects. *Biometrika 96*(1), 187–199.

Dehejia, R. H. and S. Wahba (1999). Causal effects in nonexperimental studies: Reevaluating the evaluation of training programs. *Journal of the American Statistical Association 94*(448), 1053–1062.

Doksum, K. (1974). Empirical probability plots and statistical inference for nonlinear models in the two-sample case. *Annals of Statistics 2*(2), 267–277.

Donald, S. G. and Y.-C. Hsu (2014). Estimation and inference for distribution functions and quantile functions in treatment effect models. *Journal of Econometrics 178*(3), 383–397.

Dong, C., G. Li, and X. Feng (2019). Lack-of-fit tests for quantile regression models. *Journal of the Royal Statistical Society, Series B 81*(3), 629–648.

Escanciano, J. and S. Goh (2014). Specification analysis of linear quantile models. *Journal of Econometrics 178*(2), 495–507.

Firpo, S. (2007). Efficient semiparametric estimation of quantile treatment effects. *Econometrica 75*(1), 259–276.

Firpo, S., A. F. Galvao, and S. Song (2017). Measurement errors in quantile regression models. *Journal of Econometrics 198*(1), 146–164.

Frölich, M. and B. Melly (2013). Unconditional quantile treatment effects under endogeneity. *Journal of Business & Economic Statistics 31*(3), 346–357.

Hall, P. (1984). Central limit theorem for integrated square error of multivariate nonparametric density estimators. *Journal of Multivariate Analysis 14*(1), 1–16.

Heckman, J., H. Ichimura, J. Smith, and P. Todd (1998). Characterizing selection bias using experimental data. *Econometrica 66*(5), 1017–1098.

Heckman, J. J. and R. Robb (1985). Alternative methods for evaluating the impact of interventions: An overview. *Journal of Econometrics 30*(1-2), 239–267.

Heckman, J. J., J. Smith, and N. Clements (1997). Making the most out of programme evaluations and social experiments: Accounting for heterogeneity in programme impacts. *Review of Economic Studies 64*(4), 487–535.

Hirano, K., G. W. Imbens, and G. Ridder (2003). Efficient estimation of average treatment effects using the estimated propensity score. *Econometrica 71*(4), 1161–1189.

Koenker, R. (2005). *Quantile Regression*. Cambridge University Press, Cambridge, UK.

Koenker, R. and G. Bassett (1978). Regression quantiles. *Econometrica 46*, 33–50.

Koenker, R. and Z. Xiao (2004). Unit root quantile autoregression inference. *Journal of the American Statistical Association 99*(467), 775–787.

Kramer, M. S. (1987). Intrauterine growth and gestational duration determinants. *Pediatrics 80*(4), 502–511.

LaLonde, R. J. (1995). The promise of public sector-sponsored training programs. *Journal of Economic Perspectives 9*(2), 149–168.

Lee, S., R. Okui, and Y.-J. Whang (2017). Doubly robust uniform confidence band for the conditional average treatment effect function. *Journal of Applied Econometrics 32*(7), 1207–1225.

Lee, S., K. Song, and Y.-J. Whang (2015). Uniform asymptotics for nonparametric quantile regression with an application to testing monotonicity. *ArXiv preprint arXiv:1506.05337*, 1–30.

Lee, S., K. Song, and Y.-J. Whang (2018). Testing for a general class of functional inequalities. *Econometric Theory 34*(5), 1018–1064.

Lee, S. and Y.-J. Whang (2009). Nonparametric tests of conditional treatment effects. *Cemmap Working Paper CWP36/09 and Cowles Foundation Discussion Paper 1740, Yale University*, 1–68.

Lehmann, E. L. (1975). *Nonparametrics : Statistical Methods Based on Ranks*. Holden-Day, San Francisco.

Pollard, D. (1984). *Convergence of Stochastic Processes*. Springer-Verlag, New York.

Rubin, D. B. (1974). Estimating causal effects of treatments in randomized and nonrandomized studies. *Journal of Educational Psychology 66*(5), 688–701.

Tang, S. (2020). Recent developments in modeling quantile treatment effects. *Applied Mathematics - A Journal of Chinese Universities 35*(2), 220–243.

Tang, S., Z. Cai, Y. Fang, and M. Lin (2021). A new quantile treatment effect model for studying smoking effect on birth weight during mother's pregnancy. *Journal of Management Science and Engineering 6*(3), 336–343.

Zhou, N., X. Guo, and L. Zhu (2021). The role of propensity score structure in asymptotic efficiency of estimated conditional quantile treatment effect. *Scandinavian Journal of Statistics https://doi.org/10.1111/sjos.12539*, 1–26.

Appendix: Mathematical Proofs

Note that this appendix provides some key steps for proving Theorems 1, 2 and 3, together with some auxiliary lemmas with their detailed proofs as well as some notations given in Supplement.

Proof of Theorem 1: Let $\Delta_\tau(z)$ be the partially conditional QTE conditional on $Z_i = z$ and let $\delta_\tau = \int \Delta_\tau(z) f_Z(z)\, dz$. Then,

$$J_n = \int \Big(\widehat{\Delta}_\tau(z) - \hat{\delta}_\tau\Big)^2 dz = \int \Big[\Big(\widehat{\Delta}_\tau(z) - \Delta_\tau(z)\Big) + \Big(\delta_\tau - \hat{\delta}_\tau\Big) + \Big(\Delta_\tau(z) - \delta_\tau\Big)\Big]^2 dz,$$

where $\hat{\delta}_\tau = \frac{1}{n}\sum_{i=1}^n \widehat{\Delta}_\tau(Z_i)$. Following the proof of Theorem 1 in Cai and Xiao (2012), it is easy to show that $\hat{\delta}_\tau$ is a $\sqrt{n}$-consistent estimate of δ_τ under Assumptions 1-6. Under the null hypothesis H_0, $\Delta_\tau(z) - \delta_\tau \equiv 0$, thus, by Lemma 3,

$$\begin{aligned} J_n &= \int \Big(\frac{1}{n}\sum_{i=1}^n \gamma_n(Y_i, X_i, D_i; z) + e_n\Big)^2 dz \\ &= \int \Big(\frac{1}{n}\sum_{i=1}^n \gamma_n(Y_i, X_i, D_i; z)\Big)^2 dz + e_n^2 + 2e_n \int \frac{1}{n}\sum_{i=1}^n \gamma_n(Y_i, X_i, D_i; z) dz \\ &:= J_{n,1} + J_{n,2} + J_{n,3}, \end{aligned}$$

where $\gamma_n(Y_i, X_i, D_i; z) = \varrho_{n,1,\tau}(Y_i, X_i, D_i; z) - E\varrho_{n,1,\tau}(Y_i, X_i, D_i; z) - \varrho_{n,0,\tau}(Y_i, X_i, D_i; z) + E\varrho_{n,0,\tau}(Y_i, X_i, D_i; z)$ and $e_n = O_p\Big(\max\Big\{\frac{\ln n}{\sqrt{n}}, \frac{\sqrt{\ln n}}{(nh)^{3/4}}\Big\}\Big)$. It is easy to verify that $n\sqrt{h}\, J_{n,2} = o_p(1)$. Also, by noting that $E\big(\gamma_n(Y_i, X_i, D_i; z)\big) = 0$, we have

$$n\sqrt{h}\, J_{n,3} = n\sqrt{h} \cdot e_n \cdot \frac{1}{n}\sum_{i=1}^n \int \gamma_n(Y_i, X_i, D_i; z) dz = n\sqrt{h} \cdot e_n \cdot O_p(n^{-1/2}) = o_p(1).$$

Thus, an application of Lemma 5 leads to

$$n\sqrt{h}\big(J_n - \mu_J\big) = n\sqrt{h}\big(J_{n,1} - \mu_J + J_{n,2} + J_{n,3}\big) \xrightarrow{D} \mathcal{N}(0, \sigma_J^2).$$

Now, we consider the case under the alternative hypothesis H_1. Under H_1, it is easy to show that $J_n - \mu_J = \int \big(\Delta_\tau(z) - \delta_\tau\big)^2 dz + o_p(1)$. Since $\int \big(\Delta_\tau(z) - \delta_\tau\big)^2 dz$ is a positive constant under H_1, so that

$$n\sqrt{h}\big(J_n - \mu_J\big) \xrightarrow{p} +\infty.$$

This completes the proof of Theorem 1. $\square$

Proof of Theorem 2: Under the local alternative $H_{1n} : \Delta_\tau(z) = \delta_\tau + \rho_n \cdot \zeta(z)$ with $\rho_n = n^{-1/2}h^{-1/4}$, we have

$$
\begin{aligned}
J_n &= \int \Big(\widehat{\Delta}_\tau(z) - \hat{\delta}_\tau\Big)^2 dz = \int \Big[\Big(\widehat{\Delta}_\tau(z) - \Delta_\tau(z)\Big) + \Big(\delta_\tau - \hat{\delta}_\tau\Big) + \Big(\Delta_\tau(z) - \delta_\tau\Big)\Big]^2 dz \\
&= \int \Big[\Big(\widehat{\Delta}_\tau(z) - \Delta_\tau(z)\Big) + O_p(1/\sqrt{n}) + \rho_n \cdot \zeta(z)\Big]^2 dz \\
&= \int \Big(\widehat{\Delta}_\tau(z) - \Delta_\tau(z)\Big)^2 dz + \rho_n^2 \cdot \int \zeta^2(z)dz + O_p\Big(\frac{1}{n}\Big) + 2O_p\Big(\frac{1}{\sqrt{n}}\Big) \cdot \int \Big(\widehat{\Delta}_\tau(z) - \Delta_\tau(z)\Big)dz \\
&\quad + 2O_p\Big(\frac{1}{\sqrt{n}}\Big) \cdot \rho_n \cdot \int \zeta(z)dz + \rho_n \cdot \int \zeta(z)\Big(\widehat{\Delta}_\tau(z) - \Delta_\tau(z)\Big)dz \\
&:= J_n^{(1)} + J_n^{(2)} + J_n^{(3)} + J_n^{(4)} + J_n^{(5)} + J_n^{(6)}.
\end{aligned}
$$

By noting that $\rho_n = n^{-1/2}h^{-1/4}$,

$$
\begin{aligned}
\int \Big(\widehat{\Delta}_\tau(z) - \Delta_\tau(z)\Big)dz &= \int \frac{1}{n}\sum_{i=1}^{n} \gamma_n(Y_i, X_i, D_i; z)dz + O_p\left(\max\left\{\frac{\ln n}{\sqrt{n}}, \frac{\sqrt{\ln n}}{(nh)^{3/4}}\right\}\right) \\
&= O_p\Big(\frac{1}{\sqrt{n}}\Big) + O_p\left(\max\left\{\frac{\ln n}{\sqrt{n}}, \frac{\sqrt{\ln n}}{(nh)^{3/4}}\right\}\right),
\end{aligned}
$$

and

$$
\int \zeta(z)\Big(\widehat{\Delta}_\tau(z) - \Delta_\tau(z)\Big)dz = O_p\Big(\frac{1}{\sqrt{n}}\Big) + O_p\left(\max\left\{\frac{\ln n}{\sqrt{n}}, \frac{\sqrt{\ln n}}{(nh)^{3/4}}\right\}\right),
$$

it is easy to show that $n\sqrt{h}J_n^{(k)} = o_p(1)$ for $3 \le k \le 6$. Then, using the result in Theorem 1, we have

$$
n\sqrt{h}\big(J_n - \mu_J\big) = n\sqrt{h}\big(J_n^{(1)} - \mu_J\big) + n\sqrt{h}J_n^{(2)} + o_p(1) \xrightarrow{D} \mathcal{N}\Big(\int \zeta^2(z)dz, \sigma_J^2\Big).
$$

This completes the proof of Theorem 2. $\square$

Now, before considering the proof of Theorem 3, first, let P denote the distribution of $\{(Y_i(0), Y_i(1), X_i, D_i)\}_{i=1}^n$ and use P^* to denote the Bootstrap distribution, which is the distribution of $\{(Y_i^*, X_i^*, D_i^*)\}_{i=1}^n$, conditional on $\{(Y_i, X_i, D_i)\}_{i=1}^n$. Also, we use E^* and Var^* to denote the expectation and variance with respect to P^*, respectively. Furthermore, following Lee et al. (2015), let $S_1, S_2, \cdots$ be a sequence a random variables and $a_1, a_2, \cdots$

be a sequence of positive real numbers. Finally, define $S_n = o_{p^*}(a_n)$ if for any $\varepsilon > 0$ and $\epsilon > 0$, $\lim_{n\to\infty} P\big\{P^*(|S_n/a_n| > \epsilon) > \varepsilon\big\} = 0$. Similarly, $S_n = O_{p^*}(a_n)$ means that if for any $\varepsilon > 0$ and $\epsilon > 0$, there exists $M > 0$ such that $\limsup_{n\to\infty} P\big\{P^*(|S_n/a_n| > M) > \varepsilon\big\} < \epsilon$.

Proof of Theorem 3: It is easy to see that we have the following decomposition

$$
\begin{aligned}
J_n^* &= \int \Big(\big(\widehat{\Delta}_\tau^*(z) - \widehat{\delta}_\tau^*\big) - \big(\widehat{\Delta}_\tau(z) - \widehat{\delta}_\tau\big) \Big)^2 dz \\
&= \int \Big(\widehat{\Delta}_\tau^*(z) - \widehat{\Delta}_\tau(z) \Big)^2 dz + \int \big(\widehat{\delta}_\tau^* - \widehat{\delta}_\tau\big)^2 dz - 2\int \big(\widehat{\delta}_\tau^* - \widehat{\delta}_\tau\big)\big(\widehat{\Delta}_\tau^*(z) - \widehat{\Delta}_\tau(z)\big) dz \\
&= \int \Big(\widehat{\Delta}_\tau^*(z) - \widehat{\Delta}_\tau(z) \Big)^2 dz + O_{p^*}(1/n) - 2O_{p^*}(1/\sqrt{n}) \int \big(\widehat{\Delta}_\tau^*(z) - \widehat{\Delta}_\tau(z)\big) dz \\
&:= \mathcal{Q}_{n,1} + \mathcal{Q}_{n,2} + \mathcal{Q}_{n,3}.
\end{aligned}
$$

For the term $\mathcal{Q}_{n,1}$, we have

$$
\begin{aligned}
\mathcal{Q}_{n,1} &= \int \Big(\widehat{\Delta}_\tau^*(z) - \widehat{\Delta}_\tau(z) \Big)^2 dz \\
&= \int \Big(\frac{1}{n} \sum_{i=1}^n \big(\psi_{n,1,\tau}(R_i^*; z) - \psi_{n,0,\tau}(R_i^*; z) \big) \Big)^2 dz + O_{p^*}(\eta_n^2) \\
&\quad + 2O_{p^*}(\eta_n) \cdot \int \Big[\frac{1}{n} \sum_{i=1}^n \Big(\psi_{n,1,\tau}(R_i^*; z) - \psi_{n,0}^*(R_i^*; z) \Big) \Big] dz \\
&:= \mathcal{Q}_{n,1}^{(1)} + \mathcal{Q}_{n,1}^{(2)} + \mathcal{Q}_{n,1}^{(3)},
\end{aligned}
$$

where $\eta_n = \max\left\{ \frac{\ln n}{\sqrt{n}}, \frac{\sqrt{\ln n}}{(nh)^{3/4}} \right\}$. It is easy to check that $n\sqrt{h}\mathcal{Q}_{n,1}^{(2)} = o_{p^*}(1)$. Also, from Lemma 10, we know that $n\sqrt{h}\mathcal{Q}_{n,1}^{(3)} = o_{p^*}(1)$. Therefore, an application of Lemma 9 leads to

$$
n\sqrt{h}(\mathcal{Q}_{n,1} - \mu_J)/\sigma_J = n\sqrt{h}(\mathcal{Q}_{n,1}^{(1)} - \mu_J)/\sigma_J + o_{p^*}(1) \longrightarrow \mathcal{N}(0,1)
$$

in distribution in probability. For the term $\mathcal{Q}_{n,2}$, it is easy to see that $n\sqrt{h}\mathcal{Q}_{n,2} = o_{p^*}(1)$. By using Lemma 10 again, we also know that $n\sqrt{h}\mathcal{Q}_{n,3} = o_{p^*}(1)$. Finally, we have

$$
n\sqrt{h}(J_n^* - \mu_J)/\sigma_J = n\sqrt{h}(\mathcal{Q}_{n,1} - \mu_J)/\sigma_J + o_{p^*}(1) \longrightarrow \mathcal{N}(0,1)
$$

in distribution in probability. Because $\mathcal{N}(0,1)$ is a continuous distribution, by Polyā's theorem in Bhattacharya and Rao (1986), we obtain Theorem 3. $\square$

SUPPLEMENTARY MATERIAL

This supplement provides some lemmas for proving the main theorems in the paper, entitled **"A Nonparametric Test for Testing Heterogeneity in Conditional Quantile Treatment Effects"**.

Define $W_0(X_i, D_i) = \frac{1-D_i}{1-p(X_i)}$ and $W_1(X_i, D_i) = \frac{D_i}{p(X_i)}$. We also let $\widehat{W}_{n,0}(X_i, D_i) = \frac{1-D_i}{1-\hat{p}_n(X_i)}$ and $\widehat{W}_{n,1}(X_i, D_i) = \frac{D_i}{\hat{p}_n(X_i)}$, where $\hat{p}_n(x) = p(x; \hat{\theta}_n)$ is the parametric estimate of the propensity score function $p(x)$ using (X_i, D_i), $i = 1, \cdots, n$. To prove Theorem 1, we first provide the following lemmas.

Lemma 1. *Suppose Assumptions 2.1-2.5 hold. Then,*

$$\sup_{z\in\mathcal{Z}}\left|\bar{q}_{j,\tau}(z) - q_{j,\tau}(z) - \frac{1}{n}\sum_{i=1}^{n}\left[\varrho_{n,j,\tau}(Y_i, X_i, D_i; z) - E\varrho_{n,j,\tau}(Y_i, X_i, D_i; z)\right]\right| = O_p\left(\frac{\sqrt{\ln n}}{(nh)^{3/4}}\right)$$

for $j = 0$ *and* 1, *where*

$$\bar{q}_{j,\tau}(z) = \arg\min_{q}\sum_{i=1}^{n} W_j(X_i, D_i) h K_{h,i}(z)\rho_\tau(Y_i; q)$$

and

$$\varrho_{n,j,\tau}(Y_i, X_i, D_i; z) = -S^{-1}_{n,j,\tau}(z) W_j(X_i, D_i) K_h\big(Z_i - z\big)\varphi_\tau(Y_i; q_{j,\tau}(z))$$

with $\rho_\tau(y; q) = \rho_\tau(y-q) = (y-q)(\tau - I\{y \le q\})$, $\varphi_\tau(y;\theta) = \tau - I\{y \le \theta\}$, $K_{h,i}(z) = K\big((Z_i - z)/h\big)/h$ and

$$S_{n,j,\tau}(z) = \int K(u) f_{Y(j)|Z}\big(q_{j,\tau}(z)|z+hu\big) f_Z(z+hu)du = f_{Y(j)|Z}\big(q_{j,\tau}(z)|z\big) f_Z(z) + O(h^2),\ j = 0,\ 1.$$

Proof of Lemma 1: This result can be proved following the proof of Theorem 1 in Lee et al. (2015). □

Lemma 2. *Suppose that Assumptions 2.1-2.6 are satisfied, then*

$$\sup_{z\in\mathcal{Z}}\left|\hat{q}_{j,\tau}(z) - \bar{q}_{j,\tau}(z)\right| = O_p\left\{\max\left\{\frac{\ln n}{\sqrt{n}}, \frac{\sqrt{\ln n}}{(nh)^{3/4}}\right\}\right\}, \quad j = 0,\ 1.$$

Proof of Lemma 2: For $j = 0$ and 1, define cumulative distribution functions

$$\bar{F}_{n,j}(y \,|\, z) = \frac{\sum_{i=1}^{n} K_{h,i}(z) W_j(X_i, D_i) I\{Y_i \le y\}}{\sum_{i=1}^{n} K_{h,i}(z) W_j(X_i, D_i)}$$

and

$$\widehat{F}_{n,j}(y \,|\, z) = \frac{\sum_{i=1}^{n} K_{h,i}(z) \widehat{W}_{n,j}(X_i, D_i) I\{Y_i \le y\}}{\sum_{i=1}^{n} K_{h,i}(z) \widehat{W}_{n,j}(X_i, D_i)}.$$

Then $\bar{q}_{j,\tau^*}(z) = \inf\{y : \bar{F}_{n,j}(y \,|\, z) \ge \tau^*\}$ and $\hat{q}_{j,\tau^*}(z) = \inf\{y : \widehat{F}_{n,j}(y \,|\, z) \ge \tau^*\}$ for $0 < \tau^* < 1$. By the definition of quantile, we have

$$\left| \bar{F}_{n,j}\big(\bar{q}_{j,\tau^*}(z) \,|\, z\big) - \tau^* \right| \le \max_{i=1,\cdots,n} \left\{ \frac{K_{h,i}(z) W_j(X_i, D_i) I\{Y_i \le y\}}{\sum_{i=1}^{n} K_{h,i}(z) W_j(X_i, D_i)} \right\}$$

for any $0 < \tau^* < 1$, hence $\bar{F}_{n,j}\big(\bar{q}_{j,\tau^*}(z) \,|\, z\big) = \tau^* + O_p(1/nh)$. By using $\sup_{x \in \mathcal{X}} \big| p(x, \hat{\theta}_n) - p(x, \theta_0) \big| = O_p(n^{-1/2})$ as in Assumption 2.6, it is also easy to show that

$$\sup_{y \in \mathcal{Y}_j} \sup_{z \in \mathcal{Z}} \left| \widehat{F}_{n,j}(y \,|\, z) - \bar{F}_{n,j}(y \,|\, z) \right| = O_p(n^{-1/2}),$$

where $\mathcal{Y}_j$ is the support of $Y(j)$. Let $c_n = \max\left\{ \frac{\ln n}{\sqrt{n}}, \frac{(\ln n)^2}{nh} \right\}$. Then,

$$\begin{aligned} \widehat{F}_{n,j}\big(\bar{q}_{j,\tau+c_n}(z) \,|\, z\big) &= \bar{F}_{n,j}\big(\bar{q}_{j,\tau+c_n}(z) \,|\, z\big) + O_p(1/\sqrt{n}) \\ &= \tau + c_n + O_p(1/nh) + O_p(1/\sqrt{n}) > \tau \end{aligned} \tag{S.1}$$

in probability as $n \to \infty$. Similarly,

$$\begin{aligned} \widehat{F}_{n,j}\big(\bar{q}_{j,\tau-c_n}(z) \,|\, z\big) &= \bar{F}_{n,j}\big(\bar{q}_{j,\tau-c_n}(z) \,|\, z\big) + O_p(1/\sqrt{n}) \\ &= \tau - c_n + O_p(1/nh) + O_p(1/\sqrt{n}) < \tau \end{aligned} \tag{S.2}$$

in probability as $n \to \infty$. Combining (S.1) and (S.2), we have

$$P\big(\bar{q}_{j,\tau-c_n}(z) \le \hat{q}_{j,\tau}(z) \le \bar{q}_{j,\tau+c_n}(z)\big) \to 1 \quad \text{as } n \to \infty.$$

Also, one has $\bar{q}_{j,\tau-c_n}(z) \le \bar{q}_{j,\tau}(z) \le \bar{q}_{j,\tau+c_n}(z)$ by definition, and it follows that

$$P\Big(\sup_{z\in\mathcal{Z}}\big|\hat{q}_{j,\tau}(z)-\bar{q}_{j,\tau}(z)\big| \le \sup_{z\in\mathcal{Z}}\big|\bar{q}_{j,\tau+c_n}(z)-\bar{q}_{j,\tau-c_n}(z)\big|\Big) \to 1 \quad \text{as } n\to\infty.$$

Next, we consider the order of $\sup_{z\in\mathcal{Z}}\big|\bar{q}_{j,\tau+c_n}(z)-\bar{q}_{j,\tau-c_n}(z)\big|$. Recall that $S_{n,j,\tau^*}(z) = \int K(u)f_{Y(j)|Z}\big(q_{j,\tau^*}(z)|z+hu\big)f_Z(z+hu)du$ for $j=0$ and 1. It is easy to show that $\sup_{z\in\mathcal{Z}}|S_{n,j,\tau\pm c_n}(z)-S_{n,j,\tau}(z)| = O(c_n)$. By using the Bahadur representation of $\bar{q}_{j,\tau^*}(z)$ provided by Lemma 1, it follows that

$$\begin{aligned}
& \sup_{z\in\mathcal{Z}}\big|\bar{q}_{j,\tau+c_n}(z)-\bar{q}_{j,\tau-c_n}(z)\big| \\
\le\ & \sup_{z\in\mathcal{Z}}\big|q_{j,\tau+c_n}(z)-q_{j,\tau-c_n}(z)\big| + O_p\Big\{\frac{\sqrt{\ln n}}{(nh)^{3/4}}\Big\} \\
+\ & \sup_{z\in\mathcal{Z}}\Big|S^{-1}_{n,j,\tau}(z)\frac{1}{n}\sum_{i=1}^{n}K_{h,i}(z)W_j(X_i,D_i)\times\big(I\{Y_i\le q_{j,\tau-c_n}(z)\}-I\{Y_i\le q_{j,\tau+c_n}(z)\}\big)+O_p(c_n)\Big| \\
& +\sup_{z\in\mathcal{Z}}\Big|S^{-1}_{n,j,\tau}(z)E\Big[K_{h,i}(z)W_j(X_i,D_i)\big(I\{Y_i\le q_{j,\tau-c_n}(z)\}-I\{Y_i\le q_{j,\tau+c_n}(z)\}\big)\Big]+O(c_n)\Big| \\
\le\ & \sup_{z\in\mathcal{Z}}\big|q_{j,\tau+c_n}(z)-q_{j,\tau-c_n}(z)\big| \\
& +\sup_{z\in\mathcal{Z}}\Big|S^{-1}_{n,j,\tau}(z)\frac{1}{n}\sum_{i=1}^{n}K_{h,i}(z)W_j(X_i,D_i)\times\big(I\{Y_i\le q_{j,\tau-c_n}(z)\}-I\{Y_i\le q_{j,\tau+c_n}(z)\}\big)\Big| \\
& +\sup_{z\in\mathcal{Z}}\Big|S^{-1}_{n,j,\tau}(z)E\Big[K_{h,i}(z)W_j(X_i,D_i)\big(I\{Y_i\le q_{j,\tau-c_n}(z)\}-I\{Y_i\le q_{j,\tau+c_n}(z)\}\big)\Big]\Big| \\
& +O(c_n)+O_p\Big\{\frac{\sqrt{\ln n}}{(nh)^{3/4}}\Big\} \\
:=\ & \mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3+O_p(c_n)+O_p\Big\{\frac{\sqrt{\ln n}}{(nh)^{3/4}}\Big\}.
\end{aligned}$$

For $\mathcal{M}_1$, note that $F_{Y(j)|Z}(q_{j,\tau+c_n}(z)|z) = \tau + c_n$ and $F_{Y(j)|Z}(q_{j,\tau-c_n}(z)|z) = \tau - c_n$ under Assumption 2.3. Thus,

$$2c_n = F_{Y(j)|Z}(q_{j,\tau+c_n}(z)|z) - F_{Y(j)|Z}(q_{j,\tau-c_n}(z)|z) = f_{Y(j)|Z}(q_n^*|z)\big(q_{j,\tau+c_n}(z)-q_{j,\tau-c_n}(z)\big),$$

where q_n^* is a point between $q_{j,\tau-c_n}(z)$ and $q_{j,\tau+c_n}(z)$, which implies that

$$\mathcal{M}_1 = \sup_{z\in\mathcal{Z}}\big|q_{j,\tau+c_n}(z)-q_{j,\tau-c_n}(z)\big| = O(c_n)$$

by the assumption that $f_{Y(j)|Z}(q|z)$ is uniformly bounded away from zero in a neighborhood of $q_{j,\tau}(z)$. For $\mathcal{M}_2$, since $q_{j,\tau-c_n}(z) \le q_{j,\tau}(z) \le q_{j,\tau+c_n}(z)$ and $\sup_{z\in\mathcal{Z}} \big|q_{j,\tau+c_n}(z) - q_{j,\tau-c_n}(z)\big| = O(c_n)$, there exists a constant A which does not rely on z, such that

$$\begin{aligned}
\mathcal{M}_2 &\le \sup_{z\in\mathcal{Z}} \Big| S_{n,j,\tau}^{-1}(z) \frac{1}{n} \sum_{i=1}^{n} K_{h,i}(z) W_j(X_i, D_i) I\{q_{j,\tau}(z) - Ac_n \le Y_i \le q_{j,\tau}(z) + Ac_n\} \Big| \\
&\le \sup_{z\in\mathcal{Z}} \Big| S_{n,j,\tau}^{-1}(z) \frac{1}{n} \sum_{i=1}^{n} \Big[K_{h,i}(z) W_j(X_i, D_i) I\{q_{j,\tau}(z) - Ac_n \le Y_i \le q_{j,\tau}(z) + Ac_n\} \\
&\quad - E\Big(K_{h,i}(z) W_j(X_i, D_i) I\{q_{j,\tau}(z) - Ac_n \le Y_i \le q_{j,\tau}(z) + Ac_n\}\Big)\Big]\Big| \\
&\quad + \sup_{z\in\mathcal{Z}} \Big| S_{n,j,\tau}^{-1}(z) E\Big(K_{h,i}(z) W_j(X_i, D_i) I\{q_{j,\tau}(z) - Ac_n \le Y_i \le q_{j,\tau}(z) + Ac_n\}\Big)\Big| \\
&:= \mathcal{M}_{2,1} + \mathcal{M}_{2,2}.
\end{aligned}$$

Using Assumption 2.3, we have that

$$\begin{aligned}
& E\Big[\Big(K\big((Z_i - z)/h\big) W_j(X_i, D_i) I\{q_{j,\tau}(z) - Ac_n \le Y_i \le q_{j,\tau}(z) + Ac_n\}\Big)^2\Big] \\
= \ & E\Big\{ K^2\big((Z_i - z)/h\big) p(X_i)^{-j} (1 - p(X_i))^{j-1} E\Big[I\{q_{j,\tau}(z) - Ac_n \le Y_i(j) \le q_{j,\tau}(z) + Ac_n\} \big| X_i \Big]\Big\} \\
= \ & O(hc_n).
\end{aligned}$$

Also, note that $\big\{K\big((Z_i - z)/h\big) W_j(X_i, D_i) I\{q_{j,\tau}(z) - Ac_n \le Y_i \le q_{j,\tau}(z) + Ac_n\} : z \in \mathcal{Z}\big\}$ is Euclidean for a constant envelope, which together with $\frac{\ln n}{nhc_n} = o(1)$ implies the conditions required by Theorem II.37 of Pollard (1984) are met. Hence, by Theorem II.37 of Pollard (1984),

$$\begin{aligned}
&\sup_{z\in\mathcal{Z}} \Big| \frac{1}{n} \sum_{i=1}^{n} \Big[K\big((Z_i - z)/h\big) W_j(X_i, D_i) I\{q_{j,\tau}(z) - Ac_n \le Y_i \le q_{j,\tau}(z) + Ac_n\} \\
&\qquad - E\Big(K\big((Z_i - z)/h\big) W_j(X_i, D_i) I\{q_{j,\tau}(z) - Ac_n \le Y_i \le q_{j,\tau}(z) + Ac_n\}\Big)\Big]\Big| = o_p(hc_n),
\end{aligned}$$

together with the fact $S_{n,j,\tau}(z)$ is bounded away from zero, we have that $\mathcal{M}_{2,1} = \frac{1}{h} \cdot o_p(hc_n) = o_p(c_n)$. It is also easy to show that $\mathcal{M}_{2,2} = O(c_n)$. Hence, $\mathcal{M}_2 = O_p(c_n)$. Similar to the proof of $\mathcal{M}_{2,2} = O(c_n)$, we can also show that $\mathcal{M}_3 = O(c_n)$. Therefore,

$$\sup_{z\in\mathcal{Z}} \big| \bar{q}_{j,\tau+c_n}(z) - \bar{q}_{j,\tau-c_n}(z) \big| = O_p(c_n) + O_p\Big\{ \frac{\sqrt{\ln n}}{(nh)^{3/4}} \Big\}.$$

Together with (5), we have

$$\sup_{z\in\mathcal{Z}}\left|\hat{q}_{j,\tau}(z)-\bar{q}_{j,\tau}(z)\right| = O_p(c_n) + O_p\left\{\frac{\sqrt{\ln n}}{(nh)^{3/4}}\right\} = O_p\left\{\max\left\{\frac{\ln n}{\sqrt{n}}, \frac{\sqrt{\ln n}}{(nh)^{3/4}}\right\}\right\}.$$

This completes the proof. □

Lemma 3. *Under Assumptions 2.1-2.6 and $nh^2 \to \infty$, we have*

$$\sup_{z\in\mathcal{Z}}\left|\hat{q}_{j,\tau}(z)-q_{j,\tau}(z)-\frac{1}{n}\sum_{i=1}^{n}\left[\varrho_{n,j,\tau}(Y_i,X_i,D_i;z)-E\big(\varrho_{n,j,\tau}(Y_i,X_i,D_i;z)\big)\right]\right| = O_p\left\{\max\left\{\frac{\ln n}{\sqrt{n}}, \frac{\sqrt{\ln n}}{(nh)^{3/4}}\right\}\right\}$$

for $j=0$ and 1.

Proof of Lemma 3: The result comes from Lemma 1 and Lemma 2. □

Lemma 4. *Let $R_1, R_2, \cdots$ be an i.i.d. sequence. Suppose that the U-statistic $U_n = \sum_{1\le i<j\le n} H_n(R_i,R_j)$ with symmetric variable function H_n is centered (i.e., $E[H_n(R_1,R_2)] = 0$) and degenerated (i.e., $E[H_n(R_1,R_2)|R_1=z_1]=0$ almost surely for all z_1). Then, if*

$$\lim_{n\to\infty}\frac{E\Big[E^2\big[H_n(R_1,R_3)H_n(R_2,R_3)\,\big|\,R_1,R_2\big]\Big]+n^{-1}E\big[H_n^4(R_1,R_2)\big]}{E^2\big[H_n^2(R_1,R_2)\big]} = 0,$$

we have that as $n\to\infty$,

$$\frac{2^{1/2}}{n\sigma_n}U_n \xrightarrow{D} \mathcal{N}(0,1),$$

where $\sigma_n^2 = E\big[H_n^2(R_1,R_2)\big]$.

Proof of Lemma 4: The result is given by Theorem 1 in Hall (1984). □

Lemma 5. *Suppose the conditions required by Theorem 1 are satisfied. Then,*

$$n\sqrt{h}\Bigg\{\int\Big[\frac{1}{n}\sum_{i=1}^{n}\big(\varrho_{n,1,\tau}(Y_i,X_i,D_i;z)-E\varrho_{n,1,\tau}(Y_i,X_i,D_i;z)$$
$$-\varrho_{n,0,\tau}(Y_i,X_i,D_i;z)+E\varrho_{n,0,\tau}(Y_i,X_i,D_i;z)\big)\Big]^2dz-\mu_J\Bigg\} \xrightarrow{D} \mathcal{N}(0,\sigma_J^2),$$

where

$$\mu_J = \frac{1}{nh}\int K^2(s)ds\int\left\{\frac{\mu_1(z;z)}{f^2_{Y(1)|Z}(q_{1,\tau}(z)|z)}+\frac{\mu_0(z;z)}{f^2_{Y(0)|Z}(q_{0,\tau}(z)|z)}\right\}\frac{1}{f_Z(z)}dz,$$

and

$$\sigma_J^2 = 2\int\Big(\int K(t)K(t+s)dt\Big)^2 ds \int\left\{\frac{\mu_1(u;u)}{f^2_{Y(1)|Z}(q_{1,\tau}(u)|u)} + \frac{\mu_0(u;u)}{f^2_{Y(0)|Z}(q_{0,\tau}(u)|u)}\right\}^2 \frac{1}{f_Z^2(u)}du,$$

with

$$\mu_0(z;u) = E\left[\frac{1}{1-p(X_i)}\big(I\{Y_i(0)\le q_{0,\tau}(u)\}-\tau\big)^2\big|Z_i=z\right],$$

and

$$\mu_1(z;u) = E\left[\frac{1}{p(X_i)}\big(I\{Y_i(1)\le q_{1,\tau}(u)\}-\tau\big)^2\big|Z_i=z\right].$$

Proof of Lemma 5: For simplicity, we let

$$\begin{aligned}\gamma_n(Y_i,X_i,D_i;z) = \varrho_{n,1,\tau}(Y_i,X_i,D_i;z) - E\varrho_{n,1,\tau}(Y_i,X_i,D_i;z)\\ -\varrho_{n,0,\tau}(Y_i,X_i,D_i;z) + E\varrho_{n,0,\tau}(Y_i,X_i,D_i;z).\end{aligned}$$

Then,

$$\begin{aligned}&\int\Big(\frac{1}{n}\sum_{i=1}^n \gamma_n(Y_i,X_i,D_i;z)\Big)^2 dz \qquad\qquad (S.3)\\ =\;& 2n^{-2}\sum_{1\le i<k\le n}\int \gamma_n(Y_i,X_i,D_i;z)\gamma_n(Y_k,X_k,D_k;z)dz + n^{-2}\sum_{i=1}^n\int \gamma_n^2(Y_i,X_i,D_i;z)dz\\ :=\;& I_{n,1}+I_{n,2}.\end{aligned}$$

First, we consider the term $I_{n,1}$. Let $R_i=(Y_i,X_i,D_i)$ and define

$$H_n(R_i,R_k) = \frac{2}{n^2}\int \gamma_n(R_i;z)\gamma_n(R_k;z)dz.$$

Then, $I_{n,1} = \sum_{1\le i<k\le n} H_n(R_i,R_k)$ is a centered and degenerated U-statistic. Thus,

$$\begin{aligned}&E\big[H_n(R_i,R_k)^2\big] \qquad\qquad (S.4)\\ =\;& E\left[\frac{4}{n^4}\int\int \gamma_n(R_i;u)\gamma_n(R_k;u)\,\gamma_n(R_i;v)\gamma_n(R_k;v)\,dudv\right]\\ =\;& \frac{4}{n^4}\int\int E\big[\gamma_n(R_i;u)\gamma_n(R_i;v)\gamma_n(R_k;u)\gamma_n(R_k;v)\big]\,dudv\\ =\;& \frac{4}{n^4}\int\int E^2\big[\gamma_n(R_i;u)\gamma_n(R_i;v)\big]\,dudv.\end{aligned}$$

It is easy to find that

$$E\big[S_{n,j,\tau}(z)^{-1}W_j(X_i,D_i)\varphi_\tau(Y_i;q_{j,\tau}(z))|Z_i=z\big]=0.$$

Hence,

$$\begin{aligned}
&E\big[\varrho_{n,j,\tau}(Y_i,X_i,D_i;z)\big] \qquad\qquad \text{(S.5)}\\
&\qquad = 0-\frac{1}{S_{n,j,\tau}(z)}f_Z'(z)\frac{\partial F_{Y(j)|Z}(q_{j,\tau}(z)|u)}{\partial u}\Big|_{u=z}h^2\int s^2K(s)ds+o(h^2)=O(h^2)
\end{aligned}$$

uniformly in z for $j=0$ and 1. Also note that $D_i(1-D_i)=0$, then,

$$\begin{aligned}
& E\Big[\gamma_n(R_i;u)\gamma_n(R_i;v)\Big] \qquad\qquad \text{(S.6)}\\
= \;& E\Big[\big(\varrho_{n,1,\tau}(R_i;u)-\varrho_{n,0,\tau}(R_i;u)\big)\big(\varrho_{n,1,\tau}(R_i;v)-\varrho_{n,0,\tau}(R_i;v)\big)\Big]\\
&\qquad -E\Big[\varrho_{n,1,\tau}(R_i;u)-\varrho_{n,0,\tau}(R_i;u)\Big]E\Big[\varrho_{n,1,\tau}(R_i;v)-\varrho_{n,0,\tau}(R_i;v)\Big]\\
= \;& S_{n,1,\tau}^{-1}(u)S_{n,1,\tau}^{-1}(v)E\Big[K_h\big(Z_i-u\big)K_h\big(Z_i-v\big)\frac{D_i}{p^2(X_i)}\varphi_\tau(Y_i;q_{1,\tau}(u))\varphi_\tau(Y_i;q_{1,\tau}(v))\Big]\\
&+S_{n,0,\tau}^{-1}(u)S_{n,0,\tau}^{-1}(v)E\Big[K_h\big(Z_i-u\big)K_h\big(Z_i-v\big)\frac{1-D_i}{(1-p(X_i))^2}\varphi_\tau(Y_i;q_{0,\tau}(u))\varphi_\tau(Y_i;q_{0,\tau}(v))\Big]+O(h^4)\\
= \;& S_{n,1,\tau}^{-1}(u)S_{n,1,\tau}^{-1}(v)E\Big[K_h\big(Z_i-u\big)K_h\big(Z_i-v\big)\frac{1}{p(X_i)}\varphi_\tau(Y_i;q_{1,\tau}(u))\varphi_\tau(Y_i;q_{1,\tau}(v))\Big]\\
&+S_{n,0,\tau}^{-1}(u)S_{n,0,\tau}^{-1}(v)E\Big[K_h\big(Z_i-u\big)K_h\big(Z_i-v\big)\frac{1}{1-p(X_i)}\varphi_\tau(Y_i;q_{0,\tau}(u))\varphi_\tau(Y_i;q_{0,\tau}(v))\Big]+O(h^4)\\
= \;& S_{n,1,\tau}^{-1}(u)S_{n,1,\tau}^{-1}(v)E\Big[K_h\big(Z_i-u\big)K_h\big(Z_i-v\big)\kappa_1(Z_i;u,v)\Big]\\
&+S_{n,0,\tau}^{-1}(u)S_{n,0,\tau}^{-1}(v)E\Big[K_h\big(Z_i-u\big)K_h\big(Z_i-v\big)\kappa_0(Z_i;u,v)\Big]+O(h^4)\\
= \;& \frac{1}{h}S_{n,1,\tau}^{-1}(u)S_{n,1,\tau}^{-1}(v)\int K(t)K\Big(t+\frac{u-v}{h}\Big)\kappa_1(u+ht;u,v)f_Z(u+ht)dt\\
&+\frac{1}{h}S_{n,0,\tau}^{-1}(u)S_{n,0,\tau}^{-1}(v)\int K(t)K\Big(t+\frac{u-v}{h}\Big)\kappa_0(u+ht;u,v)f_Z(u+ht)dt+O(h^4),
\end{aligned}$$

where

$$\kappa_1(z;u,v)=E\Big[\frac{1}{p(X_i)}\varphi_\tau(Y_i(1);q_{1,\tau}(u))\varphi_\tau(Y_i(1);q_{1,\tau}(v))\big|Z_i=z\Big],$$

and

$$\kappa_0(z;u,v)=E\Big[\frac{1}{1-p(X_i)}\varphi_\tau(Y_i(0);q_{0,\tau}(u))\varphi_\tau(Y_i(0);q_{0,\tau}(v))\big|Z_i=z\Big],$$

with $\varphi_\tau(y;q) = I(y \le q) - \tau$. Thus,

$$
\begin{aligned}
& E^2\Big[\gamma_n(R_i;u)\gamma_n(R_i;v)\Big] \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(S.7)} \\
= \;& \frac{1}{h^2} S_{n,1,\tau}^{-2}(u) S_{n,1,\tau}^{-2}(v) \Big(\int K(t) K\Big(t + \frac{u-v}{h}\Big)\kappa_1(u+ht;u,v) f_Z(u+ht)dt\Big)^2 \\
& + \frac{1}{h^2} S_{n,0,\tau}^{-2}(u) S_{n,0,\tau}^{-2}(v) \Big(\int K(t) K\Big(t + \frac{u-v}{h}\Big)\kappa_0(u+ht;u,v) f_Z(u+ht)dt\Big)^2 \\
& + \frac{2}{h^2} S_{n,1,\tau}^{-1}(u) S_{n,1,\tau}^{-1}(v) S_{n,0,\tau}^{-1}(u) S_{n,0,\tau}^{-1}(v) \int K(t) K\Big(t + \frac{u-v}{h}\Big)\kappa_1(u+ht;u,v) f_Z(u+ht)dt \\
& \times \int K(t) K\Big(t + \frac{u-v}{h}\Big)\kappa_0(u+ht;u,v) f_Z(u+ht)dt \\
& + 2O(h^3) S_{n,1,\tau}^{-1}(u) S_{n,1,\tau}^{-1}(v) \int K(t) K\Big(t + \frac{u-v}{h}\Big)\kappa_1(u+ht;u,v) f_Z(u+ht)dt \\
& + 2O(h^3) S_{n,0,\tau}^{-1}(u) S_{n,0,\tau}^{-1}(v) \int K(t) K\Big(t + \frac{u-v}{h}\Big)\kappa_0(u+ht;u,v) f_Z(u+ht)dt + O(h^8).
\end{aligned}
$$

An application of (S.4) and some straightforward calculations imply that

$$
\begin{aligned}
E\big[H_n(R_i,R_k)^2\big] \;&=\; \frac{4}{n^4} \int\int E^2\big[\gamma_n(R_i;u)\gamma_n(R_i;v)\big]\, dudv \qquad\qquad\qquad\qquad\qquad \text{(S.8)} \\
&=\; \frac{4}{n^4 h}\Big\{\int\Big(\int K(t)K(t+s)dt\Big)^2 ds \cdot \Big[\int S_{n,1,\tau}^{-4}(u)\kappa_1^2(u;u,u) f_Z^2(u)du \\
&\quad + \int S_{n,0,\tau}^{-4}(u)\kappa_0^2(u;u,u) f_Z^2(u)du \\
&\quad + 2\int S_{n,1,\tau}^{-2}(u) S_{n,0,\tau}^{-2}(u)\kappa_1(u;u,u)\kappa_0(u;u,u) f_Z^2(u)du\Big] + o(1)\Big\}.
\end{aligned}
$$

This, coupled with $S_{n,j,\tau}(z) = f_Z(z) f_{Y(j)|Z}(q_{j,\tau}(z)|z) + O(h^2)$ for $j = 0$ and 1, yields

$$
\begin{aligned}
E\big[H_n(R_1,R_2)^2\big] \;&=\; \frac{4}{n^4 h}\Big(\int\Big(\int K(t)K(t+s)dt\Big)^2 ds \\
&\quad \times \int\Big\{\frac{\kappa_1(u;u,u)}{f_{Y(1)|Z}^2(q_{1,\tau}(u)|u)} + \frac{\kappa_0(u;u,u)}{f_{Y(0)|Z}^2(q_{0,\tau}(u)|u)}\Big\}^2 \frac{1}{f_Z^2(u)} du + o(1)\Big) \\
&=\; \frac{2}{n^4 h}\big(\sigma_J^2 + o(1)\big). \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(S.9)}
\end{aligned}
$$

Similarly, by straightforward calculations, we can obtain

$$
E\Big[E^2\big[H_n(R_1,R_3)H_n(R_2,R_3)\,\big|\,R_1,R_2\big]\Big] = O\Big(\Big(\frac{1}{n^2h^2}\Big)^4 h^7\Big),
$$

and

$$E\big[H_n(R_1,R_2)^4\big]=O\bigg(\Big(\frac{1}{n^2h^2}\Big)^4h^5\bigg).$$

Thus, the condition

$$\lim_{n\to\infty}\frac{E\Big[E^2\big[H_n(R_1,R_3)H_n(R_2,R_3)\,\big|\,R_1,R_2\big]\Big]+n^{-1}E\big[H_n(R_1,R_2)^4\big]}{\Big(E\big[H_n(R_1,R_2)^2\big]\Big)^2}=0$$

in Lemma 4 is satisfied, so that

$$\frac{\sqrt{2}}{nE^{1/2}\big[H_n(R_1,R_2)^2\big]}\,I_{n,1}\xrightarrow{D}\mathcal{N}(0,1),$$

or equivalently,

$$n\sqrt{h}\,I_{n,1}\xrightarrow{D}\mathcal{N}(0,\sigma_J^2). \tag{S.10}$$

Now, we move to the term $I_{n,2}=n^{-2}\sum_{i=1}^{n}\int\gamma_n^2(Y_i,X_i,D_i;z)dz$. Note that

$$\begin{aligned}
&E\big[\gamma_n^2(Y_i,X_i,D_i;z)\big]\\
=\;& E\big[\varrho_{n,1,\tau}(Y_i,X_i,D_i;z)\big]^2+E\big[\varrho_{n,0,\tau}(Y_i,X_i,D_i;z)\big]^2+O(h^4)\\
=\;& S_{n,1,\tau}^{-2}(z)E\bigg[\frac{1}{h^2}K^2\Big(\frac{Z_i-z}{h}\Big)\frac{D_i}{p^2(X_i)}(I\{Y_i\le q_{1,\tau}(z)\}-\tau)^2\bigg]\\
&\quad+S_{n,0,\tau}^{-2}(z)E\bigg[\frac{1}{h^2}K^2\Big(\frac{Z_i-z}{h}\Big)\frac{1-D_i}{(1-p(X_i))^2}(I\{Y_i\le q_{0,\tau}(z)\}-\tau)^2\bigg]+O(h^4)\\
=\;& S_{n,1,\tau}^{-2}(z)E\bigg[\frac{1}{h^2}K^2\Big(\frac{Z_i-z}{h}\Big)\frac{1}{p(X_i)}(I\{Y_i(1)\le q_{1,\tau}(z)\}-\tau)^2\bigg]\\
&\quad+S_{n,0,\tau}^{-2}(z)E\bigg[\frac{1}{h^2}K^2\Big(\frac{Z_i-z}{h}\Big)\frac{1}{1-p(X_i)}(I\{Y_i(0)\le q_{0,\tau}(z)\}-\tau)^2\bigg]+O(h^4)\\
=\;& S_{n,1,\tau}^{-2}(z)E\bigg[\frac{1}{h^2}K^2\Big(\frac{Z_i-z}{h}\Big)\mu_1(Z_i;z)\bigg]+S_{n,0,\tau}^{-2}(z)E\bigg[\frac{1}{h^2}K^2\Big(\frac{Z_i-z}{h}\Big)\mu_0(Z_i;z)\bigg]+O(h^4)\\
=\;& S_{n,1,\tau}^{-2}(z)\bigg[\frac{1}{h}\Big(\mu_1(z;z)f_Z(z)\int K^2(s)ds+O(h)\Big)\bigg]\\
&\quad+S_{n,0,\tau}^{-2}(z)\bigg[\frac{1}{h}\Big(\mu_0(z;z)f_Z(z)\int K^2(s)ds+O(h)\Big)\bigg]+O(h^4)\\
=\;& \frac{1}{h}\bigg\{\int K^2(s)ds\cdot\Big(S_{n,1,\tau}^{-2}(z)\mu_1(z;z)+S_{n,0,\tau}^{-2}(z)\mu_0(z;z)\Big)f_Z(z)\bigg\}+O(1),
\end{aligned}$$

coupled with $S_{n,j,\tau}(z) = f_Z(z)f_{Y(j)|Z}(q_{j,\tau}(z)|z) + O(h^2)$ for $j = 0$ and 1, we have

$$\begin{aligned} E\big(I_{n,2}\big) &= \frac{1}{n}\int E\big[\gamma_n^2(Y_i, X_i, D_i; z)\big]dz \qquad (S.11) \\ &= \frac{1}{nh}\Big\{\int K^2(s)ds \cdot \int \Big(S_{n,1,\tau}^{-2}(z)\mu_1(z;z) + S_{n,0,\tau}^{-2}(z)\mu_0(z;z)\Big)f_Z(z)dz\Big\} + O\Big(\frac{1}{n}\Big) \\ &= \frac{1}{nh}\int K^2(s)ds \cdot \int \bigg\{\frac{\mu_1(z;z)}{f_{Y(1)|Z}^2(q_{1,\tau}(z)|z)} + \frac{\mu_0(z;z)}{f_{Y(0)|Z}^2(q_{0,\tau}(z)|z)}\bigg\}\frac{1}{f_Z(z)}dz + O\Big(\frac{1}{n}\Big) \\ &= \mu_J + O\Big(\frac{1}{n}\Big). \end{aligned}$$

Furthermore,

$$\begin{aligned} \mathrm{Var}\big(n\sqrt{h}I_{n,2}\big) &= E\Big\{n\sqrt{h}\big[I_{n,2} - E(I_{n,2})\big]\Big\}^2 \qquad (S.12) \\ &= n^{-1}h\Big\{E\Big[\Big(\int \gamma_n^2(Y_i, X_i, D_i; z)dz\Big)^2\Big] - E^2\Big[\int \gamma_n^2(Y_i, X_i, D_i; z)dz\Big]\Big\} \\ &= n^{-1}h\Big\{E\Big[\Big(\int \gamma_n^2(Y_i, X_i, D_i; z)dz\Big)^2\Big] - \Big[\int E\big[\gamma_n^2(Y_i, X_i, D_i; z)\big]dz\Big]^2\Big\} \\ &= n^{-1}h\Big\{\int\int E\Big[\gamma_n^2(Y_i, X_i, D_i; u)\gamma_n^2(Y_i, X_i, D_i; v)\Big]dudv - O(h^{-2})\Big\} \\ &= n^{-1}h\Big\{\int\int E\Big[\gamma_n^2(Y_i, X_i, D_i; u)\Big]E\Big[\gamma_n^2(Y_i, X_i, D_i; v)\Big]dudv - O(h^{-2})\Big\} \\ &= n^{-1}h \cdot O(h^{-2}) \to 0, \end{aligned}$$

together with (S.11), we have

$$n\sqrt{h}\big[I_{n,2} - \mu_J\big] = n\sqrt{h}\big[I_{n,2} - E(I_{n,2})\big] + o_p(1) = o_p(1). \qquad (S.13)$$

It follows by combining (S.3), (S.10) and (S.13) that

$$\begin{aligned} n\sqrt{h}\Big\{\int \Big[\frac{1}{n}\sum_{i=1}^n \big(&\varrho_{n,1,\tau}(Y_i, X_i, D_i; z) - E\varrho_{n,1,\tau}(Y_i, X_i, D_i; z) \\ &-\varrho_{n,0,\tau}(Y_i, X_i, D_i; z) + E\varrho_{n,0,\tau}(Y_i, X_i, D_i; z)\big)\Big]^2 dz - \mu_J\Big\} \xrightarrow{D} \mathcal{N}(0, \sigma_J^2). \quad \square \end{aligned}$$

Lemma 6. *Suppose Assumptions 2.1-2.5 are satisfied. Then,*

$$\sup_{z\in\mathcal{Z}}\left|\bar{q}^*_{j,\tau}(z)-\bar{q}_{j,\tau}(z)-\frac{1}{n}\sum_{i=1}^{n}\left[\varrho_{n,j,\tau}(R^*_i;z)-E^*\big(\varrho_{n,j,\tau}(R^*_i;z)\big)\right]\right|=O_{p^*}\left(\frac{\sqrt{\ln n}}{(nh)^{3/4}}\right)$$

for $j=0$ and 1, where $R^*_i=(Y^*_i,X^*_i,D^*_i)'$,

$$\bar{q}^*_{j,\tau}(z)=\arg\min_q\sum_{i=1}^{n}W_j(X^*_i,D^*_i)hK\Big(\frac{Z^*_i-z}{h}\Big)\rho_\tau(Y^*_i;q),$$

and

$$\varrho_{n,j,\tau}(Y^*_i,X^*_i,D^*_i;z)=-S^{-1}_{n,j,\tau}(z)W_j(X^*_i,D^*_i)K_h\big(Z^*_i-z\big)\varphi_\tau(Y^*_i;q_{j,\tau}(z)).$$

Proof of Lemma 6: This result can be proved following the proof of Theorem 2 in Lee et al. (2015). □

Lemma 7. *Under Assumptions 2.1-2.6, then, we have*

$$\sup_{z\in\mathcal{Z}}\left|\hat{q}^*_{j,\tau}(z)-\bar{q}^*_{j,\tau}(z)\right|=O_{p^*}\left\{\max\left\{\frac{\ln n}{\sqrt{n}},\frac{\sqrt{\ln n}}{(nh)^{3/4}}\right\}\right\}$$

for $j=0$ and 1.

Proof of Lemma 7: This result can be proved by the similar arguments to that of Lemma 2 and the details are thus omitted. □

Lemma 8. *Suppose Assumptions 2.1-2.6 hold. Then, for $j=0$ and 1,*

$$\sup_{z\in\mathcal{Z}}\left|\hat{q}^*_{j,\tau}(z)-\bar{q}_{j,\tau}(z)-\frac{1}{n}\sum_{i=1}^{n}\left[\varrho_{n,j,\tau}(R^*_i;z)-E^*(\varrho_{n,j,\tau}(R^*_i;z))\right]\right|=O_{p^*}\left\{\max\left\{\frac{\ln n}{\sqrt{n}},\frac{\sqrt{\ln n}}{(nh)^{3/4}}\right\}\right\}.$$

Proof of Lemma 8: It is easy to observe that

$$\begin{aligned}&\sup_{z\in\mathcal{Z}}\left|\hat{q}^*_{j,\tau}(z)-\hat{q}_{j,\tau}(z)-\frac{1}{n}\sum_{i=1}^{n}\left[\varrho_{n,j,\tau}(R^*_i;z)-E^*(\varrho_{n,j,\tau}(R^*_i;z))\right]\right|\\ \leq\;&\sup_{z\in\mathcal{Z}}\left|\bar{q}^*_{j,\tau}(z)-\bar{q}_{j,\tau}(z)-\frac{1}{n}\sum_{i=1}^{n}\left[\varrho_{n,j,\tau}(R^*_i;z)-E^*(\varrho_{n,j,\tau}(R^*_i;z))\right]\right|\end{aligned}$$

$$
\begin{aligned}
&+\sup_{z\in\mathcal{Z}}\left|\hat{q}^*_{j,\tau}(z)-\bar{q}^*_{j,\tau}(z)\right|+\sup_{z\in\mathcal{Z}}\left|\bar{q}_{j,\tau}(z)-\widehat{q}_{j,\tau}(z)\right| \\
=\;& O_{p^*}\left\{\max\left\{\frac{\ln n}{\sqrt{n}},\frac{\sqrt{\ln n}}{(nh)^{3/4}}\right\}\right\}
\end{aligned}
$$

by Lemmas 2, 6 and 7. Therefore, the proof of Lemma 8 is completed. □

Lemma 9. *Suppose the conditions required by Theorem 3 are satisfied. Then,*

$$
\sup_{y\in\mathbb{R}}\left|P^*\left\{\frac{n\sqrt{h}}{\sigma_J}\left[\int\left(\frac{1}{n}\sum_{i=1}^n\Big(\psi_{n,1,\tau}(R^*_i;z)-\psi_{n,0,\tau}(R^*_i;z)\Big)\right)^2dz-\mu_J\right]\leq y\right\}-\Phi(y)\right|=o_p(1),
$$

where $\psi_{n,j,\tau}(R^*_i;z)=\varrho_{n,j,\tau}(Y^*_i,X^*_i,D^*_i;z)-E^*\big(\varrho_{n,j,\tau}(Y^*_i,X^*_i,D^*_i;z)\big)$ *for* $j=0$ *and* 1. *That is,*

$$
\frac{n\sqrt{h}}{\sigma_J}\left[\int\left(\frac{1}{n}\sum_{i=1}^n\Big(\psi_{n,1,\tau}(R^*_i;z)-\psi_{n,0,\tau}(R^*_i;z)\Big)\right)^2dz-\mu_J\right]
$$

converges to $\mathcal{N}(0,1)$ *in distribution in probability.*

Proof of Lemma 9: It is noticed that

$$
\begin{aligned}
&\int\left(\frac{1}{n}\sum_{i=1}^n\Big(\psi_{n,1,\tau}(R^*_i;z)-\psi_{n,0,\tau}(R^*_i;z)\Big)\right)^2dz \\
=\;&\frac{2}{n^2}\sum_{1\leq i<j\leq n}\int\Big(\psi_{n,1,\tau}(R^*_i;z)-\psi_{n,0,\tau}(R^*_i;z)\Big)\Big(\psi_{n,1,\tau}(R^*_j;z)-\psi_{n,0,\tau}(R^*_j;z)\Big)dz \\
&+\frac{1}{n^2}\sum_{i=1}^n\int\Big(\psi_{n,1,\tau}(R^*_i;z)-\psi_{n,0,\tau}(R^*_i;z)\Big)^2dz \\
:=\;&Q^*_{n,1}+Q^*_{n,2}. \qquad\qquad (S.14)
\end{aligned}
$$

We first consider the term $Q^*_{n,1}$. Define

$$
T^*_n(R^*_i,R^*_j)\;=\;\frac{2}{n^2}\int\Big(\psi_{n,1,\tau}(R^*_i;z)-\psi_{n,0,\tau}(R^*_i;z)\Big)\Big(\psi_{n,1,\tau}(R^*_j;z)-\psi_{n,0,\tau}(R^*_j;z)\Big)dz.
$$

Then, $Q^*_{n,1}$ can be written as a second-order U-statistic as follows:

$$
Q^*_{n,1}=\sum_{1\leq i<j\leq n}T^*_n(R^*_i,R^*_j).
$$

By its definition, it is easy to find that $E^*[T_n^*(R_i^*, R_j^*)] = 0$ and $E^*[T_n^*(R_i^*, R_j^*)|R_i^*] = 0$. Thus, conditional on $\{Y_i, X_i, D_i\}_{i=1}^n$, $Q_{n,1}^*$ is a second-order degenerate U-statistic. To apply Lemma 4, we need to verify the condition

$$\frac{E^*\Big[E^{*2}\big[T_n^*(R_1^*, R_3^*)T_n^*(R_2^*, R_3^*) \,\big|\, R_1^*, R_2^*\big]\Big] + n^{-1}E^*\big[T_n^{*4}(R_1^*, R_2^*)\big]}{E^{*2}\big[T_n^{*2}(R_1^*, R_2^*)\big]} = o_p(1).$$

Define

$$\begin{aligned}
\sigma_n^{*2} \quad &:= \quad E^*\big(T_n^{*2}(R_1^*, R_2^*)\big) \qquad \text{(S.15)}\\
&= \quad \frac{4}{n^4}\int\int E^*\Big[\Big(\psi_{n,1,\tau}(R_1^*; u) - \psi_{n,0,\tau}(R_1^*; u)\Big)\Big(\psi_{n,1,\tau}(R_2^*; u) - \psi_{n,0,\tau}(R_2^*; u)\Big)\\
&\quad \times\Big(\psi_{n,1,\tau}(R_1^*; v) - \psi_{n,0,\tau}(R_1^*; v)\Big)\Big(\psi_{n,1,\tau}(R_2^*; v) - \psi_{n,0,\tau}(R_2^*; v)\Big)\Big]dudv\\
&= \quad \frac{4}{n^4}\int\int E^{*2}\Big[\Big(\psi_{n,1,\tau}(R_1^*; u) - \psi_{n,0,\tau}(R_1^*; u)\Big)\Big(\psi_{n,1,\tau}(R_1^*; v) - \psi_{n,0,\tau}(R_1^*; v)\Big)\Big]dudv\\
&= \quad \frac{4}{n^4}\int\int \Big[E^*\big(\psi_{n,1,\tau}(R_1^*; u)\psi_{n,1,\tau}(R_1^*; v)\big) + E^*\big(\psi_{n,0,\tau}(R_1^*; u)\psi_{n,0,\tau}(R_1^*; v)\big)\\
&\quad -E^*\big(\psi_{n,1,\tau}(R_1^*; u)\psi_{n,0,\tau}(R_1^*; v)\big) - E^*\big(\psi_{n,1,\tau}(R_1^*; v)\psi_{n,0,\tau}(R_1^*; u)\big)\Big]^2 dudv.
\end{aligned}$$

From (S.5) and some calculations, we have

$$\begin{aligned}
E^*\big(\varrho_{n,j,\tau}(R_1^*; u)\big) \quad &= \quad \frac{1}{n}\sum_{i=1}^n \varrho_{n,j,\tau}(R_i; u) \qquad \text{(S.16)}\\
&= \quad O_p\Bigg(E\big(\varrho_{n,j,\tau}(R_i; u)\big) + \frac{1}{\sqrt{n}}\mathrm{Var}^{1/2}\big(\varrho_{n,j,\tau}(R_i; u)\big)\Bigg) = O_p\Big(h^2 + \frac{1}{\sqrt{nh}}\Big)
\end{aligned}$$

uniformly in u. It follows that for $j = 0, 1$,

$$\begin{aligned}
E^*\big(\psi_{n,j,\tau}(R_1^*; u)\psi_{n,j,\tau}(R_1^*; v)\big) \quad &= \quad E^*\big(\varrho_{n,j,\tau}(R_1^*; u)\varrho_{n,j,\tau}(R_1^*; v)\big) - E^*\big(\varrho_{n,j,\tau}(R_1^*; u)\big) \cdot E^*\big(\varrho_{n,j,\tau}(R_1^*; v)\big)\\
&= \quad E^*\big(\varrho_{n,j,\tau}(R_1^*; u)\varrho_{n,j,\tau}(R_1^*; v)\big) + O_p\Big(\frac{1}{nh} + h^4\Big),
\end{aligned}$$

and

$$E^*\big(\psi_{n,1,\tau}(R_1^*; u)\psi_{n,0,\tau}(R_1^*; v)\big) = -E^*\big(\varrho_{n,1,\tau}(R_1^*; u)\big) \cdot E^*\big(\varrho_{n,0,\tau}(R_1^*; v)\big) = O_p\Big(\frac{1}{nh} + h^4\Big).$$

Hence, according to (S.15), we have

$$
\begin{aligned}
\sigma_n^{*2} &= \frac{4}{n^4}\int\int\Big[E^*\big(\psi_{n,1,\tau}(R_1^*;u)\psi_{n,1,\tau}(R_1^*;v)\big)+E^*\big(\psi_{n,0,\tau}(R_1^*;u)\psi_{n,0,\tau}(R_1^*;v)\big) \\
&\quad -E^*\big(\psi_{n,1,\tau}(R_1^*;u)\psi_{n,0,\tau}(R_1^*;v)\big)-E^*\big(\psi_{n,1,\tau}(R_1^*;v)\psi_{n,0,\tau}(R_1^*;u)\big)\Big]^2 dudv \\
&= \frac{4}{n^4}\int\int\bigg(E^*\big(\varrho_{n,1,\tau}(R_1^*;u)\varrho_{n,1,\tau}(R_1^*;v)\big)+E^*\big(\varrho_{n,0,\tau}(R_1^*;u)\varrho_{n,0,\tau}(R_1^*;v)\big)+O_p\Big(\frac{1}{nh}+h^4\Big)\bigg)^2 dudv \\
&= \frac{4}{n^4}\int\int\bigg(E^*\big(\varrho_{n,1,\tau}(R_1^*;u)\varrho_{n,1,\tau}(R_1^*;v)\big)+E^*\big(\varrho_{n,0,\tau}(R_1^*;u)\varrho_{n,0,\tau}(R_1^*;v)\big)\bigg)^2 dudv \\
&\quad +O_p\Big(\frac{1}{n^5h}+\frac{h^4}{n^4}\Big)\int\int\bigg(E^*\big(\varrho_{n,1,\tau}(R_1^*;u)\varrho_{n,1,\tau}(R_1^*;v)\big)+E^*\big(\varrho_{n,0,\tau}(R_1^*;u)\varrho_{n,0,\tau}(R_1^*;v)\big)\bigg)dudv \\
&\quad +O_p\Big(\frac{1}{n^6h^2}+\frac{h^8}{n^4}\Big) \\
&:= A_1+A_2+o_p\Big(\frac{1}{n^4h}\Big).
\end{aligned}
$$

We focus on the term A_1. Note that

$$
\begin{aligned}
& \operatorname{Var}\big[E^*\big(\varrho_{n,j,\tau}(R_1^*;u)\varrho_{n,j,\tau}(R_1^*;v)\big)\big] \\
= {} & \operatorname{Var}\Big[\frac{1}{n}\sum_{i=1}^n \varrho_{n,j,\tau}(R_i;u)\varrho_{n,j,\tau}(R_i;v)\Big] \\
\le {} & \frac{1}{n}E\Big(\varrho_{n,j,\tau}(R_i;u)\varrho_{n,j,\tau}(R_i;v)\Big)^2 \\
= {} & \frac{1}{n}E\Big(S_{n,j,\tau}^{-1}(u)S_{n,j,\tau}^{-1}(v)W_j(X_i,D_i)K_h(Z_i-u)K_h(Z_i-v)\varphi_\tau(Y_i;q_{j,\tau}(u))\varphi_\tau(Y_i;q_{j,\tau}(v))\Big)^2 \\
= {} & \frac{1}{n}\cdot\frac{1}{h^4}S_{n,j,\tau}^{-2}(u)S_{n,j,\tau}^{-2}(v)E\Big\{W_j^2(X_i,D_i)K^2\Big(\frac{Z_i-u}{h}\Big)\times K^2\Big(\frac{Z_i-v}{h}\Big)\varphi_\tau^2(Y_i;q_{j,\tau}(u))\varphi_\tau^2(Y_i;q_{j,\tau}(v))\Big\} \\
= {} & \frac{1}{n}\cdot\frac{1}{h^4}S_{n,j,\tau}^{-2}(u)S_{n,j,\tau}^{-2}(v)E\Big\{K^2\Big(\frac{Z_i-u}{h}\Big)K^2\Big(\frac{Z_i-v}{h}\Big) \\
& \times E\Big(p^{-3j}(X_i)(1-p(X_i))^{-3(1-j)}\varphi_\tau^2(Y_i(j);q_{j,\tau}(u))\varphi_\tau^2(Y_i(j);q_{j,\tau}(v))\Big|Z_i\Big)\Big\} \\
= {} & \frac{1}{n}\cdot\frac{1}{h^3}S_{n,j,\tau}^{-2}(u)S_{n,j,\tau}^{-2}(v)\int K^2(s)K^2\Big(s+\frac{u-v}{h}\Big)\ell_j(u+hs;u,v)f_Z(u+hs)ds=O_p\Big(\frac{1}{nh^3}\Big),
\end{aligned}
$$

where

$$
\ell_j(z;u,v)=E\Big(p^{-3j}(X_i)(1-p(X_i))^{-3(1-j)}\varphi_\tau^2(Y_i(j);q_{j,\tau}(u))\varphi_\tau^2(Y_i(j);q_{j,\tau}(v))\Big|Z_i=z\Big).
$$

Therefore,

$$\begin{aligned} E^*\big(\varrho_{n,j,\tau}(R_1^*;u)\varrho_{n,j,\tau}(R_1^*;v)\big) &= \frac{1}{n}\sum_{i=1}^{n}\varrho_{n,j,\tau}(R_i;u)\varrho_{n,j,\tau}(R_i;v) \\ &= E\big(\varrho_{n,j,\tau}(R_1;u)\varrho_{n,j,\tau}(R_1;v)\big) + O_p\Big(\frac{1}{\sqrt{nh^3}}\Big). \quad \text{(S.17)} \end{aligned}$$

Then,

$$\begin{aligned} A_1 &= \frac{4}{n^4}\int\int\Big(E^*\big(\varrho_{n,1,\tau}(R_1^*;u)\varrho_{n,1,\tau}(R_1^*;v)\big) + E^*\big(\varrho_{n,0,\tau}(R_1^*;u)\varrho_{n,0,\tau}(R_1^*;v)\big)\Big)^2 dudv \\ &= \frac{4}{n^4}\int\int\Big(E\big(\varrho_{n,1,\tau}(R_1;u)\varrho_{n,1,\tau}(R_1;v)\big) + E\big(\varrho_{n,0,\tau}(R_1;u)\varrho_{n,0,\tau}(R_1;v)\big) + O_p\Big(\frac{1}{\sqrt{nh^3}}\Big)\Big)^2 dudv. \end{aligned}$$

Using similar arguments as in (S.6), (S.7), (S.8) and (S.9), we have

$$\begin{aligned} A_1 &= \frac{4}{n^4}\Big[\frac{1}{h}\int\Big(\int K(t)K(t+s)dt\Big)^2 ds \\ &\quad\times\int\Big\{\frac{\kappa_1(u;u,u)}{f^2_{Y(1)|Z}(q_{1,\tau}(u)|u)} + \frac{\kappa_0(u;u,u)}{f^2_{Y(0)|Z}(q_{0,\tau}(u)|u)}\Big\}^2\frac{1}{f_Z^2(u)}du + O_p\Big(\frac{1}{\sqrt{nh^3}}\Big) + O_p\Big(\frac{1}{nh^3}\Big)\Big] \\ &= \frac{4}{n^4h}\Big[\int\Big(\int K(t)K(t+s)dt\Big)^2 ds \\ &\quad\times\int\Big\{\frac{\kappa_1(u;u,u)}{f^2_{Y(1)|Z}(q_{1,\tau}(u)|u)} + \frac{\kappa_0(u;u,u)}{f^2_{Y(0)|Z}(q_{0,\tau}(u)|u)}\Big\}^2\frac{1}{f_Z^2(u)}du + O_p\Big(\frac{1}{\sqrt{nh}}\Big) + O_p\Big(\frac{1}{nh^2}\Big)\Big] \\ &= \frac{2}{n^4h}\big(\sigma_J^2 + o_p(1)\big). \end{aligned}$$

It is also easy to find that $A_2 = o_p\Big(\frac{1}{n^4h}\Big)$. Thus,

$$\sigma_n^{*2} = E^*\big(T_n^{*2}(R_1^*,R_2^*)\big) = \frac{2\sigma_J^2}{n^4h} + o_p\Big(\frac{1}{n^4h}\Big).$$

Similarly, by some straightforward but tedious calculations, we can obtain that

$$E^*\Big[E^{*2}\big[T_n^*(R_1^*,R_3^*)T_n^*(R_2^*,R_3^*)\,\big|\,R_1^*,R_2^*\big]\Big] = O_p\Big(\frac{1}{n^8h}\Big),$$

and

$$E^*\big[T_n^{*4}(R_1^*,R_2^*)\big] = O_p\Big(\frac{1}{n^8h^3}\Big).$$

Therefore, the condition

$$\frac{E^*\Big[E^{*2}\big[T_n^*(R_1^*,R_3^*)T_n^*(R_2^*,R_3^*)\,\big|\,R_1^*,R_2^*\big]\Big]+n^{-1}E^*\big[T_n^{*4}(R_1^*,R_2^*)\big]}{E^{*2}\big[T_n^{*2}(R_1^*,R_2^*)\big]}=o_p(1)$$

is satisfied. From Lemma 4 we know that

$$\frac{\sqrt{2}Q_{n,1}^*}{n\sigma_n^*}\longrightarrow\mathcal{N}(0,1)$$

in distribution in probability. Since $\sigma_n^*=\frac{\sqrt{2}}{\sqrt{n^4h}}\big(\sigma_J+o_p(1)\big)$, we also have

$$\frac{n\sqrt{h}Q_{n,1}^*}{\sigma_J}\longrightarrow\mathcal{N}(0,1) \tag{S.18}$$

in distribution in probability. For the term $Q_{n,2}^*$, we have

$$\begin{aligned}
Q_{n,2}^* &= \frac{1}{n^2}\sum_{i=1}^n\int\Big(\psi_{n,1,\tau}(R_i^*;z)-\psi_{n,0,\tau}(R_i^*;z)\Big)^2dz\\
&= \frac{1}{n^2}\sum_{i=1}^n\int\Big(\big(\varrho_{n,1,\tau}(R_i^*;z)-\varrho_{n,0,\tau}(R_i^*;z)\big)-E^*\big(\varrho_{n,1,\tau}(R_i^*;z)-\varrho_{n,0,\tau}(R_i^*;z)\big)\Big)^2dz\\
&= \frac{1}{n^2}\sum_{i=1}^n\int\Big(\varrho_{n,1,\tau}(R_i^*;z)-\varrho_{n,0,\tau}(R_i^*;z)\Big)^2dz-\frac{1}{n^2}\sum_{i=1}^n\int\Big(E^*\big(\varrho_{n,1,\tau}(R_i^*;z)-\varrho_{n,0,\tau}(R_i^*;z)\big)\Big)^2dz\\
&\quad -\frac{2}{n^2}\sum_{i=1}^n\int\Big[E^*\big(\varrho_{n,1,\tau}(R_i^*;z)-\varrho_{n,0,\tau}(R_i^*;z)\big)\Big]\Big(\psi_{n,1,\tau}(R_i^*;z)-\psi_{n,0,\tau}(R_i^*;z)\Big)dz\\
&:= Q_{n,2,1}^*-Q_{n,2,2}^*-2Q_{n,2,3}^*.
\end{aligned}$$

For $Q_{n,2,1}^*$, it is easy to obtain that

$$E^*\big(Q_{n,2,1}^*\big)=\frac{1}{n^2}\sum_{i=1}^n\int\Big(\varrho_{n,1,\tau}(R_i;z)-\varrho_{n,0,\tau}(R_i;z)\Big)^2dz,$$

and

$$\begin{aligned}
\mathrm{Var}^*(Q_{n,2,1}^*) &= \mathrm{Var}^*\bigg(\frac{1}{n^2}\sum_{i=1}^n\int\Big(\varrho_{n,1}(R_i^*;z)-\varrho_{n,0}(R_i^*;z)\Big)^2dz\bigg) \qquad (S.19)\\
&= \frac{1}{n^3}\mathrm{Var}^*\bigg(\int\Big(\varrho_{n,1}^2(R_i^*;z)+\varrho_{n,0}^2(R_i^*;z)\Big)\bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n^3}E^*\Big(\int\Big(\varrho^2_{n,1}(R^*_i;z)+\varrho^2_{n,0}(R^*_i;z)\Big)dz\Big)^2 \\
&\quad -\frac{1}{n^3}\Big\{\int\Big(E^*\varrho^2_{n,1}(R^*_i;z)+E^*\varrho^2_{n,0}(R^*_i;z)\Big)dz\Big\}^2 \\
&= \frac{1}{n^3}\int\int\Big(E^*\big(\varrho^2_{n,1}(R^*_i;u)\varrho^2_{n,1}(R^*_i;v)\big)+E^*\big(\varrho^2_{n,0}(R^*_i;u)\varrho^2_{n,0}(R^*_i;v)\big)\Big)dudv \\
&\quad -\frac{1}{n^3}\Big\{\int\Big(E^*\varrho^2_{n,1}(R^*_i;z)+E^*\varrho^2_{n,0}(R^*_i;z)\Big)dz\Big\}^2.
\end{aligned}
$$

It is easy to obtain that

$$
\begin{aligned}
& E\Big(\int\Big(E^*\varrho^2_{n,1}(R^*_i;z)+E^*\varrho^2_{n,0}(R^*_i;z)\Big)dz\Big) \\
= & \int\Big[E\Big(E^*\varrho^2_{n,1}(R^*_i;z)\Big)+E\Big(E^*\varrho^2_{n,0}(R^*_i;z)\Big)\Big]dz \\
= & \frac{1}{h}\Big\{\int S^{-2}_{n,1,\tau}(z)\int K^2(s)\mu_1(z+hs;z)f_Z(z+hs)ds+S^{-2}_{n,0,\tau}(z)\int K^2(s)\mu_0(z+hs;z)f_Z(z+hs)ds\Big\} \\
= & O(1/h),
\end{aligned}
$$

which implies that

$$
\int\Big(E^*\varrho^2_{n,1}(R^*_i;z)+E^*\varrho^2_{n,0}(R^*_i;z)\Big)dz=O_p(1/h). \tag{S.20}
$$

Similarly, by straightforward calculations, we can obtain that

$$
\begin{aligned}
& E\int\int\Big(E^*\big(\varrho^2_{n,j}(R^*_i;u)\varrho^2_{n,j}(R^*_i;v)\big)\Big)dudv \\
= & \int\int E\Big(E^*\big(\varrho^2_{n,j}(R^*_i;u)\varrho^2_{n,j}(R^*_i;v)\big)\Big)dudv \\
= & \frac{1}{nh^4}\int\int S^{-2}_{n,j}(u)S^{-2}_{n,j}(v)E\Big[W^4_j(X_i,D_i)K^2\Big(\frac{Z_i-u}{h}\Big)K^2\Big(\frac{Z_i-v}{h}\Big)\varphi^2_\tau(Y_i;q_{j,\tau}(u))\varphi^2_\tau(Y_i;q_{j,\tau}(v))\Big] \\
& \times\omega(u)\omega(v)dudv+\frac{1}{n^2h^4}\sum_{i\neq k}\int\int S^{-2}_{n,j}(u)S^{-2}_{n,j}(v)E\Big[W^2_j(X_i,D_i)K^2\Big(\frac{Z_i-u}{h}\Big)\varphi^2_\tau(Y_i;q_{j,\tau}(u)\Big] \\
& \times E\Big[W^2_j(X_k,D_k)K^2\Big(\frac{Z_k-v}{h}\Big)\varphi^2_\tau(Y_k;q_{j,\tau}(v)\Big]dudv \\
= & \frac{1}{nh^3}\int\int S^{-2}_{n,j}(u)S^{-2}_{n,j}(v)\Big(\int K^2(t)K^2\Big(t+\frac{u-v}{h}\Big)\ell_j(u+ht;u,v)f_Z(u+ht)dt\Big)dudv \\
& +\frac{n(n-1)}{n^2h^2}\int\int S^{-2}_{n,j}(u)S^{-2}_{n,j}(v)\Big(\int K^2(s)\mu_j(u+hs;u)f_Z(u+hs)ds \\
& \qquad \times\int K^2(s)\mu_j(v+hs;v)f_Z(v+hs)ds\Big)dudv
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{n(n-1)}{n^2h^2}\Big\{\Big(\int K^2(s)ds\Big)^2 \int\int S_{n,j}^{-2}(u)S_{n,j}^{-2}(v)\mu_j(u;u)\mu_j(v;v)f_Z(u)f_Z(v)dudv + o(1)\Big\} \\
&\quad +O\Big(\frac{1}{nh^2}\Big) \\
&= O(1/h^2),
\end{aligned}
$$

which implies

$$
\int\int\Big(E^*\big(\varrho_{n,1}^2(R_i^*;u)\varrho_{n,1}^2(R_i^*;v)\big) + E^*\big(\varrho_{n,0}^2(R_i^*;u)\varrho_{n,0}^2(R_i^*;v)\big)\Big)dudv = O_p(1/h^2). \quad \text{(S.21)}
$$

From (S.19), (S.20) and (S.21), we have

$$
\mathrm{Var}^*(Q_{n,2,1}^*) = \frac{1}{n^3}\cdot O_p(1/h^2) - \frac{1}{n^3}\cdot O_p(1/h) = o_p\Big(\frac{1}{n^2h}\Big).
$$

Then, according to the results in (S.11) and (S.12), we obtain that

$$
E\big(Q_{n,2,1}^*\big) = E\big[E^*\big(Q_{n,2,1}^*\big)\big] = \mu_J + O(1/n)
$$

and

$$
\begin{aligned}
\mathrm{Var}\big(Q_{n,2,1}^*\big) &= \mathrm{Var}\big[E^*\big(Q_{n,2,1}^*\big)\big] + E\big[\mathrm{Var}^*\big(Q_{n,2,1}^*\big)\big] \\
&= O_p\Big(\frac{1}{n^3h^2}\Big) + o_p\Big(\frac{1}{n^2h}\Big) = o_p\Big(\frac{1}{n^2h}\Big),
\end{aligned}
$$

which lead to

$$
Q_{n,2,1}^* = \mu_J + O(1/n) + o_p\Big(\frac{1}{n\sqrt{h}}\Big) = \mu_J + o_p\Big(\frac{1}{n\sqrt{h}}\Big).
$$

By noting that $E^*\big(\varrho_{n,j,\tau}(R_1^*;u)\big) = O_p\Big(h^2 + \frac{1}{\sqrt{nh}}\Big)$ as in (S.16), it is easy to obtain that

$$
Q_{n,2,2}^* = \frac{1}{n}\cdot O_p\Big(h^4 + \frac{1}{nh}\Big) = o_p\Big(\frac{1}{n\sqrt{h}}\Big)
$$

and

$$
Q_{n,2,3}^* = \frac{1}{n}\cdot O_p\Big(h^2 + \frac{1}{\sqrt{nh}}\Big)\cdot O_p(1/h) = o_p\Big(\frac{1}{n\sqrt{h}}\Big)
$$

since $nh^2 \to \infty$. Thus, we have

$$Q^*_{n,2} = Q^*_{n,2,1} - Q^*_{n,2,2} - 2Q^*_{n,2,3} = \mu_J + o_p\Big(\frac{1}{n\sqrt{h}}\Big). \tag{S.22}$$

Combining (S.14), (S.18) and (S.22), we complete the proof of Lemma 9. □

Lemma 10. *Suppose the conditions required by Theorem 3 are satisfied. Then,*

$$\int \Big(\widehat{\Delta}^*_\tau(z) - \widehat{\Delta}_\tau(z)\Big)dz = O_{p^*}(1/\sqrt{n}) + O_{p^*}\left\{ \max\left\{\frac{\ln n}{\sqrt{n}}, \frac{\sqrt{\ln n}}{(nh)^{3/4}}\right\}\right\}.$$

Proof of Lemma 10: According to Lemma 8, we know that

$$\begin{aligned}\int \Big(\widehat{\Delta}^*_\tau(z) - \widehat{\Delta}_\tau(z)\Big)dz &= \int \frac{1}{n}\sum_{i=1}^n \Big(\psi_{n,1,\tau}(R^*_i;z) - \psi_{n,0,\tau}(R^*_i;z)\Big)dz \\ &\quad + O_{p^*}\left\{ \max\left\{\frac{\ln n}{\sqrt{n}}, \frac{\sqrt{\ln n}}{(nh)^{3/4}}\right\}\right\},\end{aligned}$$

where $\psi_{n,j,\tau}(R^*_i;z) = \varrho_{n,j,\tau}(R^*_i;z) - E^*\big(\varrho_{n,j,\tau}(R^*_i;z)\big)$ for $j = 0$ and 1. Denote

$$\mathcal{M}^*_n = \frac{1}{n}\sum_{i=1}^n \int \Big(\psi_{n,1,\tau}(R^*_i;z) - \psi_{n,0,\tau}(R^*_i;z)\Big)dz.$$

Obviously, $E^*(\mathcal{M}^*_n) = 0$. We then consider $E^*(\mathcal{M}^{*2}_n)$. We have

$$\begin{aligned}E^*(\mathcal{M}^{*2}_n) &= \frac{1}{n^2}\sum_{i=1}^n E^*\Big(\int \big(\psi_{n,1,\tau}(R^*_i;z) - \psi_{n,0,\tau}(R^*_i;z)\big)dz\Big)^2 \\ &= \frac{1}{n}\int\int E^*\Big(\big(\psi_{n,1,\tau}(R^*_i;u) - \psi_{n,0,\tau}(R^*_i;u)\big)\big(\psi_{n,1,\tau}(R^*_i;v) - \psi_{n,0,\tau}(R^*_i;v)\big)\Big)dudv \\ &= \frac{1}{n}\int\int E^*\big(\varrho_{n,0,\tau}(R^*_i;u)\varrho_{n,0,\tau}(R^*_i;v)\big)dudv + \frac{1}{n}\int\int E^*\big(\varrho_{n,1,\tau}(R^*_i;u)\varrho_{n,1,\tau}(R^*_i;v)\big)dudv \\ &\quad -\frac{1}{n}\int\int E^*(\varrho_{n,0,\tau}(R^*_i;u))\cdot E^*(\varrho_{n,0,\tau}(R^*_i;v))dudv \\ &\quad +\frac{1}{n}\int\int E^*(\varrho_{n,0,\tau}(R^*_i;u))\cdot E^*(\varrho_{n,1,\tau}(R^*_i;v))dudv \\ &\quad +\frac{1}{n}\int\int E^*(\varrho_{n,1,\tau}(R^*_i;u))\cdot E^*(\varrho_{n,0,\tau}(R^*_i;v))dudv \\ &\quad -\frac{1}{n}\int\int E^*(\varrho_{n,1,\tau}(R^*_i;u))\cdot E^*(\varrho_{n,1,\tau}(R^*_i;v))dudv.\end{aligned}$$

Using the results in (S.17) and (S.6), we obtain that

$$\begin{aligned}
& \frac{1}{n}\int\int E^*\big(\varrho_{n,j,\tau}(R_i^*;u)\varrho_{n,j,\tau}(R_i^*;v)\big)dudv \\
= & \frac{1}{n}\int\int\Big[E\big(\varrho_{n,j,\tau}(R_1;u)\varrho_{n,j,\tau}(R_1;v)\big)+O_p\Big(\frac{1}{\sqrt{nh^3}}\Big)\Big]dudv \\
= & \frac{1}{n}\Big[O_p(1)+O_p\Big(\frac{1}{\sqrt{nh^3}}\Big)\Big]=O_p\Big(n^{-1}\Big).
\end{aligned}$$

Also, by noting that

$$E^*\big(\varrho_{n,j,\tau}(R_1^*;u)\big)=O_p\Big(h^2+\frac{1}{\sqrt{nh}}\Big)$$

as in (S.16), we have

$$\frac{1}{n}\int\int E^*(\varrho_{n,j,\tau}(R_i^*;u))\cdot E^*(\varrho_{n,k,\tau}(R_i^*;v))dudv=o_p\Big(n^{-1}\Big)$$

for $j,k=0,1$. Therefore,

$$E^*(\mathcal{M}_n^{*2})=O_p\big(1/n\big),$$

which implies

$$\mathcal{M}_n^*=O_{p^*}\big(1/\sqrt{n}\big).$$

This completes the proof of Lemma 10. □