A Nonparametric Test for Testing Heterogeneity in Conditional Quantile Treatment Effects* †

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Abstract: This paper proposes a nonparametric test to assess whether there exists heterogeneously quantile treatment effect for an intervention on outcome of interest across different sub-populations defined by covariates of interest. Specifically, a consistent test statistic based on the Cramér-von Mises type criterion is developed to test if the treatment has a constant quantile effect for all subpopulations defined by covariates of interest. Under some regular conditions, the asymptotic behaviors of the proposed test statistic are investigated under both the null and alternative hypotheses. Furthermore, a nonparametric Bootstrap procedure is suggested to approximate the finite-sample null distribution of the proposed test and the asymptotic validity of the proposed Bootstrap test is theoretically justified. Through Monte Carlo simulations, we demonstrate the power properties of the test in finite samples. Finally, the proposed testing approach is applied to investigating whether there exists heterogeneity for the quantile treatment effect of maternal smoking during pregnancy on infant birth weight across different age groups of mothers.

Keywords: Bootstrap; Cramér-von Mises type test; Heterogeneity test; Nonparametric quantile regression; Quantile treatment effect.

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1 Introduction

In program evaluation studies, it is important to learn about the heterogeneous impacts of policy variables on different points of the outcome distribution of interest. Examples include, but not limited to, evaluating the effects of government training programs on lower quantiles of earning distributions studied by LaLonde (1995) and Abadie, Angrist and Imbens (2002), the effects of the government-subsidized saving program on lower tails of savings distributions, and among many others. To characterize the heterogeneous effects along with the outcome distribution, quantile treatment effect (QTE), originally suggested by Doksum (1974) and Lehmann (1975) and defined as the difference between the quantiles of the marginal potential distributions of the treatment and control responses, provides a powerful tool to document such heterogeneity. In the last few decades, QTE has gained increasing popularity in economics, political science, and many other social, behavioral, and statistical sciences. Recent studies on QTE include, but not limited to, the papers by Abadie et al. (2002), Chernozhukov and Hansen (2005), Donald and Hsu (2014), Firpo (2007), Frölich and Melly (2013), and the references therein. Moreover, Tang (2020) provided a comprehensive survey on recent developments in modeling methods for QTE.

It is importantly noted that the aforementioned papers mainly focus on identification and estimation of the quantile treatment effect for the overall population or the treated group under various assumptions. It is generally believed in program evaluation literature that the effect of a treatment can be heterogeneous across different individuals as in Heckman and Robb (1985) and Heckman et al. (1997). Consequently, in many cases, researchers may be more interested in studying the effects of programs across different individuals instead of the effects for the overall population or the subpopulation of treated individuals. For example, it may be of substantive interest to investigate the heterogeneous effect of maternal smoking during pregnancy on infant birth weight across mothers with different ages. How to characterize the heterogeneity of treatment effects across different individuals is a challenge in the treatment effect literature and it has been extensively considered in the recent literature. Recently, to characterize the heterogeneous effect across different sub-populations defined by some covariates of interest, Abrevaya, Hsu and Lieli (2015) and Lee, Okui and Whang (2017) considered the partially conditional average treatment effect
(ATE). Different from Abrevaya et al. (2015) and Lee et al. (2017), to capture heterogeneities for both across-distribution and across-individuals simultaneously, Cai, Fang, Lin and Tang (2021)\(^1\) and Zhou, Guo and Zhu (2021) proposed a partially conditional quantile treatment effect (PCQTE) model, whereas Tang, Cai, Fang and Lin (2021) considered a parametric model.

Our motivation in this paper comes actually from the empirical estimation results in Cai et al. (2021) and Tang et al. (2021) by investigating the QTE of maternal smoking during pregnancy on infant birth weight across different age groups of mothers. The main findings in Cai et al. (2021) are that there is a significant negative effect of smoking on infant birth weight across all mothers’ ages and quantiles considered for both whites and blacks and there is substantial heterogeneity across different mothers’ ages for whites but not for blacks. Motivated by these estimation results, from statistical and empirical perspectives, it is interesting to test whether or not the conditional QTEs conditional on mothers’ ages, for both whites and blacks, change over mothers’ ages, in other words, whether there exists heterogeneity for the QTEs of maternal smoking on infant birth weight across different age groups of mothers for both whites and blacks. To this end, we propose in this paper a novel test to assess whether there exists heterogeneously distributional effect for an intervention on outcome of interest across different sub-populations defined by covariates of interest. Specifically, a nonparametric test is developed for testing the null hypothesis that the treatment has a constant quantile effect for all subpopulations defined by covariates of interest. In other words, there is no heterogeneity in QTEs by covariates of interest. To this end, a consistent test statistic is constructed based on the Cramér-von Mises type criterion.

Under some regular conditions, we establish the asymptotic distribution of the proposed test statistic under both the null and alternative hypotheses and investigate the power of our test against a sequence of local alternatives. However, note that to calculate the critical value of the proposed test statistic under null hypothesis, one needs to consistently estimate the conditional density of the potential outcomes conditional on covariates of interest involved in the asymptotic bias and asymptotic variance, which is not an easy

\(^1\)Please note that the working paper version of this paper in English with a modification can be downloaded at https://econpapers.repec.org/paper/kanwpaper/202005.htm.
task recognized in the literature. To overcome this problem, a nonparametric Bootstrap procedure is proposed to approximate the finite-sample null distribution of the proposed test. Furthermore, the asymptotic validity of the proposed Bootstrap test is justified. Through Monte Carlo simulations, we demonstrate the power properties of the test in finite samples. As an empirical illustration, the proposed testing approach is applied to investigating whether there exists heterogeneity for the QTE of maternal smoking during pregnancy on infant birth weight across different age groups of mothers. The testing results show that the QTE of maternal smoking on infant birth weight do change over mother’s age for all quantile levels considered for whites but not for blacks, which support the findings obtained in Cai et al. (2021).

Interestingly, this paper is related to some works in literature. For example, first, Crump, Hotz, Imbens and Mitnik (2008) developed two nonparametric tests based on series approach, in which the first is to test whether a treatment has a zero average effect for all sub-populations defined by covariates, and the second is to test whether the ATE conditional on the covariates is identical for all sub-populations, in other words, whether heterogeneity exists in ATE by covariates. Second, Lee and Whang (2009) proposed to test whether the conditional QTE is significant, conditional on the whole set of covariates. By contrast, our focus is on testing if the partially conditional QTE is a constant, in which the constant needs to be estimated. More importantly, one may be interested in studying the heterogeneous effect on some particular covariates instead of the whole set of covariates, for example, the effect of maternal smoking during pregnancy on infant birth weight across mothers with different ages. Moreover, Escanciano and Goh (2014) considered a nonparametric test for testing the specification of a linear conditional quantile function over a continuum of quantile levels and they showed that the use of an orthogonal projection on the tangent space of nuisance parameters at each quantile index can improve power and facilitate the simulation of critical values via the application of a simple multiplier Bootstrap procedure. Finally, Dong, Li and Feng (2019) introduced a new approach to assess the lack of fit for quantile regression models. They first transformed the lack-of-fit tests for parametric quantile regression models into checking the equality of two conditional distributions of covariates. Then, by applying some successful two-sample test statistics in the literature, two tests are constructed to check the lack of fit for low and high dimen-
sional quantile regression models. Finally, to calculate the p-values or critical values, they suggested adopting the wild Bootstrap procedure.

The remainder of this paper is organized as follows. Section 2 introduces the proposed test statistic and presents its asymptotic properties under the null hypothesis. Also, a Bootstrap procedure is suggested to approximate the finite-sample null distribution of the proposed test and the asymptotic validity of the proposed Bootstrap test is theoretically justified. In Sections 3 and 4, the finite sample properties of our test through Monte Carlo simulations are investigated and an empirical application is considered, respectively. Section 5 concludes the paper. Finally, the key steps for proving the theorems can be found in Appendix, together with some auxiliary lemmas with their detailed proofs given in Supplement.

2 Testing Heterogeneity for Conditional QTE

2.1 Test Statistic

Let us first introduce the model framework considered in this paper. To this end, let $D_i$ be the binary treatment variable of individual $i$ in population, where $D_i = 1$ if individual $i$ receives the treatment of interest and otherwise, $D_i = 0$. Using the potential outcome framework initialized by Rubin (1974), define $Y_i(0)$ and $Y_i(1)$ to be the potential outcomes of individual $i$ if it is in the control group or in the treated group, respectively. Also, assume that $D_i$ and $Y_i$ are observed, where $Y_i$ is the realized outcome as $Y_i = (1 - D_i) \cdot Y_i(0) + D_i \cdot Y_i(1)$. In addition, suppose that $X_i$, a $p$-dimensional vector of pre-treatment variables for individual $i$, is observed too. Therefore, throughout the paper, it is assumed that $\{Y_i(0), Y_i(1), X_i, D_i\}, i = 1, \cdots, n$, are independent and identically distributed (iid).

Let $Z_i$ be a $d$-dimensional sub-vector of $X_i$, where $1 \leq d \leq p$ and in particular, $d$ is small and much smaller than $p$ in many applications. To capture heterogeneities for both across-distribution and across-individuals simultaneously, Cai et al. (2021) and Zhou et al. (2021) considered a partially conditional quantile treatment effect model, which is defined as

$$\Delta_r(z) = q_{1,r}(z) - q_{0,r}(z),$$

(1)
where $\tau \in (0, 1)$ is the quantile level and for $j = 0$ and $q_{j, \tau}(z)$ is the $\tau$-th conditional quantile function of $Y_i(j)$ conditional on $Z_i = z$. It is important to note that for each individual in the population, only one of $Y_i(0)$ and $Y_i(1)$ is observable, so that due to the missing variable, the PCQTE parameter $\Delta_{\tau}(z)$ in (1) can not be identified without further restrictions on the data-generating distribution. To identify the functionals in (1), it is common in the treatment effect literature to assume that assignment to treatment is unconfounded and that the probability of assignment is bounded away from 0 and 1. Formally, the following assumption is imposed throughout the paper.

**Assumption 1.** Assume that

(i) (unconfoundedness) conditional on pre-treatment variables $X_i$, the potential outcomes are jointly independent from the treatment variable $D_i$, namely,

$$(Y_i(0), Y_i(1)) \perp\!
\!
\!
\perp D_i \mid X_i,$$

where $\perp\!
\!
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\perp$ indicates statistical independence, and

(ii) (overlap) for almost every $x \in \mathcal{X}$, where $\mathcal{X}$ is the support of $X_i$, there exists some small $\varepsilon > 0$ so that $\varepsilon < p(x) = P(D_i = 1 \mid X_i = x) < 1 - \varepsilon$, where $p(x)$ is called the propensity score function.

Part (i) of Assumption 1 is often referred to as the (strongly) ignorable treatment assignment, conditional independence assumption or selection on observables in the econometrics and/or statistics literature and it requires that conditional on the observed individual characteristics $X_i$, the treatment assignment $D_i$ is independent of the potential outcomes $Y_i(0)$ and $Y_i(1)$. Although it is a strong assumption, it has been extensively employed in many applied fields to study on the effect of treatments or programs, among others, see, for example, the papers by Abadie and Imbens (2006, 2016), Hirano, Imbens and Ridder (2003), Heckman et al. (1998), Dehejia and Wahba (1999), and Firpo (2007). Part (ii) of Assumption 1 states that in the population for almost all values of $X_i$, both treatment assignment levels have a positive probability of occurrence. In practice, however, there are often concerns about possible lack of common support. A common approach to address this problem is to drop observations with the propensity score close to zero or one, and focus on the treatment effect in the subpopulation with propensity score bounded away from zero and
one; see Crump et al. (2009) for more details.

Under Assumption 1, Cai et al. (2021) showed that the PCQTE function $\Delta_r(z)$ is nonparametrically identified and further proposed a semiparametric estimation procedure to estimate $\Delta_r(z)$. Specifically, the proposed estimate for $\Delta_r(z)$ in Cai et al. (2021) is given by

$$\hat{\Delta}_r(z) = \hat{q}_{1,r}(z) - \hat{q}_{0,r}(z), \quad (2)$$

where for \( j = 0 \) and \( 1 \),

$$\hat{q}_{j,r} = \arg\min_q \frac{1}{n} \sum_{i=1}^n K_h(Z_i - z) \hat{W}_{n,j}(X_i, D_i) \rho_r(Y_i - q)$$

with $\hat{W}_{n,0}(X_i, D_i) = (1 - D_i)/[1 - \hat{p}_n(X_i)]$, $\hat{W}_{n,1}(X_i, D_i) = D_i/\hat{p}_n(X_i)$, and $K_h(u) = K(u/h)/h$. Here, $K(\cdot)$ is a kernel function, $h$ is the bandwidth parameter, $\rho_r(u) = u \cdot (\tau - I\{u \leq 0\})$ is the check function as in Koenker and Bassett (1978) and Koenker (2005), and $\hat{p}_n(x)$ is the parametric estimate of $p(x)$. Furthermore, the consistency and asymptotic normality of the proposed semiparametric estimator $\hat{\Delta}_r(z)$ are also established in Cai et al. (2021). The reader is referred to the paper by Cai et al. (2021) for more details.

As discussed in the introduction, in this paper, our interest is to investigate whether there exists heterogeneity in QTEs across different sub-populations defined by covariates of interest. To this end, the following hypothesis testing problem is investigated

$$H_0 : \Delta_r(z) = \delta_r \text{ for all } z \in Z \quad \text{versus} \quad H_1 : \Delta_r(z) \neq \delta_r \text{ for some } z \in Z \quad (3)$$

for some constant $\delta_r$, where $Z$ is the support of $Z_i$. Under the null hypothesis, the partially conditional quantile effect of the treatment is a constant and under the alternative, the PCQTE varies across different sub-populations defined by $Z_i$. Furthermore, the above testing setting in (3) can be generalized to the following testing problem

$$H_0 : \Delta_r(z) = \Delta_{r,0}(z, \theta_r) \quad \text{versus} \quad H_1 : \Delta_r(z) \neq \Delta_{r,0}(z, \theta_r), \quad (4)$$

where $\Delta_{r,0}(z, \theta_r)$ is a known function with unknown parameter $\theta_r$. The purpose of the test in (4) is to see whether $\Delta_r(z)$ has a particular parametric form, say, a linear function as in
In order to test whether the hypothesis testing problem formulated in (3) holds or not, the test statistic is constructed based on Cramér-von Mises criterion as follows. To this end, let

\[ J = \int \left( \Delta_r(z) - \delta_r \right)^2 dz \geq 0, \]

where the integral is taken over \( Z \). Note that \( J = 0 \) if and only if the null hypothesis in (3) is true. Hence, a test statistic using the sample analogue of \( J \) is defined by

\[ J_n = \int \left( \hat{\Delta}_r(z) - \hat{\delta}_r \right)^2 dz, \]

where \( \hat{\Delta}_r(z) \) is the semiparametric estimator (2) of \( \Delta_r(z) \), and \( \hat{\delta}_r \) is a \( \sqrt{n} \)-consistent estimator for \( \delta_r \), such as the estimator

\[ \hat{\delta}_r = \frac{1}{n} \sum_{i=1}^{n} \hat{\Delta}_r(Z_i) \]

proposed in Cai and Xiao (2012). Note that by following the same idea in Cai and Xiao (2012), one can show easily that \( \hat{\delta}_r \) in (5) a is \( \sqrt{n} \)-consistent estimator of \( \delta_r \).

**Remark 1.** If \( Z \) is taken to be \( X \) in (3), then, the hypothesis testing problem in (3) collapses into testing whether the conditional QTE is a constant for all values of the covariates. Different from our setting, Crump et al. (2008) considered to testing whether the conditional ATE is a constant or zero for all values of the covariates. However, note that even though the conditional ATE is equal to a constant, the conditional QTE may not be a constant. Consequently, our paper complements and extends the paper by Crump et al. (2008) on testing whether there exists treatment effect heterogeneity across different sub-populations defined by covariates.

**Remark 2.** Besides the testing issues displayed in (3) and (4), one may be interested in testing

\[ H_0 : \Delta_r(z) \leq 0 \ (\text{or} \ \geq 0) \ \text{for all} \ z \in Z, \]

which leads to studying the stochastic dominance between \( Y(0) \) and \( Y(1) \) for all \( Z \). Recently, Lee, Song and Whang (2015, 2018) developed a general method for testing inequality
restrictions on nonparametric functions using a Bootstrap procedure. Hence, the procedure as in Lee et al. (2015, 2018) may be used here to test the null hypothesis formulated in (6) being true or not. These extensions are beyond the scope of this paper but certainly worth pursuing in future research.

2.2 Limiting Distribution of Test Statistic $J_n$

This subsection is devoted to investigating the asymptotic properties of the proposed test statistic $J_n$. Although the asymptotic theory for $J_n$ can be obtained for any $d$-dimensional $Z_i$ with $d \ll p$, the result is presented only for $d = 1$ to save notation throughout the rest of this paper. As pointed out by Abrevaya et al. (2015), the case for $d = 1$ is the most relevant case in practice. Before studying the asymptotic properties of the proposed test statistic $J_n$, the following technical assumptions are needed, list below.

Assumption 2. (Distributions of $X_i$ and $Z_i$) $X_i$ has a compact support $X$ and the density function of $X_i$, $f_X(x)$, satisfies $\inf_{x \in X} f_X(x) \geq c$ for some $c > 0$. Furthermore, the density function of $Z_i$, $f_Z(z)$ is twice continuously differentiable in $Z$.

Assumption 3. (i) The conditional density function $f_{Y(j)|X}(y|x)$ is continuous and bounded on the support of $Y_i(j)$ and $X_i$ for $j = 0$ and 1. (ii) The conditional density function $f_{Y(j)|Z}(y|z)$ is continuous and uniformly bounded away from zero in a neighborhood of $q_{j,r}(z)$ for $j = 0$ and 1. It is twice differentiable with respect to $z$, and its first derivative with respect to $y$ is continuous and bounded on the support of $Y_i(j)$ and $Z_i$.

Assumption 4. For $j = 0$ and 1, the conditional quantile function $q_{j,r}(z)$ is continuously differentiable on the support of $Z$ with bounded second order derivatives.

Assumption 5. (Kernel and bandwidth) (i) The kernel function $K(u)$ is a symmetric, continuously differentiable probability density function with compact support, $[-1,1]$, say. (ii) $h \to 0$, $nh^2 \to \infty$ and $nh^4 \to 0$ as $n \to \infty$.

Assumption 6. (Parametric propensity score function) Suppose the propensity score function has a parametric form $p(x) = p(x; \theta_0)$ with a fixed dimensional parameter $\theta_0$. Also, assume that the estimated propensity score function $\hat{p}_n(x) = p(x; \hat{\theta}_n)$ satisfies $\sup_{x \in X} |p(x; \hat{\theta}_n) - p(x; \theta_0)| = O_p(n^{-1/2})$. 

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The restriction imposed on the distribution of $X_i$ in Assumption 2 is commonly used in the literature on treatment effect evaluation; see Hirano et al. (2003), Abadie and Imbens (2006, 2016), Firpo (2007), Abrevaya et al. (2015), and among others. Assumption 3 guarantees the conditional quantile function $q_{j;\tau}(z)$ for $j = 0$ and $1$ is unique and well defined and the smoothness conditions imposed are easily satisfied in practice. The smoothness conditions on the conditional quantile function $q_{j;\tau}(z)$ for $j = 0$ and $1$ imposed in Assumption 4 are also easily satisfied in practice. Assumption 5 on kernel function and bandwidth is frequently assumed in the literature on nonparametric estimation. Many commonly used kernel functions, such as the Epanechnikov kernel, satisfy the requirements. Assumption 6 typically holds for standard parametric estimation methods under reasonably mild regularity conditions.

Under the assumptions listed above, we now can state our main result on the asymptotic properties of the proposed test statistic $J_n$ and its proof can be found in Appendix. For easy presentation, first, define some notations as follows. Let $\mu_0(z; u) = E\{[I\{Y_i(0) \leq q_0,\tau(u)\} - \tau]/[1 - p(X_i)] | Z_i = z\}$ and $\mu_1(z; u) = E\{[I\{Y_i(1) \leq q_1,\tau(u)\} - \tau]/p(X_i) | Z_i = z\}$. Then, we have the following asymptotic results.

**Theorem 1.** Suppose that Assumptions 1-6 are satisfied. Then, under the null hypothesis $H_0$ in (3), one has

$$n \sqrt{h} (J_n - \mu_J) \xrightarrow{D} N(0, \sigma_J^2),$$

where

$$\mu_J = \frac{\nu_0(K)}{nh} \int \left\{ \frac{\mu_1(z; z)}{f_{Y(1)|Z}(q_{1,\tau}(z)|Z)} + \frac{\mu_0(z; z)}{f_{Y(0)|Z}(q_{0,\tau}(z)|Z)} \right\} \frac{1}{f_Z(z)} dz,$$

and with $\nu_0(K) = \int K^2(u) du$,

$$\sigma_J^2 = 2 \int \left( \int K(t) K(t + s) dt \right)^2 ds \int \left\{ \frac{\mu_1(u; u)}{f_{Y(1)|Z}(q_{1,\tau}(u)|u)} + \frac{\mu_0(u; u)}{f_{Y(0)|Z}(q_{0,\tau}(u)|u)} \right\}^2 \frac{1}{f_Z^2(u)} du,$$

and under the alternative hypothesis $H_1$,

$$n \sqrt{h} (J_n - \mu_J) \xrightarrow{p} +\infty.$$  (7)
Following Theorem 1, an asymptotic significance level $\alpha_0$ test is to reject $H_0$ if $n\sqrt{h}(J_n - \mu_J)/\sigma_J > C_{\alpha_0}$, where $C_{\alpha_0}$ is the $\alpha_0$ upper-quantile of the standard normal distribution. Clearly, (7) implies that the proposed test is consistent. Note that to the best of our knowledge, the above asymptotic result for testing nonparametric QTE is new in the literature.

In addition to testing the null hypothesis against fixed alternatives, it is of interest to consider testing power for local departures from the null. Suppose that $\delta_\tau$ is estimated using (5), we focus on a set of Pitman alternatives represented by

$$H_{1n} : \Delta_\tau(z) = \delta_\tau + \rho_n \cdot \zeta(z),$$

where $\rho_n = n^{-1/2}h^{-1/4} \to 0$ as $n \to \infty$ and the function $\zeta(z)$ satisfies

$$\int \zeta(z)f_Z(z)\,dz = 0 \quad \text{and} \quad 0 < \int \zeta^2(z)\,dz < \infty.$$

The following theorem shows that our test can distinguish alternatives $H_{1n}$ that get closer to $H_0$ at rate $n^{-1/2}h^{-1/4}$ while maintaining a constant power level.

**Theorem 2.** Under Assumptions 1-6, suppose that the local alternative (8) converges to the null in the sense that $\rho_n = n^{-1/2}h^{-1/4}$. Then,

$$\frac{n\sqrt{h}}{\sigma_J}(J_n - \mu_J) \xrightarrow{D} \mathcal{N}\left(\sigma_J^{-1}\int \zeta^2(z)\,dz, 1\right).$$

Clearly, it follows from Theorem 2 that under the local alternative (8),

$$P \left( n\sqrt{h}(J_n - \mu_J)/\sigma_J > C_{\alpha_0} \right) \to 1 - \Phi \left( C_{\alpha_0} - \sigma_J^{-1}\int \zeta^2(z)\,dz \right),$$

where $\Phi$ is the cumulative distribution function of the standard normal distribution. This indicates that the testing power for local alternative (8) converges to a constant greater than the significance level $\alpha_0$. 

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2.3 A Nonparametric Bootstrap Test

Theorem 1 provides the asymptotic null distribution of the test $J_n$. Consequently, one can perform tests for $H_0$ by comparing the value of $J_n$ with its asymptotic critical value. However, as expected, it can not be used directly for an accurate calculation of critical values. This is because the test based on the asymptotic distribution might be sensitive to the choice of bandwidth $h$ and the consistent estimation of $\mu_J$ and $\sigma_J^2$ in small samples. In particular, it is well known in the quantile regression literature that the consistent estimation of the conditional density of $Y_i(j)$ given $Z_i$ involved in $\mu_J$ and $\sigma_J^2$ is not an easy task; see, for example, Koenker and Xiao (2004) and Cai and Xu (2008). To overcome this difficulty, following Chen et al. (2003) and Firpo et al. (2017), although other types of Bootstrap methods such as the multiplier Bootstrap in Escanciano and Goh (2014) and the wild Bootstrap in Dong et al. (2019) can be used, for simplicity, here a nonparametric Bootstrap procedure is proposed to determine the $p$-value for $J_n$. It involves the following steps.

1. Generate the $i$th Bootstrap sample by drawing samples from the original sample \{(Y_i, X_i, D_i)\}_{i=1}^n$ with replacement, denoted by \{(Y^*_i, X^*_i, D^*_i)\}_{i=1}^n$.

2. Compute the Bootstrap test statistic

$$J^*_n = \int \left((\hat{\Delta}^*_r(z) - \hat{\delta}^*_r) - (\hat{\Delta}_r(z) - \hat{\delta}_r)\right)^2 dz,$$

where $\hat{\Delta}^*_r(z)$ and $\hat{\delta}^*_r = \sum_{i=1}^n \hat{\Delta}^*_r(Z^*_i)/n$ are estimated using the Bootstrapping sample \{(Y^*_i, X^*_i, D^*_i)\}_{i=1}^n$, and $\hat{\Delta}_r(z)$ and $\hat{\delta}_r = \sum_{i=1}^n \hat{\Delta}_r(Z_i)/n$ are computed based on the original data.

3. Repeat steps (1) and (2) a large number of times, say, $B$ times, to obtain $\{J^{*(j)}_n\}_{j=1}^B$.

4. Reject $H_0$ at significance level $\alpha_0$ if $J_n$ exceeds the $(1 - \alpha_0)$-th sample quantile of $\{J^{*(j)}_n\}_{j=1}^B$.

Define $\bar{J}^*_n = n\sqrt{h}(J^*_n - \mu_J)/\sigma_J$. The following theorem justifies the asymptotic validity of the Bootstrap test with its proof given in Appendix.
Theorem 3. Suppose the same conditions as in Theorem 1 are satisfied. Then under $H_0$ or $H_1$, we have

$$\sup_{y \in \mathbb{R}} \left| \mathbb{P} \left( \mathcal{J}_n^* \leq y \mid \{Y_i, X_i, D_i\}_{i=1}^n \right) - \Phi(y) \right| = o_p(1),$$

where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal distribution.

Theorem 3 states that the Bootstrap statistic $\mathcal{J}_n^* = n\sqrt{h}(J_n^* - \mu_J)/\sigma_J$ converges to $\mathcal{N}(0, 1)$ in distribution in probability. It is important to note that Theorem 3 holds true regardless of whether the null hypothesis is true or not. Therefore, when the null hypothesis is true, the Bootstrap test procedure leads to asymptotically correct size, because conditional on the data, the Bootstrap statistic $J_n^*$ has the same asymptotic distribution as $J_n$. When the null hypothesis is false, because the test statistic $n\sqrt{h}(J_n - \mu_J)/\sigma_J$ diverges to $+\infty$ as the sample size $n$ goes to infinity as shown in Theorem 1, whereas the Bootstrap critical value is still finite, the Bootstrap procedure leads to a consistent test.

3 Monte Carlo Studies

In this section, we investigate the finite sample performance of the proposed test $J_n$ by means of simulation studies. The goal is to assess the accuracy and power of the proposed test for moderate sample sizes in various scenarios.

Let the data generating process (DGP) be:

$$Y(0) = \gamma_0 \sqrt{U_0} X_2 \quad \text{and} \quad Y(1) = \lambda \cdot \rho_n \cdot X_1 + \gamma_1 \sqrt{U_1} X_2,$$

where $\rho_n = n^{-1/2}h^{-1/4}$, $\gamma_0 = 1.0$, $\gamma_1 = 1.5$, $U_0$ and $U_1$ independently follow the $U[0, 1]$ distribution, $X_1$ and $X_2$ are independently generated from $U[-1, 1]$ and Beta(3,1), respectively, and the propensity score function is

$$P(D = 1|X_1, X_2) = \frac{\exp\{-0.5 + X_1 + X_2\}}{1 + \exp\{-0.5 + X_1 + X_2\}}.$$

Finally, the conditional variable $Z$ is taken to be $X_1$. Under this setting, by straightforward calculations, the conditional quantile function for $Y(j)$ for $j = 0$ and 1, conditional on $Z = z$, is given by $q_{0,r}(z) = \gamma_0 a_r$ and $q_{1,r}(z) = \lambda \rho_n z + \gamma_1 a_r$, respectively, where $a_r$ is the
unique solution of equation $-2a^3 + 3a^2 - \tau = 0$ within the interval $(0, 1)$. Therefore, the PCQTE is

$$\Delta_r(z) = \lambda \rho_n z + (\gamma_1 - \gamma_0)a_r,$$

where $\lambda$ in the above equation takes different values in the experiment so that we can investigate empirical sizes and local power curves of the test statistic $J_n$ indexed by $\lambda$. It is easy to see that $\Delta_r(z)$ is equal to a constant only when $\lambda = 0$, which corresponds to the null hypothesis. The Bootstrap procedure outlined in Section 2.3 is used to determine the critical value. The number of Bootstrap replications is set as $B = 599$. To examine the size and local power performance of the test statistic $J_n$, three different sample sizes $n = 400$, $n = 800$ and $n = 1600$ are considered. To check the sensitivity of the test with respect to different values of the bandwidth $h$, motivated by the conditions in Theorem 1, $h = cn^{-1/3}$ is used with $c = 0.5, 1.0$ and $2.0$. Finally, three quantiles levels, namely, $\tau = 0.25, 0.5$ and $\tau = 0.75$, are considered. The empirical sizes and local powers of the test $J_n$ are computed using 1,000 simulations under the nominal size $\alpha = 5\%$, respectively.

The empirical sizes of the test $J_n$ based on Bootstrap critical value are reported in Table 1. It can be seen that the empirical sizes converge to their nominal sizes as the sample size $n$ increases. Particularly, when the sample size increases to 1600, the test $J_n$ performs well in all cases considered. Also, one can observe that the choice of the bandwidth $h$ seems to have little influence on empirical sizes.

Next, Figures 1-3 display the estimated local power curves with nominal size $\alpha = 5\%$ of the test $J_n$ for different quantile levels and different choices of the bandwidth. In general, the test $J_n$ performs reasonably powerful in detecting the deviation from the null hypothesis in all cases considered. Specifically, it can be seen from these figures that the test $J_n$ has power against local alternatives converging to the null at the rate of $\rho_n = n^{-1/2}h^{-1/4}$. Moreover, it is not surprising that the powers increase quickly with the value of $\lambda$ increasing. It is also noticed from these figures that the bandwidth $h$ in a certain range seems to have little impact on the power of the test.
Table 1: Empirical sizes of $J_n$ (nominal size $\alpha = 5\%$)

<table>
<thead>
<tr>
<th>$\lambda = 0$</th>
<th>$n$</th>
<th>$\tau = 0.25$</th>
<th>$\tau = 0.5$</th>
<th>$\tau = 0.75$</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>400</td>
<td>0.030</td>
<td>0.034</td>
<td>0.031</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>0.040</td>
<td>0.037</td>
<td>0.045</td>
</tr>
<tr>
<td></td>
<td>1600</td>
<td>0.047</td>
<td>0.053</td>
<td>0.044</td>
</tr>
</tbody>
</table>

$h = 0.5n^{-1/3}$

<table>
<thead>
<tr>
<th>$\lambda = 0$</th>
<th>$n$</th>
<th>$\tau = 0.25$</th>
<th>$\tau = 0.5$</th>
<th>$\tau = 0.75$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>400</td>
<td>0.043</td>
<td>0.043</td>
<td>0.045</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>0.058</td>
<td>0.052</td>
<td>0.055</td>
</tr>
<tr>
<td></td>
<td>1600</td>
<td>0.044</td>
<td>0.053</td>
<td>0.053</td>
</tr>
</tbody>
</table>

$h = 1.0n^{-1/3}$

<table>
<thead>
<tr>
<th>$\lambda = 0$</th>
<th>$n$</th>
<th>$\tau = 0.25$</th>
<th>$\tau = 0.5$</th>
<th>$\tau = 0.75$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>400</td>
<td>0.034</td>
<td>0.036</td>
<td>0.038</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>0.056</td>
<td>0.056</td>
<td>0.054</td>
</tr>
<tr>
<td></td>
<td>1600</td>
<td>0.053</td>
<td>0.051</td>
<td>0.053</td>
</tr>
</tbody>
</table>

$h = 2.0n^{-1/3}$

Local Power Curves for Test Statistic $J_n$

Figure 1: Local power curves for test statistic $J_n$ with $\rho_n = n^{-1/2}h^{-1/4}$ and bandwidth $h = 0.5n^{-1/3}$ and nominal size $\alpha = 5\%$. 
Figure 2: Local power curves for test statistic $J_n$ with $\rho_n = n^{-1/2}h^{-1/4}$ and bandwidth $h = 1.0n^{-1/3}$ and nominal size $\alpha = 5\%$.

Figure 3: Local power curves for test statistic $J_n$ with $\rho_n = n^{-1/2}h^{-1/4}$ and bandwidth $h = 2.0n^{-1/3}$ and nominal size $\alpha = 5\%$. 

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4 A Real Example

In this section, the proposed testing approach is applied to investigating whether there exists heterogeneity for the QTE of maternal smoking during pregnancy on infant birth weight across different age groups of mothers. To this end, we use the same dataset as in Abrevaya et al. (2015), composed of vital statistics collected by the North Carolina State Center Health Services between 1988 and 2002, accessible through the Odum Institute at the University of North Carolina. As in Abrevaya et al. (2015), our sample is limited to first-time mothers and as routine in the literature, we also treat blacks and whites as separate populations throughout. The number of observations is 157,989 for the blacks group and 433,558 for the whites group.

It is generally recognized that low infant birth weight is associated with health and human capital development throughout life as argued by Black et al. (2007) and Almond and Currie (2011), and maternal smoking during pregnancy is considered to be the most important preventable negative cause of low birth weight; see Kramer (1987) for more discussions. Recently, there have been several studies in the literature to explore how the effect of maternal smoking during pregnancy on infant birth weight varies across different values of the mother’s ages by using program evaluation approach. In particular, Abrevaya et al. (2015) and Lee et al. (2017) considered the ATE of maternal smoking on infant birth weight conditional on different mothers’ ages, and found different degrees of heterogeneity by age. The main qualitative finding in Abrevaya et al. (2015) and Lee et al. (2017) is that smoking has a more severe impact at higher ages. Different from the studies by Abrevaya et al. (2015) and Lee et al. (2017), Cai et al. (2021) considered the QTE of mothers’ smoking status during pregnancy on infant birth weight conditional on different mothers' ages, and found that the QTEs for the quantile levels considered seem to change significantly over ages only for whites but not for blacks. Motivated by the estimation results in Cai et al. (2021), it is further interesting to test statistically whether or not the partially conditional QTEs, for whites and blacks, change over mothers’ ages. Therefore, our interest in this section is to test whether the QTE of maternal smoking on infant birth weight varies across different age groups of mothers using the proposed testing approach in Section 2.

Since our interest is in exploring whether the QTE of maternal smoking during preg-
nancy on infant birth weight changes across different age groups of mothers, hence the conditional variable $Z$ is the mother’s age. In addition, the treatment variable $D$ is a binary variable which takes value 1 if the mother smokes and 0 otherwise. The outcome variable of interest $Y$ is the infant birth weight measured in grams. Also, in this example, $Y(0)$ denotes the infant birth weight for the untreated (no-smoking) group and $Y(1)$ stands for the infant birth weight for the treated (smoking) group.

To explore the treatment effect heterogeneity of mothers’ smoking on infant birth weight using the proposed testing approach in Section 2, one needs to find certain baseline covariates such that the unconfoundedness assumption holds true, that is, the potential infant birth weight outcomes are independent of the smoking decision conditional on the baseline covariates. In this paper, we use the same set of covariates $X$ as in Abrevaya et al. (2015), which includes the mother’s age, education, month of first prenatal visit, number of prenatal visits, and indicators for the baby’s gender, the mother’s marital status, whether the father’s age is missing, gestational diabetes, hypertension, amniocentesis, taking ultrasound exams, previous (terminated) pregnancies, and alcohol use; see Abrevaya et al. (2015) for the detailed discussion.

To use the proposed testing approach, another problem is how to estimate the unknown propensity score function $p(x)$. Following Abrevaya et al. (2015) and Cai et al. (2021), here a logit model is used to estimate the propensity score function $p(x)$ and the explanatory variables used in the logit model consist of all the elements of $X$, the square of the mother’s age, and the interaction terms between the mother’s age and all other elements of $X$. Finally, the partially conditional QTE is estimated for mothers aged between 20 and 30 for both whites and blacks.

Table 2 displays the testing results for testing whether the partially conditional QTE changes over mother’s age. It can be seen clearly from Table 2 that one should reject the null hypothesis for whites for all quantiles considered at 5%. This means that PCQTEs do change over mother’s age for all quantile levels considered at the significance level $\alpha = 5\%$ for whites. But, for blacks, there is a strong evidence to support the homogeneity of PCQTE over mother’s age for all quantile levels considered. These testing results support strongly the empirical findings obtained in Cai et al. (2021).
Table 2: Test results for testing if PCQTE function changes over mother’s age.

<table>
<thead>
<tr>
<th>Quantile level</th>
<th>Test statistic $J_n$ (Bootstrap p-value)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Whites</td>
</tr>
<tr>
<td>$\tau$</td>
<td></td>
</tr>
<tr>
<td>0.10</td>
<td>0.022</td>
</tr>
<tr>
<td>0.25</td>
<td>0.002</td>
</tr>
<tr>
<td>0.50</td>
<td>0.033</td>
</tr>
<tr>
<td>0.75</td>
<td>0.035</td>
</tr>
<tr>
<td>0.90</td>
<td>0.042</td>
</tr>
</tbody>
</table>

5 Conclusion

Motivated by investigating whether or not the conditional QTEs conditional on mothers’ ages, for both whites and blacks, change over mothers’ ages, in other words, whether there exists heterogeneity for the QTEs of maternal smoking on infant birth weight across different age groups of mothers for whites and blacks, we propose a nonparametric versus constant test under quantile regression setting, which is applied to assessing whether there exists heterogeneously distributional effect for an intervention on outcome of interest across different sub-populations defined by covariates of interest. To test whether the null hypotheses of interest holds true or not, a consistent test statistic is proposed based on the Cramér-von Mises type criterion. To the best of our knowledge, it is believed that this test is novel in the QTE literature. Under some regular conditions, we establish the asymptotic distribution of the proposed test statistic under the null hypothesis and its consistency against fixed alternatives. We also study the power of our test against a sequence of local alternatives. Also, we propose a Bootstrap procedure to approximate the finite-sample null distribution of the proposed test. Furthermore, the asymptotic validity of the proposed Bootstrap test is also established.

Finally, some extensions of our paper can be considered. For example, the first is to consider the hypothesis formulated in (4) which is a test for nonparametric versus parametric. Second, one might be interesting in extending our results to time series cases, which has a potential in a wide range of applications. Such extensions can be warranted as a future research.
References


Appendix: Mathematical Proofs

Note that this appendix provides some key steps for proving Theorems 1, 2 and 3, together with some auxiliary lemmas with their detailed proofs as well as some notations given in Supplement.

**Proof of Theorem 1:** Let $\Delta_r(z)$ be the partially conditional QTE conditional on $Z_i = z$ and let $\delta_r = \int \Delta_r(z)f_Z(z)\,dz$. Then,

$$J_n = \int \left(\Delta_r(z) - \hat{\delta}_r\right)^2\,dz = \int \left[\left(\Delta_r(z) - \Delta_r(z)\right) + \left(\delta_r - \hat{\delta}_r\right) + \left(\Delta_r(z) - \delta_r\right)\right]^2\,dz,$$

where $\hat{\delta}_r = \frac{1}{n} \sum_{i=1}^n \Delta_r(Z_i)$. Following the proof of Theorem 1 in Cai and Xiao (2012), it is easy to show that $\hat{\delta}_r$ is a $\sqrt{n}$-consistent estimate of $\delta_r$ under Assumptions 1-6. Under the null hypothesis $H_0$, $\Delta_r(z) - \delta_r \equiv 0$, thus, by Lemma 3,

$$J_n = \int \left(\frac{1}{n} \sum_{i=1}^n \gamma_n(Y_i, X_i, D_i; z) + e_n\right)^2\,dz = \int \left(\frac{1}{n} \sum_{i=1}^n \gamma_n(Y_i, X_i, D_i; z)\right)^2\,dz + e_n^2 + 2e_n \int \frac{1}{n} \sum_{i=1}^n \gamma_n(Y_i, X_i, D_i; z)\,dz,$$

where $\gamma_n(Y_i, X_i, D_i; z) = \varrho_{n,1}\varrho(Y_i, X_i, D_i; z) - E\varrho_{n,1}\varrho(Y_i, X_i, D_i; z) - \varrho_{n,0}\varrho(Y_i, X_i, D_i; z) + E\varrho_{n,0}\varrho(Y_i, X_i, D_i; z)$ and $e_n = O_p\left(\max\left\{\frac{\ln n}{\sqrt{n}}, \frac{\ln n}{(nh)^{1/2}}\right\}\right)$. It is easy to verify that $n\sqrt{h}J_n,2 = o_p(1)$. Also, by noting that $E(\gamma_n(Y_i, X_i, D_i; z)) = 0$, we have

$$n\sqrt{h}J_n,3 = n\sqrt{h} \cdot e_n \cdot \frac{1}{n} \sum_{i=1}^n \int \gamma_n(Y_i, X_i, D_i; z)\,dz = n\sqrt{h} \cdot e_n \cdot O_p(n^{-1/2}) = o_p(1).$$

Thus, an application of Lemma 5 leads to

$$n\sqrt{h}(J_n - \mu_j) = n\sqrt{h}(J_{n,1} - \mu_j + J_{n,2} + J_{n,3}) \xrightarrow{D} \mathcal{N}(0, \sigma_j^2).$$

Now, we consider the case under the alternative hypothesis $H_1$. Under $H_1$, it is easy to show that $J_n - \mu_j = \int (\Delta_r(z) - \delta_r)^2\,dz + o_p(1)$. Since $\int (\Delta_r(z) - \delta_r)^2\,dz$ is a positive constant under $H_1$, so that

$$n\sqrt{h}(J_n - \mu_j) \xrightarrow{p} +\infty.$$
This completes the proof of Theorem 1. □

**Proof of Theorem 2:** Under the local alternative $H_{1n} : \Delta_r(z) = \delta_r + \rho_n \cdot \zeta(z)$ with $ho_n = n^{-1/2}h^{-1/4}$, we have

$$J_n = \int \left( \hat{\Delta}_r(z) - \Delta_r(z) \right)^2 dz = \int \left[ \left( \hat{\Delta}_r(z) - \Delta_r(z) \right) + \left( \delta_r - \hat{\delta}_r \right) + \left( \Delta_r(z) - \delta_r \right) \right]^2 dz$$

$$= \int \left[ \left( \hat{\Delta}_r(z) - \Delta_r(z) \right) + O_p(1/\sqrt{n}) + \rho_n \cdot \zeta(z) \right]^2 dz$$

$$= \int \left( \hat{\Delta}_r(z) - \Delta_r(z) \right)^2 dz + \rho_n^2 \cdot \int \zeta^2(z) dz + O_p\left( \frac{1}{n} \right) + 2O_p\left( \frac{1}{\sqrt{n}} \right) \cdot \int \left( \hat{\Delta}_r(z) - \Delta_r(z) \right) dz$$

$$+ 2O_p\left( \frac{1}{\sqrt{n}} \right) \cdot \rho_n \cdot \int \zeta(z) dz + \rho_n \cdot \int \zeta(z) \left( \hat{\Delta}_r(z) - \Delta_r(z) \right) dz$$

$$:= J_n^{(1)} + J_n^{(2)} + J_n^{(3)} + J_n^{(4)} + J_n^{(5)} + J_n^{(6)}.$$

By noting that $\rho_n = n^{-1/2}h^{-1/4}$,

$$\int \left( \hat{\Delta}_r(z) - \Delta_r(z) \right) dz = \int \frac{1}{n} \sum_{i=1}^{n} \gamma_n(Y_i, X_i, D_i; z) dz + O_p \left( \max \left\{ \frac{\ln n}{\sqrt{n}}, \frac{\sqrt{\ln n}}{(nh)^{3/4}} \right\} \right)$$

$$= O_p\left( \frac{1}{\sqrt{n}} \right) + O_p \left( \max \left\{ \frac{\ln n}{\sqrt{n}}, \frac{\sqrt{\ln n}}{(nh)^{3/4}} \right\} \right),$$

and

$$\int \zeta(z) \left( \hat{\Delta}_r(z) - \Delta_r(z) \right) dz = O_p\left( \frac{1}{\sqrt{n}} \right) + O_p \left( \max \left\{ \frac{\ln n}{\sqrt{n}}, \frac{\sqrt{\ln n}}{(nh)^{3/4}} \right\} \right),$$

it is easy to show that $n\sqrt{h}J_n^{(k)} = o_p(1)$ for $3 \leq k \leq 6$. Then, using the result in Theorem 1, we have

$$n\sqrt{h}(J_n - \mu_J) = n\sqrt{h}(J_n^{(1)} - \mu_J) + n\sqrt{h}J_n^{(2)} + o_p(1) \xrightarrow{D} \mathcal{N} \left( \int \zeta^2(z) dz, \sigma_J^2 \right).$$

This completes the proof of Theorem 2. □

Now, before considering the proof of Theorem 3, first, let $P$ denote the distribution of $\{(Y_i(0), Y_i(1), X_i, D_i)\}_{i=1}^{n}$ and use $P^*$ to denote the Bootstrap distribution, which is the distribution of $\{(Y_i^*, X_i^*, D_i^*)\}_{i=1}^{n}$, conditional on $\{(Y_i, X_i, D_i)\}_{i=1}^{n}$. Also, we use $E^*$ and $\text{Var}^*$ to denote the expectation and variance with respect to $P^*$, respectively. Furthermore, following Lee et al. (2015), let $S_1, S_2, \cdots$ be a sequence a random variables and $a_1, a_2, \cdots$
Lemma 10, we know that 

By using Lemma 10 again, we also know that 

\( Q \) in distribution in probability. For the term 

\( 0 \) and 

\( \epsilon > 0 \), there exists \( M > 0 \) such that 

\( \limsup_{n \to \infty} P\{P^*(|S_n/a_n| > M) > \epsilon\} < \epsilon \).

**Proof of Theorem 3:** It is easy to see that we have the following decomposition

\[
J_n^* = \int \left( (\hat{\Delta}_r^*(z) - \hat{\delta}_r^*) - (\Delta_r(z) - \delta_r) \right)^2 dz
\]

\[
= \int (\hat{\Delta}_r^*(z) - \Delta_r(z))^2 dz + \int (\hat{\delta}_r^* - \delta_r)^2 dz - 2 \int (\hat{\delta}_r^* - \delta_r)(\hat{\Delta}_r^*(z) - \Delta_r(z)) dz
\]

\[
= \int (\hat{\Delta}_r^*(z) - \Delta_r(z))^2 dz + O_{p^*}(1/n) - 2O_{p^*}(1/\sqrt{n}) \int (\hat{\Delta}_r^*(z) - \Delta_r(z)) dz
\]

\[
:= Q_{n,1} + Q_{n,2} + Q_{n,3}.
\]

For the term \( Q_{n,1} \), we have

\[
Q_{n,1} = \int (\hat{\Delta}_r^*(z) - \Delta_r(z))^2 dz
\]

\[
= \int \left( \frac{1}{n} \sum_{i=1}^{n} \left( \psi_{n,1,r}(R_i^*; z) - \psi_{n,0,r}(R_i^*; z) \right) \right)^2 dz + O_{p^*}(\eta_n^2)
\]

\[
+ 2O_{p^*}(\eta_n) \cdot \int \left[ \frac{1}{n} \sum_{i=1}^{n} \left( \psi_{n,1,r}(R_i^*; z) - \psi_{n,0}(R_i^*; z) \right) \right] dz
\]

\[
:= Q_{n,1}^{(1)} + Q_{n,1}^{(2)} + Q_{n,1}^{(3)}.
\]

where \( \eta_n = \max \left\{ \frac{\ln n}{\sqrt{n}}, \frac{\sqrt{\ln n}}{n} \right\} \). It is easy to check that 

\( n\sqrt{h}Q_{n,1}^{(2)} = o_{p^*}(1) \). Also, from Lemma 10, we know that 

\( n\sqrt{h}Q_{n,1}^{(3)} = o_{p^*}(1) \). Therefore, an application of Lemma 9 leads to

\( n\sqrt{h}(Q_{n,1} - \mu_J)/\sigma_J = n\sqrt{h}(Q_{n,1}^{(1)} - \mu_J)/\sigma_J + o_{p^*}(1) \to \mathcal{N}(0, 1) \)

in distribution in probability. For the term \( Q_{n,2} \), it is easy to see that 

\( n\sqrt{h}Q_{n,2} = o_{p^*}(1) \). By using Lemma 10 again, we also know that 

\( n\sqrt{h}Q_{n,3} = o_{p^*}(1) \). Finally, we have

\( n\sqrt{h}(J_n^* - \mu_J)/\sigma_J = n\sqrt{h}(Q_{n,1} - \mu_J)/\sigma_J + o_{p^*}(1) \to \mathcal{N}(0, 1) \)

in distribution in probability. Because \( \mathcal{N}(0, 1) \) is a continuous distribution, by Polyá’s theorem in Bhattacharya and Rao (1986), we obtain Theorem 3. □
SUPPLEMENTARY MATERIAL

This supplement provides some lemmas for proving the main theorems in the paper, entitled “A Nonparametric Test for Testing Heterogeneity in Conditional Quantile Treatment Effects”.

Define $W_0(X_i, D_i) = \frac{1-D_i}{1-p(X_i)}$ and $W_1(X_i, D_i) = \frac{D_i}{p(X_i)}$. We also let $\hat{W}_{n,0}(X_i, D_i) = \frac{1-D_i}{1-\hat{p}_n(X_i)}$ and $\hat{W}_{n,1}(X_i, D_i) = \frac{D_i}{\hat{p}_n(X_i)}$, where $\hat{p}_n(x) = p(x; \theta_n)$ is the parametric estimate of the propensity score function $p(x)$ using $(X_i, D_i), i = 1, \cdots, n$. To prove Theorem 1, we first provide the following lemmas.

**Lemma 1.** Suppose Assumptions 2.1-2.5 hold. Then,

$$\sup_{z \in Z} \left| \bar{q}_{j, \tau}(z) - q_{j, \tau}(z) - \frac{1}{n} \sum_{i=1}^{n} \left[ \epsilon_{n,j, \tau}(Y_i, X_i, D_i; z) - E\epsilon_{n,j, \tau}(Y_i, X_i, D_i; z) \right] \right| = O_p \left( \frac{\sqrt{\ln n}}{(nh)^{3/4}} \right)$$

for $j = 0$ and 1, where

$$\bar{q}_{j, \tau}(z) = \arg \min_{q} \sum_{i=1}^{n} W_j(X_i, D_i) h K_{h, i}(z) \rho_{\tau}(Y_i; q)$$

and

$$\epsilon_{n,j, \tau}(Y_i, X_i, D_i; z) = -S_{n,j, \tau}^{-1}(z) W_j(X_i, D_i) h K_{h, i}(Z_i - z) \varphi_{\tau}(Y_i; q_{j, \tau}(z))$$

with $\rho_{\tau}(y; q) = \rho_{\tau}(y - q) = (y - q)(\tau - I\{y \leq q\})$, $\varphi_{\tau}(y; \theta) = \tau - I\{y \leq \theta\}$, $K_{h, i}(z) = K((Z_i - z)/h)/h$ and

$$S_{n,j, \tau}(z) = \int K(u) f_{Y(j)|Z}(q_{j, \tau}(z)|z+hu) f_Z(z+hu) du = f_{Y(j)|Z}(q_{j, \tau}(z)|z) f_Z(z) + O(h^2), j = 0, 1.$$

**Proof of Lemma 1:** This result can be proved following the proof of Theorem 1 in Lee et al. (2015). □

**Lemma 2.** Suppose that Assumptions 2.1-2.6 are satisfied, then

$$\sup_{z \in Z} \left| \bar{q}_{j, \tau}(z) - \tilde{q}_{j, \tau}(z) \right| = O_p \left\{ \max \left\{ \frac{\ln n}{\sqrt{n}}, \frac{\sqrt{\ln n}}{(nh)^{3/4}} \right\} \right\}, \quad j = 0, 1.$$
Proof of Lemma 2: For \( j = 0 \) and \( 1 \), define cumulative distribution functions

\[
F_{n,j}(y \mid z) = \frac{\sum_{i=1}^{n} K_{h,i}(z) W_{j}(X_i, D_i) I\{Y_i \leq y\}}{\sum_{i=1}^{n} K_{h,i}(z) W_{j}(X_i, D_i)}
\]

and

\[
\hat{F}_{n,j}(y \mid z) = \frac{\sum_{i=1}^{n} K_{h,i}(z) \hat{W}_{n,j}(X_i, D_i) I\{Y_i \leq y\}}{\sum_{i=1}^{n} K_{h,i}(z) \hat{W}_{n,j}(X_i, D_i)}.
\]

Then \( \tilde{q}_{j,\tau^*}(z) = \inf\{y : \hat{F}_{n,j}(y \mid z) \geq \tau^*\} \) and \( \hat{q}_{j,\tau^*}(z) = \inf\{y : \hat{F}_{n,j}(y \mid z) \geq \tau^*\} \) for \( 0 < \tau^* < 1 \). By the definition of quantile, we have

\[
\left| \hat{F}_{n,j}(\tilde{q}_{j,\tau^*}(z) \mid z) - \tau^* \right| \leq \max_{i=1, \ldots, n} \left\{ \frac{K_{h,i}(z) W_{j}(X_i, D_i) I\{Y_i \leq y\}}{\sum_{i=1}^{n} K_{h,i}(z) W_{j}(X_i, D_i)} \right\}
\]

for any \( 0 < \tau^* < 1 \), hence \( \hat{F}_{n,j}(\tilde{q}_{j,\tau^*}(z) \mid z) = \tau^* + O_p(1/\sqrt{n}) \). By using \( \sup_{x \in \mathcal{X}} |p(x, \hat{\theta}_n) - p(x, \theta_0)| = O_p(n^{-1/2}) \) as in Assumption 2.6, it is also easy to show that

\[
\sup_{y \in \mathcal{Y}_j} \sup_{z \in \mathcal{Z}} \left| \hat{F}_{n,j}(y \mid z) - \check{F}_{n,j}(y \mid z) \right| = O_p(n^{-1/2}),
\]

where \( \mathcal{Y}_j \) is the support of \( Y(j) \). Let \( c_n = \max\left\{ \frac{\ln n}{\sqrt{n}}, \frac{(\ln n)^2}{n} \right\} \). Then,

\[
\hat{F}_{n,j}(\tilde{q}_{j,\tau+c_n}(z) \mid z) = \hat{F}_{n,j}(\check{q}_{j,\tau+c_n}(z) \mid z) + O_p(1/\sqrt{n})
\]

\[
= \tau + c_n + O_p(1/\sqrt{n}) + O_p(1/\sqrt{n}) > \tau \tag{S.1}
\]

in probability as \( n \to \infty \). Similarly,

\[
\hat{F}_{n,j}(\check{q}_{j,\tau-c_n}(z) \mid z) = \hat{F}_{n,j}(\check{q}_{j,\tau-c_n}(z) \mid z) + O_p(1/\sqrt{n})
\]

\[
= \tau - c_n + O_p(1/\sqrt{n}) + O_p(1/\sqrt{n}) < \tau \tag{S.2}
\]

in probability as \( n \to \infty \). Combining (S.1) and (S.2), we have

\[
P(\tilde{q}_{j,\tau-c_n}(z) \leq \check{q}_{j,\tau}(z) \leq \tilde{q}_{j,\tau+c_n}(z)) \to 1 \text{ as } n \to \infty.
\]
Also, one has \( \bar{q}_{j,\tau - c_n}(z) \leq \bar{q}_{j,\tau}(z) \leq \bar{q}_{j,\tau + c_n}(z) \) by definition, and it follows that

\[
P \left( \sup_{z \in \mathcal{Z}} |\bar{q}_{j,\tau}(z) - \bar{q}_{j,\tau}(z)| \leq \sup_{z \in \mathcal{Z}} |\bar{q}_{j,\tau + c_n}(z) - \bar{q}_{j,\tau - c_n}(z)| \right) \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty.
\]

Next, we consider the order of \( \sup_{z \in \mathcal{Z}} |\bar{q}_{j,\tau + c_n}(z) - \bar{q}_{j,\tau - c_n}(z)| \). Recall that \( S_{n,j,\tau,\ast}(z) = \int K(u) f_{Y(j)|Z}(q_{j,\tau,\ast}(z)|z + hu) f_Z(z + hu) du \) for \( j = 0 \) and \( 1 \). It is easy to show that \( \sup_{z \in \mathcal{Z}} |S_{n,j,\tau,\ast}(z) - S_{n,j,\tau}(z)| = O(c_n) \). By using the Bahadur representation of \( \bar{q}_{j,\tau,\ast}(z) \) provided by Lemma 1, it follows that

\[
\begin{align*}
\sup_{z \in \mathcal{Z}} |\bar{q}_{j,\tau + c_n}(z) - \bar{q}_{j,\tau - c_n}(z)| &\leq \sup_{z \in \mathcal{Z}} |q_{j,\tau + c_n}(z) - q_{j,\tau - c_n}(z)| + O_p \left\{ \frac{\sqrt{\ln n}}{(nh)^{3/4}} \right\} \\
&+ \sup_{z \in \mathcal{Z}} \left| S_{n,j,\tau,\ast}(z) - S_{n,j,\tau}(z) \right| \\
&+ \sup_{z \in \mathcal{Z}} \left| S_{n,j,\tau}(z) E \left[ K_{h,i}(z) W_j(X_i, D_i) (I\{Y_i \leq q_{j,\tau - c_n}(z)\} - I\{Y_i \leq q_{j,\tau - c_n}(z)\}) \right] \right| \\
&+ O(c_n) + O_p \left\{ \frac{\sqrt{\ln n}}{(nh)^{3/4}} \right\} \\
&:= M_1 + M_2 + M_3 + O_p(c_n) + O_p \left\{ \frac{\sqrt{\ln n}}{(nh)^{3/4}} \right\}.
\end{align*}
\]

For \( M_1 \), note that \( F_{Y(j)|Z}(q_{j,\tau + c_n}(z)|z) = \tau + c_n \) and \( F_{Y(j)|Z}(q_{j,\tau - c_n}(z)|z) = \tau - c_n \) under Assumption 2.3. Thus,

\[
2c_n = F_{Y(j)|Z}(q_{j,\tau + c_n}(z)|z) - F_{Y(j)|Z}(q_{j,\tau - c_n}(z)|z) = f_{Y(j)|Z}(q_n^*|z) (q_{j,\tau - c_n}(z) - q_{j,\tau - c_n}(z)),
\]

where \( q_n^* \) is a point between \( q_{j,\tau - c_n}(z) \) and \( q_{j,\tau + c_n}(z) \), which implies that

\[
M_1 = \sup_{z \in \mathcal{Z}} |q_{j,\tau + c_n}(z) - q_{j,\tau - c_n}(z)| = O(c_n)
\]

by the assumption that \( f_{Y(j)|Z}(q|z) \) is uniformly bounded away from zero in a neighbor-
hood of $q_{j,\tau}(z)$. For $\mathcal{M}_2$, since $q_{j,\tau-c_n}(z) \leq q_{j,\tau}(z) \leq q_{j,\tau+c_n}(z)$ and $\sup_{z \in \mathcal{Z}} |q_{j,\tau+c_n}(z) - q_{j,\tau-c_n}(z)| = O(c_n)$, there exists a constant $A$ which does not rely on $z$, such that

$$\mathcal{M}_2 \leq \sup_{z \in \mathcal{Z}} \left| S_{n,j,\tau}^{-1}(z) \frac{1}{n} \sum_{i=1}^{n} K_{h,i}(z) W_j(X_i, D_i) I\{q_{j,\tau}(z) - Ac_n \leq Y_i \leq q_{j,\tau}(z) + Ac_n\} \right|$$

$$\leq \sup_{z \in \mathcal{Z}} \left| S_{n,j,\tau}^{-1}(z) \frac{1}{n} \sum_{i=1}^{n} \left[ K_{h,i}(z) W_j(X_i, D_i) I\{q_{j,\tau}(z) - Ac_n \leq Y_i \leq q_{j,\tau}(z) + Ac_n\} \right. \right.$$ 

$$- \left. E\left(K_{h,i}(z) W_j(X_i, D_i) I\{q_{j,\tau}(z) - Ac_n \leq Y_i \leq q_{j,\tau}(z) + Ac_n\}\right) \right|$$

$$+ \sup_{z \in \mathcal{Z}} \left| S_{n,j,\tau}^{-1}(z) E\left(K_{h,i}(z) W_j(X_i, D_i) I\{q_{j,\tau}(z) - Ac_n \leq Y_i \leq q_{j,\tau}(z) + Ac_n\}\right) \right|$$

$$:= \mathcal{M}_{2,1} + \mathcal{M}_{2,2}.$$

Using Assumption 2.3, we have that

$$E\left[ \left( K((Z_i - z)/h) W_j(X_i, D_i) I\{q_{j,\tau}(z) - Ac_n \leq Y_i \leq q_{j,\tau}(z) + Ac_n\} \right)^2 \right]$$

$$= E\left\{ K^2((Z_i - z)/h) p(X_i)^{-1}(1 - p(X_i))^{-1} E\left[I\{q_{j,\tau}(z) - Ac_n \leq Y_i(j) \leq q_{j,\tau}(z) + Ac_n\} | X_i \right] \right\}$$

$$= O(h c_n).$$

Also, note that $\{K((Z_i - z)/h) W_j(X_i, D_i) I\{q_{j,\tau}(z) - Ac_n \leq Y_i \leq q_{j,\tau}(z) + Ac_n\} : z \in \mathcal{Z}\}$ is Euclidean for a constant envelope, which together with $\frac{\ln n}{nh c_n} = o(1)$ implies the conditions required by Theorem II.37 of Pollard (1984) are met. Hence, by Theorem II.37 of Pollard (1984),

$$\sup_{z \in \mathcal{Z}} \left| \frac{1}{n} \sum_{i=1}^{n} K((Z_i - z)/h) W_j(X_i, D_i) I\{q_{j,\tau}(z) - Ac_n \leq Y_i \leq q_{j,\tau}(z) + Ac_n\} \right.$$ 

$$- \left. E\left(K((Z_i - z)/h) W_j(X_i, D_i) I\{q_{j,\tau}(z) - Ac_n \leq Y_i \leq q_{j,\tau}(z) + Ac_n\}\right) \right| = o_p(h c_n),$$

together with the fact $S_{n,j,\tau}(z)$ is bounded away from zero, we have that $\mathcal{M}_{2,1} = \frac{1}{n} \cdot o_p(h c_n) = o_p(c_n)$. It is also easy to show that $\mathcal{M}_{2,2} = O(c_n)$. Hence, $\mathcal{M}_2 = O_p(c_n)$. Similar to the proof of $\mathcal{M}_{2,2} = O(c_n)$, we can also show that $\mathcal{M}_3 = O(c_n)$. Therefore,

$$\sup_{z \in \mathcal{Z}} \left| \tilde{q}_{j,\tau+c_n}(z) - \tilde{q}_{j,\tau-c_n}(z) \right| = O_p(c_n) + O_p\left\{ \frac{\sqrt{\ln n}}{(nh)^{3/4}} \right\}. $$
Together with (5), we have

$$
\sup_{z \in \mathbb{Z}} |\hat{q}_{j,\tau}(z) - \bar{q}_{j,\tau}(z)| = O_p(c_n) + O_p\left\{ \frac{\sqrt{\ln n}}{(nh)^{3/4}} \right\} = O_p\left\{ \max\left\{ \frac{\ln n}{\sqrt{n}}, \frac{\sqrt{\ln n}}{(nh)^{3/4}} \right\} \right\}.
$$

This completes the proof. □

**Lemma 3.** Under Assumptions 2.1-2.6 and \( nh^2 \to \infty \), we have

$$
\sup_{z \in \mathbb{Z}} |\hat{q}_{j,\tau}(z) - q_{j,\tau}(z)| - \frac{1}{n} \sum_{i=1}^{n} \left[ q_{n,j,\tau}(Y_i, X_i, D_i; z) - E(q_{n,j,\tau}(Y_i, X_i, D_i; z)) \right] = O_p\left\{ \max\left\{ \frac{\ln n}{\sqrt{n}}, \frac{\sqrt{\ln n}}{(nh)^{3/4}} \right\} \right\}
$$

for \( j = 0 \) and 1.

**Proof of Lemma 3:** The result comes from Lemma 1 and Lemma 2. □

**Lemma 4.** Let \( R_1, R_2, \ldots \) be an i.i.d. sequence. Suppose that the U-statistic \( U_n = \sum_{1 \leq i < j \leq n} H_n(R_i, R_j) \) with symmetric variable function \( H_n \) is centered (i.e., \( E[H_n(R_1, R_2)] = 0 \)) and degenerated (i.e., \( E[H_n(R_1, R_2)|R_1 = z_1] = 0 \) almost surely for all \( z_1 \)). Then, if

$$
\lim_{n \to \infty} \frac{E[E^2[H_n(R_1, R_3)H_n(R_2, R_3) | R_1, R_2]] + n^{-1}E[H_n^4(R_1, R_2)]}{E^2[H_n^2(R_1, R_2)]} = 0,
$$

we have that as \( n \to \infty \),

$$
\frac{2^{1/2}}{n\sigma_n} U_n \xrightarrow{d} \mathcal{N}(0, 1),
$$

where \( \sigma_n^2 = E[H_n^2(R_1, R_2)] \).

**Proof of Lemma 4:** The result is given by Theorem 1 in Hall (1984). □

**Lemma 5.** Suppose the conditions required by Theorem 1 are satisfied. Then,

$$
n\sqrt{h} \int \left[ \frac{1}{n} \sum_{i=1}^{n} \left( q_{n,1,\tau}(Y_i, X_i, D_i; z) - E(q_{n,1,\tau}(Y_i, X_i, D_i; z)) \right) - q_{n,0,\tau}(Y_i, X_i, D_i; z) + E(q_{n,0,\tau}(Y_i, X_i, D_i; z)) \right]^2 dz - \mu_J \right] \xrightarrow{d} \mathcal{N}(0, \sigma_J^2),
$$

where

$$
\mu_J = \frac{1}{nh} \int K^2(s)ds \int \left\{ \frac{\mu_1(z; z)}{f_{Y(1)|Z}(q_{1,\tau}(z)|z)} + \frac{\mu_0(z; z)}{f_{Y(0)|Z}(q_{0,\tau}(z)|z)} \right\} \frac{1}{f_Z(z)} dz,
$$

S5
and

$$\sigma_j^2 = 2 \int \left( \int K(t)K(t+s)dt \right)^2 ds \int \left\{ \frac{\mu_1(u;u)}{f_Y(1,z)(q_{1,r}(u)|u) - q_{0,r}(u)|u)} + \frac{\mu_0(u;u)}{f_Y(0,z)(q_{0,r}(u)|u)} \right\}^2 \frac{1}{f_Z(u)} du,$$

with

$$\mu_0(z;u) = E \left[ \frac{1}{1 - p(X_i)} (I\{Y_i(0) \leq q_{0,r}(u)\} - \tau)^2 | Z_i = z \right],$$

and

$$\mu_1(z;u) = E \left[ \frac{1}{p(X_i)} (I\{Y_i(1) \leq q_{1,r}(u)\} - \tau)^2 | Z_i = z \right].$$

**Proof of Lemma 5:** For simplicity, we let

$$\gamma_n(Y_i, X_i, D_i; z) = \varrho_{n,1,r}(Y_i, X_i, D_i; z) - E\varrho_{n,1,r}(Y_i, X_i, D_i; z) - \varrho_{n,0,r}(Y_i, X_i, D_i; z) + E\varrho_{n,0,r}(Y_i, X_i, D_i; z).$$

Then,

$$\int \left( \frac{1}{n} \sum_{i=1}^{n} \gamma_n(Y_i, X_i, D_i; z) \right)^2 dz = 2n^{-2} \sum_{1 \leq i < k \leq n} \int \gamma_n(Y_i, X_i, D_i; z) \gamma_n(Y_k, X_k, D_k; z) dz + n^{-2} \sum_{i=1}^{n} \int \gamma_n^2(Y_i, X_i, D_i; z) dz =: I_{n,1} + I_{n,2}.$$  \hspace{1cm} (S.3)

First, we consider the term $I_{n,1}$. Let $R_i = (Y_i, X_i, D_i)$ and define

$$H_n(R_i, R_k) = \frac{2}{n^2} \int \gamma_n(R_i; z) \gamma_n(R_k; z) dz.$$  \hspace{1cm} (S.4)

Then, $I_{n,1} = \sum_{1 \leq i < k \leq n} H_n(R_i, R_k)$ is a centered and degenerated $U$-statistic. Thus,
It is easy to find that

\[ E \left[ S_{n,j,r}(z)^{-1}W_j(X_i, D_i)\varphi_r(Y_i; q_{j,r}(z)) | Z_i = z \right] = 0. \]

Hence,

\[
E \left[ \varrho_{n,j,r}(Y_i, X_i, D_i; z) \right] = 0 - \frac{1}{S_{n,j,r}(z)} f_Z(z) \frac{\partial F_{Y(j)|Z}(q_{j,r}(z)|u)}{\partial u} \bigg|_{u=z} h^2 \int s^2 K(s) ds + o(h^2) = O(h^2)
\]

uniformly in \( z \) for \( j = 0 \) and 1. Also note that \( D_i(1 - D_i) = 0 \), then,

\[
E \left[ \gamma_n(R_i; u) \gamma_n(R_i; v) \right] = E \left[ \left( \varrho_{n,1,r}(R_i; u) - \varrho_{n,0,r}(R_i; u) \right) \left( \varrho_{n,1,r}(R_i; v) - \varrho_{n,0,r}(R_i; v) \right) \right]
\]

\[
- E \left[ \varrho_{n,1,r}(R_i; u) \varrho_{n,0,r}(R_i; u) \right] E \left[ \varrho_{n,1,r}(R_i; v) \varrho_{n,0,r}(R_i; v) \right]
\]

\[
= S_{n,1,r}^{-1}(u) S_{n,1,r}^{-1}(v) E \left[ K_h(Z_i - u) K_h(Z_i - v) \frac{D_i}{p^2(X_i)} \varphi_r(Y_i; q_{1,r}(u))\varphi_r(Y_i; q_{1,r}(v)) \right]
\]

\[
+ S_{n,0,r}^{-1}(u) S_{n,0,r}^{-1}(v) E \left[ K_h(Z_i - u) K_h(Z_i - v) \frac{1 - D_i}{(1 - p(X_i))^2} \varphi_r(Y_i; q_{0,r}(u))\varphi_r(Y_i; q_{0,r}(v)) \right] + O(h^4)
\]

\[
= S_{n,1,r}^{-1}(u) S_{n,1,r}^{-1}(v) E \left[ K_h(Z_i - u) K_h(Z_i - v) \frac{1}{p(X_i)} \varphi_r(Y_i; q_{1,r}(u))\varphi_r(Y_i; q_{1,r}(v)) \right]
\]

\[
+ S_{n,0,r}^{-1}(u) S_{n,0,r}^{-1}(v) E \left[ K_h(Z_i - u) K_h(Z_i - v) \frac{1}{1 - p(X_i)} \varphi_r(Y_i; q_{0,r}(u))\varphi_r(Y_i; q_{0,r}(v)) \right] + O(h^4)
\]

\[
= S_{n,1,r}^{-1}(u) S_{n,1,r}^{-1}(v) E \left[ K_h(Z_i - u) K_h(Z_i - v) \kappa_1(Z_i; u, v) \right]
\]

\[
+ S_{n,0,r}^{-1}(u) S_{n,0,r}^{-1}(v) E \left[ K_h(Z_i - u) K_h(Z_i - v) \kappa_0(Z_i; u, v) \right] + O(h^4)
\]

\[
= \frac{1}{h} S_{n,1,r}^{-1}(u) S_{n,1,r}^{-1}(v) \int K(t) K \left( t + \frac{u - v}{h} \right) \kappa_1(u + ht; u, v) f_Z(u + ht) dt
\]

\[
+ \frac{1}{h} S_{n,0,r}^{-1}(u) S_{n,0,r}^{-1}(v) \int K(t) K \left( t + \frac{u - v}{h} \right) \kappa_0(u + ht; u, v) f_Z(u + ht) dt + O(h^4),
\]

where

\[
\kappa_1(z; u, v) = E \left[ \frac{1}{p(X_i)} \varphi_r(Y_i(1); q_{1,r}(u))\varphi_r(Y_i(1); q_{1,r}(v)) | Z_i = z \right],
\]

and

\[
\kappa_0(z; u, v) = E \left[ \frac{1}{1 - p(X_i)} \varphi_r(Y_i(0); q_{0,r}(u))\varphi_r(Y_i(0); q_{0,r}(v)) | Z_i = z \right],
\]
with $\varphi(y; q) = I(y \leq q) - \tau$. Thus,

\[
E^2\left[\gamma_n(R_i; u)\gamma_n(R_i; v)\right] = \frac{1}{h^2} S_{n,1,\tau}(u) S_{n,1,\tau}(v) \left( \int K(t) K\left( t + \frac{u - v}{h} \right) \kappa_1(u + ht; u, v) f_Z(u + ht) dt \right)^2 + \frac{1}{h^2} S_{n,0,\tau}(u) S_{n,0,\tau}(v) \left( \int K(t) K\left( t + \frac{u - v}{h} \right) \kappa_0(u + ht; u, v) f_Z(u + ht) dt \right)^2 + \frac{2}{h^2} S_{n,1,\tau}(u) S_{n,1,\tau}(v) S_{n,0,\tau}(u) S_{n,0,\tau}(v) \int K(t) K\left( t + \frac{u - v}{h} \right) \kappa_1(u + ht; u, v) f_Z(u + ht) dt \\
\times \int K(t) K\left( t + \frac{u - v}{h} \right) \kappa_0(u + ht; u, v) f_Z(u + ht) dt
\]

An application of (S.4) and some straightforward calculations imply that

\[
E\left[H_n(R_i; R_k)^2\right] = \frac{4}{n^4} \int \int E^2\left[\gamma_n(R_i; u)\gamma_n(R_i; v)\right] dudv = \frac{4}{n^4 h} \left\{ \int \left( \int K(t) K(t + s) dt \right)^2 ds \cdot \left[ \int S_{n,1,\tau}(u) \kappa_1^2(u; u, u) f_Z^2(u) du + \int S_{n,0,\tau}(u) \kappa_0^2(u; u, u) f_Z^2(u) du + 2 \int S_{n,1,\tau}(u) S_{n,0,\tau}(u) \kappa_1(u; u, u) \kappa_0(u; u, u) f_Z^2(u) du \right] + o(1) \right\}.
\]

This, coupled with $S_{n,j,\tau}(z) = f_Z(z)f_{Y(j)\mid Z(q_{j,\tau}(z))\mid z} + O(h^2)$ for $j = 0$ and 1, yields

\[
E\left[H_n(R_1; R_2)^2\right] = \frac{4}{n^4 h} \left( \int \left( \int K(t) K(t + s) dt \right)^2 ds \times \int \left\{ \frac{\kappa_1(u; u, u)}{f_Y^2(u) f_{Y(0)\mid Z(q_{0,\tau}(u))\mid u}} + \frac{\kappa_0(u; u, u)}{f_Y^2(u) f_{Y(0)\mid Z(q_{0,\tau}(u))\mid u}} \right)^2 \frac{1}{f_Z^2(u)} du + o(1) \right) = \frac{2}{n^4 h} \left( \sigma_j^2 + o(1) \right).
\]

Similarly, by straightforward calculations, we can obtain

\[
E\left[E^2\left[H_n(R_1, R_3)H_n(R_2, R_3) \mid R_1, R_2\right]\right] = O\left(\left(\frac{1}{n^2 h^2}\right)^4 h^7\right),
\]
and
\[ E[H_n(R_1, R_2)^4] = O\left(\left(\frac{1}{n^2 h^2}\right)^4 h^5\right).\]

Thus, the condition
\[
\lim_{n \to \infty} \frac{E\left[ E^2[H_n(R_1, R_3)H_n(R_2, R_3) \mid R_1, R_2] + n^{-1}E[H_n(R_1, R_2)^4] \right]}{E[H_n(R_1, R_2)^2]^2} = 0
\]
in Lemma 4 is satisfied, so that
\[
\frac{\sqrt{2}}{nE^{1/2}[H_n(R_1, R_2)^2]} I_{n,1} \xrightarrow{D} \mathcal{N}(0, 1),
\]
or equivalently,
\[
n\sqrt{h} I_{n,1} \xrightarrow{D} \mathcal{N}(0, \sigma_j^2).
\] (S.10)

Now, we move to the term \( I_{n,2} = n^{-2} \sum_{i=1}^{n} \int \gamma_n^2(Y_i, X_i, D_i; z) dz \). Note that

\[
E[\gamma_n^2(Y_i, X_i, D_i; z)] = E[\varrho_{n,1,\tau}(Y_i, X_i, D_i; z)]^2 + E[\varrho_{n,0,\tau}(Y_i, X_i, D_i; z)]^2 + O(h^4)
\]
\[
= S_{n,1,\tau}(z) E\left[ \frac{1}{h^2} K^2\left(\frac{Z_i - z}{h}\right) \frac{D_i}{p^2(X_i)} (I\{Y_i \leq q_{1,\tau}(z)\} - \tau)^2 \right]
\]
\[
+ S_{n,0,\tau}(z) E\left[ \frac{1}{h^2} K^2\left(\frac{Z_i - z}{h}\right) \frac{1 - D_i}{1-p(X_i)^2} (I\{Y_i \leq q_{0,\tau}(z)\} - \tau)^2 \right] + O(h^4)
\]
\[
= S_{n,1,\tau}(z) E\left[ \frac{1}{h^2} K^2\left(\frac{Z_i - z}{h}\right) \frac{1}{p(X_i)} (I\{Y_i(1) \leq q_{1,\tau}(z)\} - \tau)^2 \right]
\]
\[
+ S_{n,0,\tau}(z) E\left[ \frac{1}{h^2} K^2\left(\frac{Z_i - z}{h}\right) \frac{1}{1-p(X_i)} (I\{Y_i(0) \leq q_{0,\tau}(z)\} - \tau)^2 \right] + O(h^4)
\]
\[
= S_{n,1,\tau}(z) E\left[ \frac{1}{h^2} K^2\left(\frac{Z_i - z}{h}\right) \mu_1(Z_i; z) \right] + S_{n,0,\tau}(z) E\left[ \frac{1}{h^2} K^2\left(\frac{Z_i - z}{h}\right) \mu_0(Z_i; z) \right] + O(h^4)
\]
\[
= S_{n,1,\tau}(z) \left[ \frac{1}{h} \left( \mu_1(z; z)f_Z(z) \int K^2(s) ds + O(h) \right) \right]
\]
\[
+ S_{n,0,\tau}(z) \left[ \frac{1}{h} \left( \mu_0(z; z)f_Z(z) \int K^2(s) ds + O(h) \right) \right] + O(h^4)
\]
\[
= \frac{1}{h} \left\{ \int K^2(s) ds \cdot \left( S_{n,1,\tau}(z) \mu_1(z; z) + S_{n,0,\tau}(z) \mu_0(z; z) \right) f_Z(z) \right\} + O(1),
\]
coupled with \( S_{n,j,r}(z) = f_z(z)f_{Y(j)}|Z(q_{j,r}(z)|z) + O(h^2) \) for \( j = 0 \) and 1, we have

\[
E(I_{n,2}) = \frac{1}{n} \int E[\gamma_n^2(Y_i, X_i, D_i; z)] \, dz
\]

\[
= \frac{1}{nh} \left\{ \int K^2(s) \, ds \cdot \int \left( S_{n,1,r}^{-2}(z) \mu_1(z; z) + S_{n,0,r}^{-2}(z) \mu_0(z; z) \right) f_z(z) \, dz \right\} + O\left( \frac{1}{n} \right)
\]

\[
= \frac{1}{nh} \int K^2(s) \, ds \cdot \int \left\{ \frac{\mu_1(z; z)}{f_{Y(1)}^2(z)} + \frac{\mu_0(z; z)}{f_{Y(0)}^2(z)} \right\} \frac{1}{f_z(z)} \, dz + O\left( \frac{1}{n} \right)
\]

\[
= \mu_J + O\left( \frac{1}{n} \right).
\]

Furthermore,

\[
\text{Var}(n\sqrt{h}I_{n,2}) = E \left\{ n\sqrt{h} \left[ I_{n,2} - E(I_{n,2}) \right] \right\}^2
\]

\[
= n^{-1}h \left\{ E \left[ \int \gamma_n^2(Y_i, X_i, D_i; z) \, dz \right]^2 - E^2 \left[ \int \gamma_n^2(Y_i, X_i, D_i; z) \, dz \right] \right\}
\]

\[
= n^{-1}h \left\{ \int \int E\left[ \gamma_n^2(Y_i, X_i, D_i; z) \, dz \right] - E \left[ \int \gamma_n^2(Y_i, X_i, D_i; z) \, dz \right]^2 \right\}
\]

\[
= n^{-1}h \left\{ \int \int E\left[ \gamma_n^2(Y_i, X_i, D_i; z) \, dz \right] u \right\} \mu_n^2(Y_i, X_i, D_i; v) \, dudv - O(h^{-2})
\]

\[
= n^{-1}h \left\{ \int \int E\left[ \gamma_n^2(Y_i, X_i, D_i; z) \, dz \right] u \right\} \mu_n^2(Y_i, X_i, D_i; v) \, dudv - O(h^{-2})
\]

\[
= n^{-1}h \cdot O(h^{-2}) \to 0,
\]

together with (S.11), we have

\[
n\sqrt{h}[I_{n,2} - \mu_J] = n\sqrt{h}[I_{n,2} - E(I_{n,2})] + o_p(1) = o_p(1).
\]

It follows by combining (S.3), (S.10) and (S.13) that

\[
n\sqrt{h} \left\{ \int \left[ \frac{1}{n} \sum_{i=1}^{n} \left( \varphi_{n,1,r}(Y_i, X_i, D_i; z) - E\varphi_{n,1,r}(Y_i, X_i, D_i; z) \right) - \varphi_{n,0,r}(Y_i, X_i, D_i; z) + E\varphi_{n,0,r}(Y_i, X_i, D_i; z) \right] \, dz - \mu_J \right\} \xrightarrow{D} \mathcal{N}(0, \sigma_J^2).
\]
Lemma 6. Suppose Assumptions 2.1-2.5 are satisfied. Then,

$$
\sup_{z \in \mathcal{Z}} \left| \hat{q}_{j,\tau}^* (z) - \bar{q}_{j,\tau}^* (z) - \frac{1}{n} \sum_{i=1}^{n} \left[ \varrho_{n,j,\tau}(R_i^*; z) - E^*(\varrho_{n,j,\tau}(R_i^*; z)) \right] \right| = O_p \left( \frac{\sqrt{\ln n}}{(nh)^{3/4}} \right)
$$

for $j = 0$ and $1$, where $R_i^* = (Y_i^*, X_i^*, D_i^*)'$,

$$
\hat{q}_{j,\tau}^* (z) = \arg \min_q \sum_{i=1}^{n} W_j(X_i^*, D_i^*)hK\left( \frac{Z_i^* - z}{h} \right) \rho_{\tau}(Y_i^*: q),
$$

and

$$
\varrho_{n,j,\tau}(Y_i^*, X_i^*, D_i^*; z) = -S_{n,j,\tau}^{-1}(z)W_j(X_i^*, D_i^*)Kh(Z_i^* - z)\varphi_{\tau}(Y_i^*: q_{j,\tau}(z)).
$$

Proof of Lemma 6: This result can be proved following the proof of Theorem 2 in Lee et al. (2015). □

Lemma 7. Under Assumptions 2.1-2.6, then, we have

$$
\sup_{z \in \mathcal{Z}} \left| \hat{q}_{j,\tau}^* (z) - \bar{q}_{j,\tau}^* (z) \right| = O_p \left( \max \left\{ \frac{\ln n}{\sqrt{n}}, \frac{\sqrt{\ln n}}{(nh)^{3/4}} \right\} \right)
$$

for $j = 0$ and $1$.

Proof of Lemma 7: This result can be proved by the similar arguments to that of Lemma 2 and the details are thus omitted. □

Lemma 8. Suppose Assumptions 2.1-2.6 hold. Then, for $j = 0$ and $1$,

$$
\sup_{z \in \mathcal{Z}} \left| \hat{q}_{j,\tau}^* (z) - \bar{q}_{j,\tau}^* (z) - \frac{1}{n} \sum_{i=1}^{n} \left[ \varrho_{n,j,\tau}(R_i^*; z) - E^*(\varrho_{n,j,\tau}(R_i^*; z)) \right] \right| = O_p \left( \max \left\{ \frac{\ln n}{\sqrt{n}}, \frac{\sqrt{\ln n}}{(nh)^{3/4}} \right\} \right).
$$

Proof of Lemma 8: It is easy to observe that

$$
\sup_{z \in \mathcal{Z}} \left| \hat{q}_{j,\tau}^* (z) - \bar{q}_{j,\tau}^* (z) - \frac{1}{n} \sum_{i=1}^{n} \left[ \varrho_{n,j,\tau}(R_i^*; z) - E^*(\varrho_{n,j,\tau}(R_i^*; z)) \right] \right| \leq \sup_{z \in \mathcal{Z}} \left| \hat{q}_{j,\tau}^* (z) - \bar{q}_{j,\tau}^* (z) - \frac{1}{n} \sum_{i=1}^{n} \left[ \varrho_{n,j,\tau}(R_i^*; z) - E^*(\varrho_{n,j,\tau}(R_i^*; z)) \right] \right|
$$
\[ + \sup_{z \in \mathcal{Z}} |\hat{q}_{j,r}(z) - \bar{q}_{j,r}(z)| + \sup_{z \in \mathcal{Z}} |\bar{q}_{j,r}(z) - \bar{q}_{j,r}(z)| \]
\[ = O_p\left(\max\left\{ \frac{\ln n}{\sqrt{n}}, \sqrt{\frac{\ln n}{(nh)^{3/4}}} \right\}\right) \]

by Lemmas 2, 6 and 7. Therefore, the proof of Lemma 8 is completed. \(\Box\)

**Lemma 9.** Suppose the conditions required by Theorem 3 are satisfied. Then,
\[
\sup_{y \in \mathbb{R}} P^*\left\{ \frac{n\sqrt{h}}{\sigma_J} \left[ \int \left( \frac{1}{n} \sum_{i=1}^{n} \left( \psi_{n,1,\tau}(R_i^*; z) - \psi_{n,0,\tau}(R_i^*; z) \right) \right)^2 dz - \mu_J \right] \leq y \right\} - \Phi(y) = o_p(1),
\]
where \(\psi_{n,j,\tau}(R_i^*; z) = \varrho_{n,j,\tau}(Y_i^*, X_i^*, D_i^*; z) - E^*(\varrho_{n,j,\tau}(Y_i^*, X_i^*, D_i^*; z))\) for \(j = 0\) and \(1\). That is,
\[
\frac{n\sqrt{h}}{\sigma_J} \left[ \int \left( \frac{1}{n} \sum_{i=1}^{n} \left( \psi_{n,1,\tau}(R_i^*; z) - \psi_{n,0,\tau}(R_i^*; z) \right) \right)^2 dz - \mu_J \right]
\]
converges to \(\mathcal{N}(0, 1)\) in distribution in probability.

**Proof of Lemma 9:** It is noticed that
\[
\int \left( \frac{1}{n} \sum_{i=1}^{n} \left( \psi_{n,1,\tau}(R_i^*; z) - \psi_{n,0,\tau}(R_i^*; z) \right) \right)^2 dz
\]
\[ = \frac{2}{n^2} \sum_{1 \leq i < j \leq n} \int \left( \psi_{n,1,\tau}(R_i^*; z) - \psi_{n,0,\tau}(R_i^*; z) \right) \left( \psi_{n,1,\tau}(R_j^*; z) - \psi_{n,0,\tau}(R_j^*; z) \right) dz
\]
\[ + \frac{1}{n^2} \sum_{i=1}^{n} \int \left( \psi_{n,1,\tau}(R_i^*; z) - \psi_{n,0,\tau}(R_i^*; z) \right)^2 dz
\]
\[ := Q^*_{n,1} + Q^*_{n,2}. \quad (S.14)\]

We first consider the term \(Q^*_{n,1}\). Define
\[
T^*_{n}(R_i^*, R_j^*) = \frac{2}{n^2} \int \left( \psi_{n,1,\tau}(R_i^*; z) - \psi_{n,0,\tau}(R_i^*; z) \right) \left( \psi_{n,1,\tau}(R_j^*; z) - \psi_{n,0,\tau}(R_j^*; z) \right) dz.
\]

Then, \(Q^*_{n,1}\) can be written as a second-order U-statistic as follows:
\[
Q^*_{n,1} = \sum_{1 \leq i < j \leq n} T^*_{n}(R_i^*, R_j^*).
\]
By its definition, it is easy to find that \( E^*[T_n^*(R_i^*, R_j^*)] = 0 \) and \( E^*[T_n^*(R_i^*, R_j^*)|R_i^*] = 0 \). Thus, conditional on \( \{Y_i, X_i, D_i\}_{i=1}^n, Q_{n,1}^* \) is a second-order degenerate U-statistic. To apply Lemma 4, we need to verify the condition

\[
\frac{E^*[E^2[T_n^*(R_1^*, R_2^*)]|R_1^*, R_2^*]}{E^*[T_n^2(R_1^*, R_2^*)]} + n^{-1}E^*[T_n^4(R_1^*, R_2^*)] = o_p(1).
\]

Define

\[
\sigma_n^2 := E^*(T_n^2(R_1^*, R_2^*)) \tag{S.15}
\]

\[
= \frac{4}{n^4} \int \int E^* \left[ (\psi_{n,1,\tau}(R_1^*; u) - \psi_{n,0,\tau}(R_1^*; u))(\psi_{n,1,\tau}(R_2^*; u) - \psi_{n,0,\tau}(R_2^*; u)) \right. \\
\times \left. (\psi_{n,1,\tau}(R_1^*; v) - \psi_{n,0,\tau}(R_1^*; v))(\psi_{n,1,\tau}(R_2^*; v) - \psi_{n,0,\tau}(R_2^*; v)) \right] dudv
\]

\[
= \frac{4}{n^4} \int \int E^* \left[ (\psi_{n,1,\tau}(R_1^*; u) - \psi_{n,0,\tau}(R_1^*; u))(\psi_{n,1,\tau}(R_1^*; v) - \psi_{n,0,\tau}(R_1^*; v)) \right] dudv
\]

\[
= \frac{4}{n^4} \int \int \left[ E^*(\psi_{n,1,\tau}(R_1^*; u)\psi_{n,1,\tau}(R_1^*; v)) + E^*(\psi_{n,0,\tau}(R_1^*; u)\psi_{n,0,\tau}(R_1^*; v)) \\
- E^*(\psi_{n,1,\tau}(R_1^*; u)\psi_{n,0,\tau}(R_1^*; v)) - E^*(\psi_{n,1,\tau}(R_1^*; v)\psi_{n,0,\tau}(R_1^*; u)) \right] dudv.
\]

From (S.5) and some calculations, we have

\[
E^*(\varrho_{n,j,\tau}(R_i^*; u)) = \frac{1}{n} \sum_{i=1}^n \varrho_{n,j,\tau}(R_i; u) \tag{S.16}
\]

\[
= O_p \left( E(\varrho_{n,j,\tau}(R_i; u)) + \frac{1}{\sqrt{n}} \text{Var}^{1/2}(\varrho_{n,j,\tau}(R_i; u)) \right) = O_p \left( h^2 + \frac{1}{\sqrt{nh}} \right)
\]

uniformly in \( u \). It follows that for \( j = 0, 1 \),

\[
E^*(\psi_{n,j,\tau}(R_1^*; u)\psi_{n,j,\tau}(R_1^*; v)) = E^*(\varrho_{n,j,\tau}(R_1^*; u)\varrho_{n,j,\tau}(R_1^*; v)) - E^*(\varrho_{n,j,\tau}(R_1^*; u)) \cdot E^*(\varrho_{n,j,\tau}(R_1^*; v))
\]

\[
= E^*(\varrho_{n,j,\tau}(R_1^*; u)\varrho_{n,j,\tau}(R_1^*; v)) + O_p \left( \frac{1}{nh} + h^4 \right),
\]

and

\[
E^*(\psi_{n,1,\tau}(R_1^*; u)\psi_{n,0,\tau}(R_1^*; v)) = -E^*(\varrho_{n,1,\tau}(R_1^*; u)) \cdot E^*(\varrho_{n,0,\tau}(R_1^*; v)) = O_p \left( \frac{1}{nh} + h^4 \right).
\]
Hence, according to (S.15), we have

\[
\sigma_n^2 = \frac{4}{n^4} \int \int \left[ E^* \left( \psi_{n,1,\tau}(R_1^*; u) \psi_{n,1,\tau}(R_1^*; v) \right) + E^* \left( \psi_{n,0,\tau}(R_1^*; u) \psi_{n,0,\tau}(R_1^*; v) \right) \\
- E^* \left( \psi_{n,1,\tau}(R_1^*; u) \psi_{n,0,\tau}(R_1^*; v) \right) - E^* \left( \psi_{n,1,\tau}(R_1^*; v) \psi_{n,0,\tau}(R_1^*; u) \right) \right]^2 du dv
\]

\[
= \frac{4}{n^4} \int \int \left( E^* \left( \varrho_{n,1,\tau}(R_1^*; u) \varrho_{n,1,\tau}(R_1^*; v) \right) + E^* \left( \varrho_{n,0,\tau}(R_1^*; u) \varrho_{n,0,\tau}(R_1^*; v) \right) + O_p \left( \frac{1}{n h} + h^4 \right) \right)^2 du dv
\]

\[
= \frac{4}{n^4} \int \int \left( E^* \left( \varrho_{n,1,\tau}(R_1^*; u) \varrho_{n,1,\tau}(R_1^*; v) \right) + E^* \left( \varrho_{n,0,\tau}(R_1^*; u) \varrho_{n,0,\tau}(R_1^*; v) \right) \right)^2 du dv
\]

\[+ O_p \left( \frac{1}{n h^2} + \frac{h^4}{n^4} \right) \int \int \left( E^* \left( \varrho_{n,1,\tau}(R_1^*; u) \varrho_{n,1,\tau}(R_1^*; v) \right) + E^* \left( \varrho_{n,0,\tau}(R_1^*; u) \varrho_{n,0,\tau}(R_1^*; v) \right) \right) du dv
\]

\[+ O_p \left( \frac{1}{n^4 h^2} + \frac{h^8}{n^4} \right) := A_1 + A_2 + o_p \left( \frac{1}{n^4 h^2} \right).
\]

We focus on the term \( A_1 \). Note that

\[
\text{Var} \left[ E^* \left( \varrho_{n,j,\tau}(R_1^*; u) \varrho_{n,j,\tau}(R_1^*; v) \right) \right] = \text{Var} \left[ \frac{1}{n} \sum_{i=1}^{n} \varrho_{n,j,\tau}(R_i; u) \varrho_{n,j,\tau}(R_i; v) \right]
\]

\[\leq \frac{1}{n} E \left( \varrho_{n,j,\tau}(R_i; u) \varrho_{n,j,\tau}(R_i; v) \right)^2
\]

\[= \frac{1}{n} E \left( S_{n,j,\tau}(u) S_{n,j,\tau}(v) W_j(X_i, D_i) K_h(Z_i - u) K_h(Z_i - v) \varphi_{\tau}(Y_i; q_{j,\tau}(u)) \varphi_{\tau}(Y_i; q_{j,\tau}(v)) \right)^2
\]

\[= \frac{1}{n^2} \cdot \frac{1}{h^4} S_{n,j,\tau}(u) S_{n,j,\tau}(v) E \left\{ W_j^2(X_i, D_i) K^2 \left( \frac{Z_i - u}{h} \right) \times K^2 \left( \frac{Z_i - v}{h} \right) \varphi_{\tau}^2(Y_i; q_{j,\tau}(u)) \varphi_{\tau}^2(Y_i; q_{j,\tau}(v)) \right\}
\]

\[= \frac{1}{n^2} \cdot \frac{1}{h^4} S_{n,j,\tau}(u) S_{n,j,\tau}(v) E \left\{ K^2 \left( \frac{Z_i - u}{h} \right) K^2 \left( \frac{Z_i - v}{h} \right)
\]

\[\times E \left( p^{-3j}(X_i)(1 - p(X_i))^{-3(1-j)} \varphi_{\tau}^2(Y_i(j); q_{j,\tau}(u)) \varphi_{\tau}^2(Y_i(j); q_{j,\tau}(v)) \big| Z_i \right) \right\}
\]

\[= \frac{1}{n^2} \cdot \frac{1}{h^3} S_{n,j,\tau}(u) S_{n,j,\tau}(v) \int K^2(s) K^2 \left( s + \frac{u - v}{h} \right) \ell_j(u + hs; u, v) f_Z(u + hs) ds = O_p \left( \frac{1}{n h^3} \right),
\]

where

\[
\ell_j(z; u, v) = E \left( p^{-3j}(X_i)(1 - p(X_i))^{-3(1-j)} \varphi_{\tau}^2(Y_i(j); q_{j,\tau}(u)) \varphi_{\tau}^2(Y_i(j); q_{j,\tau}(v)) \big| Z_i = z \right).
\]
Therefore,

\[ E^*(\varphi_{n,j,\tau}(R_1^*; u)\varphi_{n,j,\tau}(R_1^*; v)) = \frac{1}{n} \sum_{i=1}^{n} \varphi_{n,j,\tau}(R_i^*; u)\varphi_{n,j,\tau}(R_i^*; v) \]

\[ = E(\varphi_{n,j,\tau}(R_1^*; u)\varphi_{n,j,\tau}(R_1^*; v)) + O_p\left( \frac{1}{\sqrt{nh^3}} \right). \] \tag{S.17}

Then,

\[
A_1 = \frac{4}{n^4} \int \int \left( E^*(\varphi_{1,\tau}(R_1^*; u)\varphi_{1,\tau}(R_1^*; v)) + E^*(\varphi_{0,\tau}(R_1^*; u)\varphi_{0,\tau}(R_1^*; v)) \right)^2 du dv
\]

\[ = \frac{4}{n^4} \int \int \left( E(\varphi_{1,\tau}(R_1^*; u)\varphi_{1,\tau}(R_1^*; v)) + E(\varphi_{0,\tau}(R_1^*; u)\varphi_{0,\tau}(R_1^*; v)) + O_p\left( \frac{1}{nh^3} \right) \right)^2 du dv.
\]

Using similar arguments as in (S.6), (S.7), (S.8) and (S.9), we have

\[
A_1 = \frac{4}{n^4} \left[ \frac{1}{\bar{h}} \int \left( \int K(t)K(t+s) dt \right)^2 ds \right.
\]

\[
\times \frac{1}{f_2^2(u)} \left( \int \frac{\kappa_1(u; u, u)}{f_2^2(u)} + \frac{\kappa_0(u; u, u)}{f_2^2(u)} \right)^2 du + O_p\left( \frac{1}{\sqrt{nh^3}} \right) + O_p\left( \frac{1}{nh^3} \right)
\]

\[ = \frac{4}{n^4} \left[ \frac{1}{\bar{h}} \int \left( \int K(t)K(t+s) dt \right)^2 ds \right.
\]

\[
\times \frac{1}{f_2^2(u)} \left( \int \frac{\kappa_1(u; u, u)}{f_2^2(u)} + \frac{\kappa_0(u; u, u)}{f_2^2(u)} \right)^2 du + O_p\left( \frac{1}{\sqrt{nh^3}} \right) + O_p\left( \frac{1}{nh^3} \right)
\]

\[ = \frac{2}{n^4} \left( \sigma_n^2 + o_p(1) \right).
\]

It is also easy to find that \( A_2 = o_p\left( \frac{1}{n^4h} \right) \). Thus,

\[ \sigma_n^2 = E^*(T_n^{*2}(R_1^*, R_2^*)) = \frac{2\sigma_n^2}{n^4h} + o_p\left( \frac{1}{n^4h} \right). \]

Similarly, by some straightforward but tedious calculations, we can obtain that

\[ E^*\left[ E^{*2}[T_n^{*2}(R_1^*, R_3^*)T_n^{*2}(R_2^*, R_3^*) | R_1^*, R_2^*] \right] = O_p\left( \frac{1}{n^8h^3} \right), \]

and

\[ E^*\left[ T_n^{*4}(R_1^*, R_2^*) \right] = O_p\left( \frac{1}{n^8h^3} \right), \]
Therefore, the condition

\[
\frac{E^* \left[ E^{*2} \left[ T_n^*(R_1^*, R_3^*) T_n^*(R_2^*, R_3^*) \mid R_1^*, R_2^* \right] \right] + n^{-1} E^* \left[ T_n^{*4}(R_1^*, R_2^*) \right]}{E^{*2} \left[ T_n^{*2}(R_1^*, R_2^*) \right]} = o_p(1)
\]

is satisfied. From Lemma 4 we know that

\[
\frac{\sqrt{2} Q_{n,1}^*}{n \sigma_n^*} \xrightarrow{d} \mathcal{N}(0, 1)
\]

in distribution in probability. Since \( \sigma_n^* = \frac{\sqrt{2}}{\sqrt{n^3}} (\sigma_J + o_p(1)) \), we also have

\[
\frac{n \sqrt{\bar{n}} Q_{n,1}^*}{\sigma_J} \xrightarrow{d} \mathcal{N}(0, 1) \quad (S.18)
\]

in distribution in probability. For the term \( Q_{n,2}^* \), we have

\[
Q_{n,2}^* = \frac{1}{n^2} \sum_{i=1}^{n} \int \left( \psi_{n,1,\tau}(R_i^*; z) - \psi_{n,0,\tau}(R_i^*; z) \right)^2 dz
\]

\[
= \frac{1}{n^2} \sum_{i=1}^{n} \int \left( \varrho_{n,1,\tau}(R_i^*; z) - \varrho_{n,0,\tau}(R_i^*; z) - E^* (\varrho_{n,1,\tau}(R_i^*; z) - \varrho_{n,0,\tau}(R_i^*; z)) \right)^2 dz
\]

\[
= \frac{1}{n^2} \sum_{i=1}^{n} \int \left( \varrho_{n,1,\tau}(R_i^*; z) - \varrho_{n,0,\tau}(R_i^*; z) \right)^2 dz - \frac{1}{n^2} \sum_{i=1}^{n} \int \left( E^* (\varrho_{n,1,\tau}(R_i^*; z) - \varrho_{n,0,\tau}(R_i^*; z)) \right)^2 dz
\]

\[
- \frac{2}{n^2} \sum_{i=1}^{n} \int \left[ E^* (\varrho_{n,1,\tau}(R_i^*; z) - \varrho_{n,0,\tau}(R_i^*; z)) \right] \left( \psi_{n,1,\tau}(R_i^*; z) - \psi_{n,0,\tau}(R_i^*; z) \right) dz
\]

\[
:= Q_{n,2,1}^* - Q_{n,2,2}^* - 2Q_{n,2,3}^*.
\]

For \( Q_{n,2,1}^* \), it is easy to obtain that

\[
E^* (Q_{n,2,1}^*) = \frac{1}{n^2} \sum_{i=1}^{n} \int \left( \varrho_{n,1,\tau}(R_i^*; z) - \varrho_{n,0,\tau}(R_i^*; z) \right)^2 dz,
\]

and

\[
\text{Var}^* (Q_{n,2,1}^*) = \text{Var}^* \left( \frac{1}{n^2} \sum_{i=1}^{n} \int \left( \varrho_{n,1}(R_i^*; z) - \varrho_{n,0}(R_i^*; z) \right)^2 dz \right) \quad (S.19)
\]

\[
= \frac{1}{n^3} \text{Var}^* \left( \int \left( \varrho_{n,1}^2(R_i^*; z) + \varrho_{n,0}^2(R_i^*; z) \right) \right)
\]
It is easy to obtain that

\[
E \left( \int \left( E^* g_{n,1}^2(R_i^*; z) + E^* g_{n,0}^2(R_i^*; z) \right) dz \right)
\]

\[
= \int \left[ E \left( E^* g_{n,1}^2(R_i^*; z) \right) + E \left( E^* g_{n,0}^2(R_i^*; z) \right) \right] dz
\]

\[
= \frac{1}{h} \left\{ \int S_{n,1}^{-2}(z) K^2(s) \mu_1(z + hs; z) f_Z(z + hs) ds + S_{n,0}^{-2}(z) K^2(s) \mu_0(z + hs; z) f_Z(z + hs) ds \right\}
\]

\[
= O(1/h),
\]

which implies that

\[
\int \left( E^* g_{n,1}^2(R_i^*; z) + E^* g_{n,0}^2(R_i^*; z) \right) dz = O_p(1/h).
\]  

(S.20)

Similarly, by straightforward calculations, we can obtain that

\[
E \int \int \left( E^* (g_{n,j}^2(R_i^*; u) g_{n,j}^2(R_i^*; v)) \right) dudv
\]

\[
= \int \int \left( E^* (g_{n,j}^2(R_i^*; u) g_{n,j}^2(R_i^*; v)) \right) dudv
\]

\[
= \frac{1}{nh^4} \int \int S_{n,j}^{-2}(u) S_{n,j}^{-2}(v) E \left[ W_j^2(X_i, D_i) K^2 \left( \frac{Z_i - u}{h} \right) K^2 \left( \frac{Z_i - v}{h} \right) \phi^2(\gamma; q_j, \tau(u)) \phi^2(\gamma; q_j, \tau(v)) \right]
\]

\[
\times \omega(u) \omega(v) dudv + \frac{1}{n^2h^4} \sum_{i \neq k} \int \int S_{n,j}^{-2}(u) S_{n,j}^{-2}(v) E \left[ W_j^2(X_i, D_i) K^2 \left( \frac{Z_i - u}{h} \right) \phi^2(\gamma; q_j, \tau(u)) \right]
\]

\[
\times E \left[ W_j^2(X_k, D_k) K^2 \left( \frac{Z_k - v}{h} \right) \phi^2(\gamma; q_j, \tau(v)) \right] dudv
\]

\[
= \frac{1}{nh^3} \int \int S_{n,j}^{-2}(u) S_{n,j}^{-2}(v) \left( \int K^2(t) K^2 \left( t + \frac{u - v}{h} \right) \ell_j(u + ht; u, v) f_Z(u + ht) dt \right) dudv
\]

\[
+ \frac{n(n - 1)}{n^2h^2} \int \int S_{n,j}^{-2}(u) S_{n,j}^{-2}(v) \left( \int K^2(s) \mu_j(u + hs; u) f_Z(u + hs) ds \right)
\]

\[
\times \int K^2(s) \mu_j(v + hs; v) f_Z(v + hs) ds \right) dudv
\]

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\[
\frac{n(n-1)}{n^2 h^2} \left\{ \left( \int K^2(s) ds \right)^2 \int \int S_{n,j}^{-2}(u)S_{n,j}^{-2}(v) \mu_j(u; u) \mu_j(v; v) f_Z(u) f_Z(v) du dv + o(1) \right\} \\
+ O\left( \frac{1}{nh^2} \right) \\
= O\left( \frac{1}{h^2} \right),
\]

which implies

\[
\int \int \left( E^* (q_{n,1}^2(R_i^*; u) q_{n,1}^2(R_i^*; v)) + E^* (q_{n,0}^2(R_i^*; u) q_{n,0}^2(R_i^*; v)) \right) du dv = O_p(1/h^2). \tag{S.21}
\]

From (S.19), (S.20) and (S.21), we have

\[
\text{Var}^* (Q_{n,2,1}^*) = \frac{1}{n^3} \cdot O_p(1/h^2) - \frac{1}{n^3} \cdot O_p(1/h) = o_p\left( \frac{1}{n^2 h} \right).
\]

Then, according to the results in (S.11) and (S.12), we obtain that

\[
E(Q_{n,2,1}^*) = E\left[ E^* (Q_{n,2,1}^*) \right] = \mu_j + O(1/n)
\]

and

\[
\text{Var}(Q_{n,2,1}^*) = \text{Var}\left[ E^* (Q_{n,2,1}^*) \right] + E\left[ \text{Var}^* (Q_{n,2,1}^*) \right] \\
= O_p\left( \frac{1}{n^3 h^2} \right) + o_p\left( \frac{1}{n^2 h} \right) = o_p\left( \frac{1}{n^2 h} \right),
\]

which lead to

\[
Q_{n,2,1}^* = \mu_j + O(1/n) + o_p\left( \frac{1}{n\sqrt{h}} \right) = \mu_j + o_p\left( \frac{1}{n\sqrt{h}} \right).
\]

By noting that \( E^*(\varphi_{n,j,R}(R_1^*; u)) = O_p\left( h^2 + \frac{1}{\sqrt{n}h} \right) \) as in (S.16), it is easy to obtain that

\[
Q_{n,2,2}^* = \frac{1}{n} \cdot O_p\left( h^4 + \frac{1}{n h} \right) = o_p\left( \frac{1}{n\sqrt{h}} \right)
\]

and

\[
Q_{n,2,3}^* = \frac{1}{n} \cdot O_p\left( h^2 + \frac{1}{\sqrt{n}h} \right) \cdot O_p(1/h) = o_p\left( \frac{1}{n\sqrt{h}} \right)
\]

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since \( nh^2 \rightarrow \infty \). Thus, we have

\[
Q^*_{n,2} = Q^*_{n,2,1} - Q^*_{n,2,2} - 2Q^*_{n,2,3} = \mu_J + o_p\left(\frac{1}{n^{3/4}h}\right). \tag{S.22}
\]

Combining (S.14), (S.18) and (S.22), we complete the proof of Lemma 9. \( \Box \)

**Lemma 10.** Suppose the conditions required by Theorem 3 are satisfied. Then,

\[
\int \left( \Delta^*_\tau(z) - \tilde{\Delta}_\tau(z) \right) dz = O_p^*(1/\sqrt{n}) + O_p^* \left\{ \max \left\{ \frac{\ln n}{\sqrt{n}}, \frac{\sqrt{\ln n}}{(nh)^{3/4}} \right\} \right\}.
\]

**Proof of Lemma 10:** According to Lemma 8, we know that

\[
\int \left( \Delta^*_\tau(z) - \tilde{\Delta}_\tau(z) \right) dz = \int \frac{1}{n} \sum_{i=1}^{n} \left( \psi_{n,1,\tau}(R^*_i; z) - \psi_{n,0,\tau}(R^*_i; z) \right) dz
\]

\[
+ O_p^* \left\{ \max \left\{ \frac{\ln n}{\sqrt{n}}, \frac{\sqrt{\ln n}}{(nh)^{3/4}} \right\} \right\},
\]

where \( \psi_{n,j,\tau}(R^*_i; z) = \varrho_{n,j,\tau}(R^*_i; z) - E^*(\varrho_{n,j,\tau}(R^*_i; z)) \) for \( j = 0 \) and \( 1 \). Denote

\[
\mathcal{M}^*_n = \frac{1}{n} \sum_{i=1}^{n} \int \left( \psi_{n,1,\tau}(R^*_i; z) - \psi_{n,0,\tau}(R^*_i; z) \right) dz.
\]

Obviously, \( E^*(\mathcal{M}^*_n) = 0 \). We then consider \( E^*(\mathcal{M}^*_n^2) \). We have

\[
E^*(\mathcal{M}^*_n^2) = \frac{1}{n^2} \sum_{i=1}^{n} E^* \left( \int \left( \psi_{n,1,\tau}(R^*_i; z) - \psi_{n,0,\tau}(R^*_i; z) \right) dz \right)^2
\]

\[
= \frac{1}{n} \int \int E^* \left( \left( \psi_{n,1,\tau}(R^*_i; u) - \psi_{n,0,\tau}(R^*_i; u) \right) \left( \psi_{n,1,\tau}(R^*_i; v) - \psi_{n,0,\tau}(R^*_i; v) \right) \right) dudv
\]

\[
+ \frac{1}{n} \int \int E^* \left( \varrho_{n,0,\tau}(R^*_i; u) \varrho_{n,0,\tau}(R^*_i; v) \right) dudv + \frac{1}{n} \int \int E^* \left( \varrho_{n,1,\tau}(R^*_i; u) \varrho_{n,1,\tau}(R^*_i; v) \right) dudv
\]

\[
- \frac{1}{n} \int \int E^* \left( \varrho_{n,0,\tau}(R^*_i; u) \right) \cdot E^* \left( \varrho_{n,0,\tau}(R^*_i; v) \right) dudv
\]

\[
+ \frac{1}{n} \int \int E^* \left( \varrho_{n,0,\tau}(R^*_i; u) \right) \cdot E^* \left( \varrho_{n,1,\tau}(R^*_i; v) \right) dudv
\]

\[
+ \frac{1}{n} \int \int E^* \left( \varrho_{n,1,\tau}(R^*_i; u) \right) \cdot E^* \left( \varrho_{n,0,\tau}(R^*_i; v) \right) dudv
\]

\[
- \frac{1}{n} \int \int E^* \left( \varrho_{n,1,\tau}(R^*_i; u) \right) \cdot E^* \left( \varrho_{n,1,\tau}(R^*_i; v) \right) dudv.
\]
Using the results in (S.17) and (S.6), we obtain that

\[
\frac{1}{n} \int \int E^*(\varrho_{n,j,r}(R_i^*; u) \varrho_{n,j,r}(R_i^*; v)) dudv
= \frac{1}{n} \int \int \left[ E(\varrho_{n,j,r}(R_1^*; u) \varrho_{n,j,r}(R_1^*; v)) + O_p \left( \frac{1}{\sqrt{nh^3}} \right) \right] dudv
= \frac{1}{n} \left[ O_p(1) + O_p \left( \frac{1}{\sqrt{nh^3}} \right) \right] = O_p \left( n^{-1} \right).
\]

Also, by noting that

\[
E^*(\varrho_{n,j,r}(R_1^*; u)) = O_p \left( h^2 + \frac{1}{\sqrt{nh^3}} \right)
\]

as in (S.16), we have

\[
\frac{1}{n} \int \int E^*(\varrho_{n,j,r}(R_i^*; u)) \cdot E^*(\varrho_{n,k,r}(R_i^*; v)) dudv = o_p \left( n^{-1} \right)
\]

for \( j, k = 0, 1 \). Therefore,

\[
E^*(\mathcal{M}_n^{*2}) = O_p \left( 1/n \right),
\]

which implies

\[
\mathcal{M}_n^{*} = O_p \left( 1/\sqrt{n} \right).
\]

This completes the proof of Lemma 10. □