A Functional-Coefficient VAR Model for Dynamic Quantiles with an Application to Constructing Financial Network

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Abstract

The degree of interdependences among holdings of financial sectors and its varying patterns play important roles in forming systemic risks within a financial system. In this article, we propose a VAR model of conditional quantiles with functional coefficients to construct a novel class of dynamic network system, of which the interdependences among tail risks such as Value-at-Risk are allowed to vary with a variable of general economy. Methodologically, we develop an easy-to-implement two-stage procedure to estimate functionals in the dynamic network system by the local linear smoothing technique. We establish the consistency and the asymptotic normality of the proposed estimator under time series settings. The simulation studies are conducted to show that our new methods work fairly well. The potential of the proposed estimation procedures is demonstrated by an empirical study of constructing and estimating a new type of dynamic financial network.

Keywords: Conditional quantile models; Dynamic financial network; Functional coefficient models; Nonparametric estimation; VAR modeling.

JEL Classification: C14, C58, C45, G32.

1 Introduction

Since the seminal work of Koenker and Bassett (1978), quantile regression, also called conditional quantile or regression quantile or dynamic quantile, has become an increasingly popular tool for risk analysis in many economics fields such as labor economics, macroeconomics and financial risk management; see, for instance, White, Kim and Manganelli (2015), Abrian and Brunnermeier (2016), Härdle, Wang and Yu (2016), Zhu, Wang, Wang and Härdle (2019) and the references therein. It is well known that when the distribution of the dependent variable has heavy-tails, heteroscedasticity, and/or outliers, the quantile regression is more reliable than mean regression models. The reader is referred to the review papers by Koenker (2005) and Koenker, Chernozhukov, He and Peng (2017) for more applications of quantile regression.

Among developments of quantile methods in the econometrics literature, dynamic quantile models have attracted intensively attentions in the recent years. The previous researches in this area were mainly motivated by estimating Value-at-Risk (VaR), which is essentially a procedure of estimating lower-tail conditional quantile of financial return distribution. The early work includes the autoregressive models for conditional quantiles (CaViaR) as in Engle and Manganelli (2004), the dynamic additive quantile models proposed in Gourieroux and Jasiak (2008), and conditional quantile estimation for generalized autoregressive conditional heteroscedasticity (GARCH)-type models studied by Xiao and Koenker (2009), and among others. In addition, dynamic quantile models are naturally suitable for capturing the dependence between the lower-tail conditional quantile of the distribution of financial returns and its lag or other covariates (also called tail dependence). For example, White et al. (2015) proposed an innovative method to estimate directly the sensitivity of VaR of a given financial institution to shocks to the whole financial system by constructing a vector autoregressive (VAR) model for dynamics of quantiles, while Härdle et al. (2016) developed a model to describe the network relationship among VaRs of financial institutions by a flexible nonparametric quantile model with L_1-penalty. Finally, Zhu et al. (2019) constructed a quantile autoregressive model that embeds the observed dependency structure in a dynamic network. The tail dependence is in particular important in reflecting the risk interdependence and contains network information in a financial system. To the best of our knowledge, much of existing literature assumed constant tail dependence in their models or focused on the response of conditional quantile to endogenous variables or shocks. However, numerous studies have documented temporal changes of risk interdependence in financial time series and discussed their possible origins and relation to spillover effects; see, for example, Billio, Getmansky, Lo and Pelizzon (2012), Diebold and Yílmaz (2014), Härdle et al. (2016), Yang and Zhou (2017), Liu, Ji and Fan (2017), Ando and Bai (2020) and the references therein. The driving force for the variations of risk interdependence may be the institutional changes or the policy interventions, such as the changes of exchange rate systems and the U.S. quantitative easing policy. With these backgrounds, it is desirable to consider modeling the interaction between varying patterns of tail dependence and macroeconomic circumstances. These theoretical and empirical studies inspire us to build a more general framework to capture the time-varying interdependences among conditional quantiles.

In this article, we propose a nonparametric approach involving multivariate dynamic quantile models with nonlinear structures. Different from previous studies, we capture nonlinearities in data by using a functional coefficient setting, which allows coefficients of the multivariate dynamic quantile models to vary with a smoothing variable. Since coefficients of dynamic quantile models play an important role in reflecting interdependences among conditional quantiles, under our model setup, one can easily illustrate the variation of tail dependence and its relation with the variable which is of interest. To interpret features of varying interdependences within various conditional quantiles, we form a vector autoregressive (VAR) model with functional coefficients where the quantiles of several random variables depend on lagged quantiles and other lagged covariates. For this reason, this model is termed as a functional-coefficient VAR model for dynamic quantiles (FCVAR-DQ) and is presented in (1) later. In an effort to study nonlinear relationship between the quantile of response variable and its covariates, various smoothing techniques (e.g., kernel methods, splines, and their variants) have been used to estimate the nonparametric quantile regression for both independent and time series data, to name just a few, He and Ng (1999), Honda (2000, 2004), Wei and He (2006), Kim (2007), Cai and Xu (2008), Kong, Linton and Xia (2010), Qu and Yoon (2015), and Li, Li and Li (2020). Among many kinds of methods, we adapt one of modeling methods to analyze dynamic quantiles, termed the functional coefficient modeling approach. Compared with existing literature, our approach is different mainly in three parts. First, we provide a kernel-based estimation framework for a new type of dynamic quantile models, which impose relatively less restriction on model's structures. Second, our model admits nonlinearities of tail dependence, which can be ignored by dynamic quantile models with fixed coefficients. Third, the proposed model allows for studying interaction between tail dependence and the variable which is of interest.

One of our motivations for this study comes from analyzing the dynamic mechanism of financial network in international equity markets. It is well documented in the literature that financial systems contain enormous numbers of institutions that interplay with each other. These interactions form a financial network in which a node represents each institution and a linkage between two nodes acts as an observable or unobservable interaction of some forms between two institutions. Also, it is well-established that the possibility of major financial distress is closely related to the degree of correlation among the assets of institutions and how sensitive they are to the changes in economic conditions. Based on these intuitions, provided that the node of a network is represented by the VaR of returns of institutions' assets or of market indexes, one may construct a financial network that can capture interdependences among VaRs within the financial system. Since VaRs and interdependences among them appear to be unobservable in practice, as addressed in Sewell and Chen (2015), Zhu et al. (2019), and Bräuning and Koopman (2020), it is unnecessarily feasible to apply commonly known technologies that have access to the binary data with observed network structures for estimating the risk network formed by VaRs. An influential precedent of analyzing the network topology of unobservable connectedness of risk attributes to Diebold and Yílmaz (2014), who constructed a risk network based on forecast error variance decompositions of classical VAR models and studied the volatility connectedness by methods of network analysis. Compared to the literature thus far, we consider capturing unobserved interconnectedness of tail risk among institutions in the dynamic network, which can not be achieved by models with observed network data and by measuring conditional correlation in Diebold and Yílmaz (2014). Moreover, in order to illustrate overall patterns of time-varying network of risk across institutions, the main interest in this paper lies in modeling the relationship between the general states of economy and a financial network formed by VaRs of world major market index's return series. More specifically, we allow interdependences among VaRs of market index's return series to vary with a smoothing variable of economic status to capture the dynamic changes. Recent studies found increasingly numbers of evidences to show that the variation of risk interdependence not only reveals the behavior of spillover effects of risk but also contains the information about the stability of financial systems; see, e.g., Acemoglu, Ozdaglar and Tahbaz-Salehi (2015). Both practitioners and policymakers may be interested in knowing how a financial network changes with the macroeconomic climate or financial market circumstances, and the way to evaluate the influences of economic policies to the whole network within the financial market. Extensive reviews about financial network can be found in Diebold and Yílmaz (2014) and Härdle et al. (2016). The empirical study in this paper shows that FCVAR-DQ models are suitable for estimating a novel class of dynamic financial network. A detailed analysis of this class of financial network is reported in Section 4.

Lastly, our contributions to the literature can be summarized as follows. First, the model setting in this paper (see (1) later) is general enough to nest many well-known dynamic quantile models in the literature; see, for example, CaViaR models proposed by Engle and Manganelli (2004) and Xiao and Koenker (2009), threshold CaViaR in Gerlach, Chen and Chan (2011), and static vector autoregressive (VAR) for VaR models constructed by White et al. (2015). Second, by allowing coefficients to vary with a smoothing variable, FCVAR-DQ models provide a new tool to estimate the relationship between the interdependence of risk and the state variable of economy or time. Third, a large sample theory for the proposed estimators is established to construct confidence intervals for functional coefficients in the empirical study.

The rest of this paper is organized as follows. In Section 2, the model setup, properties and two-stage estimation procedures are presented for the FCVAR-DQ model. In addition, a large sample theory for the proposed estimators and a consistent estimator of the asymptotic covariance matrix are also investigated in this section. A Monte Carlo simulation for the proposed estimation procedures is discussed in Section 3 and corresponding results are reported in Appendix. In Section 4, our models are applied to constructing a novel class of financial networks based on the real example. Finally, the conclusion is given in Section 5 and all the technical proofs are gathered in Appendix, Supplements B and C.

Throughout this article, T represents transpose of a vector or a matrix; $0_{a\times b}$ stands for a $a \times b$ matrix of zeros and I_a is a $a \times a$ identity matrix; $\|\cdot\|$ is denoted as the Euclidean norm (L_2-norm) and $\|B\|_F$ is the Frobenius norm of a matrix B.

2 Functional-Coefficient VAR Model for Dynamic Quantiles

2.1 Model Setup

Let u_{it}, a scalar dependent variable, be the ith observation at time t for $1 \leq i \leq \kappa$ and $1 \leq t \leq n$, $\mathcal{F}_{i,t-1}$ represent information set up to time $t-1$ for $1 \leq i \leq \kappa$, and let $q_{\tau,t,i}$ be the τth conditional quantile of u_{it} given $\mathcal{F}_{i,t-1}$. Then, we study the following functional-coefficient VAR model for dynamic quantiles, termed as FCVAR-DQ model, given by

$$q_{\tau,t,i} = \gamma_{i0,\tau}(Z_{it}) + \sum_{s=1}^{q} \boldsymbol{\gamma}_{i,s,\tau}^T(Z_{it})\boldsymbol{q}_{\tau,t-s} + \sum_{l=1}^{p} \boldsymbol{\beta}_{i,l,\tau}^T(Z_{it})\mathbb{U}_{t-l} \tag{1}$$

for some p and q, where $\boldsymbol{q}_{\tau,t} = (q_{\tau,t,1}, \ldots, q_{\tau,t,\kappa})^T$ and $\mathbb{U}_t$ is a $\kappa_1 \times 1$ vector of covariates, including possibly some or all of $\{u_{it}\}_{i=1}^{\kappa}$ and/or some exogenous information $\{v_{it}\}$. In addition, $\gamma_{i0,\tau}(\cdot)$ is a scalar function and is allowed to depend on τ, both $\boldsymbol{\gamma}_{i,s,\tau}(\cdot) = (\gamma_{si1,\tau}(\cdot), \ldots, \gamma_{si\kappa,\tau}(\cdot))^T$ and $\boldsymbol{\beta}_{i,l,\tau}(\cdot) = (\beta_{li1,\tau}(\cdot), \ldots, \beta_{li\kappa_1,\tau}(\cdot))^T$ are $\kappa \times 1$ and $\kappa_1 \times 1$ vectors of functional coefficients, respectively, and they are allowed to depend on τ too. Here, Z_{it} is an observable scalar smoothing variable, which might be one part of $\mathbb{U}_{t-l}$ and/or time or other exogenous variables $\{v_{it}\}$ or their lagged variables. Of course, Z_{it} can be an economic index to characterize economic activities. Also, note that Z_{it} can be set as a multivariate variable. In such a case, the estimation procedures and the related theory for the univariate case still hold for multivariate case, but more complicated notations are involved and models with Z_{it} in very high dimension are often not practically useful due to the "curse of dimensionality". In addition, note that similar to the setting of the multi-quantile CaViaR (MQ-CaViaR) model as in White, Kim and Manganelli (2008), one may further generalize model (1) by allowing τ in $q_{\tau,t,i}$ to vary across different equations, only with mild changes on asymptotic theory in this paper. Thus, in order to meet our empirical motivation, all of $\tau's$ in model (1) are the same throughout this article.

Importantly, in the case of estimating dynamic financial network in our empirical studies, by following White et al. (2015), we consider only the tail dependence between current state and the state of one-period lagged, and take $\mathbb{U}_t$ to be $\mathbb{U}_t = (|u_{1t}|, \ldots, |u_{\kappa t}|)^T$ with $|\cdot|$ representing absolute value. Furthermore, the smoothing variable Z_{it} varies only across different time periods but keeps constant over individual units. Therefore, in this paper, for easy exposition, our focus is on the simple case that $q = p = 1$, $\mathbb{U}_t = (|u_{1t}|, \ldots, |u_{\kappa t}|)^T$, and $Z_{it} = Z_t$ for all $1 \le i \le \kappa$. Then, model (1) can be rewritten as

$$q_{\tau,t,i} = \boldsymbol{g}_{i,\tau}^T(Z_t)\boldsymbol{X}_t, \tag{2}$$

where $\boldsymbol{g}_{i,\tau}(\cdot) = (\gamma_{i0,\tau}(\cdot), \gamma_{i1,\tau}(\cdot), \ldots, \gamma_{i\kappa,\tau}(\cdot), \beta_{i1,\tau}(\cdot), \ldots, \beta_{i\kappa,\tau}(\cdot))^T$ is a $(2\kappa+1)\times 1$ vector of functional coefficients and $\boldsymbol{X}_t = (1, q_{\tau,t-1,1}, \ldots, q_{\tau,t-1,\kappa}, |u_{1(t-1)}|, \ldots, |u_{\kappa(t-1)}|)^T$.

It is worthwhile to note that if $q_{\tau,t,i}$ in model (2) is defined as VaR of return u_{it}, then $\{\gamma_{ij,\tau}(Z_t)\}_{i=1,j=1}^{\kappa}$ in model (2) becomes the sensitivity of VaR of returns for one portfolio at time t to that of another at time $t-1$. With these functional coefficients $\{\gamma_{ij,\tau}(Z_t)\}_{i=1,j=1}^{\kappa}$, define the following $\kappa \times \kappa$ matrix

$$\boldsymbol{\Gamma}_\tau(Z_t) = \begin{pmatrix} \gamma_{11,\tau}(Z_t) & \gamma_{12,\tau}(Z_t) & \ldots & \gamma_{1\kappa,\tau}(Z_t) \\ \gamma_{21,\tau}(Z_t) & \gamma_{22,\tau}(Z_t) & \ldots & \gamma_{2\kappa,\tau}(Z_t) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{\kappa 1,\tau}(Z_t) & \gamma_{\kappa 2,\tau}(Z_t) & \ldots & \gamma_{\kappa\kappa,\tau}(Z_t) \end{pmatrix}. \tag{3}$$

Then, (2) can be expressed as a matrix form, which, indeed, is a functional coefficient vector autoregressive (FCVAR) model for $\boldsymbol{q}_{\tau,t}$ with exogenous variables,

$$\boldsymbol{q}_{\tau,t} = \boldsymbol{\gamma}_{0,\tau}(Z_t) + \boldsymbol{\Gamma}_\tau(Z_t)\,\boldsymbol{q}_{\tau,t-1} + \boldsymbol{\Gamma}_{\beta,\tau}(Z_t)\,\mathbb{U}_{t-1},$$

where $\boldsymbol{\gamma}_{0,\tau}(Z_t)$ and $\boldsymbol{\Gamma}_{\beta,\tau}(Z_t)$ are defined obviously. Therefore, $\boldsymbol{\Gamma}_\tau(Z_t)$ in (3) can serve as a dynamic network system changing with both τ and some information variable Z_t. Notice that the general setting in the dynamic network system (3) covers some famous network models for characterizing financial risk system, including the one formed by VAR for VaR model in White et al. (2015), which assumes the tail dependence $\{\gamma_{ij,\tau}(Z_t)\}_{i=1,j=1}^{\kappa}$ to be constant and the static financial network in Abrian and Brunnermeier (2016) and Härdle et al. (2016) as special cases.

Remark 2.1. *(Special Cases) The proposed FCVAR-DQ model* (1) *is related to the papers by Engle and Manganelli (2004) and Xiao and Koenker (2009), which discussed the relation between modeling dynamic structures of conditional quantiles and conditional volatility of returns. Indeed, if u_{it} in* (1) *takes a simple form as $u_{it} = \sigma_{it}\,\varepsilon_{it}$, where σ_{it}^2 is the conditional variance of u_{it} and ε_{it} is an independent and identically distributed (i.i.d.) sequence of random variables with mean zero and unit variance, then, $q_{\tau,t,i} = \sigma_{it}F_{\varepsilon}^{-1}(\tau)$, where $F_{\varepsilon}(\cdot)$ is the distribution function of ε_{it}. Furthermore, if $u_{it} = \sigma_{it}\,\varepsilon_{it}$ is generated from a functional coefficient multivariate GARCH (p, q)-type process for κ ($\kappa \geq 1$) returns extended from the setting in Taylor (1986) as follows*

$$\sigma_{it} = \gamma_{i0}(Z_{it}) + \sum_{s=1}^{q} \boldsymbol{\gamma}_{i,s}^T(Z_{it})\boldsymbol{\Sigma}_{t-s} + \sum_{l=1}^{p} \boldsymbol{\beta}_{i,l}^T(Z_{it})\mathbb{U}_{t-l}, \tag{4}$$

where $\boldsymbol{\Sigma}_t = (\sigma_{it}, \ldots, \sigma_{\kappa t})^T$ and $\mathbb{U}_t = (|u_{1t}|, \ldots, |u_{\kappa t}|)^T$, then, (1) *reduces to following dynamic quantile model:*

$$q_{\tau,t,i} = \gamma_{i0,\tau}(Z_{it}) + \sum_{s=1}^{q} \boldsymbol{\gamma}_{i,s}^T(Z_{it})\boldsymbol{q}_{\tau,t-s} + \sum_{l=1}^{p} \boldsymbol{\beta}_{i,l,\tau}^T(Z_{it})\mathbb{U}_{t-l}, \tag{5}$$

where $\gamma_{i0,\tau}(\cdot) = \gamma_{i0}(\cdot)F_{\varepsilon}^{-1}(\tau)$, $\boldsymbol{\gamma}_{i,s}(\cdot) = (\gamma_{si1}(\cdot), \ldots, \gamma_{si\kappa}(\cdot))^T$ and $\boldsymbol{\beta}_{i,l,\tau}(\cdot) = (\beta_{li1,\tau}(\cdot), \ldots, \beta_{li\kappa,\tau}(\cdot))^T$ with $\beta_{lij,\tau}(\cdot) = \beta_{lij}(\cdot)F_{\varepsilon}^{-1}(\tau)$. Notice that if $\gamma's$ and $\beta's$ in (5) *are constant, model* (5) *becomes to those in Engle and Manganelli (2004) and Xiao and Koenker (2009). For details, the reader is referred to the aforementioned papers.*

Remark 2.2. *(Monotonicity). The issue of monotonicity has been frequently discussed for the quantile autoregression model. A specific case for the monotonicity of* (1) *to hold is that* $\{\gamma_{i,s,\tau}(Z_t)\}_{i=1,s=1}^{\kappa,q}$ *are all monotone increasing functions with respect to* τ*, and* $\mathbb{U}_t$ *is a positive random vector. In other cases, the assumption of monotonicity can be satisfied by conducting certain data transformation techniques; see Koenker and Xiao (2006) and Fan and Fan (2006) for detailed discussions.*

Remark 2.3. *(Selection of* Z_t*). Of importance is to choose an appropriate smoothing variable* Z_t *in applying functional-coefficient VAR model for dynamic quantiles in* (2)*. Knowledge on physical background or economic theory of the data may be very helpful, as we have witnessed in modeling the real data in Section 4 by choosing* Z_t *to be the first difference of daily log series of the U.S. dollar index. Without any prior information, it is pertinent to choose* Z_t *in terms of some data-driven methods such as the Akaike information criterion, cross-validation, and other criteria. Ideally, we would choose* Z_t *as a linear function of given explanatory variables according to some optimal criterion or an economic index based on economic theory or background. Nevertheless, here we would recommend using a simple and practical approach proposed by Cai, Fan and Yao (2000) in practice.*

2.2 Properties of the FCVAR-DQ Model

In order to apply our estimation procedures, one has to show that the model given in (1) can be approximated by functional-coefficient moving average (MA(∞)) representation. To this end, for convenience of presentation, we first let $\kappa = \kappa_1$, $\mathbb{U}_t = (|u_{1t}|, \ldots, |u_{\kappa t}|)^T$ and $Z_{it} = Z_t$ for all $1 \leq i \leq \kappa$ in model (1). Then, we can rewrite (1) into a autoregression process of order 1 as follows

$$\mathbb{X}_t = \boldsymbol{\mu}(Z_t) + \boldsymbol{A}_{e_t}(Z_t)\mathbb{X}_{t-1} + \boldsymbol{D}_{e_t}(Z_t), \tag{6}$$

where $\mathbb{X}_t = (\mathbb{U}_t^T, \ldots, \mathbb{U}_{t-p+1}^T, \boldsymbol{q}_{\tau,t}^T, \ldots, \boldsymbol{q}_{\tau,t-q+1}^T)^T$. In addition, let $\{e_{it}\}$ be a sequence of mutually i.i.d. standard uniform random variables on the set of $[0,1]$ for $i = 1, \ldots, \kappa$, denote $\boldsymbol{A}_{e_t}(Z_t)$ as a $\kappa(p+q) \times \kappa(p+q)$ matrix as follows:

$$\begin{pmatrix} A_{1,e_t}(Z_t) & A_{2,e_t}(Z_t) & \ldots & A_{p-1,e_t}(Z_t) & A_{p,e_t}(Z_t) & B_{1,e_t}(Z_t) & B_{2,e_t}(Z_t) & \ldots & B_{q-1,e_t}(Z_t) & B_{q,e_t}(Z_t) \\ I_\kappa & 0_{\kappa\times\kappa} & \ldots & 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} & \ldots & 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} \\ 0_{\kappa\times\kappa} & I_\kappa & \ldots & 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} & \ldots & 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} & \ldots & I_\kappa & 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} & \ldots & 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} \\ A_{1,\tau}(Z_t) & A_{2,\tau}(Z_t) & \ldots & A_{p-1,\tau}(Z_t) & A_{p,\tau}(Z_t) & B_{1,\tau}(Z_t) & B_{2,\tau}(Z_t) & \ldots & B_{q-1,\tau}(Z_t) & B_{q,\tau}(Z_t) \\ 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} & \ldots & 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} & I_\kappa & 0_{\kappa\times\kappa} & \ldots & 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} \\ 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} & \ldots & 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} & I_\kappa & \ldots & 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} & \ldots & 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} & 0_{\kappa\times\kappa} & \ldots & I_\kappa & 0_{\kappa\times\kappa} \end{pmatrix}.$$

Here, for $i, j = 1, \ldots, \kappa$, $l = 1, \ldots, p$ and $s = 1, \ldots, q$, $A_{l,e_t}(Z_t)$ is a matrix with entries $\beta_{lij,e_{it}}(Z_t)$ and $B_{s,e_t}(Z_t)$ is a matrix with entries $\gamma_{sij,e_{it}}(Z_t)$, where $\beta_{lij,e_{it}}(Z_t)$ and $\gamma_{sij,e_{it}}(Z_t)$ are unknown functions of Z_t and e_{it} (from $\mathbb{R}^2$ to $\mathbb{R}$). Similarly, $A_{l,\tau}(Z_t)$ is a matrix with entries $\beta_{lij,\tau}(Z_t)$ and $B_{s,\tau}(Z_t)$ is a matrix with entries $\gamma_{sij,\tau}(Z_t)$ for given τ. Furthermore, $\boldsymbol{\mu}(Z_t) = (E_e(\boldsymbol{\Omega}_{e_t}^T(Z_t)), 0, \ldots, 0, \boldsymbol{\Omega}_\tau^T(Z_t), 0, \ldots, 0)^T$, $E_e(\boldsymbol{\Omega}_{e_t}(Z_t)) = (E_e(\gamma_{10,e_{1t}}(Z_t)), \ldots, E_e(\gamma_{\kappa 0,e_{\kappa t}}(Z_t)))^T$ and $\boldsymbol{\Omega}_\tau(Z_t) = (\gamma_{10,\tau}(Z_t), \ldots, \gamma_{\kappa 0,\tau}(Z_t))^T$, where $E_e(\cdot)$ is denoted as taking expectation on e_{it} for any fixed Z_t, and $\gamma_{i0,e_{it}}(Z_t)$ and $\gamma_{i0,\tau}(Z_t)$ are defined in a similar way as foregoing functional coefficients, respectively. In addition, for $1 \le i \le \kappa$, $\boldsymbol{D}_{e_t}(Z_t) = (\check{\gamma}_{10,e_{1t}}(Z_t), \ldots, \check{\gamma}_{\kappa 0,e_{\kappa t}}(Z_t), 0_{1\times\kappa(p+q-1)})^T$, where $\check{\gamma}_{i0,e_{it}}(Z_t) = \gamma_{i0,e_{it}}(Z_t) - E_e(\gamma_{i0,e_{it}}(Z_t))$. We assume that $\boldsymbol{D}_{e_t}(Z_t)$ can be written as $\boldsymbol{D}_{e_t}(Z_t) = \check{\boldsymbol{\gamma}}(Z_t)\boldsymbol{D}_{e_t}$, where $\boldsymbol{D}_{e_t} = (\check{\gamma}_{10,e_{1t}}, \ldots, \check{\gamma}_{\kappa 0,e_{\kappa t}}, 0_{1\times\kappa(p+q-1)})^T$ with $\check{\gamma}_{i0,e_{it}}$ being defined in Assumption A1 later. Finally, $\check{\boldsymbol{\gamma}}(Z_t)$ is a matrix of unknown coefficient functions of Z_t.

Remark 2.4. *Notice that when setting* Z_t *as a smoothing variable, the equations corresponding to* $(\kappa p+1)$*-th, ...,* $(\kappa p+\kappa)$*-th rows of (6) are exactly the model (1) with* $\mathbb{U}_t = (|u_{1t}|, \ldots, |u_{\kappa t}|)^T$*, while the ith row of (6) with* $i = 1, \ldots, \kappa$ *is written as*

$$u_{it} = \gamma_{i0,e_{it}}(Z_t) + \sum_{s=1}^{q} \boldsymbol{\gamma}_{i,s,e_{it}}^T(Z_t)\boldsymbol{q}_{\tau,t-s} + \sum_{l=1}^{p} \boldsymbol{\beta}_{i,l,e_{it}}^T(Z_t)\mathbb{U}_{t-l}, \tag{7}$$

where $\mathbb{U}_t = (|u_{1t}|, \ldots, |u_{\kappa t}|)^T$*, elements of* $\boldsymbol{\gamma}_{i,s,e_{it}}(\cdot) = (\gamma_{si1,e_{it}}(\cdot), \ldots, \gamma_{si\kappa,e_{it}}(\cdot))^T$ *and* $\boldsymbol{\beta}_{i,l,e_{it}}(\cdot) = (\beta_{li1,e_{it}}(\cdot), \ldots, \beta_{li\kappa,e_{it}}(\cdot))^T$ *have the same definitions as that in matrices* $B_{s,e_t}(Z_t)$ *and* $A_{l,e_t}(Z_t)$*, respectively. Note that equation (7) is similar to the random coefficient autoregressive (RCA) model discussed in Koenker and Xiao (2006), it can be shown that the conditional quantile function of* u_{it} *in (7) given* Z_t*,* $\boldsymbol{q}_{\tau,t-s}$ *and* $\mathbb{U}_{t-l}$ *is model (1) with* $\mathbb{U}_t = (|u_{1t}|, \ldots, |u_{\kappa t}|)^T$*. Given these relations, one can conclude that* $\mathbb{U}_t$ *and* $\boldsymbol{q}_{\tau,t}$ *jointly follow a VAR process of order 1 in (6), which is similar to the nonparametric additive models in Cai and Masry (2000) and the generalized polynomial RCA vector models in Carrasco and Chen (2002). Also note that if* $\check{\gamma}_{i0,e_{it}}$ *in* $\boldsymbol{D}_{e_t}$ *is* $F^{-1}(e_{it})$*, where* $F(\cdot)$ *is a distribution function, then,* $\check{\gamma}_{i0,e_{it}}$ *is an innovation random variable with distribution function being* $F(\cdot)$*. Thus, the heteroscedasticity can be imposed in model (6) by the assumption of* $\boldsymbol{D}_{e_t}(Z_t) = \check{\gamma}(Z_t)\boldsymbol{D}_{e_t}$*, which is a common setting among literature.*

In addition, similar to the notation in Dahlhaus and Polonik (2009), let

$$V(\alpha(x)) = \sup\left\{\sum_{k=1}^{d} |\alpha(x_k) - \alpha(x_{k-1})| : \mathfrak{a} \le x_0 < \cdots < x_d \le \mathfrak{c}, d \in \boldsymbol{N}\right\}$$

be the total variation of a function $\alpha(\cdot)$ on a closed interval $[\mathfrak{a}, \mathfrak{c}]$. Subsequently, for all $Z_t \in [\mathfrak{a}, \mathfrak{c}]$ and all $e_{it} \in [0,1]$, define $\mathcal{B}_\tau(\mathcal{L}) = I_\kappa - \sum_{s=1}^{q} B_{s,\tau}(Z_t)\mathcal{L}^s$ and $\mathcal{A}_{e_t}(\mathcal{L}) = \sum_{l=1}^{p} A_{l,e_t}(Z_t)\mathcal{L}^l$ as matrices, where each entry is a lag polynomial and $\mathcal{L}$ denotes the lag operator.

Assumption A.

A1: *Each entry of* $\check{\boldsymbol{\gamma}}(Z_t)$ *is second order continuously differentiable on* $[\mathfrak{a}, \mathfrak{c}]$. $\check{\boldsymbol{\gamma}}(Z_t)\check{\boldsymbol{\gamma}}^T(Z_t)$ *is a positive-definite matrix uniformly on* $[\mathfrak{a}, \mathfrak{c}]$. *Moreover,* $\{\check{\gamma}_{i0,e_{it}}\}$ *are i.i.d. random variables with mean 0 and finite variance. Finally,* $E\|\boldsymbol{D}_{e_t}\|^2 < \infty$ *and* $E\|\boldsymbol{\mu}(Z_t)\| < \infty$.

A2: *For* $i, j = 1, \ldots, \kappa$ *and for each fixed* $Z_t \in [\mathfrak{a}, \mathfrak{c}]$, $\beta_{lij,e_{it}}(Z_t)$ *and* $\gamma_{sij,e_{it}}(Z_t)$ *are functions of* e_{it} *and are continuous in a neighborhood of* $\tau \in (0, 1)$, *respectively. The total variations of each element of* $\check{\boldsymbol{\gamma}}(\cdot)$, $V(\gamma_{sij,\tau}(\cdot))$ *and* $V(\beta_{lij,\tau}(\cdot))$, *are bounded for* $l = 1, \ldots, p$, $s = 1, \ldots, q$ *and for given* τ.

A3: *For all* $Z_t \in [\mathfrak{a}, \mathfrak{c}]$ *and all* $e_{it} \in [0, 1]$, $\mathcal{B}_\tau(\mathcal{L}) \neq \mathcal{A}_{e_t}(\mathcal{L}) \neq 0$ *for all* $0 < |\mathcal{L}| \leq 1 + \delta$ *and for some* $\delta > 0$.

Remark 2.5. *The first part of Assumption A1 is usually imposed in literature on heteroscedasticity, while the second part is similar to the condition given by Koenker and Xiao (2006) for parametric RCA models and the boundedness condition in Carrasco and Chen (2002). The boundedness of total variation in Assumption A2 guarantees a certain smoothness of coefficients in the direction of* Z_t*, see Dahlhaus and Polonik (2009) for more discussions. Assumption A3 is an invertibility condition, which ensures that all eigenvalue of matrix* $\boldsymbol{A}_{e_t}(Z_t)$ *have modulus less than 1 for all* Z_t *and all* e_{it}*. Undoubtedly, Assumption A1-A3 could be weakened, but this attempt goes beyond the scope of this paper.*

From the following proposition with its proof given in Appendix, it guarantees that (6) can be approximated by a functional-coefficient MA(∞) representation.

Proposition 2.1. *Under Assumption A1-A3, there exists a functional-coefficient MA*(∞) *representation for (6) as follows*

$$\mathbb{X}_t = \boldsymbol{V}_t(l) + \sum_{l=0}^{\infty} \boldsymbol{C}_{e,t}(l)\boldsymbol{D}_{e_{t-l}}, \tag{8}$$

where $\boldsymbol{V}_t(l) = \boldsymbol{\mu}(Z_t) + \sum_{l=1}^{\infty}\left(\prod_{\jmath=0}^{l-1}\boldsymbol{A}_{e_{t-\jmath}}(Z_{t-\jmath})\right)\boldsymbol{\mu}(Z_{t-l})$, $\boldsymbol{C}_{e,t}(0) = \check{\boldsymbol{\gamma}}(Z_t)$ *and*

$$\boldsymbol{C}_{e,t}(l) = \prod_{\jmath=0}^{l-1}\boldsymbol{A}_{e_{t-\jmath}}(Z_{t-\jmath})\check{\boldsymbol{\gamma}}(Z_{t-l})$$

for $l > 0$*, with* $\alpha_{ijt}(l)$ *being each entry of matrix* $\boldsymbol{C}_{e,t}(l)$ *and satisfying the following condition:*

$$\max_{t\geq 1}|\alpha_{ijt}(l)| \leq C\rho^l \tag{9}$$

for some positive constant $\rho < 1$ *and for* $l > 0$*. Moreover, there exists a functional-coefficient MA(*∞*) representation*

$$\widetilde{\mathbb{X}}_t = \boldsymbol{V}(Z_t) + \sum_{l=0}^{\infty}\boldsymbol{C}_{l,e_t}(Z_t)\boldsymbol{D}_{e_{t-l}} \tag{10}$$

with $\boldsymbol{V}(Z_t) = \boldsymbol{\mu}(Z_t) + \sum_{l=1}^{\infty}(\boldsymbol{A}_{e_t}(Z_t))^l\boldsymbol{\mu}(Z_t)$ *and* $\boldsymbol{C}_{l,e_t}(Z_t) = (\boldsymbol{A}_{e_t}(Z_t))^l\check{\boldsymbol{\gamma}}(Z_t)$*, such that*

$$\max_{t\geq 1}E\|\mathbb{X}_t - \widetilde{\mathbb{X}}_t\| = O(n^{-1}).$$

2.3 Two-stage Estimation Procedure

Since the estimation procedures for (1) and (2) are the same, we aim at estimating functional coefficients $\boldsymbol{g}_{i,\tau}(\cdot)$ in the model defined in (2) for simplicity. Because $q_{\tau,t-1,i}$ in $\boldsymbol{X}_t$ depends on unknown functional coefficients $\boldsymbol{g}_{i,\tau}(\cdot)$, model (2) is more complicated than functional coefficient models with observed data. Our procedures consist of two steps, the first is to estimate latent $q_{\tau,t-1,i}$, and then we perform locally weighted estimation for functional coefficients using the estimated $q_{\tau,t-1,i}$ from the first step. In this paper, we only focus on estimating functional coefficients in (2), rather than jointly forecasting $q_{\tau,t,i}$ or doing impulse response analysis. So, it is sufficient to estimate $\boldsymbol{g}_{i,\tau}(\cdot)$ in an equation-by-equation way for different i. Thus, by abuse of notation, i will be dropped in what follows.

Given (1), (2) and (6), Assumption A3 ensures the invertibility of $\mathcal{B}_\tau(\mathcal{L})$. Let $\mathcal{A}_\tau(\mathcal{L}) = \sum_{l=1}^{p} A_{l,\tau}(Z_t)\mathcal{L}^l$. Then, by Proposition 2.1, $\mathcal{B}_\tau(\mathcal{L})^{-1}(\mathbf{\Omega}_\tau(Z_t) + \mathcal{A}_\tau(\mathcal{L}))$ can be represented by a matrix series $C_{0,\tau}(Z_t)\mathbf{\Omega}_\tau(Z_t) + \sum_{l=1}^{\infty} C_{l,\tau}(Z_t)\mathcal{L}^l$ for all Z_t. Now, let $\alpha_{0,\tau}(\cdot)$ be the ith row of matrix $C_{0,\tau}(Z_t)\mathbf{\Omega}_\tau(Z_t)$ and $\boldsymbol{\alpha}_{l,\tau}(\cdot) = (\alpha_{l1,\tau}(\cdot), \ldots, \alpha_{l\kappa,\tau}(\cdot))^T$ be the ith row of matrix $C_{l,\tau}(Z_t)$. Therefore, with the definitions of $\alpha_{0,\tau}(\cdot)$ and $\boldsymbol{\alpha}_{l,\tau}(\cdot)$, we can first approximate the latent $q_{\tau,t}$ by using a functional-coefficient quantile function:

$$q_{\tau,t} = \alpha_{0,\tau}(Z_t) + \sum_{l=1}^{\infty} \boldsymbol{\alpha}_{l,\tau}^T(Z_t)\mathbb{U}_{t-l}, \tag{11}$$

where $\mathbb{U}_t = (|u_{1t}|, \ldots, |u_{\kappa t}|)^T$ and each element of $\boldsymbol{\alpha}_{l,\tau}(\cdot)$ decreases at a geometric rate; that is, there exist positive constants $\mathfrak{b} < 1$ and c, such that $\max_{t\geq 1} |\alpha_{lij,\tau}(Z_t)| \leq c\mathfrak{b}^l$ for $i, j = 1, \ldots, \kappa$. Denote $\boldsymbol{\alpha}_\tau(\cdot) = (\alpha_{0,\tau}(\cdot), \boldsymbol{\alpha}_{1,\tau}^T(\cdot), \ldots, \boldsymbol{\alpha}_{m,\tau}^T(\cdot))^T$. Since $\alpha_{lij,\tau}(\cdot)$ decreases geometrically, by choosing an appropriate $m_n = m(n) = m$, we study following truncated equation (12) with increasing dimension of covariates:

$$q_{\tau,t} = \alpha_{0,\tau}(Z_t) + \sum_{l=1}^{m_n} \boldsymbol{\alpha}_{l,\tau}^T(Z_t)\mathbb{U}_{t-l} \equiv \boldsymbol{W}_t^T\boldsymbol{\alpha}_\tau(Z_t) = q_\tau(Z_t, \boldsymbol{W}_t), \tag{12}$$

where $\boldsymbol{W}_t = (1, \mathbb{U}_{t-1}^T, \ldots, \mathbb{U}_{t-m}^T)^T$ are covariates. Note that (12) can be regarded as an approximation of (11) and is similar to the model in Cai and Xu (2008). Under smoothness condition of coefficient functions $\boldsymbol{\alpha}_\tau(\cdot)$ presented later in Assumption B1 in Section 2.4, for any given grid point $z_0 \in \mathbb{R}$, when Z_t is in a neighborhood of z_0, $\boldsymbol{\alpha}_\tau(Z_t)$ can be approximated by a polynomial function as

$$\boldsymbol{\alpha}_\tau(Z_t) \approx \sum_{r=0}^{w} \boldsymbol{\alpha}_\tau^{(r)}(z_0)(Z_t - z_0)^r/r!,$$

where $\approx$ denotes the approximation by ignoring the higher orders and $\boldsymbol{\alpha}_\tau^{(r)}(\cdot)$ is the rth derivative of $\boldsymbol{\alpha}_\tau(\cdot)$. Thus,

$$q_{\tau,t} \approx \sum_{r=0}^{w} \boldsymbol{W}_t^T \boldsymbol{\delta}_{r,\tau}(Z_t - z_0)^r,$$

where $\boldsymbol{\delta}_{r,\tau} = \boldsymbol{\alpha}_\tau^{(r)}(z_0)/r!$. Hence, $\hat{\boldsymbol{\delta}} = \text{argmin}_{\boldsymbol{\delta}} Q(\boldsymbol{\delta})$, where $Q(\boldsymbol{\delta})$ is the locally weighted loss function for fixed κ, given by

$$Q(\boldsymbol{\delta}) = \sum_{t=m+1}^{n} \rho_\tau\{u_t - \sum_{r=0}^{w} \boldsymbol{W}_t^T \boldsymbol{\delta}_r (Z_t - z_0)^r\} K_{h_1}(Z_t - z_0), \tag{13}$$

where $\rho_\tau(y) = y[\tau - I(y < 0)]$ is called the "check" (loss) function, $I(A)$ is the indicator function of any set A, $K(\cdot)$ is a kernel function, $K_{h_1}(u) = K(u/h_1)/h_1$, and $h_1 = h_1(n)$ is a sequence of positive numbers tending to zero and controls the amount of smoothing used in estimation. In practice, if we smooth locally around Z_t and consider a local linear estimation, the loss function (13) becomes

$$G(\boldsymbol{\delta}) = \sum_{s \neq t}^{n} \rho_\tau\{u_s - \sum_{r=0}^{1} \boldsymbol{W}_s^T \boldsymbol{\delta}_r (Z_s - Z_t)^r\} K_{h_1}(Z_s - Z_t). \tag{14}$$

After yielding $\hat{\boldsymbol{\delta}}_{0,\tau}$ at τ by minimizing (14), $q_{\tau,t}$ can be estimated by

$$\hat{q}_{\tau,t} = \boldsymbol{W}_t^T \hat{\boldsymbol{\delta}}_{0,\tau}.$$

Remark 2.6. *(Truncation parameter $m(n)$). Welsh (1989) and He and Shao (2000) studied nonlinear M-estimation with increasing parametric dimension and discussed the possible expansion rate for the number of parameters $m(n)$. As for the quantile estimation for functional coefficient models with increasing dimension of covariates, Tang et al. (2013) considered estimation and variable selection for high-dimensional quantile varying coefficient models based on B-spline approach. They showed that the oracle property for varying coefficients can be preserved when $m_n^2 \log(p_n m_n)/n \to 0$, where p_n is* *the dimension of covariates and m_n is a parameter associated with degree of polynomial and internal knots. In this step, we are interested in studying varying interdependences among conditional quantiles, rather than determining the optimal number for m. In addition, we focus on estimating (12) using kernel-based approaches, which is necessary in order to obtain asymptotic properties for functional coefficients. Under Assumption B9 in Section 2.4, it will suffice to consider a truncation m as a sufficiently large constant multiple of $n^{1/7}$.*

To summarize, we propose the following two-step procedures for estimating $\boldsymbol{g}_\tau(\cdot)$:
Step One: Choosing the truncation parameter $m = cn^{1/7}$ for some $c > 0$ and estimating $\hat{\boldsymbol{\delta}}_{0,\tau}$ at each Z_t by minimizing (14). Then, approximating latent $q_{\tau,t}$ by $\hat{q}_{\tau,t} = \boldsymbol{W}_t^T \hat{\boldsymbol{\delta}}_{0,\tau}$.
Step Two: Having obtained $\hat{q}_{\tau,t}$ and given

$$\hat{\boldsymbol{X}}_t = (1, \hat{q}_{\tau,t-1,1}, \ldots, \hat{q}_{\tau,t-1,\kappa}, |u_{1(t-1)}|, \ldots, |u_{\kappa(t-1)}|)^T,$$

we can estimate $\boldsymbol{g}_\tau(\cdot)$ by a local linear estimation method; see Cai and Xu (2008) for details. In particular, let $\varsigma = 1$, minimize the following loss function $G(\Theta)$ at any given grid point $z_0 \in \mathbb{R}$ to obtain $\hat{\Theta}$, where

$$G(\Theta) = \sum_{t=1}^{n} \rho_\tau\{u_t - \sum_{r=0}^{\varsigma} \hat{\boldsymbol{X}}_t^T \Theta_{r,\tau}(Z_t - z_0)^r\} K_{h_2}(Z_t - z_0) \tag{15}$$

and $\Theta_{r,\tau} = \boldsymbol{g}_\tau^{(r)}(\cdot)/r!$. Similar to (14), $K_{h_2}(u) = K(u/h_2)/h_2$ and h_2 is the bandwidth used for this step, which is different from the bandwidth h_1 used in (14); see Remark 2.7 later in Section 2.4 for more discussions. Further improvement can be achieved by applying iteration to the foregoing two-stage procedures.

2.4 Large Sample Theory

To study the asymptotic distribution of the nonparametric quantile estimator, we impose some technical conditions as follows.

Assumption B.

B1: *Each entry in the vector* $\boldsymbol{\alpha}_\tau(\cdot)$ *is* $(w+1)th$ *order continuously differentiable in a neighborhood of* z_0 *for any* z_0*; Similarly, each entry in the vector* $\boldsymbol{g}_\tau(\cdot)$ *is* $(\varsigma+1)th$ *order continuously differentiable in a neighborhood of* z_0 *for any* z_0.

B2: $f_z(z)$ *is a continuously marginal density of* Z *and* $f_z(z_0) > 0$.

B3: *The distribution of* u *given* Z *and* $\boldsymbol{W}$ *has an everywhere positive conditional density* $f_{u|Z,\boldsymbol{W}}(\cdot)$*, which is bounded and satisfies the Lipschitz continuity condition. The kernel function* $K(\cdot)$ *is a bounded, symmetric density with a bounded support region. Let* $\mu_2 = \int \nu^2 K(\nu)d\nu$ *and* $\nu_0 = \int K^2(\nu)d\nu$.

B4: $\{(u_t, Z_t)\}$ *is a strictly stationary sequence with* α*-mixing coefficient* $\alpha(t)$ *which satisfies* $\sum_{t=1}^{\infty} t^{\iota}\alpha^{(\delta-2)/\delta}(t) < \infty$ *for some positive real number* $\delta > 2$ *and* $\iota > (\delta-2)/\delta$.

B5: *There exist (small) positive constants* $\varpi_1 > 0$ *and* $\varpi_2 > 0$ *such that*

$$P\{\max_{1\le t\le n} u_t^2 > n^{\varpi_1}\} \le \exp(-n^{\varpi_2}).$$

B6: *Let* $\boldsymbol{B}_n = \frac{1}{n}\sum_{t=m+1}^{n} \boldsymbol{W}_t\boldsymbol{W}_t^T$ *and denote the maximum and minimum eigenvalues of* $\boldsymbol{B}_n$ *as* $\lambda_{\max}(\boldsymbol{B}_n)$ *and* $\lambda_{\min}(\boldsymbol{B}_n)$*. Then,* $\liminf_{n\to\infty}\lambda_{\min}(\boldsymbol{B}_n) > 0$, $\limsup_{n\to\infty}\lambda_{\max}(\boldsymbol{B}_n) < \infty$*. It is assumed that* $\max_{1\le t\le n}\|\boldsymbol{W}_t\|^2 \le Cm$.

B7: $\boldsymbol{D}(z_0) \equiv E[\boldsymbol{W}_t\boldsymbol{W}_t^T|Z_t = z_0]$ *is positive-definite and continuous in a neighborhood of* z_0 *and* $\boldsymbol{D}^*(z_0) \equiv E[\boldsymbol{W}_t\boldsymbol{W}_t^T f_{u|Z,\boldsymbol{W}}(q_\tau(z_0,\boldsymbol{W}_t))|Z_t = z_0]$ *is positive-definite and continuous in a neighborhood of* z_0.

B8: *Let* $\boldsymbol{X}_{t1} = (1, \mathbb{U}_{t-1}^T)^T$*. Then,* $E\|\boldsymbol{X}_{t1}\|^{2\delta^*} < \infty$ *with* $\delta^* > \delta$.

B9: *The bandwidth* h_1 *satisfies* $h_1 \to 0$, $nh_1 \to \infty$*; The bandwidth* h_2 *satisfies* $h_2 \propto n^{-1/5}$, $h_2 \to 0$, $nh_2 \to \infty$*. In addition,* $h_1 = o(h_2)$ *and* $mh_1 \to 0$.

B10: $f(\boldsymbol{w},\boldsymbol{\omega}|\boldsymbol{u}_0,\boldsymbol{u}_\ell;\ell) \le H < \infty$ *for* $\ell \ge 1$*, where* $f(\boldsymbol{w},\boldsymbol{\omega}|\boldsymbol{u}_0,\boldsymbol{u}_\ell;\ell)$ *is the conditional density of* (Z_0, Z_ℓ) *given* $(\mathbb{U}_0 = \boldsymbol{u}_0, \mathbb{U}_\ell = \boldsymbol{u}_\ell)$.

B11: $n^{1/2-\delta/4} h_2^{\delta/\delta^*-1/2-\delta/4} = O(1)$.

Remark 2.7. *Assumptions B1-B3 are common in nonparametric literature. Assumption B4 is a standard assumption for α-mixing. Assumption B5 can be implied when the maximum of u_t^2 follows a generalized extreme value distribution, which is generally satisfied for weakly dependent data; see also Xiao and Koenker (2009). Assumption B6 guarantees the asymptotic behavior of regression estimators with increasing dimension of covariates, which is similar to Welsh (1989). Assumptions B7 and B8 are commonly required for the model identification and to ensure the convergence of $\boldsymbol{B}_n$ to $E[\boldsymbol{W}_t \boldsymbol{W}_t^T]$ when $\boldsymbol{W}_t$ is α-mixing, respectively. The assumption $h_1 = o(h_2)$ in Assumption B9 is about the under-smoothing at the step one, which is common for the two-stage nonparametric estimation approaches; see also Cai (2002) and Cai and Xiao (2012) for more discussions. The assumption $mh_1 \to 0$ in B9 is necessary for the proof of stochastic equi-continuity. Assumption B10 is very standard and used for the proof under mixing conditions. Assumption B11 allows one to verify standard Lindeberg-Feller conditions for asymptotic normality of the proposed estimators in the proof of Theorems 2.1; see Cai and Xu (2008) for details on nonparametric quantile regressions models for time series.*

It is necessary to discuss more about the strictly stationarity and α-mixing conditions in Assumption B4. By Pham (1986), a geometrically ergodic time series is a α-mixing sequence. Meanwhile, it is well-known that an ergodic Markov process initiated from its invariant distribution is (strictly) stationary. Thus, geometrical ergodicity plays an important role in establishing strictly stationarity and α-mixing properties. Indeed, sufficient conditions for the ergodicity of nonlinear time series have been studied extensively in literature. For example, Chen and Tsay (1993) provided sufficient conditions of geometrical ergodicity for functional-coefficient autoregressive (FAR) models. In addition, An and Chen (1997) and An and Huang (1996) surveyed various sufficient conditions for the ergodicity of nonlinear autoregressive models. In Supplement C, we show that under some regularity conditions, model (1) can generate a strictly stationary and α-mixing process. The derivation of these two properties is of independent interest, since the strictly stationarity and α-mixing assumptions are only imposed for proving the asymptotic theory and the main contribution of this article lies in estimating a new class of dynamic network. Therefore, we give conditions that imply these important probabilistic properties and corresponding proofs in Supplement C for space saving.

Theorem 2.1. *(Asymptotic Normality) Under Assumptions B1-B11, we have*

$$\sqrt{nh_2}\left(\hat{\boldsymbol{g}}_\tau(z_0)-\boldsymbol{g}_\tau(z_0)-\frac{h_2^2\mu_2}{2}\boldsymbol{g}_\tau^{(2)}(z_0)+o_p(h_2^2)\right) \xrightarrow{d} \mathcal{N}(0,(\Omega^*(z_0))^{-1}\boldsymbol{\Xi}(z_0)(\Omega^*(z_0))^{-1}/f_z(z_0)).$$

Here, $\Omega(z_0) \equiv E[\boldsymbol{X}_t\boldsymbol{X}_t^T|Z_t = z_0]$, $\Omega^*(z_0) \equiv E[\boldsymbol{X}_t\boldsymbol{X}_t^T f_{u|Z,\boldsymbol{X}}(q_\tau(z_0,\boldsymbol{X}_t))|Z_t = z_0]$ *with* $q_\tau(z_0,\boldsymbol{X}_t) = \boldsymbol{g}_\tau^T(z_0)\boldsymbol{X}_t$ *and* $f_{u|Z,\boldsymbol{X}}(\cdot)$ *satisfying Assumption B3. In addition,* $\boldsymbol{\Xi}(z_0) \equiv \tau(1-\tau)\nu_0[\Omega(z_0) + \Gamma_{20}\boldsymbol{D}^*(z_0)^{-1}\Gamma_{20}^T]$, *where*

$$\Gamma_{20} \equiv E\left\{ (f_{u|Z,\boldsymbol{W}}(q_\tau(z_0,\boldsymbol{W}_t))\boldsymbol{X}_t^T\boldsymbol{g}_\tau(z_0)) \begin{pmatrix} 0_{1\times(\kappa m+1)} \\ \boldsymbol{W}_t^T \\ \vdots \\ \boldsymbol{W}_t^T \\ 0_{\kappa\times(\kappa m+1)} \end{pmatrix} \middle| Z_t = z_0 \right\}$$

is a $(2\kappa+1)\times(\kappa m+1)$ *matrix.*

Remark 2.8. *Notice that in Theorem 2.1, the asymptotic bias does not depend on* h_1. *Indeed, since the estimation in the step one is under-smoothed by Assumption B9, the part that relies on* h_1 *in the asymptotic bias term disappears. However, different from the conventional nonparametric estimation,* $\boldsymbol{\Xi}(z_0)$ *in Theorem 2.1 contains* $\boldsymbol{D}^*(z_0)$. *This formation of asymptotic variance appears because of the fact that* $\hat{\boldsymbol{X}}_t$ *contains* $\hat{q}_{\tau,t-1}$, *which is estimated in the step one of our two-stage approaches and therefore includes* *information of $\boldsymbol{W}_t$. Similar results of asymptotic variance were also obtained by Xiao and Koenker (2009), which can be seen as a nature of any two-stage approach; see, for example, Cai, Das, Xiong and Wu (2006) for details.*

Remark 2.9. *(Bandwidth Selection) Finally, we would like to address how to select the bandwidth h_2 at the second step. It is well known that the bandwidth plays an essential role in the trade-off between reducing bias and variance. In view of* (15)*, it is about selecting the bandwidth in the context of estimating the coefficient functions in the quantile regression. Therefore, we recommend the method proposed in Cai and Xu (2008) for selecting h_2 in* (15).

2.5 Inferences

For the purpose of constructing pointwise confidence intervals, we turn to discussing how to obtain consistent estimator of the asymptotic covariance matrix. Toward this end, we need to estimate $\Omega(z_0)$, $\Omega^*(z_0)$, $\boldsymbol{D}^*(z_0)$ and Γ_{20} consistently. For this purpose, define

$$\hat{\Gamma}_{20} = \begin{pmatrix} 0_{1\times(\kappa m+1)} \\ \frac{1}{n}\sum_{t=1}^{n}(w_{1t}\hat{\boldsymbol{X}}_t^T\hat{\boldsymbol{g}}_\tau(z_0))\boldsymbol{W}_t^T K_{h_2}(Z_t - z_0) \\ \vdots \\ \frac{1}{n}\sum_{t=1}^{n}(w_{1t}\hat{\boldsymbol{X}}_t^T\hat{\boldsymbol{g}}_\tau(z_0))\boldsymbol{W}_t^T K_{h_2}(Z_t - z_0) \\ 0_{\kappa\times(\kappa m+1)} \end{pmatrix},$$

where $w_{1t} = I(\boldsymbol{W}_t^T\hat{\boldsymbol{\alpha}}_\tau(z_0) - \delta_{1n} < u_t \le \boldsymbol{W}_t^T\hat{\boldsymbol{\alpha}}_\tau(z_0) + \delta_{1n})/(2\delta_{1n})$ for any $\delta_{1n} \to 0$. In Appendix, it shows that

$$\hat{\Gamma}_{20} = f_z(z_0)\Gamma_{20} + o_p(1). \tag{16}$$

As for $\Omega(z_0)$, $\Omega^*(z_0)$ and $\boldsymbol{D}^*(z_0)$, define

$$\hat{\Omega}(z_0) = \frac{1}{n}\sum_{t=1}^{n}\hat{\boldsymbol{X}}_t\hat{\boldsymbol{X}}_t^T K_{h_2}(Z_t - z_0),$$

$$\hat{\Omega}^*(z_0) = \frac{1}{n}\sum_{t=1}^{n} w_{2t}\hat{\boldsymbol{X}}_t\hat{\boldsymbol{X}}_t^T K_{h_2}(Z_t - z_0)$$

and

$$\hat{\boldsymbol{D}}^*(z_0) = \frac{1}{n}\sum_{t=1}^{n} w_{1t}\boldsymbol{W}_t\boldsymbol{W}_t^T K_{h_1}(Z_t - z_0),$$

where $w_{2t} = I(\hat{\boldsymbol{g}}_\tau^T(z_0)\hat{\boldsymbol{X}}_t - \delta_{2n} < u_t \le \hat{\boldsymbol{g}}_\tau^T(z_0)\hat{\boldsymbol{X}}_t + \delta_{2n})/(2\delta_{2n})$ for any $\delta_{2n} \to 0$ as $n \to \infty$. Similar to the proof in Cai and Xu (2008), one can show that $\hat{\Omega}(z_0) = f_z(z_0)\Omega(z_0) + o_p(1)$, $\hat{\Omega}^*(z_0) = f_z(z_0)\Omega^*(z_0) + o_p(1)$ and $\hat{\boldsymbol{D}}^*(z_0) = f_z(z_0)\boldsymbol{D}^*(z_0) + o_p(1)$. Therefore, a consistent estimate of $(\Omega^*(z_0))^{-1}\boldsymbol{\Xi}(z_0)(\Omega^*(z_0))^{-1}/f_z(z_0)$ can be given by $(\hat{\Omega}^*(z_0))^{-1}\hat{\boldsymbol{\Xi}}(z_0)(\hat{\Omega}^*(z_0))^{-1}$, where $\hat{\boldsymbol{\Xi}} = \tau(1-\tau)\nu_0[\hat{\Omega}(z_0) + \hat{\Gamma}_{20}\hat{\boldsymbol{D}}^*(z_0)^{-1}\hat{\Gamma}_{20}^T]$.

3 A Monte Carlo Study

In this section, we provide a simulation example to exam the performance of our two-stage estimations for functional coefficients. In this example, the bandwidth is selected based on a rule-of-thumb idea similar to the procedure in Cai and Xiao (2012) as follows. First, we use a data-driven bandwidth selector as suggested in Cai and Xu (2008) to obtain an initial bandwidth denoted by $\hat{h}_0$ which should be $O(n^{-1/5})$. At step one, the bandwidth should be under-smoothed. Therefore, by following the idea in Cai (2002) and Cai and Xiao (2012) for two-step approaches, we take the bandwidth as $\hat{h}_1 = A_0 \times \hat{h}_0$ with $A_0 = n^{-1/10}$ so that $\hat{h}_1$ satisfies Assumption B9. At step two, we choose optimal bandwidth $\hat{h}_2$ by the nonparametric AIC criterion as in Cai and Xu (2008). Finally, the Epanechnikov kernel $K(x) = 0.75(1-x^2)I(|x| \le 1)$ is used.

In this example, the data are generated from the following process:

$$\begin{pmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \\ u_{4t} \end{pmatrix} = \begin{pmatrix} \sigma_{1t} & 0 & 0 & 0 \\ 0 & \sigma_{2t} & 0 & 0 \\ 0 & 0 & \sigma_{3t} & 0 \\ 0 & 0 & 0 & \sigma_{4t} \end{pmatrix} \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \end{pmatrix},$$

and

$$\begin{pmatrix} \sigma_{1t} \\ \sigma_{2t} \\ \sigma_{3t} \\ \sigma_{4t} \end{pmatrix} = \begin{pmatrix} \gamma_{10}(Z_t) \\ \gamma_{20}(Z_t) \\ \gamma_{30}(Z_t) \\ \gamma_{40}(Z_t) \end{pmatrix} + \begin{pmatrix} \gamma_{11,\epsilon_{1t}}(Z_t) & \gamma_{12,\chi_{1t}}(Z_t) & \gamma_{13,\epsilon_{1t}}(Z_t) & \gamma_{14,\chi_{1t}}(Z_t) \\ \gamma_{21,\epsilon_{2t}}(Z_t) & \gamma_{22,\chi_{2t}}(Z_t) & \gamma_{23,\epsilon_{2t}}(Z_t) & \gamma_{24,\chi_{2t}}(Z_t) \\ \gamma_{31,\epsilon_{3t}}(Z_t) & \gamma_{32,\chi_{3t}}(Z_t) & \gamma_{33,\epsilon_{3t}}(Z_t) & \gamma_{34,\chi_{3t}}(Z_t) \\ \gamma_{41,\epsilon_{4t}}(Z_t) & \gamma_{42,\chi_{4t}}(Z_t) & \gamma_{43,\epsilon_{4t}}(Z_t) & \gamma_{44,\chi_{4t}}(Z_t) \end{pmatrix} \begin{pmatrix} \sigma_{1(t-1)} \\ \sigma_{2(t-1)} \\ \sigma_{3(t-1)} \\ \sigma_{4(t-1)} \end{pmatrix}$$
$$+ \begin{pmatrix} \beta_{11}(Z_t) & \beta_{12}(Z_t) & \beta_{13}(Z_t) & \beta_{14}(Z_t) \\ \beta_{21}(Z_t) & \beta_{22}(Z_t) & \beta_{23}(Z_t) & \beta_{24}(Z_t) \\ \beta_{31}(Z_t) & \beta_{32}(Z_t) & \beta_{33}(Z_t) & \beta_{34}(Z_t) \\ \beta_{41}(Z_t) & \beta_{42}(Z_t) & \beta_{43}(Z_t) & \beta_{44}(Z_t) \end{pmatrix} \begin{pmatrix} |u_{1(t-1)}| \\ |u_{2(t-1)}| \\ |u_{3(t-1)}| \\ |u_{4(t-1)}| \end{pmatrix}.$$

Here, $\gamma_{10}(z) = \gamma_{30}(z) = 1.5\exp(-3(z+1)^2) + \exp(-8(z-1)^2)$, $\gamma_{20}(z) = \gamma_{40}(z) = 1.5\exp(-3(z-1)^2) + \exp(-8(z+1)^2)$. In addition, when $j = 1$ and 3:

$$\begin{aligned} \gamma_{1j,\epsilon_{1t}}(z) &= \left(\frac{0.15\exp(2z)}{1+\exp(2z)}\right)\epsilon_{1t}, \\ \gamma_{2j,\epsilon_{2t}}(z) &= (0.1\sin(0.8\pi z) + 0.1)\epsilon_{2t}, \\ \gamma_{3j,\epsilon_{3t}}(z) &= (0.1\sin(0.8\pi z) + 0.1)\epsilon_{3t}, \\ \gamma_{4j,\epsilon_{4t}}(z) &= (0.1\cos(0.8\pi z) + 0.1)\epsilon_{4t}, \\ \beta_{1j}(z) &= \frac{0.15\exp(2z)}{1+\exp(2z)}, \\ \beta_{2j}(z) &= 0.1\sin(0.8\pi z) + 0.1, \\ \beta_{3j}(z) &= 0.1\sin(0.8\pi z) + 0.1, \\ \beta_{4j}(z) &= 0.1\cos(0.8\pi z) + 0.1, \end{aligned}$$

where $\epsilon_{it} = 0.2U_{it}^2 + 0.8$ with $U_{it} \sim$ i.i.d. Uniform $[0,1]$ for $i = 1,2,3,4$. Moreover, when

$j = 2$ and 4:

$$\begin{aligned}
\gamma_{1j,\chi_{1t}}(z) &= (0.1\sin(0.8\pi z) + 0.1)\chi_{1t}, \\
\gamma_{2j,\chi_{2t}}(z) &= \left(\frac{0.15\exp(2z)}{1+\exp(2z)}\right)\chi_{2t}, \\
\gamma_{3j,\chi_{3t}}(z) &= (0.1\cos(0.8\pi z) + 0.1)\chi_{3t}, \\
\gamma_{4j,\chi_{4t}}(z) &= (0.1\sin(0.8\pi z) + 0.1)\chi_{4t}, \\
\beta_{1j}(z) &= 0.1\sin(0.8\pi z) + 0.1, \\
\beta_{2j}(z) &= \frac{0.15\exp(2z)}{1+\exp(2z)}, \\
\beta_{3j}(z) &= 0.1\cos(0.8\pi z) + 0.1, \\
\beta_{4j}(z) &= 0.1\sin(0.8\pi z) + 0.1,
\end{aligned}$$

where $\chi_{it} = 0.2\exp(U_{it}) + 0.8$ with $U_{it} \sim$ i.i.d. Uniform $[0,1]$ for $i = 1,2,3,4$. Finally, ε_{it} are mutually i.i.d. from $\mathcal{N}(0,1)$ for $i = 1,2,3,4$. Thus, our working model is:

$$\begin{pmatrix} q_{\tau,t,1} \\ q_{\tau,t,2} \\ q_{\tau,t,3} \\ q_{\tau,t,4} \end{pmatrix} = \begin{pmatrix} \gamma_{10,\tau}(Z_t) \\ \gamma_{20,\tau}(Z_t) \\ \gamma_{30,\tau}(Z_t) \\ \gamma_{40,\tau}(Z_t) \end{pmatrix} + \begin{pmatrix} \gamma_{11,\tau}(Z_t) & \gamma_{12,\tau}(Z_t) & \gamma_{13,\tau}(Z_t) & \gamma_{14,\tau}(Z_t) \\ \gamma_{21,\tau}(Z_t) & \gamma_{22,\tau}(Z_t) & \gamma_{23,\tau}(Z_t) & \gamma_{24,\tau}(Z_t) \\ \gamma_{31,\tau}(Z_t) & \gamma_{32,\tau}(Z_t) & \gamma_{33,\tau}(Z_t) & \gamma_{34,\tau}(Z_t) \\ \gamma_{41,\tau}(Z_t) & \gamma_{42,\tau}(Z_t) & \gamma_{43,\tau}(Z_t) & \gamma_{44,\tau}(Z_t) \end{pmatrix} \begin{pmatrix} q_{\tau,t-1,1} \\ q_{\tau,t-1,2} \\ q_{\tau,t-1,3} \\ q_{\tau,t-1,4} \end{pmatrix}$$
$$+ \begin{pmatrix} \beta_{11,\tau}(Z_t) & \beta_{12,\tau}(Z_t) & \beta_{13,\tau}(Z_t) & \beta_{14,\tau}(Z_t) \\ \beta_{21,\tau}(Z_t) & \beta_{22,\tau}(Z_t) & \beta_{23,\tau}(Z_t) & \beta_{24,\tau}(Z_t) \\ \beta_{31,\tau}(Z_t) & \beta_{32,\tau}(Z_t) & \beta_{33,\tau}(Z_t) & \beta_{34,\tau}(Z_t) \\ \beta_{41,\tau}(Z_t) & \beta_{42,\tau}(Z_t) & \beta_{43,\tau}(Z_t) & \beta_{44,\tau}(Z_t) \end{pmatrix} \begin{pmatrix} |u_{1(t-1)}| \\ |u_{2(t-1)}| \\ |u_{3(t-1)}| \\ |u_{4(t-1)}| \end{pmatrix},$$

where Z_t is generated from Uniform $[-2,2]$ independently. Notice that our working model corresponds to model (1) with $\kappa = 4$, $\mathbb{U}_t = (|u_{1t}|, |u_{2t}|, |u_{3t}|, |u_{4t}|)^T$, $q = p = 1$ and $Z_{it} = Z_t$. Also, note that $\gamma_{i0,\tau}(\cdot) = \gamma_{i0}(\cdot)\Phi^{-1}(\tau)$, $\gamma_{i1,\tau}(\cdot) = \gamma_{i1}(\cdot)(0.2\tau^2 + 0.8)$, $\gamma_{i3,\tau}(\cdot) =$

$\gamma_{i3}(\cdot)(0.2\tau^2 + 0.8)$, while $\gamma_{i2,\tau}(\cdot) = \gamma_{i2}(\cdot)(0.2\exp(\tau) + 0.8)$, $\gamma_{i4,\tau}(\cdot) = \gamma_{i4}(\cdot)(0.2\exp(\tau) + 0.8)$ and $\beta_{ij,\tau}(\cdot) = \beta_{ij}(\cdot)\Phi^{-1}(\tau)$ for $i, j = 1, 2, 3, 4$, with $\Phi(\cdot)$ being the distribution function of the standard normal. Therefore, $\gamma_{i0,\tau}(\cdot)$, $\gamma_{ij,\tau}(\cdot)$ and $\beta_{ij,\tau}(\cdot)$ are functions of τ, suggesting different covariate effects at different levels of τ.

To assess the finite sample performance of the proposed nonparametric estimators, we compute the mean absolute deviation error (MADE) for $\hat{\gamma}_{i0,\tau}(\cdot)$, $\hat{\gamma}_{ij,\tau}(\cdot)$ and $\hat{\beta}_{ij,\tau}(\cdot)$, defined as

$$\text{MADE}(\gamma) = \frac{1}{n_0}\sum_{k}^{n_0} |\hat{\gamma}_\tau(z_k) - \gamma_\tau(z_k)|, \text{ and } \text{MADE}(\beta_{ij,\tau}) = \frac{1}{n_0}\sum_{k}^{n_0} |\hat{\beta}_{ij,\tau}(z_k) - \beta_{ij,\tau}(z_k)|,$$

where $\gamma_\tau(\cdot)$ can be either $\gamma_{ij,\tau}(\cdot)$ or $\gamma_{i0,\tau}(\cdot)$, both $\hat{\gamma}_\tau(\cdot)$ and $\hat{\beta}_{ij,\tau}(\cdot)$ are local linear quantile estimates of $\gamma_\tau(\cdot)$ and $\beta_{ij,\tau}(\cdot)$, respectively, and $\{z_k = 0.1(k-1) - 1.75 : 1 \le k \le n_0 = 36\}$ are the grid points. Also note that in this example, $q_{\tau,t,i} = \sigma_{it} F_\varepsilon^{-1}(\tau) = 0$ when $\tau = 0.5$, which leads the quantile regression problem to be ill-posed so that the results for $\tau = 0.5$ are omitted. Therefore, we only consider τ's level to be 0.05, 0.15, 0.85 and 0.95 and the sample sizes are $n = 500$, 1500 and 4000. For each setting, we replicate simulation 500 times and compute the median and standard deviation (in parentheses) of 500 MADE values. The results are summarized in Tables 1-4 in Appendix.

One can see from Tables 1-4 that both median and standard deviation of 500 MADE values steadily decrease as the sample size increases for all four values of τ. Moreover, the performances for $\gamma_{i0,\tau}(\cdot)$ and $\beta_{ij,\tau}(\cdot)$ at $\tau = 0.15$ and 0.85 are slightly better than those for $\tau = 0.05$ and 0.95. This observation is because of the sparsity of data in the tailed regions, which is similar to that in Cai and Xu (2008). Nevertheless, since the data that are used to estimate $\gamma_{ij,\tau}(\cdot)$ at $\tau = 0.05$ and 0.95 are conditional quantiles, the distributional information at tailed regions is preserved, which may reduce the problem of data sparsity. For this reason, the performances for $\gamma_{ij,\tau}(\cdot)$ at $\tau = 0.15$ and 0.85 are not necessarily superior to that for $\tau = 0.05$ and 0.95. In general, the results of this simulated experiment demonstrate that the proposed procedure is reliable and works fairly well.

4 A Real Example

4.1 Empirical Models

In this section, the proposed model and estimation methods are applied to constructing and estimating a new class of dynamic financial network in international equity markets. Different from existing literatures, the interdependences of this class of network vary with a smoothing variable of general economy. To capture the intertemporal transition of risk and avoid endogeneity, we consider the interaction between current and one-day lagged VaR. In particular, we define each linkage between a pair of VaRs in our network as the sensitivity of VaR of returns of one market index at time t to that of another at time $t-1$. Therefore, our network can be written as following equation system:

$$\mathrm{VaR}_{it} = \boldsymbol{\gamma}_{i,\tau}^{T}(Z_{t-1})\mathrm{VaR}_{t-1}, \quad i = 1, 2, \ldots, \kappa, \tag{17}$$

where $\mathrm{VaR}_{t-1} = (\mathrm{VaR}_{1(t-1)}, \ldots, \mathrm{VaR}_{\kappa(t-1)})^T$ is a vector of VaRs for all market index returns at time $t-1$ and VaR_{it} is the VaR of the ith market index return at time t, which is described as follows

$$\mathrm{VaR}_{it} = -\inf\{u \in \mathbb{R} : P(u_{it} > u|\mathcal{F}_{i,t-1}) \leq 1-\tau\} = -\inf\{u \in \mathbb{R} : F(u|\mathcal{F}_{i,t-1}) > \tau\}$$

for $i = 1, 2, \cdots, \kappa$ at a given $\tau \in (0,1)$. Here, $\mathcal{F}_{i,t-1}$ is the information set to present all information of the ith return available at time $t-1$ and $F(\cdot|\mathcal{F}_{i,t-1})$ represents the conditional distribution function of u_{it} given $\mathcal{F}_{i,t-1}$. In addition, Z_{t-1} is a smoothing variable of general economy and $\boldsymbol{\gamma}_{i,\tau}(\cdot) = (\gamma_{i1,\tau}(\cdot), \ldots, \gamma_{i\kappa,\tau}(\cdot))^T$ is a $\kappa \times 1$ vector of functional coefficients. Then, we extract the quantile estimation of functional coefficients from equation system (17) and construct a weighted matrix $|\hat{\mathbf{\Gamma}}_\tau(Z_{t-1})|$ as our financial network as follows:

$$|\hat{\mathbf{\Gamma}}_\tau(Z_{t-1})| = \begin{pmatrix} |\hat{\gamma}_{11,\tau}(Z_{t-1})| & |\hat{\gamma}_{12,\tau}(Z_{t-1})| & \dots & |\hat{\gamma}_{1\kappa,\tau}(Z_{t-1})| \\ |\hat{\gamma}_{21,\tau}(Z_{t-1})| & |\hat{\gamma}_{22,\tau}(Z_{t-1})| & \dots & |\hat{\gamma}_{2\kappa,\tau}(Z_{t-1})| \\ \vdots & \vdots & \ddots & \vdots \\ |\hat{\gamma}_{\kappa 1,\tau}(Z_{t-1})| & |\hat{\gamma}_{\kappa 2,\tau}(Z_{t-1})| & \dots & |\hat{\gamma}_{\kappa\kappa,\tau}(Z_{t-1})| \end{pmatrix}.$$

In this matrix $|\hat{\mathbf{\Gamma}}_\tau(Z_{t-1})|$, $|\hat{\gamma}_{ij,\tau}(Z_{t-1})|$ represents the absolute value of the sensitivity of VaR of return for the market index j at time t to that of return for the index i at time $t-1$, under τ-th quantile level, and is driven by the smoothing variable Z_{t-1}. Here, taking absolute value on each $\hat{\gamma}_{ij,\tau}(Z_{t-1})$ enables us to calculate and analyze indicators of connectedness, and details will be reported in Section 4.3 later. Thus, matrix $|\hat{\mathbf{\Gamma}}_\tau(Z_{t-1})|$ is useful to capture risk interdependence and how it changes with a smoothing variable Z_{t-1}. Notice that entries of weighted matrix $|\hat{\mathbf{\Gamma}}_\tau(Z_{t-1})|$ correspond to the absolute value of the estimated $\gamma_{ij,\tau}(\cdot)$ in the network model (3), so our two-stage procedures can be applied here for direct estimation of the interdependence among VaRs of returns for the market indexes. In general, the proposed framework is particularly suitable to investigate the dynamic characteristics of risk spillover across world market indexes under the changes of economic circumstance.

It is necessary to mention that our interest here is in studying how an observable and time-dependent circumstance of general economy affects risk interdependence among financial network, rather than exploring unobservable common factors that determine the quantile co-movement in financial markets as in Ando and Bai (2020). In addition, Yang and Zhou (2017) leaded an empirical study of the interaction between quantitative easing in the U.S. and patterns in volatility spillover using network-based spillover indices extended from Diebold and Yílmaz (2014). Their work provided an insight in the relation between macroeconomic variables and risk network, though the econometric model that can describe this relation is far from being revealed. Meanwhile, Chen, Härdle and Okhrin (2019) studied the risk propagation and dynamics in the network by developing a panel quantile autoregression involving network effects that are quantified through a time-varying adjacency matrix. Their model is effective for network analysis and geographic comparison, yet difficult to model the interplay between the network variation and states of general economy.

4.2 Data

Our dataset includes the daily series between January 5, 2006 and February 10, 2021 for four major world equity market indexes: the U.K. FTSE 100 Index, the Japanese Nikkei 225 Index, the U.S. S&P 500 Composite Index and the Chinese Shanghai Composite Index. We model the ith index's return series $u_{it} = 10\log(\Pi_{it}/\Pi_{i(t-1)})$, where $i = 1, 2, 3, 4$ correspond to the four aforementioned market indexes in turn and Π_{it} is ith index level on the tth day. The studies of world market indexes help to explore the dynamic of risk dependences in the global financial market, and the time range of data includes the financial crisis in the U.S. in 2008, the European sovereign debt crisis of 2011-2012 and the COVID-19 pandemic starting from 2019. The daily series of four market indexes are downloaded in Yahoo Finance and the estimation sample sizes are 3254. Although it is feasible to introduce more kinds of market index into the equation system (17), due to the computational burdens, we only consider risk co-dependences among four major markets' indexes.

As for the smoothing variable Z_t, we choose $Z_t = 10\log(\Upsilon_t/\Upsilon_{t-1})$, where Υ_t is the U.S. dollar index on the tth day and can be downloaded from the Federal Reserve Bank of St. Louis. The U.S. dollar index measures value of U.S. dollar against the currencies of a broad group of major U.S. trading partners, higher values of the index indicate a stronger U.S. dollar. This choice of smoothing variable is reasonable, because the exchange rate has been regarded as an important factor associated with international transmission of risk in many empirical studies. For instance, Menkhoff, Sarno, Schmelling and Schrimpf (2012) discussed the relation between innovations in global foreign exchange volatility and excess returns arising from strategies of carry trade, through which the risk spillover transmits from one country to others. In addition, Yang and Zhou (2017) showed that volatility spillover intensity increases with U.S. dollar depreciation. We do not claim that the U.S. dollar index is the only choice for smoothing variable, but we choose the U.S. dollar index because it contains more information about risk transmission among international equity markets. It is desirable to consider other variables of economic status as the smoothing variable and this may be left in a future study.

4.3 Estimation Results

The empirical analysis in this section includes two steps: First, we estimate $\gamma_{ij,\tau}(Z_{t-1})$ for each market index in the equation system (17) under $\tau = 0.05$. Second, we use the estimated value of $\gamma_{ij,\tau}(Z_{t-1})$ to construct the weighted matrix $|\hat{\mathbf{\Gamma}}_\tau(Z_{t-1})|$, and do network analysis on this matrix. Before exploring the weighted matrix $|\hat{\mathbf{\Gamma}}_\tau(Z_{t-1})|$, it is important to exam whether each $\gamma_{ij,\tau}(Z_{t-1})$ in (17) varies significantly with Z_{t-1} or not. To this end, we estimate each $\gamma_{ij,\tau}(Z_{t-1})$ and corresponding 95% pointwise confidence intervals. Figure 1 in Appendix depicts the corresponding estimation results, in which ij-th panel represents the result for $\gamma_{ij,\tau}(\cdot)$, respectively. The black solid line in each panel of Figure 1 represents the estimates of the $\gamma_{ij,\tau}(\cdot)$ for $1 \leq i \leq 4$ and $1 \leq j \leq 4$ in (17) along various values of Z_{t-1} under $\tau = 0.05$, and the red dashed lines are 95% pointwise confidence intervals for each estimate without bias correction. From Figure 1, we clearly see that these coefficient functions vary significantly over the interval [-0.075, 0.075], which means that we can not use fixed-coefficient dynamic quantiles models to fit the data.

Next, we consider analyzing weighted matrix $|\hat{\mathbf{\Gamma}}_\tau(Z_{t-1})|$, in which each entry is $|\gamma_{ij,\tau}(Z_{t-1})|$. By studying the dynamic of this weighted matrix $|\hat{\mathbf{\Gamma}}_\tau(Z_{t-1})|$ driven by Z_{t-1}, one can find how the risk interdependence behaves among the four major markets under conditions of U.S. dollar depreciation or appreciation. To simplify the notation, Z_{t-1} and τ are dropped from $|\hat{\gamma}_{ij,\tau}(Z_{t-1})|$ and $|\hat{\gamma}_{ji,\tau}(Z_{t-1})|$ in the weighted matrix $|\hat{\mathbf{\Gamma}}_\tau(Z_{t-1})|$ in what follows. So, $|\hat{\gamma}_{ji}|$ in the weighted matrix $|\hat{\mathbf{\Gamma}}_\tau(Z_{t-1})|$ represents the intensity of influence from the risk of market index i at time $t-1$ to that of market index j at time t. For the purpose of visualization, by following Härdle et al. (2016), we first define the levels of connectedness. The connectedness with respect to incoming links is defined as $\sum_{i=1}^{4} |\hat{\gamma}_{ji}|$, which is the strength of the influence of all indexes' VaR at time $t-1$ to the VaR of market index j at time t. Analogously, the connectedness with respect to outgoing links is defined as $\sum_{i=1}^{4} |\hat{\gamma}_{ij}|$, which is the strength of the influence of index j's VaR at time $t-1$ to the VaRs of all indexes at time t. Here, $i, j = 1, 2, 3, 4$ correspond to the four aforementioned market indexes in turn. The connectedness with respect to incoming links measures the risk spillover that was emitted from all four market indexes one day ago and is received currently by a certain market index; the connectedness with respect to outgoing links measures the risk spillover emitted from a certain market index one day ago and is received currently by all market indexes. Intuitively, the connectedness with respect to incoming links measures exposures of individual indexes to systemic shocks from the financial network, while the connectedness with respect to outgoing links measures contributions of individual indexes for risk events in the network. Other than connectedness with respect to incoming links and outgoing links, we also analyze the total connectedness in the global market, which is equal to $\sum_{j=1}^{4}\sum_{i=1}^{4} |\hat{\gamma}_{ij}|$ and indicates the total risk spillover in the global market, see Härdle et al. (2016) for more applications about these types of connectedness.

Figures 2 and 3 in Appendix present corresponding results along the same values of Z_{t-1}, under $\tau = 0.05$. In Figure 2, each panel displays the connectedness with respect to both incoming and outgoing links subject to U.S. dollar change. The solid line in each panel represents values of connectedness with respect to outgoing links and the dashed line indicates values of connectedness with respect to incoming links. For Figure 3, the vertical axis measures the total connectedness appeared in international equity markets. The horizontal axises in both figures are the same as that in each panel of Figure 1. Figure 2 shows that the curves of all four major market indexes vary greatly over the interval [-0.075, 0.075] and exhibit U-shaped. In particular, when the U.S. dollar experiences appreciation and during the "bad times" of the market (when Z_{t-1} is large and $\tau = 0.05$), domestic prices of commodity in Europe, Japan and China may increase, which pose risks on domestic companies. Then, all investors who invested corporations in European, Japanese and Chinese markets suffer from loss of returns, causing both curves of incoming and outgoing links to go up in all three markets. For the U.S. market, U.S. assets may become favorable among global investors during the U.S. dollar appreciation, while investors in the U.S. market who invested corporations in the rest of the world face loss of returns. These two forces lead the U.S. market to be both more influential to the global market and to be influenced by global market more easily, respectively. Thus, both curves in the panel of S&P 500 index increase. As for the case when U.S. dollar depreciated, profits of investment on domestic corporations in European, Japanese and Chinese markets may increase, which lead the total amount of investment in these three markets to grow. As a result, both types of curves in all three markets, as well as the curve of incoming links in the U.S. market increase. Nevertheless, global investors who invested assets in the U.S. market subject to adverse situation, which results in an upward movement of curve of outgoing links of S&P 500 index.

It is interesting that in the European and Japanese markets, during the U.S. dollar appreciation (Z_{t-1} is large), the curve of outgoing links dominates that of incoming links. These dynamic patterns in the European and Japanese markets may be explained by the so called "carry trade". The carry trade refers to borrowing a low-yielding asset and buying a higher-yielding foreign asset to earn the interest rate differential plus the expected foreign currency appreciation. Due to the relatively lower interest rate in the European and Japanese markets within our time span of study, as Z_{t-1} is large, carry traders who borrowed low-yielding assets from Japanese or European markets and bought assets from the U.S. market enjoy the increase of excess returns to carry trade. This increase of excess returns may attract more carry traders to borrow Japanese or European assets and thus, make these two markets more influential to the global market. For this reason, the curve of outgoing links become larger than that of incoming links in these two markets. While in the U.S. market, since the price of risky assets rely heavily on the demand of carry trade during U.S. dollar appreciation, it becomes much easier for the U.S. market to be affected by the global market. Therefore, the curve of incoming links dominates that of outgoing links in the U.S. market.

On the other hand, during U.S. dollar depreciation, carry traders who borrowed Japanese or European assets may be unable to repay due to the significant loss of returns, which cause the Japanese or European markets to become more vulnerable. Consequently, the curve of incoming links in both Japan and Europe increases drastically relative to the curve of outgoing links. Yet in the U.S. market, the price of risky assets affect the solvency of carry traders in the world, which let the U.S. market become more influential to the world. Thus, the curve of outgoing links rises compared to that of incoming links of S&P 500 index. As for the Chinese market, when U.S. dollar depreciated, corporations associated with export subject to harmful impact. Under this unfavorable environment, investors in China may be more willing to invest assets from outside of Chinese market. This trend magnifies the influence of global risk events on the Chinese market, causing the curve of incoming links dominates that of outgoing links.

Figure 3 sheds light on the variation of risk spillover in the global financial market. Observed that in Figure 3, the total connectedness of all four market indexes demonstrates an U-shaped and asymmetric pattern. It means that total risk spillover in the four major markets decrease when Z_{t-1} becomes larger within the interval [-0.075, -0.025]. As Z_{t-1} exceeds -0.025, the risk spillover intensity is magnified. In general, Figure 3 shows that the response of total risk spillover to the U.S. dollar change switches its pattern at a certain threshold of the U.S. dollar change, which is a relatively new result in literature.

5 Conclusion

In this paper, we investigate a functional coefficient vector autoregressive model for conditional quantiles. A two-stage kernel method is proposed to estimate coefficients functionals and the properties of asymptotic normality for the proposed estimators are established. The simulation results show that our new methods of estimation work fairly well. In addition, there is little literatures regarding the relationship between the variation of financial network and the general state of economy. Based on our two-stage estimation approaches, the proposed framework allows to study how specific state of economy influences the network characteristics of risk spillover in a financial system.

There are several issues still worth of further studies. First, it is interesting to visualize the topological change of our financial network and to measure the transition of risk spillover among different market indexes when the general economy is shifting. Technically, these studies can be realized by our econometric model. Second, the asymptotic properties of functional coefficients in our model provide solid theory to test the abnormal variation of financial network. Third, it is meaningful to allow for cross-sectional dependence in the current model. Although some methods have been developed to deal with cross-sectional dependence in the literature of conditional mean models, due to the nature of conditional quantile model, it is not obvious to extend these under the quantile setting. Finally, if Z_t in (2) is time, then the model in (2) provides a good start for studying conditional quantile estimation of ARCH- and GARCH-type models with time-varying parameters; see, for example, the papers by Dahlhaus and Subba Rao (2006) and Chen and Hong (2016) for the time-varying GARCH type models. We leave these important issues as future research topics.

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References

Acemoglu, D., Ozdaglar, A. and Tahbaz-Salehi, A. (2015). Systemic Risk and Stability in Financial Networks. *American Economic Review*, 105(2), 564-608.

Adrian, T. and Brunnermeier, M.K. (2016). CoVaR. *American Economic Review*, 106(7), 1705-1741.

An, H. Z., and Chen, S.G. (1997). A Note on the Ergodicity of Nonlinear Autoregressive Models. *Statistics and Probability Letters*, 34, 365-372.

An, H.Z. and Huang, F.C. (1996). The Geometrical Ergodicity of Nonlinear Autoregressive Models. *Statistica Sinica*, 6, 943-956.

Ando, T. and Bai, J. (2020). Quantile Co-Movement in Financial Markets: A Panel Quantile Model with Unobserved Heterogeneity. *Journal of the American Statistical Association*, 115(529), 266-279.

Billio, M., Getmansky, M., Lo., A.W. and Pelizzon, L. (2012). Econometric Measures of Connectedness and Systemic Risk in the Finance and Insurance Sectors. *Journal of Financial Economics*, 104(3), 535-559.

Bräuning, F. and Koopman, S,J. (2020). The Dynamic Factor Network Model with an Application to International Trade. *Journal of Econometrics*, 216, 494-515.

Cai, Z. (2002). Two-step Likelihood Estimation Procedure for Varying-Coefficient Models. *Journal of Multivariate Analysis*, 82, 189-209.

Cai, Z., Das, M., Xiong, H. and Wu, X. (2006). Functional Coefficient Instrumental Variables Models. *Journal of Econometrics*, 133(1), 207-241.

Cai, Z., Fan, J. and Yao, Q. (2000). Functional-Coefficient Regression Models for Nonlinear Time Series. *Journal of the American Statistical Association*, 95(451), 941-956.

Cai, Z. and Masry, E. (2000). Nonparametric Estimation in Nonlinear ARX Time Series Models: Local Linear Fitting and Projections. *Econometric Theory*, 16(4), 465-501.

Cai, Z. and Xiao, Z. (2012). Semiparametric Quantile Regression Estimation in Dynamic Models with Partially Varying Coefficients. *Journal of Econometrics*, 167, 413-425.

Cai, Z. and Xu, X. (2008). Nonparametric Quantile Estimations for Dynamic Smooth Coefficient Models. *Journal of the American Statistical Association*, 103(484), 1595-1608.

Carrasco, M. and Chen, X. (2002). Mixing and Moment Properties of Various GARCH and Stochastic Volatility Models. *Econometric Theory*, 18(1), 17-39.

Chen, B. and Hong, Y. (2016). Detecting for Smooth Structural Changes in GARCH Models. *Econometric Theory*, 32(3), 740-791.

Chen, C.Y.-H., Härdle, W.K. and Okhrin, Y. (2019). Tail Event Driven Networks of SIFIs. *Journal of Econometrics*, 208(1), 282-298.

Chen, R. and Tsay, R.S. (1993). Functional Coefficient Autoregressive Model. *Journal of the American Statistical Association*, 88(421), 298-308.

Chen, W.Y. and Gerlach, R.H. (2019). Semi-parametric GARCH via Bayesian Model Averaging. *Journal of Business & Economic Statistics* with DOI: https://doi.org/10.1080/07350015.2019.1668796.

Dahlhaus, R. and Polonik, W. (2009). Empirical Spectral Processes for Locally Stationary Time Series. *Bernoulli*, 15(1), 1-39.

Dahlhaus, R. and S. Subba Rao. (2006). Statistical Inference for Time-varying ARCH Process. *Annals of Statistics*, 34(3), 1075-1114.

Diebold, F.X. and Yílmaz, K. (2014). On the Network Topology of Variance Decompositions: Measuring the Connectedness of Financial Firms. *Journal of Econometrics*, 182(1), 119-134.

Engle, R.F. and Manganelli, S. (2004). CaViaR: Conditional Autoregressive Value at Risk by Regression Quantiles. *Journal of Business & Economic Statistics*, 22(4), 367-381.

Fan, J. and Fan, Y. (2006). Issues on Quantile Autoregression. *Journal of the American Statistical Association*, 101(475), 991-994.

Gerlach, R., Chen, C.W.S. and Chan, N. Y.C. (2011). Bayesian Time-Varying Quantile Forecasting for Value-at-Risk in Financial Markets. *Journal of Business & Economic Statistics*, 29(4), 481-492.

Gourieroux, C. and Jasiak, J. (2008). Dynamic Quantile Models. *Journal of Econometrics*, 147(1), 198-205.

Hall, P. and Heyde, C.C. (1980). *Martingale Limit Theory and Its Application.* Academic Press: New York.

Härdle, W.K., Wang, W. and Yu, L. (2016). TENET: Tail-Event Driven NETwork Risk. *Journal of Econometrics*, 192(2), 499-513.

He, X. and Ng, P. (1999). Quantile Splines with Several Covariates. *Journal of Statistical Planning and Inference*, 75(2), 343-352.

He, X. and Shao, Q. (2000). On Parameters of Increasing Dimensions. *Journal of Multivariate Analysis*, 73(1), 120-135.

He, X. and Shi, P. (1996). Bivariate Tensor-Product B-splines in a Partly Linear Model. *Journal of Multivariate Analysis*, 58(2), 162-181.

Honda, T. (2000). Nonparametric Estimation of a Conditional Quantile for α-mixing Processes. *Annals of the Institute of Statistical Mathematics*, 52(3), 459-470.

Honda, T. (2004). Quantile Regression in Varying Coefficient Models. *Journal of Statistical Planning and Inferences*, 121(1), 113-125.

Kim, M. (2007). Quantile Regression With Varying Coefficients. *The Annals of Statistics*, 35(1), 92-108.

Koenker, R. (2005). *Quantile Regression*. Cambridge University Press, New York, NY.

Koenker, R. and Bassett, G.W. (1978). Regression Quantiles. *Econometrica*, 46(1), 33-50.

Koenker, R., Chernozhukov, V., He, X., and Peng, L. (2017). *Handbook of Quantile Regression*. Chapman and Hall/CRC: Boca Raton, FL.

Koenker, R. and Xiao, Z. (2006). Quantile Autoregression. *Journal of the American Statistical Association*, 101(475), 980-990.

Koenker, R. and Zhao, Q. (1996). Conditional Quantile Estimation and Inference for ARCH Models. *Econometric Theory*, 12(5), 793-813.

Kong, E., Linton, O. and Xia, Y. (2010). Uniform Bahadur Representation for Local Polynomial Estimates of M-Regression and Its Application to the Additive Model. *Econometric Theory*, 26(5), 1529-1564.

Li, D., Li, Q. and Li, Z. (2020). Nonparametric Quantile Regression Estimation With Mixed Discrete and Continuous Data. *Journal of Business & Economic Statistics* with DOI: https://doi.org/10.1080/07350015.2020.1730856.

Liu, B.Y., Ji, Q. and Fan, Y. (2017). A New Time-Varying Optimal Copula Model Identifying the Dependence Across Markets. *Quantitative Finance*, 17(1), 1-17.

Menkhoff, L., Sarno, L., Schmelling, M. and Schrimpf, A. (2012). Carry Trades and Global Foreign Exchange Volatility. *Journal of Finance*, 67(2), 681-718.

Pham, D.T. (1986). The Mixing Property of Bilinear and Generalized Random Coefficient Autoregressive Models. *Stochastic Processes and Their Applications*, 23(2), 291-300.

Qu, Z. and Yoon, J. (2015). Nonparametric Estimation and Inference on Conditional Quantile Processes. *Journal of Econometrics*, 185(1), 1-19.

Ruppert, D. and Carroll, R.J. (1980). Trimmed Least Squares Estimation in the Linear Model. *Journal of the American Statistical Association*, 75(372), 828-838.

Sewell, D.K., and Chen, Y. (2015). Latent Space Models for Dynamic Networks. *Journal of the American Statistical Association*, 110(512), 1646-1657.

Taylor, S. (1986). *Modeling Financial Time Series.* Wiley: New York.

Tang, Y., Song, X., Wang, H.J. and Zhu, Z. (2013). Variable Selection in High-Dimensional Quantile Varying Coefficient Models. *Journal of Multivariate Analysis*, 122(1), 115-132.

Wei, Y. and He, X. (2006). Conditional Growth Charts (with discussion). *The Annals of Statistics*, 34(4), 2069-2097.

Welsh, A. (1989). On M-Processes and M-Estimation. *The Annals of Statistics*, 17(1), 337-361.

White, H., Kim, T.H. and Manganelli, S. (2008). Modeling Autoregressive Conditional Skewness and Kurtosis With Multi-quantile CAViaR. In: Russell, J., Watson, M. (Eds.), *Volatility and Time Series Econometrics*: A Festschrift in Honor of Robert F. Engle.

White, H., Kim, T.H. and Manganelli, S. (2015). VAR for VaR: Measuring Tail Dependence Using Multivariate Regression Quantiles. *Journal of Econometrics*, 187(1), 169-188.

Xiao, Z. and Koenker, R. (2009). Conditional Quantile Estimation for Generalized Autoregressive Conditional Heteroscedasticity Models. *Journal of the American Statistical Association*, 104(488), 1696-1712.

Xu, X. (2005). Semi-parametric and Nonparametric Estimation for Dynamic Quantile Regression Models. *Ph.D. Dissertation*, Department of Mathematics and Statistics, University of North Carolina at Charlotte.

Yang, Z. and Zhou, Y. (2017). Quantitative Easing and Volatility Spillovers Across Countries and Asset Classes. *Management Science*, 63(2), 333-354.

Zhu, X., Wang, W., Wang, H. and Härdle, W.K. (2019). Network Quantile Autoregression. *Journal of Econometrics*, 212(1), 345-358.

Appendix: Mathematical Proofs, Tables and Figures

In this appendix, we give certain lemmas, with their detailed proofs given in Supplement B, that are useful for proving and presenting mathematical proofs for the proposition and theorems in the paper. In addition, in Supplement C, it shows that under some regularity conditions, the model in (1) can generate a strictly stationary and α-mixing process. Finally, all tables and figures are also gathered in this appendix.

Proof of Proposition 2.1:

Proof. By recursively substituting in (6), we have

$$\mathbb{X}_t = \boldsymbol{V}_t(l) + \sum_{l=1}^{\infty} \left(\prod_{\jmath=0}^{l-1} \boldsymbol{A}_{e_{t-\jmath}}(Z_{t-\jmath}) \check{\gamma}(Z_{t-l}) \right) \boldsymbol{D}_{e_{t-l}} + \check{\gamma}(Z_t)\boldsymbol{D}_{e_t},$$

where $\boldsymbol{V}_t(l) = \boldsymbol{\mu}(Z_t) + \sum_{l=1}^{\infty} \left(\prod_{\jmath=0}^{l-1} \boldsymbol{A}_{e_{t-\jmath}}(Z_{t-\jmath}) \right) \boldsymbol{\mu}(Z_{t-l})$. Denote $\lambda_{\max}(\boldsymbol{A}_{e_t}(Z_t))$ as the largest eigenvalue in absolute value of matrix $\boldsymbol{A}_{e_t}(Z_t)$. By simple algebra, we have $\det(\boldsymbol{A}_{e_t}(Z_t) - \lambda I_{\kappa(p+q)}) = \lambda^{p+q}\{I_\kappa - \sum_{s=1}^{q} \lambda^{-s} B_{s,\tau}(Z_t) - \sum_{l=1}^{p} \lambda^{-l} A_{l,e_t}(Z_t)\}$. Then, similar to the proof of Proposition 2.4 of Dahlhaus and Polonik (2009), under Assumption A3, it follows that $\lambda_{\max}(\boldsymbol{A}_{e_t}(Z_t)) < 1/(1+\delta)$ for all Z_t and all e_{it}, and for some $\delta > 0$. Following the techniques in Dahlhaus and Polonik (2009), for every $\epsilon > 0$, for every $Z_t \in [\mathfrak{a}, \mathfrak{c}]$ and for every $e_{it} \in [0, 1]$, we have

$$\|\boldsymbol{A}_{e_t}(Z_t)\|_F \leq \lambda_{\max}(\boldsymbol{A}_{e_t}(Z_t)) + \epsilon.$$

Since $\beta_{lij,\tau}(\cdot)$ and $\gamma_{sij,\tau}(\cdot)$ are functions of bounded variation, there exists for all $\epsilon_1 > 0$ and $\epsilon_2 > 0$ a finite partition of intervals $K_1 \cup, \ldots, \cup K_d = [\mathfrak{a}, \mathfrak{c}]$ such that $|\beta_{lij,\tau}(u) - \beta_{lij,\tau}(v)| < \epsilon_1$ and $|\gamma_{sij,\tau}(u) - \gamma_{sij,\tau}(v)| < \epsilon_2$, for all l and all s whenever u and v are in the same K_k. Thus, k (and the partition) can be chosen such that

$$\|\boldsymbol{A}_{e_t}(v)\|_F \leq \rho \equiv \left(1 + \frac{\delta}{2}\right)^{-1} < 1$$

for all $v \in K_k$. We now define $L_k \equiv \{\jmath \geq 0 : Z_{t-\jmath} \in K_k\}$ and $L_{k,l} \equiv L_k \cap \{0, \ldots, l-1\}$. Then,

$$|\alpha_{ijt}(l)| \leq \left\| \prod_{\jmath=0}^{l-1} \boldsymbol{A}_{e_{t-\jmath}}(Z_{t-\jmath}) \check{\boldsymbol{\gamma}}(Z_{t-l}) \right\|_F \leq \prod_{k=1}^{d} \left\| \prod_{\jmath \in L_{k,l}} \boldsymbol{A}_{e_{t-\jmath}}(Z_{t-\jmath}) \right\|_F \left\| \check{\boldsymbol{\gamma}}(Z_{t-l}) \right\|_F$$
$$\leq C c_0^d \left(\prod_{k=1}^{d} \rho^{|L_{k,l}|} \right) \leq C\rho^l.$$

Thus, we have proven (9).

Before going further, we first need to check whether the functional-coefficient MA(∞) representations (8) and (10) are valid or not. Indeed, since $\max_{t\geq 1} \| \prod_{\jmath=0}^{l-1} \boldsymbol{A}_{e_{t-\jmath}}(Z_{t-\jmath}) \|_F \leq C\rho^l$, under Assumption A1 and by choosing sufficiently large $C > 0$, we have

$$\|E(\mathbb{X}_t)\| \leq \|E(\boldsymbol{V}_t(l))\| + \sum_{l=1}^{\infty} \left(\max_{t\geq 1} \left\| \prod_{\jmath=0}^{l-1} \boldsymbol{A}_{e_{t-\jmath}}(Z_{t-\jmath}) \right\|_F E(\check{\boldsymbol{\gamma}}(Z_{t-l}) \boldsymbol{D}_{e_{t-l}}) \right) + \|E(\check{\boldsymbol{\gamma}}(Z_t) \boldsymbol{D}_{e_t})\|$$
$$\leq C \sum_{l=1}^{\infty} \max_{t\geq 1} \left\| \prod_{\jmath=0}^{l-1} \boldsymbol{A}_{e_{t-\jmath}}(Z_{t-\jmath}) \right\|_F$$
$$\leq C \sum_{l=1}^{\infty} \rho^l < \infty$$

and

$$\|Var(\mathbb{X}_t)\|_F \leq C \left\| \sum_{l=1}^{\infty} E \left(\prod_{\jmath=0}^{l-1} \boldsymbol{A}_{e_{t-\jmath}}(Z_{t-\jmath}) \right) \boldsymbol{D}_{e_{t-l}}(Z_{t-l}) \boldsymbol{D}_{e_{t-l}}^T(Z_{t-l}) \left(\prod_{\jmath=0}^{l-1} \boldsymbol{A}_{e_{t-\jmath}}(Z_{t-\jmath}) \right)^T \right\|_F$$
$$\leq C \left\| \sum_{l=1}^{\infty} \max_{t\geq 1} \left(\prod_{\jmath=0}^{l-1} \boldsymbol{A}_{e_{t-\jmath}}(Z_{t-\jmath}) \right) E(\boldsymbol{D}_{e_{t-l}} \boldsymbol{D}_{e_{t-l}}^T) \max_{t\geq 1} \left(\prod_{\jmath=0}^{l-1} \boldsymbol{A}_{e_{t-\jmath}}(Z_{t-\jmath}) \right)^T \right\|_F$$
$$\leq C \sum_{l=1}^{\infty} \max_{t\geq 1} \left\| \prod_{\jmath=0}^{l-1} \boldsymbol{A}_{e_{t-\jmath}}(Z_{t-\jmath}) \right\|_F^2$$
$$\leq C \sum_{l=1}^{\infty} \rho^{2l} < \infty.$$

By similar derivations, one can also show that $\|E(\widetilde{\mathbb{X}}_t)\| < \infty$ and $\|Var(\widetilde{\mathbb{X}}_t)\|_F < \infty$.

Next, we begin to show that $\max_{t\geq 1} E\|\mathbb{X}_t - \widetilde{\mathbb{X}}_t\| = O(n^{-1})$. Notice that by Assumption A2, for every $\epsilon > 0$, $|\beta_{lij,e_{it}}(Z_t) - \beta_{lij,\tau}(Z_t)| < \epsilon$ in a neighborhood of τ. Then, for sufficiently large C_1,

$$\begin{aligned}
&|\beta_{lij,e_{i(t-k)}}(Z_{t-k}) - \beta_{lij,e_{it}}(Z_t)| \\
\leq&|\beta_{lij,e_{i(t-k)}}(Z_{t-k}) - \beta_{lij,\tau}(Z_{t-k})| + |\beta_{lij,\tau}(Z_{t-k}) - \beta_{lij,\tau}(Z_t)| + |\beta_{lij,e_{it}}(Z_t) - \beta_{lij,\tau}(Z_t)| \\
\leq&C_1|\beta_{lij,\tau}(Z_{t-k}) - \beta_{lij,\tau}(Z_t)|.
\end{aligned}$$

Similarly, for sufficiently large C_2, we have $|\gamma_{sij,e_{i(t-k)}}(Z_{t-k}) - \gamma_{sij,e_{it}}(Z_t)| \leq C_2|\gamma_{sij,\tau}(Z_{t-k}) - \gamma_{sij,\tau}(Z_t)|$. Then,

$$\begin{aligned}
&\max_{t\geq 1} \|\boldsymbol{C}_{e,t}(l) - \boldsymbol{C}_{l,e_t}(Z_t)\|_F \\
\leq&\max_{t\geq 1} \sum_{k=1}^{l-1} \left\| \left((\boldsymbol{A}_{e_t}(Z_t))^k \left(\boldsymbol{A}_{e_{t-k}}(Z_{t-k}) - \boldsymbol{A}_{e_t}(Z_t) \right) \prod_{\jmath=k+1}^{l-1} \boldsymbol{A}_{e_{t-\jmath}}(Z_{t-\jmath}) \right) \right\|_F \left\| \check{\boldsymbol{\gamma}}(Z_{t-l}) \right\|_F \\
&+ \max_{t\geq 1} \left\| (\boldsymbol{A}_{e_t}(Z_t))^l \right\|_F \left\| \check{\boldsymbol{\gamma}}(Z_{t-l}) - \check{\boldsymbol{\gamma}}(Z_t) \right\|_F \\
\leq&C \sum_{k=1}^{l-1} \frac{k}{n}\rho^{l-1} + C\frac{l}{n}\rho^l,
\end{aligned}$$

which implies that

$$\begin{aligned}
\max_{t\geq 1} E\|\mathbb{X}_t - \widetilde{\mathbb{X}}_t\| \leq& \max_{t\geq 1} E\left\| \sum_{l=1}^{\infty} \left(\prod_{\jmath=0}^{l-1} \boldsymbol{A}_{e_{t-\jmath}}(Z_{t-\jmath}) \right) \boldsymbol{\mu}(Z_{t-l}) - (\boldsymbol{A}_{e_t}(Z_t))^l \boldsymbol{\mu}(Z_t) \right\| \\
&+ \max_{t\geq 1} \left\| \sum_{l=1}^{\infty} (\boldsymbol{C}_{e,t}(l) - \boldsymbol{C}_{l,e_t}(Z_t)) \right\|_F \cdot E\|\boldsymbol{D}_{e_t}\| \\
\leq&C \sum_{l=1}^{\infty} \left(2\sum_{k=1}^{l-1} \frac{k}{n}\rho^{l-1} + C\frac{l}{n}\rho^l \right) = O(n^{-1}).
\end{aligned}$$

Therefore, Proposition 2.1 is proved. □

Lemma A.1. *Let $\hat{\beta}$ be the minimizer of the function $\sum_{t=1}^n \omega_t \rho_\tau(u_t - X_t^T\beta)$, where $\omega_t > 0$. Then, $\|\sum_{t=1}^n \omega_t X_t \psi_\tau(u_t - X_t^T\hat{\beta})\| \leq \dim(X)\max_{t\leq n}\|\omega_t X_t\|$.*

Proof. The proof follows from Ruppert and Carroll (1980). □

Now, some notations are introduced here to make a convenient presentation of our Bahadur results given in Theorem A.6 (below). In Lemmas A.2 - A.5 as well as Theorem A.6, τ is dropped from $\boldsymbol{\alpha}_\tau(z_0)$ and write h_1 as h for simplicity. Let $a_n = (nh)^{-1/2}$, $\boldsymbol{\vartheta}_0 = a_n^{-1}(\boldsymbol{\delta}_0 - \boldsymbol{\alpha}(z_0))$, $\boldsymbol{\vartheta}_1 = ha_n^{-1}(\boldsymbol{\delta}_1 - \boldsymbol{\alpha}^{(1)}(z_0))$. $\boldsymbol{H} = diag\{I, hI\}$, $\boldsymbol{\vartheta} = a_n^{-1}\boldsymbol{H}\begin{pmatrix}\boldsymbol{\delta}_0 - \boldsymbol{\alpha}(z_0)\\ \boldsymbol{\delta}_1 - \boldsymbol{\alpha}^{(1)}(z_0)\end{pmatrix}$. Let $\boldsymbol{W}_t^* = \begin{pmatrix}\boldsymbol{W}_t\\ z_{th}\boldsymbol{W}_t\end{pmatrix}$, where $z_{th} = (Z_t - z_0)/h$. Also, define $u_t^* = u_t - \boldsymbol{W}_t^T[\boldsymbol{\alpha}(z_0) + \boldsymbol{\alpha}^{(1)}(z_0)(Z_t - z_0)]$. Therefore,

$$\hat{\boldsymbol{\vartheta}} = \arg\min_{\boldsymbol{\vartheta}} \sum_{t=m+1}^{n} \rho_\tau(u_t^* - a_n\boldsymbol{\vartheta}^T\boldsymbol{W}_t^*)K(z_{th}) \equiv \arg\min_{\boldsymbol{\vartheta}} G(\boldsymbol{\vartheta}). \tag{18}$$

The derivative of $G(\boldsymbol{\vartheta})$ with respect to $\boldsymbol{\vartheta}$ (except at point $u_t^* = a_n\boldsymbol{\vartheta}^T\boldsymbol{W}_t^*$) is given by

$$T_n(\boldsymbol{\vartheta}) = a_n \sum_{t=m+1}^{n} \psi_\tau(u_t^* - a_n\boldsymbol{\vartheta}^T\boldsymbol{W}_t^*)\boldsymbol{W}_t^*K(z_{th}). \tag{19}$$

For notational convenience, set $\boldsymbol{\zeta} = a_n\boldsymbol{\vartheta}$. Then, (19) becomes

$$T_n(\boldsymbol{\zeta}) = a_n \sum_{t=m+1}^{n} \psi_\tau(u_t^* - \boldsymbol{\zeta}^T\boldsymbol{W}_t^*)\boldsymbol{W}_t^*K(z_{th}). \tag{20}$$

Lemma A.2. *Under Assumptions B1 – B11, one has* $\|\hat{\boldsymbol{\zeta}}\|^2 = O_p(m/nh)$.

Proof. The proof can be found in Supplement B. □

In the next two lemmas, we need to show stochastic equicontinuity corresponding to $T_n(\boldsymbol{\zeta}) - T_n(0) - E[T_n(\boldsymbol{\zeta}) - T_n(0)]$, so that we can derive the local Bahadur representation for $\sqrt{nh}\hat{\boldsymbol{\zeta}}$. In particular, define $D_m = \{\boldsymbol{\zeta} : \|\boldsymbol{\zeta}\| \leq C(m/nh)^{1/2}\}$ for each fixed $0 < C < \infty$.

Lemma A.3. *Under Assumptions B1 – B11, for any* $a \in \mathbb{R}^{2(\kappa m+1)}$ *satisfying* $\|a\| =$

$O(1)$, *one has*

$$\sup_{\boldsymbol{\zeta}\in D_m} |a^T\{T_n(\boldsymbol{\zeta}) - T_n(0) - E[T_n(\boldsymbol{\zeta}) - T_n(0)]\}| = o_p(1). \tag{21}$$

Proof. The proof can be found in Supplement B. □

Lemma A.4. *Under Assumptions B1 – B11, for any* $a \in \mathbb{R}^{2(\kappa m+1)}$ *satisfying* $\|a\| = O(1)$, *one has*

$$\sup_{\boldsymbol{\zeta}\in D_m} \|a^T\{E[T_n(\boldsymbol{\zeta}) - T_n(0)] + f_z(z_0)\boldsymbol{D}_1^*(z_0)\sqrt{nh}\boldsymbol{\zeta}\}\| = o(1), \tag{22}$$

where $\boldsymbol{D}_1^*(z_0) = diag\{\boldsymbol{D}^*(z_0), \mu_2\boldsymbol{D}^*(z_0)\}$.

Proof. The proof can be found in Supplement B. □

Lemma A.5. *Let* $S_t = \psi_\tau(u_t^*)\boldsymbol{W}_t^*K(z_{th})$. *Under Assumptions B1 – B11, one has*

$$E[S_1] = \frac{h^3 f_z(z_0)}{2}\begin{pmatrix}\mu_2\boldsymbol{D}^*(z_0)\boldsymbol{\alpha}^{(2)}(z_0)\\ 0\end{pmatrix} + o(h^3),$$

and

$$Var[S_1] = h\tau(1-\tau)f_z(z_0)\boldsymbol{D}_1(z_0) + o(h), \tag{23}$$

where $\boldsymbol{D}_1(z_0) = diag\{\nu_0\boldsymbol{D}(z_0), \nu_2\boldsymbol{D}(z_0)\}$ *with* $\nu_2 = \int \nu^2K^2(\nu)d\nu$. *Further,*

$$Var[T_n(0)] \to \tau(1-\tau)f_z(z_0)\boldsymbol{D}_1(z_0). \tag{24}$$

Therefore, $\|T_n(0)\| = O_p(1)$.

Proof. The proof can be found in Supplement B. □

Theorem A.6. *(Bahadur representation) Under Assumptions B1 – B11, one has,*

$$\sqrt{nh_1}\hat{\boldsymbol{\zeta}} = \frac{1}{\sqrt{nh_1}f_z(z_0)}(\boldsymbol{D}_1^*(z_0)^{-1})\sum_{t=m+1}^{n}\psi_\tau(u_t^*)\boldsymbol{W}_t^*K(z_{th_1}) + o_p(1),$$

where $\boldsymbol{D}_1^*(z_0) = \begin{pmatrix} 1 & 0 \\ 0 & \mu_2 \end{pmatrix} \otimes \boldsymbol{D}^*(z_0)$.

Proof. By Lemma A.2, $\|\hat{\boldsymbol{\zeta}}\| = O_p((m/nh)^{1/2})$. On the other hand, by Lemmas A.3, A.4 and A.5, $T_n(\boldsymbol{\zeta})$ satisfies $\|T_n(0)\| = O_p(1)$ and $\sup_{\|\boldsymbol{\zeta}\| \le C(m/nh)^{1/2}} |a^T\{T_n(\boldsymbol{\zeta}) + D\sqrt{nh}\boldsymbol{\zeta} - T_n(0)\}| = o_p(1)$ with $D = f_z(z_0)\boldsymbol{D}_1^*(z_0)$. In order to show $\|T_n(\hat{\boldsymbol{\zeta}})\| = o_p(1)$, it follows from Assumption B9 and Lemma A.1 that

$$\|T_n(\hat{\boldsymbol{\zeta}})\| = a_n \left\| \sum_{t=m+1}^{n} \psi_\tau(u_t^* - \hat{\boldsymbol{\zeta}}^T \boldsymbol{W}_t^*)\boldsymbol{W}_t^* K(z_{th}) \right\| = m a_n \max_{t \le n} \|\boldsymbol{W}_t^* K(z_{th})\| = O(m^{3/2}(nh)^{-1/2}) = o(1).$$

Then, replacing a by $D^{-1}a$, the theorem is proved. □

Lemma A.7. *Define* $K_{nL} = \{(\Delta, \xi) : \|\xi\| \le L, \|\Delta\| \le M\}$ *for some* $0 < M < \infty$, *and let* $V_n(\Delta, \xi)$ *be a vector that satisfies (i)* $-\Delta^T V_n(\lambda\Delta, \xi) \ge -\Delta^T V_n(\Delta, \xi)$ *for* $\lambda \ge 1$ *and* $\|\xi\| \le L$, $0 < L < \infty$, *and (ii)*

$$\sup_{(\Delta,\xi)\in K_{nL}} \|V_n(\Delta, \xi) + D\Delta - A_n\| = o_p(1),$$

where $\|A_n\| = O_p(1)$ *and* D *is a positive-definite matrix. Suppose that* Δ_n *is a vector such that* $\|V_n(\Delta_n, \xi_n)\| = o_p(1)$. *Then, one has* $\|\Delta_n\| = O_p(1)$ *and* $\Delta_n = D^{-1}A_n + o_p(1)$.

Proof. The proof follows from Koenker and Zhao (1996) and Conditions (i) and (ii) that $V_n(\Delta_n, \xi_n) + D\Delta_n - A_n = o_p(1)$. This completes the proof. □

To show Lemmas A.8 and A.9 later, τ is dropped from $\boldsymbol{g}_\tau(z_0)$ and h_2 is written as h for simplicity. For the notational convenience again, let $\boldsymbol{\theta}_0 = a_n^{-1}(\Theta_0 - \boldsymbol{g}(z_0))$ and $\boldsymbol{\theta}_1 = h a_n^{-1}(\Theta_1 - \boldsymbol{g}^{(1)}(z_0))$. Then, $\boldsymbol{\theta} = a_n^{-1}\boldsymbol{H}\begin{pmatrix} \Theta_0 - \boldsymbol{g}(z_0) \\ \Theta_1 - \boldsymbol{g}^{(1)}(z_0) \end{pmatrix}$. Now, let $\boldsymbol{\delta}_r(Z_t) = \boldsymbol{\alpha}^{(r)}(Z_t)/r!$ and denote $\xi = \sqrt{nh}(\boldsymbol{\delta}_0(Z_t) - \boldsymbol{\alpha}(Z_t))$. Then, $\hat{\xi} = \sqrt{nh}(\hat{\boldsymbol{\delta}}_0(Z_t) - \boldsymbol{\alpha}(Z_t))$ is the estimation of ξ. Thus, it is obvious that $\xi = \boldsymbol{\vartheta}_0$ given $Z_t = z_0$ by recalling that $\boldsymbol{\vartheta}_0 = \sqrt{nh}(\boldsymbol{\delta}_0 - \boldsymbol{\alpha}(z_0))$. For convenience of analysis, we rewrite $\hat{\boldsymbol{X}}_t \equiv \boldsymbol{X}_t(\hat{\xi}) \equiv \boldsymbol{X}_t(\boldsymbol{\alpha}(Z_t) + (nh)^{-1/2}\hat{\xi})$ because it contains $\hat{q}_{\tau,t} = \boldsymbol{W}_t^T\hat{\boldsymbol{\delta}}_0$. Similarly, $\boldsymbol{X}_t(\xi) \equiv \boldsymbol{X}_t(\boldsymbol{\alpha}(Z_t) + (nh)^{-1/2}\xi)$, $\boldsymbol{X}_t^*(\xi) \equiv \boldsymbol{X}_t^*(\boldsymbol{\alpha}(Z_t) +$

$(nh)^{-1/2}\xi)$ and $\hat{\boldsymbol{X}}_t^* \equiv \boldsymbol{X}_t^*(\hat{\xi}) \equiv \boldsymbol{X}_t^*(\boldsymbol{\alpha}(Z_t) + (nh)^{-1/2}\hat{\xi})$, where $\boldsymbol{X}_t^*(\xi) = \begin{pmatrix} \boldsymbol{X}_t(\xi) \\ z_{th}\boldsymbol{X}_t(\xi) \end{pmatrix}$ and $\boldsymbol{X}_t^*(\hat{\xi}) = \begin{pmatrix} \boldsymbol{X}_t(\hat{\xi}) \\ z_{th}\boldsymbol{X}_t(\hat{\xi}) \end{pmatrix}$. Hence,

$$\left(\frac{\partial \boldsymbol{X}_t(\xi)}{\partial \xi}\right) = a_n \begin{pmatrix} 0_{1\times(\kappa m+1)} \\ \boldsymbol{W}_t^T \\ \vdots \\ \boldsymbol{W}_t^T \\ 0_{\kappa\times(\kappa m+1)} \end{pmatrix} = a_n \boldsymbol{U}.$$

Denote $v_t^* = u_t - \boldsymbol{X}_t^T(\xi)[\boldsymbol{g}(z_0) + \boldsymbol{g}^{(1)}(z_0)(Z_t - z_0)]$ and $v_t^{**} = u_t - \boldsymbol{X}_t^T[\boldsymbol{g}(z_0) + \boldsymbol{g}^{(1)}(z_0)(Z_t - z_0)]$. Similar to Xiao and Koenker (2009), it can be easily shown that $v_t^{**} = u_t - q_\tau(z_0, \boldsymbol{X}_t) = u_t - q_\tau(z_0, \boldsymbol{W}_t) = u_t^*$. Again, define $A_m = \{\boldsymbol{\theta} : \|\boldsymbol{\theta}\| \le M\}$ and $B_m = \{\xi : \|\xi\| \le L\}$ for some $0 < M < \infty$ and for some $0 < L < \infty$, Therefore,

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \sum_{t=1}^{n} \rho_\tau(v_t^* - a_n \boldsymbol{\theta}^T \boldsymbol{X}_t^*(\hat{\xi}))K(z_{th}) \equiv \arg\min_{\boldsymbol{\theta}} J(\boldsymbol{\theta}). \tag{25}$$

Now, define vector functions of $\boldsymbol{\theta}$ and ξ

$$V_n(\boldsymbol{\theta}, \xi) = a_n \sum_{t=1}^{n} \psi_\tau(v_t^* - a_n \boldsymbol{\theta}^T \boldsymbol{X}_t^*(\xi))\boldsymbol{X}_t^*(\xi)K(z_{th}), \tag{26}$$

where $\psi_\tau(x) = \tau - I(x < 0)$. In the next three lemmas, we show that $V_n(\boldsymbol{\theta}, \xi)$ satisfies Lemma A.7, so that we can derive the local Bahadur representation for $\hat{\boldsymbol{\theta}}$.

Lemma A.8. *Under the assumptions in Theorem 2.1, one has*

$$\sup_{\xi \in B_m, \boldsymbol{\theta} \in A_m} \|V_n(\boldsymbol{\theta}, \xi) - V_n(0,0) - E[V_n(\boldsymbol{\theta}, \xi) - V_n(0,0)]\| = o_p(1). \tag{27}$$

Proof. The proof can be found in Supplement B. □

Lemma A.9. *Under the assumptions in Theorem 2.1, one has*

$$\sup_{\xi\in B_m,\boldsymbol{\theta}\in A_m} \|E[V_n(\boldsymbol{\theta},\xi)-V_n(0,0)]+f_z(z_0)\Omega_1^*(z_0)\boldsymbol{\theta}-f_z(z_0)\Gamma_{20}^*\boldsymbol{\vartheta}_0\| = o(1), \tag{28}$$

where $\Omega_1^*(z_0)=diag\{\Omega^*(z_0),\mu_2\Omega^*(z_0)\}$ *and* $\Gamma_{20}^* = \begin{pmatrix}1\\0\end{pmatrix}\otimes\Gamma_{20}$.

Proof. The proof can be found in Supplement B. □

Lemma A.10. *Let* $D_{m/K}=(I_{(\kappa m+1)}\vdots\, 0_{(\kappa m+1)\times(\kappa m+1)})$ *and*

$$B_t=\psi_\tau(u_t^*)[\boldsymbol{X}_t^*K(z_{th_2})+\Gamma_{20}^*D_{m/K}(\boldsymbol{D}_1^*(z_0))^{-1}\boldsymbol{W}_t^{**}K(z_{th_1})],$$

where $\boldsymbol{W}_t^{**}=(h_2/h_1)^{1/2}\boldsymbol{W}_t^*$. *In addition, denote* $H_t^*=\Gamma_{20}^*D_{m/K}(\boldsymbol{D}_1^*(z_0))^{-1}\boldsymbol{W}_t^*$. *Under the assumptions in Theorem 2.1, one has*

$$\begin{aligned}E[B_1]=&\frac{h_2^3f_z(z_0)}{2}\begin{pmatrix}\mu_2\Omega^*(z_0)\boldsymbol{g}_\tau^{(2)}(z_0)\\0\end{pmatrix}+\frac{h_1^{5/2}h_2^{1/2}f_z(z_0)}{2}\Gamma_{20}^*D_{m/K}\begin{pmatrix}\mu_2\boldsymbol{\alpha}_\tau^{(2)}(z_0)\\0\end{pmatrix}\\&+o(h_2^3)+o(h_1^{5/2}h_2^{1/2}),\end{aligned} \tag{29}$$

and

$$Var[B_1]=h_2\tau(1-\tau)f_z(z_0)\left\{\begin{pmatrix}\nu_0&0\\0&\nu_2\end{pmatrix}\otimes\Omega(z_0)+\boldsymbol{H}\right\}+o(h_2), \tag{30}$$

where $\boldsymbol{H}=\begin{pmatrix}\nu_0&0\\0&0\end{pmatrix}\otimes\Gamma_{20}\boldsymbol{D}^*(z_0)^{-1}\Gamma_{20}^T$. *Further,*

$$Var[V_n(0,0)]=\tau(1-\tau)f_z(z_0)\left\{\begin{pmatrix}\nu_0&0\\0&\nu_2\end{pmatrix}\otimes\Omega(z_0)+\boldsymbol{H}\right\}+o(1). \tag{31}$$

Therefore, $\|V_n(0,0)\|=O_p(1)$.

Proof. This proof is similar to the proof of Lemma A.4 in Cai and Xu (2008). First, we calculate $E[B_1]$ to obtain

$$E[B_1] = E[\psi_\tau(u_1^*)\boldsymbol{X}_1^* K(z_{1h_2})] + \Gamma_{20}^* D_{m/K} E[\psi_\tau(u_1^*)(\boldsymbol{D}_1^*(z_0))^{-1}\boldsymbol{W}_1^{**}K(z_{1h_1})] \equiv Q_1 + Q_2.$$

Similar to the proof of Lemma A.5, one can easily obtain that

$$Q_1 = \frac{h_2^3}{2} f_z(z_0)\{\begin{pmatrix}\mu_2\\0\end{pmatrix} \otimes \Omega^*(z_0)\}\boldsymbol{g}_\tau^{(2)}(z_0) + o(h_2^3)$$

with the detail omitted. For Q_2, similarly,

$$\begin{aligned}Q_2 \equiv&\Gamma_{20}^* D_{m/K} E[\psi_\tau(u_1^*)(\boldsymbol{D}_1^*(z_0))^{-1}\boldsymbol{W}_1^{**}K(z_{1h_1})]\\ =&\frac{h_1^2}{2}\Big(\frac{h_2^{1/2}}{h_1^{1/2}}\Gamma_{20}^* D_{m/K} E[f_{u|Z,\boldsymbol{W}}(q_\tau(z_0,\boldsymbol{W}_1) + h_1 z_{1h_1}\boldsymbol{\alpha}^{(1)}(z_0)^T\boldsymbol{W}_1 + \xi\Lambda(h_1, z_0, Z_1, \boldsymbol{W}_1)|Z_1, \boldsymbol{W}_1)]\\ &\times \boldsymbol{D}_1^*(z_0)^{-1}\begin{pmatrix}1\\z_{1h_1}\end{pmatrix}\boldsymbol{D}(Z_1)\boldsymbol{\alpha}^{(2)}(z_0 + \varsigma h_1 z_{1h_1})z_{1h_1}^2 K(z_{1h_1})\Big)\\ =&\frac{h_1^{5/2}h_2^{1/2}}{2} f_z(z_0)\Gamma_{20}^* D_{m/K}\begin{pmatrix}\mu_2\boldsymbol{\alpha}_\tau^{(2)}(z_0)\\0\end{pmatrix}(1+o(1)).\end{aligned} \tag{32}$$

As for $E[B_1 B_1^T]$, similar to the derivation in Lemma A.5,

$$\begin{aligned}&E[B_1B_1^T]\\ =&E\Big(\psi_\tau^2(u_1^*)[\boldsymbol{X}_1^*\boldsymbol{X}_1^{*T}K^2(z_{1h_2}) + (H_1^*\boldsymbol{X}_1^{*T} + \boldsymbol{X}_1^*H_1^{*T})K(z_{1h_1})K(z_{1h_2}) + H_1^*H_1^{*T}K^2(z_{1h_1})]\Big)\\ =&(2\tau-1)E\Big([\tau - I_{\{u_1^*<0\}}]\{\boldsymbol{X}_1^*\boldsymbol{X}_1^{*T}K^2(z_{1h_2}) + (H_1^*\boldsymbol{X}_1^{*T} + \boldsymbol{X}_1^*H_1^{*T})K(z_{1h_1})K(z_{1h_2}) + H_1^*H_1^{*T}K^2(z_{1h_1})\}\Big)\\ &+\tau(1-\tau)E\Big(\boldsymbol{X}_1^*\boldsymbol{X}_1^{*T}K^2(z_{1h_2}) + (H_1^*\boldsymbol{X}_1^{*T} + \boldsymbol{X}_1^*H_1^{*T})K(z_{1h_1})K(z_{1h_2}) + H_1^*H_1^{*T}K^2(z_{1h_1})\Big)\\ \equiv&P^{(1)} + P^{(2)}.\end{aligned}$$

It is not difficult to show that $P^{(1)} = o(h_2^2)$. As for $P^{(2)}$,

$$
\begin{aligned}
P^{(2)} \equiv& \tau(1-\tau)E[\boldsymbol{X}_1^*\boldsymbol{X}_1^{*T}K^2(z_{1h_2})] + \tau(1-\tau)E[(H_1^*\boldsymbol{X}_1^{*T} + \boldsymbol{X}_1^*H_1^{*T}) \\
&\times K(z_{1h_1})K(z_{1h_2})] + \tau(1-\tau)E[H_1^*H_1^{*T}K^2(z_{1h_1})] \\
\equiv& P^{(21)} + P^{(22)} + P^{(23)}.
\end{aligned}
$$

We first focus on $P^{(22)}$. A simple algebra gives that

$$
\begin{aligned}
P^{(22)} \equiv& \tau(1-\tau)E[(H_1^*\boldsymbol{X}_1^{*T} + \boldsymbol{X}_1^*H_1^{*T})K(z_{1h_1})K(z_{1h_2})] \\
=& (h_2/h_1)^{1/2}\tau(1-\tau)E\Bigg\{\begin{pmatrix} \nu_0^{-1}\Gamma_{20}\boldsymbol{D}^*(z_0)^{-1} & 0 \\ 0 & 0 \end{pmatrix}(\boldsymbol{W}_1^*\boldsymbol{X}_1^{*T})K(z_{1h_1})K(z_{1h_2})\Bigg\} \\
&+ (h_2/h_1)^{1/2}\tau(1-\tau)E\Bigg\{(\boldsymbol{X}_1^*\boldsymbol{W}_1^{*T})\begin{pmatrix} \nu_0^{-1}(\Gamma_{20}\boldsymbol{D}^*(z_0)^{-1})^T & 0 \\ 0 & 0^T \end{pmatrix}K(z_{1h_1})K(z_{1h_2})\Bigg\} \\
=& (h_2/h_1)^{1/2}\tau(1-\tau)E\Bigg\{\begin{pmatrix} \nu_0^{-1}\Gamma_{20}\boldsymbol{D}^*(z_0)^{-1}\boldsymbol{W}_1\boldsymbol{X}_1^T & \nu_0^{-1}z_{1h_2}\Gamma_{20}\boldsymbol{D}^*(z_0)^{-1}\boldsymbol{W}_1\boldsymbol{X}_1^T \\ 0 & 0 \end{pmatrix} \\
&\times K(z_{1h_1})K(z_{1h_2})\Bigg\} + (h_2/h_1)^{1/2}\tau(1-\tau)E\Bigg\{\begin{pmatrix} \nu_0^{-1}\boldsymbol{X}_1\boldsymbol{W}_1^T(\Gamma_{20}\boldsymbol{D}^*(z_0)^{-1})^T & 0 \\ \nu_0^{-1}z_{1h_2}\boldsymbol{X}_1\boldsymbol{W}_1^T(\Gamma_{20}\boldsymbol{D}^*(z_0)^{-1})^T & 0^T \end{pmatrix} \\
&\times K(z_{1h_1})K(z_{1h_2})\Bigg\} \\
=& \nu_0^{-1}(h_2/h_1)^{1/2}\tau(1-\tau)E\Bigg\{\begin{pmatrix} 1 & z_{1h_2} \\ 0 & 0 \end{pmatrix} \otimes (\Gamma_{20}\boldsymbol{D}^*(z_0)^{-1}\boldsymbol{W}_1)\boldsymbol{X}_1^TK(z_{1h_1})K(z_{1h_2})\Bigg\} \\
&+ \nu_0^{-1}(h_2/h_1)^{1/2}\tau(1-\tau)E\Bigg\{\begin{pmatrix} 1 & 0 \\ z_{1h_2} & 0 \end{pmatrix} \otimes \boldsymbol{X}_1(\Gamma_{20}\boldsymbol{D}^*(z_0)^{-1}\boldsymbol{W}_1)^TK(z_{1h_1})K(z_{1h_2})\Bigg\} \\
=& O(h_1^{1/2}h_2^{3/2}) = o(h_2).
\end{aligned}
$$

Similarly,

$$
P^{(21)} \equiv \tau(1-\tau)E[\boldsymbol{X}_1^*\boldsymbol{X}_1^{*T}K^2(z_{1h_2})] = h_2\tau(1-\tau)f_z(z_0)\begin{pmatrix} \nu_0 & 0 \\ 0 & \nu_2 \end{pmatrix} \otimes \Omega(z_0)(1+o(1)) \quad (33)
$$

and

$$
\begin{aligned}
P^{(23)} \equiv& (h_2/h_1)\tau(1-\tau)E[H_1^* H_1^{*T} K^2(z_{1h_1})] \\
=& (h_2/h_1)\tau(1-\tau)E\Bigg\{ \begin{pmatrix} \Gamma_{20}\boldsymbol{D}^*(z_0)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & z_{1h_1} \\ z_{1h_1} & z_{1h_1}^2 \end{pmatrix} \otimes \boldsymbol{D}^*(z_0) \\
& \times \begin{pmatrix} (\Gamma_{20}\boldsymbol{D}^*(z_0)^{-1})^T & 0 \\ 0 & 0^T \end{pmatrix} K^2(z_{1h_1}) \Bigg\} \\
=& h_2\tau(1-\tau)f_z(z_0) \begin{pmatrix} \nu_0 & 0 \\ 0 & 0 \end{pmatrix} \otimes (\Gamma_{20}\boldsymbol{D}^*(z_0)^{-1}\Gamma_{20}^T).
\end{aligned} \tag{34}
$$

Next, it is shown that the last part of lemma holds true,

$$
\begin{aligned}
Var[V_n(0,0)] =& \frac{1}{nh} Var(\sum_{t=1}^{n} B_t) = \frac{1}{h}[Var(B_1) + 2\sum_{\ell=1}^{n-1}(1-\frac{\ell}{n})Cov(B_1, B_{\ell+1})] \\
\le& \frac{1}{h}Var(B_1) + \frac{2}{h}\sum_{\ell=1}^{e_n-1}|Cov(B_1, B_{\ell+1})| + \frac{2}{h}\sum_{\ell=e_n}^{\infty}|Cov(B_1, B_{\ell+1})| \\
\equiv& G_1 + G_2 + G_3.
\end{aligned}
$$

By (32), (33) and (34),

$$
G_1 \to \tau(1-\tau)f_z(z_0)\Bigg\{ \begin{pmatrix} \nu_0 & 0 \\ 0 & \nu_2 \end{pmatrix} \otimes \Omega(z_0) + \boldsymbol{H} \Bigg\}.
$$

Now it remains to show that $|G_2| = o(1)$ and $|G_3| = o(1)$. First, we consider G_3. To this end, by using Davydov's inequality (see, e.g., Corollary A.2 of Hall and Heyde (1980)) and the boundedness of $\psi_\tau(\cdot)$, one has

$$
|Cov(B_1, B_{\ell+1})| \le C\alpha^{1-2/\delta}(\ell)[E|B_1|^\delta]^{2/\delta} \le Ch^{2/\delta}\alpha^{1-2/\delta}(\ell),
$$

which gives

$$G_3 \leq Ch^{2/\delta-1}\sum_{\ell=e_n}^{\infty}\alpha^{1-2/\delta}(\ell) \leq Ch^{2/\delta-1}e_n^{-w}\sum_{\ell=e_n}^{\infty}\ell^w\alpha^{1-2/\delta}(\ell) = o(h^{2/\delta-1}e_n^{-w}) = o(1),$$

by choosing e_n to satisfy $e_n^w h^{1-2/\delta} = c$. As for G_2, following the proof of Lemma 3.5 in Xu (2005), one has $|G_2| = o(1)$. These prove Lemma A.10. $\square$

Proof of Theorem 2.1:

Proof. By Lemmas A.8, A.9 and A.10, $V_n(\boldsymbol{\theta}, \xi)$ satisfies Condition (ii) in Lemma A.7; that is, $\|A_n\| = O_p(1)$ and $\sup_{\|\Delta\|\leq M, \|\xi\|\leq L}\|V_n(\Delta, \xi) + D\Delta - A_n\| = o_p(1)$ with $D = f_z(z_0)\Omega_1^*(z_0)$ and $A_n = V_n(0,0) + f_z(z_0)\Gamma_{20}^*\boldsymbol{\vartheta}_0$. To show $\|V_n(\hat{\boldsymbol{\theta}}, \hat{\xi})\| = o_p(1)$, it follows from Lemma A.1 and mean value theorem that

$$\begin{aligned}\|V_n(\hat{\boldsymbol{\theta}}, \hat{\xi})\| =& a_n\left\|\sum_{t=1}^{n}[\psi_\tau(v_t^* - a_n\hat{\boldsymbol{\theta}}^T\boldsymbol{X}_t^*(\hat{\xi}))]\boldsymbol{X}_t^*(\hat{\xi})K(z_{th_2})\right\| \leq a_n\max_{1\leq t\leq n}\|\boldsymbol{X}_t^*(\hat{\xi})K(z_{th_2})\| \\ \leq& a_n\max_{1\leq t\leq n}\|\boldsymbol{X}_t^*K(z_{th_2})\| + Ca_n^2\max_{1\leq t\leq n}\left\|\left(\frac{\partial \boldsymbol{X}_t^*(\hat{\xi})}{\partial\hat{\xi}}\bigg|_{\hat{\xi}=\hat{\xi}'}\right)LK(z_{th_2})\right\| = o(1),\end{aligned}$$

where $\hat{\boldsymbol{\theta}}$ is the minimizer of $J(\boldsymbol{\theta}, \hat{\xi})$. Finally, because $\psi_\tau(x)$ is an increasing function of x; then $-\boldsymbol{\theta}^T V_n(\lambda\boldsymbol{\theta}) = a_n\sum_{t=1}^n\psi_\tau[v_t^* + \lambda a_n(-\boldsymbol{\theta}^T\boldsymbol{X}_t^*(\xi))](-\boldsymbol{\theta}^T\boldsymbol{X}_t^*(\xi))K(z_{th_2})$ is an increasing function of λ. Thus, Condition (i) in Lemma A.7 is satisfied. Then, it follows from Theorem A.6, Lemmas A.8 and A.9 that

$$\begin{aligned}\hat{\boldsymbol{\theta}} =& \frac{(\Omega_1^*(z_0))^{-1}}{\sqrt{nh_2}f_z(z_0)}\sum_{t=1}^{n}\psi_\tau(u_t^*)[\boldsymbol{X}_t^*K(z_{th_2})] + (\Omega_1^*(z_0))^{-1}\Gamma_{20}^*\boldsymbol{\vartheta}_0 + o_p(1) \\ =& \frac{(\Omega_1^*(z_0))^{-1}}{\sqrt{nh_2}f_z(z_0)}\sum_{t=1}^{n}\psi_\tau(u_t^*)[\boldsymbol{X}_t^*K(z_{th_2}) + \Gamma_{20}^*D_{m/K}(\boldsymbol{D}_1^*(z_0))^{-1}\boldsymbol{W}_t^{**}K(z_{th_1})] + o_p(1).\end{aligned}$$

Following the proof of Theorem 1 in Cai and Xu (2008), the theorem is proved. $\square$

Proof of (16)**:**

Proof. We only focus on $\hat{\Gamma}_{20}^T$, which can be written as

$$\hat{\Gamma}_{20}^T = \frac{1}{n}\sum_{t=1}^{n}\left\{(w_{1t}\hat{\boldsymbol{X}}_t^T\hat{\boldsymbol{g}}_\tau(z_0))\begin{pmatrix}0_{(\kappa m+1)\times 1} & \boldsymbol{W}_t K_{h_2}(Z_t-z_0) & \dots & \boldsymbol{W}_t K_{h_2}(Z_t-z_0) & 0_{(\kappa m+1)\times\kappa}\end{pmatrix}\right\}$$
$$\equiv \begin{pmatrix}0_{(\kappa m+1)\times 1} & \hat{\Gamma}_{10} & \dots & \hat{\Gamma}_{10} & 0_{(\kappa m+1)\times\kappa}\end{pmatrix},$$

where $\hat{\Gamma}_{10} \equiv \frac{1}{n}\sum_{t=1}^{n}(w_{1t}\hat{\boldsymbol{X}}_t^T\hat{\boldsymbol{g}}_\tau(z_0))\boldsymbol{W}_t K_{h_2}(Z_t-z_0)$. Define

$$\Gamma_1(z_0) \equiv E[f_{u|Z,\boldsymbol{W}}(q_\tau(z_0,\boldsymbol{W}_t))(\boldsymbol{X}_t^T\boldsymbol{g}_\tau(z_0))\boldsymbol{W}_t|Z_t=z_0].$$

Clearly,

$$\Gamma_{20}^T \equiv \begin{pmatrix}0_{(\kappa m+1)\times 1} & \Gamma_1(z_0) & \dots & \Gamma_1(z_0) & 0_{(\kappa m+1)\times\kappa}\end{pmatrix}.$$

Thus, we only need to show that $\hat{\Gamma}_{10} = f_z(z_0)\Gamma_1(z_0) + o_p(1)$. By Taylor's expansion, we have

$$\begin{aligned}E[w_{1t}|Z_t,\boldsymbol{W}_t] &= (F_{u|Z,\boldsymbol{W}}(\boldsymbol{W}_t^T\hat{\boldsymbol{\alpha}}_\tau(z_0)+\delta_{1n}) - F_{u|Z,\boldsymbol{W}}(\boldsymbol{W}_t^T\hat{\boldsymbol{\alpha}}_\tau(z_0)-\delta_{1n}))/(2\delta_{1n})\\ &= f_{u|Z,\boldsymbol{W}}(\boldsymbol{W}_t^T\hat{\boldsymbol{\alpha}}_\tau(z_0)) + o_p(1).\end{aligned}$$

In addition, notice that by applying mean value theorem, there exists $\hat{\xi}' \in (0,\hat{\xi})$ such that

$$\hat{\boldsymbol{X}}_t \equiv \boldsymbol{X}_t(\hat{\xi}) = \left(\boldsymbol{X}_t + (nh_2)^{-1/2}\left(\frac{\partial \boldsymbol{X}_t(\hat{\xi})}{\partial\hat{\xi}}\bigg|_{\hat{\xi}=\hat{\xi}'}\right)\hat{\xi}\right).$$

Therefore,

$$\begin{aligned}E[\hat{\Gamma}_{10}] &= E[f_{u|Z,\boldsymbol{W}}(\boldsymbol{W}_t^T\hat{\boldsymbol{\alpha}}_\tau(z_0))(\boldsymbol{X}_t^T\boldsymbol{g}_\tau(z_0))\boldsymbol{W}_t K_{h_2}(Z_t-z_0)] + o(1)\\ &= \int f_{u|Z,\boldsymbol{W}}(\boldsymbol{W}_t^T\hat{\boldsymbol{\alpha}}_\tau(z_0))\Gamma_1(z_0+h_2z)K(z)f_z(z_0+h_2z)dz + o(1) \to f_z(z_0)\Gamma_1(z_0).\end{aligned}$$

Similar to the proof of $Var[T_n(0)]$ in Lemma A.5, we can show that $Var(\hat{\Gamma}_{10}) \to 0$. This gives us $\hat{\Gamma}_{10} = f_z(z_0)\Gamma_1(z_0) + o_p(1)$, which proves (16). The consistency of $\hat{\Omega}(z_0)$, $\hat{\Omega}^*(z_0)$

and $\hat{\boldsymbol{D}}^{*}(z_0)$ can be derived in a similar way.

□

Table 1: Simulation results for $\gamma_{10,\tau}(\cdot)$, $\gamma_{20,\tau}(\cdot)$, $\gamma_{30,\tau}(\cdot)$, $\gamma_{40,\tau}(\cdot)$, and $\gamma_{ij,\tau}(\cdot)$ for $i = 1, 2$ and for $j = 1, 2, 3, 4$.

τ	$n = 500$		$n = 1500$		$n = 4000$	
	MADE(γ_{10})	MADE(γ_{20})	MADE(γ_{10})	MADE(γ_{20})	MADE(γ_{10})	MADE(γ_{20})
0.05	0.888 (0.241)	0.839 (0.245)	0.636 (0.101)	0.644 (0.110)	0.438 (0.056)	0.362 (0.066)
0.15	0.505 (0.127)	0.511 (0.167)	0.375 (0.050)	0.337 (0.056)	0.292 (0.031)	0.259 (0.036)
0.85	0.486 (0.121)	0.487 (0.138)	0.378 (0.051)	0.351 (0.050)	0.278 (0.036)	0.252 (0.039)
0.95	0.836 (0.228)	0.750 (0.200)	0.560 (0.100)	0.437 (0.106)	0.415 (0.054)	0.341 (0.063)
	MADE(γ_{30})	MADE(γ_{40})	MADE(γ_{30})	MADE(γ_{40})	MADE(γ_{30})	MADE(γ_{40})
0.05	0.722 (0.180)	0.901 (0.195)	0.593 (0.080)	0.547 (0.092)	0.501 (0.042)	0.499 (0.051)
0.15	0.476 (0.117)	0.490 (0.124)	0.378 (0.043)	0.363 (0.049)	0.307 (0.032)	0.283 (0.032)
0.85	0.452 (0.092)	0.472 (0.147)	0.372 (0.042)	0.379 (0.047)	0.284 (0.035)	0.241 (0.040)
0.95	0.700 (0.205)	0.807 (0.212)	0.548 (0.088)	0.606 (0.089)	0.476 (0.049)	0.502 (0.059)
	MADE(γ_{11})	MADE(γ_{12})	MADE(γ_{11})	MADE(γ_{12})	MADE(γ_{11})	MADE(γ_{12})
0.05	0.166 (0.079)	0.170 (0.072)	0.131 (0.053)	0.126 (0.047)	0.087 (0.030)	0.084 (0.035)
0.15	0.134 (0.068)	0.128 (0.049)	0.107 (0.045)	0.113 (0.041)	0.095 (0.038)	0.082 (0.030)
0.85	0.147 (0.071)	0.145 (0.055)	0.116 (0.048)	0.126 (0.044)	0.083 (0.034)	0.100 (0.032)
0.95	0.176 (0.070)	0.177 (0.065)	0.129 (0.047)	0.137 (0.048)	0.102 (0.034)	0.107 (0.037)
	MADE(γ_{13})	MADE(γ_{14})	MADE(γ_{13})	MADE(γ_{14})	MADE(γ_{13})	MADE(γ_{14})
0.05	0.171 (0.088)	0.159 (0.074)	0.129 (0.053)	0.136 (0.048)	0.087 (0.037)	0.108 (0.036)
0.15	0.148 (0.067)	0.188 (0.083)	0.109 (0.051)	0.124 (0.048)	0.083 (0.035)	0.097 (0.030)
0.85	0.151 (0.070)	0.179 (0.076)	0.104 (0.041)	0.138 (0.055)	0.081 (0.031)	0.104 (0.033)
0.95	0.148 (0.064)	0.186 (0.077)	0.132 (0.051)	0.154 (0.052)	0.097 (0.034)	0.110 (0.036)
	MADE(γ_{21})	MADE(γ_{22})	MADE(γ_{21})	MADE(γ_{22})	MADE(γ_{21})	MADE(γ_{22})
0.05	0.172 (0.082)	0.166 (0.071)	0.141 (0.056)	0.121 (0.050)	0.094 (0.041)	0.082 (0.032)
0.15	0.145 (0.068)	0.109 (0.058)	0.120 (0.047)	0.098 (0.040)	0.103 (0.042)	0.074 (0.030)
0.85	0.153 (0.084)	0.116 (0.065)	0.128 (0.046)	0.097 (0.045)	0.090 (0.037)	0.082 (0.033)
0.95	0.257 (0.072)	0.219 (0.054)	0.146 (0.050)	0.130 (0.044)	0.110 (0.039)	0.084 (0.030)
	MADE(γ_{23})	MADE(γ_{24})	MADE(γ_{23})	MADE(γ_{24})	MADE(γ_{23})	MADE(γ_{24})
0.05	0.181 (0.083)	0.139 (0.064)	0.139 (0.058)	0.131 (0.056)	0.091 (0.031)	0.101 (0.031)
0.15	0.156 (0.075)	0.188 (0.106)	0.114 (0.045)	0.116 (0.047)	0.094 (0.036)	0.089 (0.033)
0.85	0.167 (0.081)	0.163 (0.088)	0.114 (0.046)	0.124 (0.056)	0.090 (0.034)	0.102 (0.039)
0.95	0.216 (0.052)	0.259 (0.068)	0.145 (0.047)	0.149 (0.050)	0.105 (0.034)	0.102 (0.038)

Table 2: Simulation results for $\gamma_{ij,\tau}(\cdot)$ for $i = 3, 4$ and for $j = 1, 2, 3, 4$.

τ	$n = 500$		$n = 1500$		$n = 4000$	
	MADE(γ_{31})	MADE(γ_{32})	MADE(γ_{31})	MADE(γ_{32})	MADE(γ_{31})	MADE(γ_{32})
0.05	0.151 (0.060)	0.147 (0.061)	0.119 (0.047)	0.108 (0.039)	0.090 (0.034)	0.079 (0.029)
0.15	0.128 (0.061)	0.117 (0.043)	0.110 (0.046)	0.098 (0.040)	0.095 (0.033)	0.077 (0.026)
0.85	0.150 (0.068)	0.128 (0.046)	0.115 (0.048)	0.112 (0.040)	0.087 (0.033)	0.091 (0.032)
0.95	0.176 (0.067)	0.163 (0.056)	0.123 (0.050)	0.122 (0.045)	0.102 (0.038)	0.094 (0.030)
	MADE(γ_{33})	MADE(γ_{34})	MADE(γ_{33})	MADE(γ_{34})	MADE(γ_{33})	MADE(γ_{34})
0.05	0.159 (0.065)	0.133 (0.055)	0.115 (0.047)	0.118 (0.047)	0.080 (0.034)	0.099 (0.033)
0.15	0.132 (0.064)	0.171 (0.070)	0.105 (0.044)	0.112 (0.044)	0.086 (0.033)	0.090 (0.038)
0.85	0.149 (0.064)	0.158 (0.067)	0.104 (0.038)	0.122 (0.046)	0.087 (0.031)	0.112 (0.034)
0.95	0.158 (0.055)	0.180 (0.066)	0.122 (0.049)	0.141 (0.049)	0.095 (0.036)	0.104 (0.037)
	MADE(γ_{41})	MADE(γ_{42})	MADE(γ_{41})	MADE(γ_{42})	MADE(γ_{41})	MADE(γ_{42})
0.05	0.311 (0.072)	0.300 (0.064)	0.132 (0.048)	0.115 (0.045)	0.088 (0.031)	0.087 (0.030)
0.15	0.135 (0.060)	0.123 (0.053)	0.109 (0.045)	0.104 (0.038)	0.097 (0.039)	0.082 (0.027)
0.85	0.212 (0.068)	0.171 (0.054)	0.119 (0.045)	0.119 (0.045)	0.097 (0.034)	0.096 (0.032)
0.95	0.173 (0.076)	0.171 (0.070)	0.128 (0.048)	0.123 (0.042)	0.097 (0.034)	0.098 (0.034)
	MADE(γ_{43})	MADE(γ_{44})	MADE(γ_{43})	MADE(γ_{44})	MADE(γ_{43})	MADE(γ_{44})
0.05	0.307 (0.064)	0.255 (0.053)	0.120 (0.050)	0.129 (0.047)	0.088 (0.034)	0.096 (0.034)
0.15	0.142 (0.066)	0.187 (0.080)	0.106 (0.041)	0.117 (0.045)	0.090 (0.032)	0.090 (0.034)
0.85	0.201 (0.062)	0.226 (0.074)	0.104 (0.040)	0.131 (0.052)	0.091 (0.033)	0.113 (0.035)
0.95	0.149 (0.066)	0.180 (0.074)	0.133 (0.048)	0.140 (0.050)	0.100 (0.033)	0.106 (0.037)

Table 3: Simulation results for $\beta_{ij,\tau}(\cdot)$ for $i = 1, 2$ and for $j = 1, 2, 3, 4$.

τ	$n = 500$		$n = 1500$		$n = 4000$	
	MADE(β_{11})	MADE(β_{12})	MADE(β_{11})	MADE(β_{12})	MADE(β_{11})	MADE(β_{12})
0.05	0.252 (0.116)	0.252 (0.120)	0.146 (0.061)	0.165 (0.066)	0.090 (0.036)	0.107 (0.032)
0.15	0.167 (0.082)	0.167 (0.075)	0.097 (0.040)	0.105 (0.042)	0.065 (0.024)	0.069 (0.026)
0.85	0.160 (0.086)	0.174 (0.076)	0.096 (0.045)	0.108 (0.041)	0.066 (0.025)	0.069 (0.030)
0.95	0.246 (0.100)	0.251 (0.103)	0.152 (0.057)	0.158 (0.064)	0.096 (0.033)	0.112 (0.036)
	MADE(β_{13})	MADE(β_{14})	MADE(β_{13})	MADE(β_{14})	MADE(β_{13})	MADE(β_{14})
0.05	0.240 (0.111)	0.255 (0.121)	0.146 (0.060)	0.167 (0.066)	0.091 (0.035)	0.109 (0.037)
0.15	0.153 (0.077)	0.172 (0.088)	0.106 (0.038)	0.094 (0.036)	0.062 (0.023)	0.072 (0.028)
0.85	0.164 (0.074)	0.182 (0.082)	0.095 (0.040)	0.106 (0.040)	0.059 (0.023)	0.070 (0.023)
0.95	0.222 (0.100)	0.249 (0.105)	0.147 (0.052)	0.162 (0.053)	0.093 (0.034)	0.107 (0.035)
	MADE(β_{21})	MADE(β_{22})	MADE(β_{21})	MADE(β_{22})	MADE(β_{21})	MADE(β_{22})
0.05	0.270 (0.116)	0.241 (0.117)	0.183 (0.067)	0.153 (0.068)	0.112 (0.035)	0.098 (0.033)
0.15	0.179 (0.090)	0.154 (0.081)	0.106 (0.038)	0.099 (0.039)	0.071 (0.024)	0.065 (0.028)
0.85	0.178 (0.086)	0.163 (0.084)	0.111 (0.044)	0.096 (0.044)	0.072 (0.027)	0.064 (0.024)
0.95	0.362 (0.090)	0.338 (0.083)	0.168 (0.053)	0.161 (0.050)	0.104 (0.035)	0.100 (0.034)
	MADE(β_{23})	MADE(β_{24})	MADE(β_{23})	MADE(β_{24})	MADE(β_{23})	MADE(β_{24})
0.05	0.261 (0.115)	0.243 (0.106)	0.166 (0.067)	0.156 (0.067)	0.103 (0.034)	0.099 (0.036)
0.15	0.176 (0.088)	0.164 (0.090)	0.102 (0.040)	0.102 (0.041)	0.068 (0.021)	0.066 (0.026)
0.85	0.173 (0.081)	0.169 (0.093)	0.107 (0.038)	0.098 (0.044)	0.069 (0.025)	0.070 (0.024)
0.95	0.359 (0.086)	0.354 (0.090)	0.160 (0.052)	0.161 (0.053)	0.102 (0.033)	0.103 (0.031)

Table 4: Simulation results for $\beta_{ij,\tau}(\cdot)$ for $i = 3, 4$ and for $j = 1, 2, 3, 4$.

τ	$n = 500$		$n = 1500$		$n = 4000$	
	MADE(β_{31})	MADE(β_{32})	MADE(β_{31})	MADE(β_{32})	MADE(β_{31})	MADE(β_{32})
0.05	0.230 (0.096)	0.211 (0.083)	0.148 (0.055)	0.134 (0.050)	0.096 (0.038)	0.094 (0.034)
0.15	0.154 (0.063)	0.138 (0.055)	0.103 (0.039)	0.089 (0.036)	0.071 (0.025)	0.061 (0.023)
0.85	0.149 (0.065)	0.139 (0.063)	0.098 (0.039)	0.089 (0.035)	0.066 (0.023)	0.064 (0.024)
0.95	0.245 (0.085)	0.222 (0.078)	0.145 (0.057)	0.137 (0.055)	0.098 (0.036)	0.094 (0.032)
	MADE(β_{33})	MADE(β_{34})	MADE(β_{33})	MADE(β_{34})	MADE(β_{33})	MADE(β_{34})
0.05	0.220 (0.088)	0.208 (0.089)	0.143 (0.055)	0.133 (0.053)	0.103 (0.030)	0.097 (0.027)
0.15	0.154 (0.066)	0.142 (0.061)	0.093 (0.036)	0.091 (0.040)	0.065 (0.025)	0.064 (0.025)
0.85	0.149 (0.061)	0.151 (0.067)	0.095 (0.037)	0.093 (0.036)	0.066 (0.025)	0.066 (0.024)
0.95	0.225 (0.078)	0.242 (0.087)	0.138 (0.053)	0.133 (0.052)	0.100 (0.034)	0.094 (0.032)
	MADE(β_{41})	MADE(β_{42})	MADE(β_{41})	MADE(β_{42})	MADE(β_{41})	MADE(β_{42})
0.05	0.425 (0.085)	0.427 (0.091)	0.141 (0.055)	0.155 (0.056)	0.099 (0.030)	0.111 (0.036)
0.15	0.161 (0.067)	0.163 (0.076)	0.096 (0.038)	0.100 (0.041)	0.068 (0.024)	0.071 (0.023)
0.85	0.219 (0.066)	0.214 (0.071)	0.096 (0.037)	0.101 (0.039)	0.070 (0.024)	0.068 (0.027)
0.95	0.240 (0.107)	0.248 (0.099)	0.147 (0.052)	0.164 (0.056)	0.096 (0.032)	0.111 (0.039)
	MADE(β_{43})	MADE(β_{44})	MADE(β_{43})	MADE(β_{44})	MADE(β_{43})	MADE(β_{44})
0.05	0.411 (0.086)	0.436 (0.085)	0.137 (0.050)	0.152 (0.054)	0.094 (0.027)	0.104 (0.034)
0.15	0.151 (0.064)	0.164 (0.066)	0.089 (0.034)	0.101 (0.039)	0.060 (0.022)	0.074 (0.026)
0.85	0.211 (0.066)	0.226 (0.065)	0.090 (0.033)	0.106 (0.039)	0.069 (0.020)	0.069 (0.025)
0.95	0.220 (0.089)	0.257 (0.107)	0.141 (0.052)	0.164 (0.057)	0.101 (0.032)	0.115 (0.035)

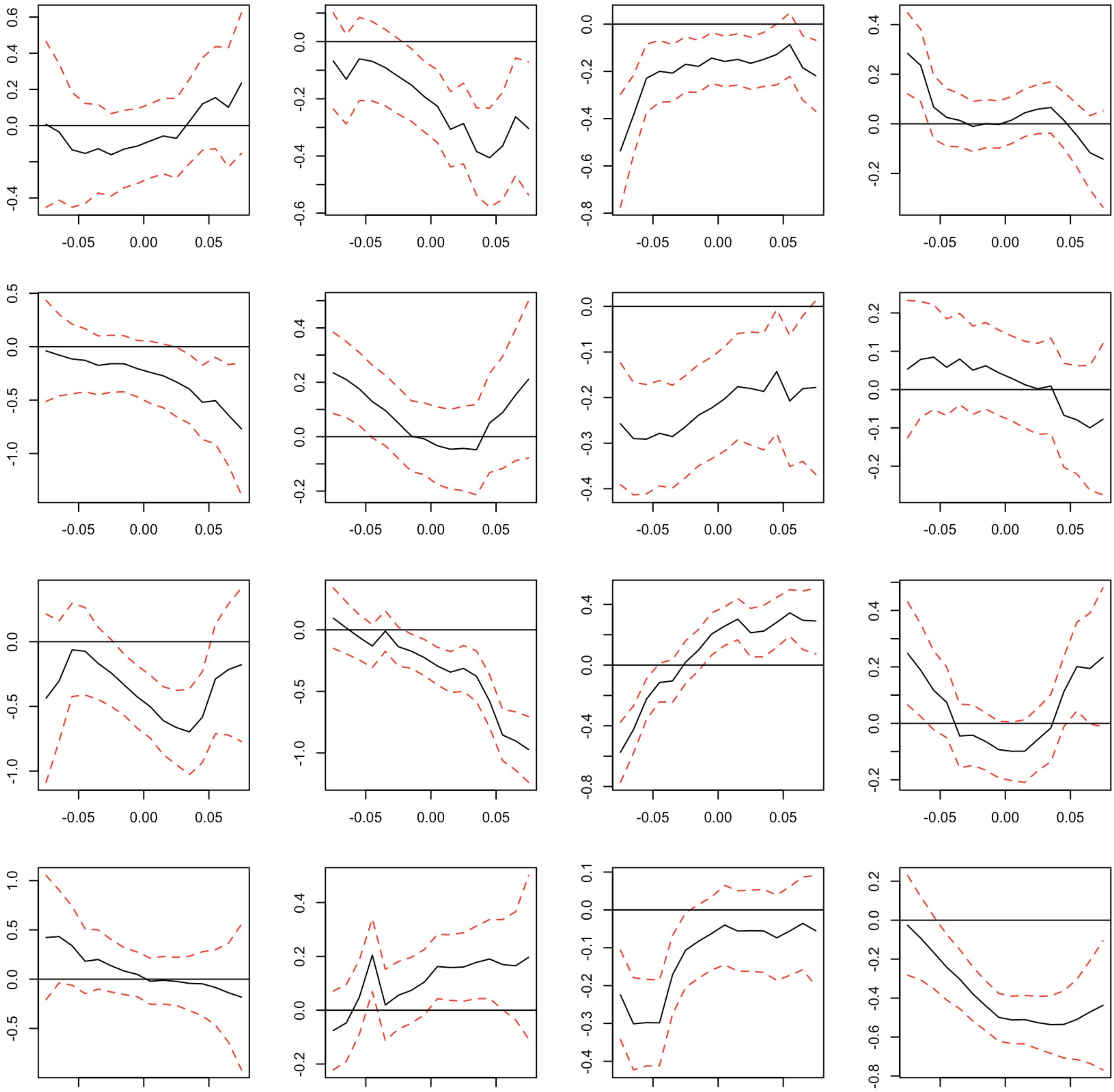

Figure 1: Plots of the estimated coefficient functions $\gamma_{ij,\tau}(\cdot)$ for $1 \leq i \leq 4$ and $1 \leq j \leq 4$ in (17) under $\tau = 0.05$ (black solid lines), in which ij-th panel represents the result for $\gamma_{ij,\tau}(\cdot)$, respectively. The red dashed lines in each panel indicate the 95% pointwise confidence interval for the estimate with the bias ignored.

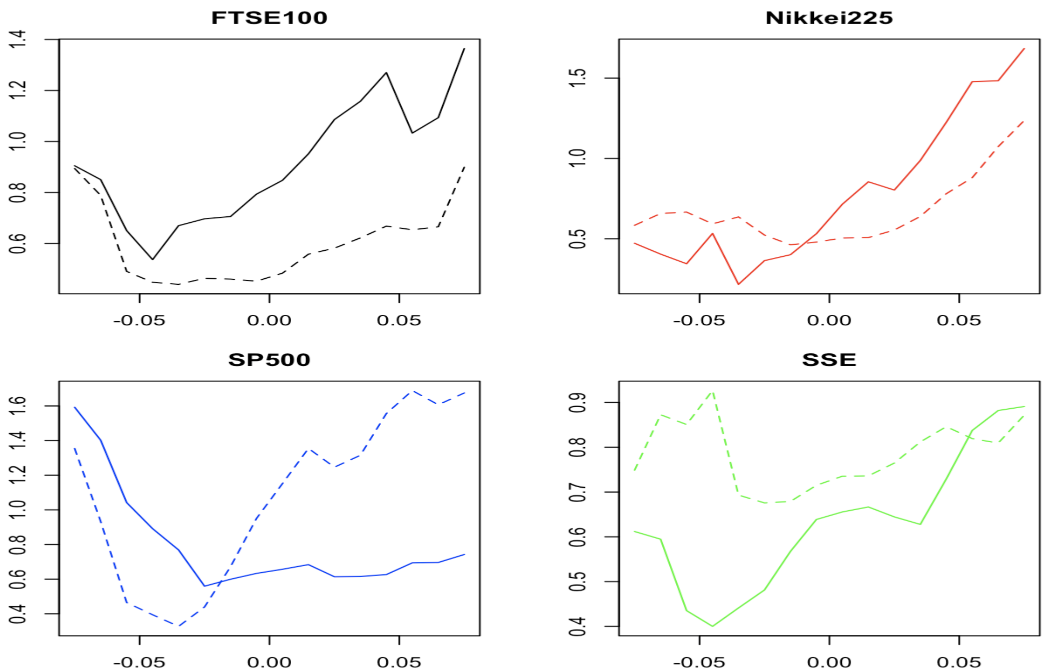

Figure 2: Connectedness with respect to outgoing links and connectedness with respect to incoming links for four market indexes. The solid line in each panel represents values of connectedness with respect to outgoing links and the dashed line in each panel indicates values of connectedness is for incoming link, with $\tau = 0.05$.

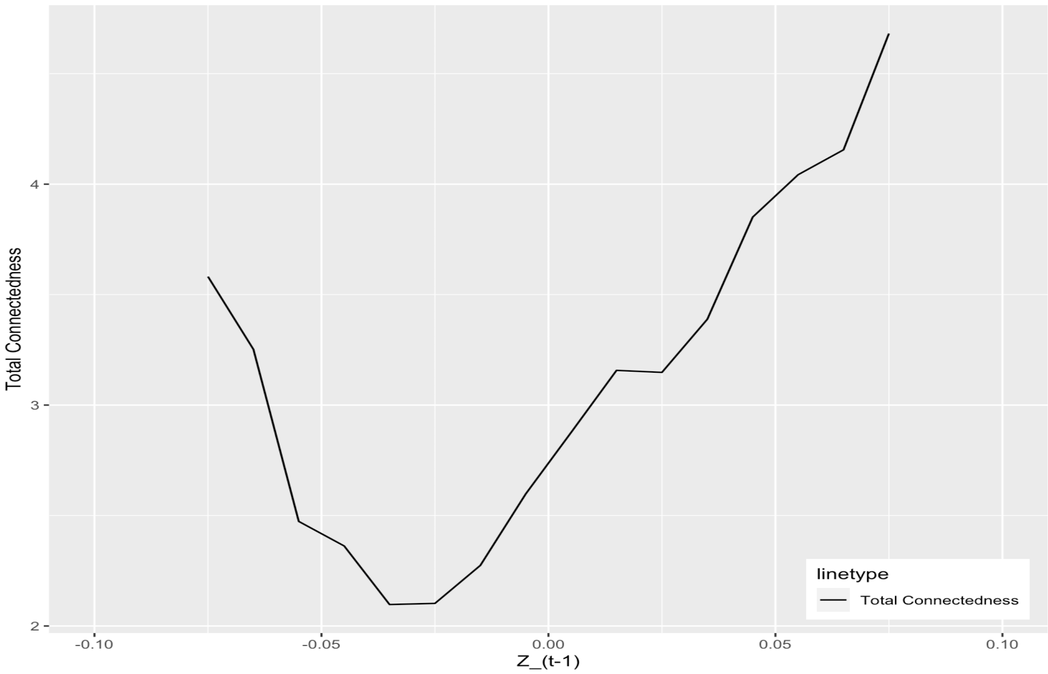

Figure 3: Total connectedness in international equity markets. $\tau = 0.05$.