Dynamic strategic complements in two stage, 2x2 games

By

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Abstract

Strategic complements are well understood for normal form games, but less so for extensive form games. There is some evidence that extensive form games with strategic complementarities are a very restrictive class of games (Echenique (2004)). We study necessary and sufficient conditions for strategic complements (defined as increasing best responses) in two stage, 2×2 games. We find that the restrictiveness imposed by quasisupermodularity and single crossing property is particularly severe, in the sense that the set of games in which payoffs satisfy these conditions has measure zero. Payoffs with these conditions require the player to be indifferent between their actions in two of the four subgames in stage two, eliminating any strategic role for their actions in these two subgames. In contrast, the set of games that exhibit strategic complements (increasing best responses) has infinite measure. This enlarges the scope of strategic complements in two stage, 2×2 games (and provides a basis for possibly greater scope in more general games). The set of subgame perfect Nash equilibria in the larger class of games continues to remain a nonempty, complete lattice. The results are easy to apply, and are robust to including dual payoff conditions and adding a third player. Examples with several motivations are included.

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1 Introduction

Strategic complements are well understood for normal form games, but less so for extensive form games. Indeed, Echenique (2004) identifies a potential concern about the possibility of strategic complements in extensive form games. Using a natural definition (each player's payoff function satisfies standard notions of quasisupermodularity and single crossing property in every subgame), he gives several examples showing that games that should intuitively exhibit strategic complementarities do not satisfy this definition. He also gives examples of simple extensive form games that cannot be made into extensive form games with strategic complements. He concludes that extensive form games with strategic complementarities are a very restrictive class of games.

We study necessary and sufficient conditions for strategic complements (defined as increasing best responses) in two stage, 2×2 games. These are games with two players, two actions for each player, and two stages of play, sometimes denoted $2 \times 2 \times 2$ games. Such games are a building block for multi-stage games and repeated games. We apply the standard framework. In the first stage, two players play a 2×2 game. After observing the outcome, they play another 2×2 game in the second stage and payoffs are realized for each possible outcome. There are sixteen terminal nodes (four outcomes in the first stage, and for each outcome in the first stage, four outcomes in the second stage). The payoff for each player is characterized by an ordered tuple in $\mathbb{R}^{16}$, and the set of all two stage, 2×2 games is identified naturally with $\mathbb{R}^{16} \times \mathbb{R}^{16}$. We use Euclidean topology and Lebesgue measure on this space.

We find that the restrictiveness imposed by the standard sufficient conditions of quasisupermodularity and single crossing property is particularly severe, in the sense that the set of two stage, 2×2 games in which payoffs satisfy these conditions has (Lebesgue) measure zero in the set of all two stage, 2×2 games (identified with $\mathbb{R}^{16} \times \mathbb{R}^{16}$). Indeed, these conditions imply that each player must be indifferent between their actions in two of the four subgames in stage two, eliminating any strategic role for their actions in these two subgames.

On the other hand, we find that necessary conditions for strategic complements are much weaker, in the sense that the set of two stage, 2×2 games with strategic complements (increasing best responses) has infinite (Lebesgue) measure in the set of all two stage, 2×2 games.[1] This enlarges the scope of strategic complements in this class of games (and provides a basis for possibly greater scope in more general extensive form games).

The results here are based on a detailed study of strategic complements in two stage, 2×2 games, and a characterization of when a player exhibits strategic complements in such games. As steps in the development of the main results, we show that strategic complements imply a particular structure for best choices in the first and second stage games. A careful analysis of best responses both on and off the path of play helps to identify necessary implications of strategic complements in three different scenarios. This is important to characterize strategic complements. The conditions we identify are easy to formulate in terms of individual player payoffs and yield (uncountably) many new extensive form games with strategic complements.

The notion of subgame strategic complements used here is consistent with the notion of increasing extended best responses in Echenique (2004), and therefore, his result that the set of subgame perfect Nash equilibria is a nonempty, complete lattice continues to hold in the larger class of games considered here, further expanding the scope of strategic complements techniques and results in extensive form games.

One could convert the extensive form game into a 16×16 normal form game and investigate strategic complements in that setting. We don't do this for the usual reason that this would collapse the dynamic structure of the model, which is a motivating feature for our work, and it would allow for Nash equilibria that are not subgame perfect.

[1]Notably, such a distinction does not hold for normal form games in general, as can be shown readily for the case of 2×2 games, where the set of games in which payoffs satisfy quasisupermodularity and single crossing properties has infinite (Lebesgue) measure.

As is well known, in one shot 2×2 matrix games, the role of strategic complements may be larger, because a 2×2 matrix game with strategic substitutes may be transformed into a game with strategic complements by reversing the order on the actions of one player. In other words, it is useful to know if our results remain true when we include the larger set of payoffs that include those sufficient for strategic substitutes. We show this is indeed the case. In other words, the set of games in which payoffs satisfy standard sufficient conditions for either strategic complements or strategic substitutes has measure zero, but the set of games in which best responses are either increasing or decreasing has infinite measure.

We explore the case of games with three players, two actions for each player, and two stages overall, denoted $3 \times 2 \times 2$ games. In the first stage, three players play a 3×2 game. After observing the outcome, they play another 3×2 game in stage two. Payoffs are realized at the terminal nodes. There are sixty four terminal nodes (eight outcomes in the first stage game, and for each outcome in the first stage, eight outcomes in the second stage). The payoff for each player is characterized by an ordered tuple in $\mathbb{R}^{64}$, and the set of all $3 \times 2 \times 2$ games is identified naturally with $\mathbb{R}^{64} \times \mathbb{R}^{64} \times \mathbb{R}^{64}$ (or equivalently, $\mathbb{R}^{192}$). We use Euclidean topology and Lebesgue measure on this space.

We find that the restrictiveness imposed by sufficient conditions of quasisupermodularity and single crossing property is even more severe here, in the sense that not only the set of $3 \times 2 \times 2$ games in which payoffs satisfy these conditions has (Lebesgue) measure zero in the set of all $3 \times 2 \times 2$ games, but more surprisingly, each player must be indifferent between their actions in six of the eight subgames in stage two, eliminating any strategic role for their actions in these six subgames. This shows that quasisupermodularity and single crossing property conditions restrict payoffs to $\mathbb{R}^{40} \times \mathbb{R}^{40} \times \mathbb{R}^{40}$ (or equivalently, $\mathbb{R}^{120}$).

On the other hand, we show with a class of examples that strategic complements (increasing best responses) in $3 \times 2 \times 2$ games allow for strategic interaction in six of the eight subgames. More generally, increasing best responses allow for payoffs in a set of infinite measure in $\mathbb{R}^{56} \times \mathbb{R}^{56} \times \mathbb{R}^{56}$ (or equivalently, $\mathbb{R}^{168}$).

As is well known, the problem of characterizing strategic complements in general extensive form games remains intractable. Some of the difficulties involved can be seen in our work. In order to have tractable models, sufficient conditions have been shown for particular classes of games, for example, those with a recursive framework (like repeated games) and using more specialized strategies like Markov strategies and conditions on transition probabilities, as described in Amir (1996) and Curtat (1996). This has been developed further by Vives (2009) and Balbus, Reffett, and Woźny (2014), providing the reader with a flavor of the complexities involved. Walker, Wooders, and Amir (2011) provide results for two player, extensive form, binary Markov games such as tennis matches where monotonicity properties are helpful to identify predicted Nash equilbria.

As two stage, 2×2 games are a basic building block for multi-stage games and infinitely repeated games, our results may provide insight to other researchers to explore more general cases. In particular, our results show the need to go beyond the natural and direct adaptation of quasisupermodularity and single crossing property as used in Echenique (2004).

In order to present ideas more concretely, we consider an explicit example in the next section. Section 3 defines the general framework and presents the main result characterizing strategic complements. Section 4 formalizes the connection to Echenique (2004), section 5 contains the results for three player games, and section 6 concludes.

2 Motivating Example

Consider an industry with two firms using production technologies with spillovers that provide an incentive for firms to coordinate on technology choice.

In stage 1, the two firms are considering to use an existing technology (A_1^0 for firm

P2

		B_1^0	B_2^0
P1	A_1^0	$3, 3$ (1)	$1, -3$ (2)
	A_2^0	$-3, 1$ (3)	$-4, -4$ (4)

Table 1: Stage 1 Game

1, B_1^0 for firm 2), or invest in an expensive new technology (A_2^0 for firm 1, B_2^0 for firm 2) that may, perhaps, scale better in the future. Staying with the cheaper technology is profitable at present and there is a stage 1 loss in going with the expensive technology. If one firm invests in the new technology, the other enjoys a spillover gain. If both firms invest, there is overinvestment leading to larger stage 1 losses for both firms. Payoffs are given in table 1. Assume that $A_1^0 \prec A_2^0$ and $B_1^0 \prec B_2^0$.

For each of the four outcomes in stage 1, there is a corresponding stage 2 game, indexed by the number in the top right hand corner of each cell in table 1, and given by the normal form in table 2. In each stage 2 game $n \in \{1, 2, 3, 4\}$, each firm decides whether to produce low (A_1^n for firm 1, B_1^n for firm 2) or produce high (A_2^n for firm 1, B_2^n for firm 2). Assume that for each stage 2 game $n \in \{1, 2, 3, 4\}$, $A_1^n \prec A_2^n$ and $B_1^n \prec B_2^n$. For notational convenience, payoffs are assumed to be discounted payoffs. The payoffs given in table 2 may be motivated as follows.

If the outcome in stage 1 is (A_1^0, B_1^0), then both firms may continue with the existing technology, produce less and make a higher profit. If one firm produces high and the other produces low, the profits of the high firm suffer from the high cost of inefficient cheaper technology, and due to spillovers, this also affects the profits of the low firm to some extent. Both firms cannot produce high in the absence of the expensive technology.

If the outcome in stage 1 is (A_1^0, B_2^0), then it is strictly dominant for firm 2 to produce high. Assuming large technological spillovers (perhaps because once invested, the better techonology is easily available to others), it may be dominant for firm 1 to produce high

(1) P2

		B_1^1	B_2^1
P1	A_1^1	15, 15	14, 5
	A_2^1	5, 14	0, 0

(2) P2

		B_1^2	B_2^2
P1	A_1^2	5, 5	4, 13
	A_2^2	13, 10	14, 19

(3) P2

		B_1^3	B_2^3
P1	A_1^3	5, 5	10, 13
	A_2^3	13, 4	19, 14

(4) P2

		B_1^4	B_2^4
P1	A_1^4	8, 8	0, 5
	A_2^4	5, 0	17, 17

Table 2: Stage 2 Games

as well. A similar scenario occurs with the roles of firm 1 and 2 reversed if the outcome in stage 1 is $(A_2^0,\ B_1^0)$.

If the outcome in stage 1 is $(A_2^0,\ B_2^0)$, then both firms may produce high and there is no signigicant benefit from technological spillovers. There is no benefit to a firm from producing less if the other firm is producing more (perhaps because the low output firm is crowded out, its more expensive technology is idle, and this may have a negative spillover for the high firm). It remains somewhat profitable if both firms coordinate on low output.

The extensive form of the overall two stage game is depicted in figure 1. The payoff at terminal nodes is sum of stage 1 payoff and stage 2 (discounted) payoff.

An alternative interpretation is in terms of brand marketing. In the first stage, each firm considers whether to use an existing lower quality brand or spend more to promote and develop a high quality brand. In the second stage, each firm decides whether to produce the low brand or the high brand. Explanations similar to the ones above may be used to motivate stage one and stage two payoffs.

In this two stage game, a strategy for player 1 is a 5-tuple $s = (s^0, s^1, s^2, s^3, s^4)$, where for each $n = 0, 1, 2, 3, 4$, $s^n \in \{A_1^n, A_2^n\}$. The strategy space for player 1 is the collection of

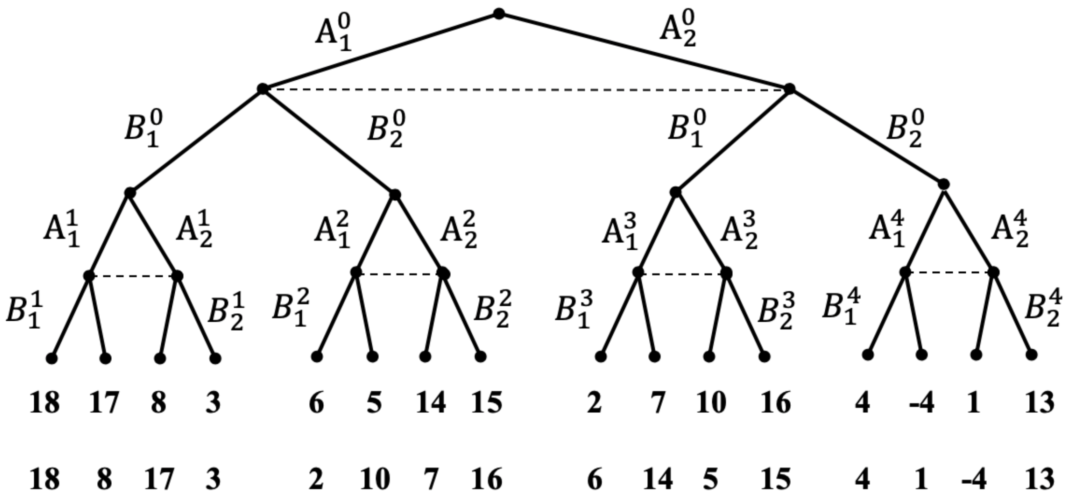


Figure 1: Extensive Form of Motivating Example

all strategies, denoted $\mathcal{S}$, and is endowed with the product order. It is a (complete) lattice in the product order.[2] Similarly, a strategy for player 2 is a 5-tuple $t = (t^0, t^1, t^2, t^3, t^4)$, where for each $n = 0, 1, 2, 3, 4$, $t^n \in \{B_1^n, B_2^n\}$. The strategy space for player 2 is the collection of all strategies, denoted $\mathcal{T}$, and is endowed with the product order. It is also a (complete) lattice in the product order. We denote payoffs from a strategy profile (s, t) by $u_1(s, t)$ for player 1 and $u_2(s, t)$ for player 2, as usual.

This makes the game into a lattice game (each player's strategy space is a lattice), and we can inquire if this game exhibits strategic complements. In other words, is the best response of one player increasing (in the lattice set order)[3] in the strategy of the other player?

Notice that the component games are very well behaved in terms of monotone comparative statics. Each of the games 0, 1, 2, and 3 has a strictly dominant action for each player, and game 4 is a classic coordination game with two strict Nash equilibria.

[2]We use standard lattice theoretic concepts. Useful references are Milgrom and Shannon (1994) and Topkis (1998).

[3]See next section for the (standard) definition.

Therefore, we may think that this is a game with strategic complements.

Indeed, as shown below in more generality, this game does exhibit strategic complements. Moreover, it is straightforward to check that this game has two subgame perfect Nash equilibria, one given by $\hat{s}^* = (A_1^0, A_1^1, A_2^2, A_2^3, A_1^4)$ and $\hat{t}^* = (B_1^0, B_1^1, B_2^2, B_2^3, B_1^4)$, and the other given by $\tilde{s}^* = (A_1^0, A_1^1, A_2^2, A_2^3, A_2^4)$ and $\tilde{t}^* = (B_1^0, B_1^1, B_2^2, B_2^3, B_2^4)$, and the set of subgame perfect Nash equilibria is a complete lattice.

Nevertheless, this game does not satisfy the definition of an extensive form game with strategic complementarities used in Echenique (2004). For example, the payoff function of player 1 is not quasisupermodular.[4] Consider $\hat{s} = (A_1^0, A_1^1, A_2^2, A_1^3, A_1^4)$, $\tilde{s} = (A_2^0, A_1^1, A_1^2, A_1^3, A_1^4)$, and $\hat{t} = (B_2^0, B_1^1, B_1^2, B_1^3, B_1^4)$. In this case, player 1 payoff is $u_1(\hat{s} \vee \tilde{s}, \hat{t}) = 4 = u_1(\tilde{s}, \hat{t})$, and therefore, quasisupermodularity implies $15 = u_1(\hat{s}, \hat{t}) \leq u_1(\hat{s} \wedge \tilde{s}, \hat{t}) = 6$, a contradiction.

As shown below in more detail, this example is one of a large class of two stage, 2×2 games that exhibit strategic complements but do not satisfy the definition used in Echenique (2004). The example shows that monotonicity may be helpful to study dynamic coordination in games. It may be hard to verify for monotonicity directly, because each player here has 32 strategies and we would need to compute best response set to each opponent strategy and verify that it increases for each higher opponent strategy. Our result (theorem 1 below) provides equivalent conditions on payoffs that are easy to apply and automatically yield increasing best responses.

3 General Framework

Consider a general two stage, 2×2 game (denoted Γ). In the first stage, a 2×2 game (denoted game 0) is played in which player 1 can take actions in $\{A_1^0, A_2^0\}$ and player 2 can take actions in $\{B_1^0, B_2^0\}$. In the second stage, another 2×2 game is played depending

[4]See section 3 for the (standard) definition of a quasisupermodular function.

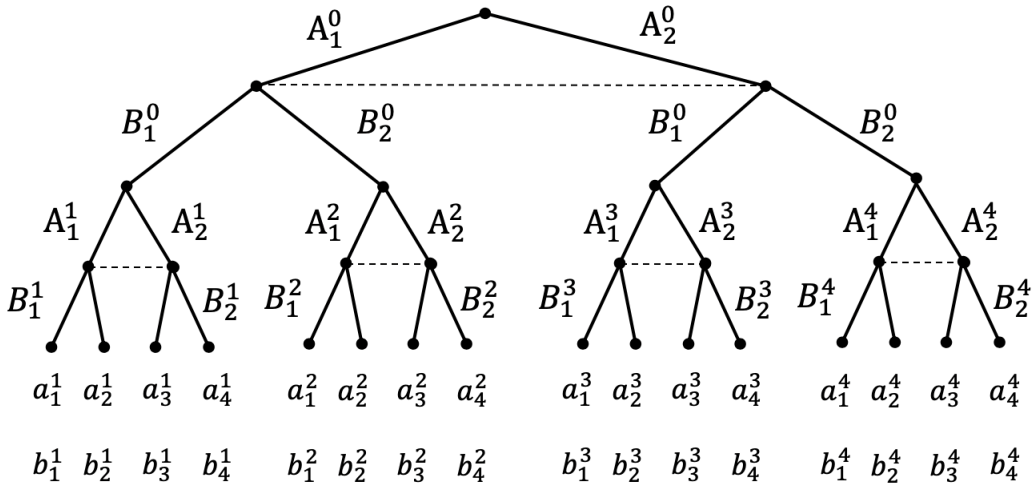


Figure 2: General Two Stage, 2×2 Game

on first stage outcome. If first stage outcome is (A_1^0, B_1^0), then game 1 is played, in which player 1 can take actions in $\{A_1^1, A_2^1\}$ and player 2 can take actions in $\{B_1^1, B_2^1\}$. If outcome is (A_1^0, B_2^0), then game 2 is played, in which player 1 can take actions in $\{A_1^2, A_2^2\}$ and player 2 can take actions in $\{B_1^2, B_2^2\}$. If outcome is (A_2^0, B_1^0), then game 3 is played, in which player 1 can take actions in $\{A_1^3, A_2^3\}$ and player 2 can take actions in $\{B_1^3, B_2^3\}$. If outcome is (A_2^0, B_2^0), then game 4 is played, in which player 1 can take actions in $\{A_1^4, A_2^4\}$ and player 2 can take actions in $\{B_1^4, B_2^4\}$. The extensive form of Γ is depicted in figure 2, with general payoffs at terminal nodes. When there is no confusion, we use the term *game* for such a two stage, 2×2 game. The set of all such games is identified naturally with $\mathbb{R}^{16} \times \mathbb{R}^{16}$. Throughout the paper, we view Euclidean space as a standard measure space with the Borel sigma-algebra and Lebesgue measure.

In each component game of a two stage, 2×2 game, suppose action 1 is lower than action 2, that is, for $n = 0, 1, 2, 3, 4$, $A_1^n \prec A_2^n$ and $B_1^n \prec B_2^n$. A strategy for player 1 is a 5-tuple $s = (s^0, s^1, s^2, s^3, s^4)$, where for each $n = 0, 1, 2, 3, 4$, $s^n \in \{A_1^n, A_2^n\}$. The strategy space for player 1 is the collection of all strategies, denoted $\mathcal{S}$, and is endowed with the product order. Notice that $\mathcal{S}$ is a complete lattice in the product order. Similarly, a strategy for player 2 is a 5-tuple $t = (t^0, t^1, t^2, t^3, t^4)$, where for each $n = 0, 1, 2, 3, 4$, $t^n \in \{B_1^n, B_2^n\}$. The strategy space for player 2 is the collection of all strategies, denoted $\mathcal{T}$, and is endowed with the product order. The strategy space $\mathcal{T}$ is a complete lattice in the product order. This makes Γ into a lattice game (each player's strategy space is a lattice). We denote payoffs from a strategy profile (s, t) as $u_1(s, t)$ for player 1 and $u_2(s, t)$ for player 2, as usual.

We shall formulate conditions under which such games exhibit strategic complements, defined in terms of increasing best responses, as usual. ***Player 1 has strategic complements***, if best response of player 1, denoted $BR^1(t)$, is increasing in t in the lattice set order (denoted $\sqsubseteq$).[5] That is, $\forall \hat{t}, \tilde{t} \in \mathcal{T},\ \hat{t} \preceq \tilde{t} \implies BR^1(\hat{t}) \sqsubseteq BR^1(\tilde{t})$. Similarly, we may define when ***player 2 has strategic complements***. The game Γ ***is a game with strategic complements***, if both players exhibit strategic complements.

Notice that strategic complements is defined for best response sets in the overall game. As shown by a closer analysis of examples in Echenique (2004) and in more detail here, this is the hard case. When we want to include strategic complements in subgames, we shall assume that second stage subgames exhibit strategic complements. As those are standard 2×2 games, conditions under which they exhibit strategic complements are well understood.

In the remainder of this section, we make the assumption that payoffs to different final outcomes (in other words, at different terminal nodes) are different. Such a two stage, 2×2 game is termed a ***game with differential payoffs to outcomes***. This assumption is sufficient to prove the results in this paper. Theoretically, the set of two stage, 2×2 games with differential payoffs to outcomes is open, dense, and has full (Lebesgue) measure[6] in $\mathbb{R}^{16} \times \mathbb{R}^{16}$ (the set of all such games).

[5]The lattice set order is the standard set order on lattices: $A \sqsubseteq B$ means that $\forall a \in A, \forall b \in B$, $a \wedge b \in A$ and $a \vee b \in B$. It is sometimes termed the Veinott set order, or the strong set order.

[6]Recall that a set has full (Lebesgue) measure if its complement has (Lebesgue) measure zero. This is different from a set with infinite (Lebesgue) measure.

Differential payoffs to outcomes has the following implications for the structure of best responses. For every $t \in T$, and for every $\hat{s}, \tilde{s} \in BR^1(t)$, the subgame reached on the path of play for profile $(\hat{s}, t)$ is the same as the subgame reached on the path of play for profile $(\tilde{s}, t)$. Moreover, the actions played by each player in the subgame reached on the path of play for profile $(\hat{s}, t)$ are the same as the actions played by each player in the subgame reached on the path of play for profile $(\tilde{s}, t)$. Furthermore, every $s \in S$ that has the same actions as $\hat{s}$ on the path of play for profile $(\hat{s}, t)$ is also a member of $BR^1(t)$. This is helpful in the proof of the following results.

The next three lemmas are important because they give necessary implications of strategic complements in the class of games studied here. Taken together they help to characterize strategic complements, as shown in theorem 1 below.

Lemma 1. *Consider a game with differential payoffs to outcomes and suppose player 1 exhibits strategic complements.*
For every $\hat{t}, \tilde{t} \in T$, for every $\hat{s} \in BR^1(\hat{t})$, and for every $\tilde{s} \in BR^1(\tilde{t})$, if $\hat{t}^0 = \tilde{t}^0$, then $\hat{s}^0 = \tilde{s}^0$.

Proof. See Appendix. □

Lemma 1 shows that in the class of games considered here, strategic complements for player 1 implies that if a fixed first stage action is part of player 1's best response to $\hat{t}$, then for every player 2 strategy $\tilde{t}$ that has the same first stage action as $\hat{t}$, every best response of player 1 must play the same fixed first stage action, and therefore, necessarily lead to the same subgame in stage two.

Lemma 2. *Consider a game with differential payoffs to outcomes and suppose player 1 exhibits strategic complements.*
(1) If there exists $\hat{t} \in T$ and $\hat{s} \in BR^1(\hat{t})$ such that $\hat{t}^0 = B_1^0$ and $\hat{s}^0 = A_2^0$, then for every $t \in T$ and for every $s \in BR^1(t)$, $s^0 = A_2^0$.

(2) If there exists $\hat{t} \in T$ *and* $\hat{s} \in BR^1(\hat{t})$ *such that* $\hat{t}^0 = B_2^0$ *and* $\hat{s}^0 = A_1^0$*, then for every* $t \in T$ *and for every* $s \in BR^1(t)$*,* $s^0 = A_1^0$*.*

Proof. See Appendix. □

Part (1) of this lemma shows that if playing the higher action in the first stage is ever a best response of player 1 to player 2 playing the lower action in the first stage, then for every player 2 strategy t, playing the higher action must be a best response of player 1. Similarly, part (2) of this lemma shows that if playing the lower action in the first stage is ever a best response of player 1 to player 2 playing the higher action in the first stage, then for every player 2 strategy t, playing the lower action must be a best response of player 1.

Lemma 3. *Consider a game with differential payoffs to outcomes and suppose player 1 exhibits strategic complements.*

(1) If there exists $\hat{t} \in T$ *and* $\hat{s} \in BR^1(\hat{t})$ *such that* $\hat{t}^0 = B_1^0$ *and* $\hat{s}^0 = A_1^0$*, then for every* $t \in T$ *and for every* $s \in BR^1(t)$*, if* $t^0 = B_1^0$ *then* $s^1 = A_1^1$*.*

(2) If there exists $\hat{t} \in T$ *and* $\hat{s} \in BR^1(\hat{t})$ *such that* $\hat{t}^0 = B_1^0$ *and* $\hat{s}^0 = A_2^0$*, then for every* $t \in T$ *and for every* $s \in BR^1(t)$*, if* $t^0 = B_1^0$ *then* $s^3 = A_1^3$*.*

(3) If there exists $\hat{t} \in T$ *and* $\hat{s} \in BR^1(\hat{t})$ *such that* $\hat{t}^0 = B_2^0$ *and* $\hat{s}^0 = A_2^0$*, then for every* $t \in T$ *and for every* $s \in BR^1(t)$*, if* $t^0 = B_2^0$ *then* $s^4 = A_2^4$*.*

(4) If there exists $\hat{t} \in T$ *and* $\hat{s} \in BR^1(\hat{t})$ *such that* $\hat{t}^0 = B_2^0$ *and* $\hat{s}^0 = A_1^0$*, then for every* $t \in T$ *and for every* $s \in BR^1(t)$*, if* $t^0 = B_2^0$ *then* $s^2 = A_2^2$*.*

Proof. See Appendix. □

Lemma 3 , building on lemmas 1 and 2, presents a very useful necessary implication of strategic complements in this setting. Whenever a particular subgame is reached on the best response path, lemma 3 locates the unique action that must be chosen in that subgame to be consistent with strategic complements. For example, statement (1) says that if subgame 1 is ever on the best response path, then whenever there is a chance to reach subgame 1 (that is, $t^0 = B_1^0$), player 1 necessarily plays A_1^0 (by lemma 1), and therefore, subgame 1 must be reached, and moreover, player 1 must play A_1^1 in subgame 1. As the lemma covers all four possible cases, this helps to characterize strategic complements in theorem 1 below. Notice that similar lemmas hold for player 2.

In order to make theorem 1 more accessible, it is useful to define when an action dominates another action, not just in a given subgame, but across subgames as well. For subgames $m, n \in \{1, 2, 3, 4\}$, and for action indices $k, \ell \in \{1, 2\}$, ***action A_k^m dominates action A_ℓ^n***, if regardless of which action player 2 plays in subgames m and n, action A_k^m in subgame m gives player 1 a higher payoff than A_ℓ^n.

In particular, a statement of the form A_1^1 *dominates* A_2^1 means that player 1 payoffs satisfy $a_1^1 > a_3^1$ and $a_2^1 > a_4^1$, a statement of the form A_1^1 *dominates* A_1^3 means that $\min\{a_1^1, a_2^1\} > \max\{a_1^3, a_2^3\}$, and a statement of the form A_1^1 *dominates* A_2^3 means that $\min\{a_1^1, a_2^1\} > \max\{a_3^3, a_4^3\}$. Consequently, the statement A_1^1 *dominates* A_2^1, A_1^3, *and* A_2^3 is equivalent to $a_1^1 > a_3^1$, $a_2^1 > a_4^1$, and $\min\{a_1^1, a_2^1\} > \max\{a_1^3, a_2^3, a_3^3, a_4^3\}$. Similar equivalences hold for the other conditions in the theorem. Here is the main theorem.

Theorem 1. *Consider a game with differential payoffs to outcomes.*
The following are equivalent.

1. *Player 1 has strategic complements*
2. *Exactly one of the following holds*
   (a) A_1^1 *dominates* A_2^1, A_1^3, *and* A_2^3, *and* A_2^2 *dominates* A_1^2, A_1^4, *and* A_2^4
   (b) A_1^1 *dominates* A_2^1, A_1^3, *and* A_2^3, *and* A_2^4 *dominates* A_1^4, A_1^2, *and* A_2^2
   (c) A_1^3 *dominates* A_2^3, A_1^1, *and* A_2^1, *and* A_2^4 *dominates* A_1^4, A_1^2, *and* A_2^2

Proof. For this proof, let $\underline{T} = \{t \in \mathcal{T} : t^0 = B_1^0\}$ and $\overline{T} = \{t \in \mathcal{T} : t^0 = B_2^0\}$.

For sufficiency, suppose player 1 has strategic complements. As case 1, suppose there exists $\hat{t} \in \underline{T}$, there exists $\hat{s} \in BR^1(\hat{t})$ such that $\hat{s}^0 = A_2^0$. Then lemma 3(2) implies that action A_1^3 dominates action A_2^3 for player 1 in subgame 3. Moreover, by lemma 1 and lemma 3(2), whenever player 2 plays B_1^0 in the first-stage game, player 1 chooses to reach subgame 3 over subgame 1, and then to play A_1^3 in subgame 3, regardless of player 2 choice in the second-stage game. Therefore, A_1^3 dominates A_1^1 and A_2^1. Furthermore, lemma 2(1) implies that for every $t \in T$ and $s \in BR^1(t)$, if $t^0 = B_2^0$, then $s^0 = A_2^0$, and therefore, lemma 3(3) implies that action A_2^4 dominates action A_1^4 for player 1 in subgame 4. Reasoning as above, A_2^4 dominates A_1^2 and A_2^2, and therefore, statement 2(c) holds.

As case 2, suppose for every $\hat{t} \in \underline{T}$, for every $\hat{s} \in BR^1(\hat{t})$, $\hat{s} = A_1^0$. Then lemma 3(1) implies that action A_1^1 dominates A_2^1 for player 1 in subgame 1, and reasoning as above, it follows that A_1^1 dominates A_1^3 and A_2^3 in subgame 3. Now consider $\overline{T}$. As subcase 1, suppose there exists $\tilde{t} \in \overline{T}$, there exists $\tilde{s} \in BR^1(\tilde{t})$ such that $\tilde{s}^0 = A_1^0$. Then lemma 3(4) implies that action A_2^2 dominates A_1^2 for player 1 in subgame 2, and that A_2^2 dominates A_1^4 and A_2^4. Therefore, statement 2(a) holds. As subcase 2, suppose for every $\tilde{t} \in \overline{T}$, for every $\tilde{s} \in BR^1(\tilde{t})$, $\tilde{s}^0 = A_2^0$. Then lemma 3(3) implies that action A_2^4 dominates A_1^4, and that A_2^4 dominates A_1^2 and A_2^2 in subgame 2. Therefore, statement 2(b) holds.

The reasoning above shows that one of the statements 2(a), 2(b), or 2(c) holds. It is easy to check that no more than one statement holds, because the statements are mutually exclusive. (In particular, A_2^2 dominates A_2^4 implies A_2^4 does not dominate A_2^2, A_1^1 dominates A_1^3 implies that A_1^3 does not dominate A_1^1, and so on.)

For necessity, suppose exactly one of 2(a), 2(b), or 2(c) holds. Suppose statement 2(a) holds. In this case, A_1^1 dominates A_2^1, A_1^3, and A_2^3 implies that for every $t \in \underline{T}$, player 1 chooses to reach subgame 1 over subgame 3 and to play A_1^1 in subgame 1. In other words, for every $t \in \underline{T}$, player 1's best response is given by

$$BR^1(t) = \{(A_1^0, A_1^1, s^2, s^3, s^4) \in \mathcal{S} : s^n \in \{A_1^n, A_2^n\}, n = 2, 3, 4\}.$$

Notice that this is a sublattice of $\mathcal{S}$. Similarly, A_2^2 dominates A_1^2, A_1^4, and A_2^4 implies that for every $t \in \overline{T}$, player 1 chooses to reach subgame 2 over subgame 4, and to play A_2^2 in subgame 2. In other words, for every $t \in \overline{T}$, player 1's best response is given by

$$BR^1(t) = \{(A_1^0, s^1, A_2^2, s^3, s^4) \in S : s^n \in \{A_1^n, A_2^n\}, n = 1, 3, 4\}.$$

Notice that this is a sublattice of $\mathcal{S}$ as well.

Now consider arbitrary $\hat{t}, \tilde{t} \in T$ such that $\hat{t} \preceq \tilde{t}$. If $\hat{t}^0 = \tilde{t}^0$, then $BR^1(\hat{t}) = BR^1(\tilde{t})$, and therefore, $BR^1(\hat{t}) \sqsubseteq BR^1(\tilde{t})$. And if $\hat{t}^0 = B_1^0$ and $\tilde{t}^0 = B_2^0$, then it is easy to check that $BR^1(\hat{t}) \sqsubseteq BR^1(\tilde{t})$. Thus, player 1 exhibits strategic complements.

The cases where statement 2(b) or 2(c) holds are proved similarly. □

As shown before the statement of theorem 1, it is easy to write the conditions in statements 2(a), 2(b), and 2(c) in terms of the corresponding payoffs, and these conditions are easy to satisfy (see proof of theorem 2 below). Therefore, this characterization yields uncountably many examples of two stage, 2×2 games with strategic complements. For reference, the motivating example in the previous section satisfies statement 2(a) of the theorem. Moreover, changing the index of player 1 and player 2 and following the same logic as above, a similar characterization holds for player 2 to have strategic complements.

The three cases described by conditions 2(a), 2(b), and 2(c) can be seen graphically in the following figures.

Condition 2(a) is represented in figure 3 as case 1. As shown in the upper panel, if player 2 plays B_1^0 in stage 1, then subgame 1 or 3 may be reached in stage 2, and A_1^1 dominates A_2^1, A_1^3, A_2^3 implies that player 1 will play A_1^0 in stage 1 and A_1^1 in stage 2, as necessitated by lemma 3 as well. In particular, all actions for player 1 in subgames 2, 3, and 4 are a part of the best response of player 1 to any strategy of player 2 that plays B_1^0.

If player 2 switches to a higher strategy B_2^0 in stage 1, then for the best response set of player 1 to move higher in the lattice set order, it cannot be that only the lower response in subgames 2, 3, and 4 is in the best response set. As subgame 3 is no longer on the

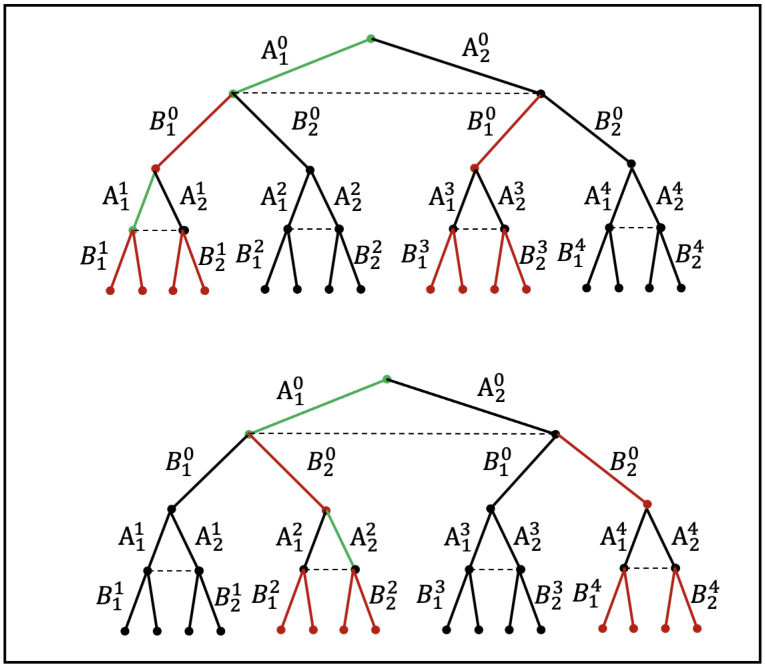


Figure 3: Theorem 1, Case 1

path of play, both actions of player 1 in subgame 3 are a part of the new best response. For subgames 2 and 4, either both actions must be part of the best response or only the higher action. Lemma 3 (using differential payoffs to outcomes) necessitates this to be A_2^2 in subgame 2, and therefore, both A_1^4 and A_2^4 in subgame 4 may be part of the best response. This is shown in the lower panel.

Similarly, condition 2(b) is represented in figure 4 as case 2, and condition 2(c) is represented in figure 5 as case 3.

As shown by the discussion before the statement of theorem 1, the conditions in theorem 1 are open conditions. This can be used to generate open sets of payoffs with strategic complements. In this sense, games with strategic complements (increasing best responses) lie in open neighborhoods of games with strategic complements. Moreover, this helps to show that the set of games with strategic complements is large in the sense of infinite Lebesgue measure, as follows.

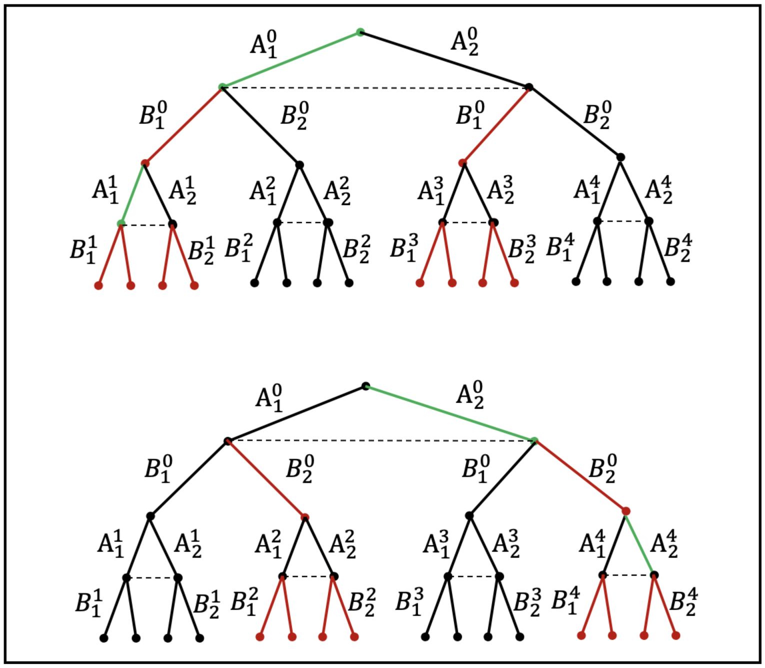


Figure 4: Theorem 1, Case 2

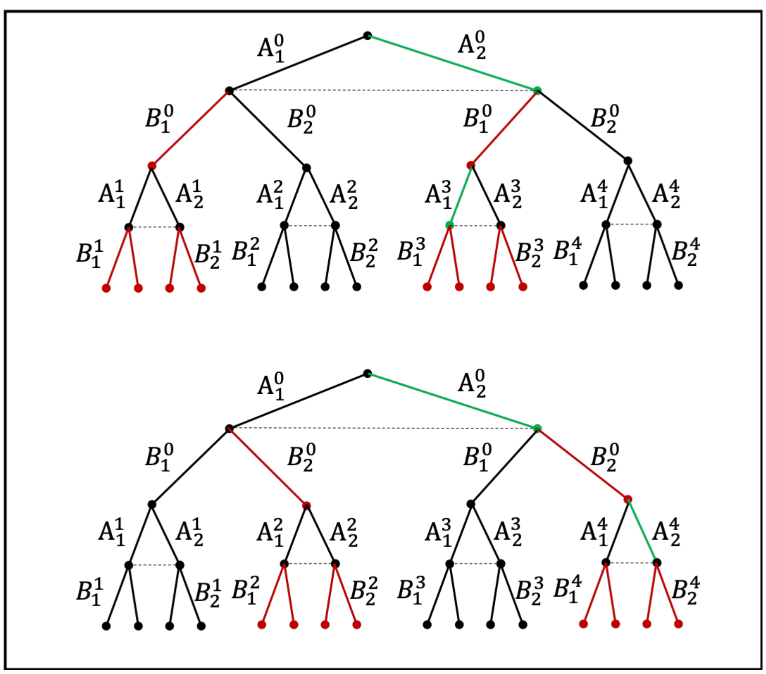


Figure 5: Theorem 1, Case 3

Theorem 2. *In the set of all two stage, 2×2 games, the set of games that satisfy strategic complements has infinite (Lebesgue) measure.*

Proof. The statement follows, because the set of games that satisfy each of the conditions 2(a), 2(b), and 2(c) in theorem 1 has infinite (Lebesgue) measure. For example, in condition 2(a), A_1^1 dominates A_2^1, A_1^3, and A_2^3 is equivalent to $a_1^1 > a_3^1$, $a_2^1 > a_4^1$, and $\min\{a_1^1, a_2^1\} > \max\{a_1^3, a_2^3, a_3^3, a_4^3\}$, and A_2^2 dominates A_1^2, A_1^4, and A_2^4 is equivalent to $a_3^2 > a_1^2$, $a_4^2 > a_2^2$, and $\min\{a_3^2, a_4^2\} > \max\{a_1^4, a_2^4, a_3^4, a_4^4\}$. Therefore, the set of payoffs satisfying condition 2(a) includes the following set.

$$\begin{aligned}
(a_1^1, a_2^1, a_3^1, a_4^1) &\in (20, +\infty) \times (15, 16) \times (13, 14) \times (10, 11) \quad &\subset \mathbb{R}^4 \\
(a_1^2, a_2^2, a_3^2, a_4^2) &\in (10, 11) \times (13, 14) \times (15, 16) \times (20, +\infty) \quad &\subset \mathbb{R}^4 \\
(a_1^3, a_2^3, a_3^3, a_4^3) &\in (0, 1) \times (2, 3) \times (5, 6) \times (8, 9) &\subset \mathbb{R}^4 \\
(a_1^4, a_2^4, a_3^4, a_4^4) &\in (0, 1) \times (2, 3) \times (5, 6) \times (8, 9) &\subset \mathbb{R}^4
\end{aligned}$$

The product of these sets has infinite Lebesgue measure in $\mathbb{R}^{16}$. Therefore, the set of games satisfying condition 2(a) has infinite measure. Consequently, the set of games in which player 1 has strategic complements has infinite measure. Similarly, it can be shown that the set of games in which player 2 has strategic complements has infinite measure. Taken together, this shows that the set of games with strategic complements has infinite (Lebesgue) measure. □

The measurable rectangle constructed in the proof above has minimum side length of one unit. It is easy to see that the example may be modified to construct a rectangle with minimum side length that is arbitrarily large.

Theorem 2 may be extended to include subgame strategic complements as follows. A two stage, 2×2 game satisfies ***subgame strategic complements***, if it exhibits strategic complements, and in each of the four 2×2 subgames in stage 2, the best response of each player is increasing (in the lattice set order) in the other player's strategy.

Corollary 1. *In the set of all two stage, 2×2 games, the set of games that satisfy subgame strategic complements has infinite (Lebesgue) measure.*

Proof. Consider the infinite measure set constructed in the proof of theorem 2 and notice that the construction satisfies subgame strategic complements. In particular, games with payoffs in that set have the property that for player 1, action A_1^1, A_2^2, A_2^3, and A_2^4 are dominant in stage two subgames 1, 2, 3, and 4, respectively. A similar statement holds for player 2. Taking these two statements together yields the result for games with strategic complements. □

Finally, the next theorem follows immediately by noting that subgame strategic complements implies increasing extended best response correspondences, as used in Echenique (2004), and to apply his corresponding result.

Theorem 3. *In every two stage, 2×2 game with subgame strategic complements, the set of subgame perfect Nash equilibria is a nonempty, complete lattice.*

Proof. Apply theorem 9 in Echenique (2004) by noting that its proof only requires nonempty, increasing best responses in every subgame, which is satisfied here. □

4 Comparison to Echenique (2004)

Echenique (2004) defines an extensive form game with strategic complementarities as an extensive form game in which each player's payoff function satisfies quasisupermodularity (in own strategy) and single crossing property in (own strategy; other players' strategy) in all subgames. For consistency in comparison, we shall first restrict the definition to the overall game and then include stage two subgames.

Player 1 payoff function $u_1 : \mathcal{S} \times \mathcal{T} \to \mathbb{R}$ is ***quasisupermodular (in s)***, if for every $t \in \mathcal{T}$ and for every $s, s' \in \mathcal{S}$, $u_1(s \wedge s', t) < (\leq)\ u_1(s, t) \implies u_1(s', t) < (\leq)\ u_1(s \vee s', t)$.

Player 1 payoff function $u_1 : \mathcal{S} \times \mathcal{T} \to \mathbb{R}$ satisfies ***single crossing property in (s; t)***, if for all $t, t' \in \mathcal{T}$ such that $t \prec t'$ and for all $s, s' \in \mathcal{S}$ such that $s \prec s'$, $u_1(s,t) < (\leq)\ u_1(s',t) \implies u_1(s,t') < (\leq)\ u_1(s',t')$. These are defined similarly for player 2 payoff function u_2. A two stage, 2×2 game satisfies ***E-payoff complementarity*** (short for payoff complementarities as used in Echenique (2004)), if the payoff function of each player is quasisupermodular in own strategy and satisfies single crossing property in (own strategy; other player strategy).

In order to state the following lemma, it is useful to recall when an action weakly dominates another action in a given subgame. For $n \in \{1,2,3,4\}$ and for $k, \ell \in \{1,2\}$, ***action A_k^n weakly dominates action A_ℓ^n***, if regardless of which action player 2 plays in subgame n, playing action A_k^n in subgame n gives player 1 a weakly higher payoff than A_ℓ^n. This definition shows that a statement like A_1^2 *weakly dominates* A_2^2 is equivalent to $a_1^2 \geq a_3^2$ and $a_2^2 \geq a_4^2$.

Lemma 4. *Consider a two stage,* 2×2 *game that satisfies E-payoff complementarity. For player 1,*
(1) A_1^2 *weakly dominates* A_2^2*, and* A_2^2 *weakly dominates* A_1^2
(2) A_1^3 *weakly dominates* A_2^3*, and* A_2^3 *weakly dominates* A_1^3

Proof. For the first statement, consider the following strategies: $\hat{s} = (A_1^0, A_1^1, A_2^2, A_1^3, A_1^4)$, $\tilde{s} = (A_2^0, A_1^1, A_1^2, A_1^3, A_1^4)$, $\hat{t} = (B_2^0, B_1^1, B_2^2, B_1^3, B_1^4)$, and $\tilde{t} = (B_2^0, B_1^1, B_1^2, B_1^3, B_1^4)$. In this case, $u_1(\hat{s} \vee \tilde{s}, \hat{t}) = a_1^4 = u_1(\tilde{s}, \hat{t})$, and therefore, quasisupermodularity implies $a_4^2 = u_1(\hat{s}, \hat{t}) \leq u_1(\hat{s} \wedge \tilde{s}, \hat{t}) = a_2^2$. Moreover, $u_1(\hat{s} \vee \tilde{s}, \tilde{t}) = a_1^4 = u_1(\tilde{s}, \tilde{t})$, and therefore, quasisupermodularity implies $a_3^2 = u_1(\hat{s}, \tilde{t}) \leq u_1(\hat{s} \wedge \tilde{s}, \tilde{t}) = a_1^2$. This shows that A_1^2 weakly dominates A_2^2.

For the other part, consider the following strategies: $\hat{s} = (A_1^0, A_1^1, A_1^2, A_1^3, A_1^4)$, $\tilde{s} = (A_1^0, A_1^1, A_2^2, A_1^3, A_1^4)$, $\hat{t} = (B_1^0, B_1^1, B_1^2, B_1^3, B_1^4)$, and $\tilde{t} = (B_2^0, B_1^1, B_1^2, B_1^3, B_1^4)$. Notice that $\hat{s} \prec \tilde{s}$ and $\hat{t} \prec \tilde{t}$. In this case, $u_1(\hat{s}, \hat{t}) = a_1^1 = u_1(\tilde{s}, \hat{t})$, and therefore, single crossing property implies $a_1^2 = u_1(\hat{s}, \tilde{t}) \leq u_1(\tilde{s}, \tilde{t}) = a_3^2$. Now consider the same $\hat{s}$ and $\tilde{s}$, and the following $\hat{t} = (B_1^0, B_1^1, B_2^2, B_1^3, B_1^4)$ and $\tilde{t} = (B_2^0, B_1^1, B_2^2, B_1^3, B_1^4)$. Again, notice that $\hat{s} \prec \tilde{s}$ and $\hat{t} \prec \tilde{t}$. In this case, $u_1(\hat{s}, \hat{t}) = a_1^1 = u_1(\tilde{s}, \hat{t})$, and therefore, single crossing property implies $a_2^2 = u_1(\hat{s}, \tilde{t}) \leq u_1(\tilde{s}, \tilde{t}) = a_4^2$. This shows that A_2^2 weakly dominates A_1^2.

The second statement is proved similarly. □

Statement 1 shows that $a_1^2 = a_3^2$ and $a_2^2 = a_4^2$, and therefore, in every two stage, 2×2 game, quasisupermodular and single crossing property require that player 1 must be indifferent between actions A_1^2 and A_2^2 in subgame 2, essentially eliminating any strategic role for player 1 actions in subgame 2. Statement 2 shows that player 1 must be indifferent between actions A_1^3 and A_2^3 in subgame 3, eliminating a strategic role for player 1 actions in subgame 3. A similar lemma holds for player 2. This yields the following theorem.

Theorem 4. *(1) In the set of two stage, 2×2 games with differential payoffs to outcomes, the set of games that satisfy E-payoff complementarity is empty.*
(2) In the set of all two stage, 2×2 games, the set of games that satisfy E-payoff complementarity has (Lebesgue) measure zero.

Proof. For the first statement, if a game satisfies E-payoff complementarity, then lemma 4(1) shows that $a_1^2 = a_3^2$ and $a_2^2 = a_4^2$, contradicting differential payoffs to outcomes.

The second statement follows, because the set of games with differential payoffs to outcomes has full (Lebesgue) measure and the first statement here shows that the set of games satisfying E-payoff complementarity lies in the complement of this set. □

Theorem 4 may be extended to include complementarity in subgames, as follows. A two stage, 2×2 game satisfies ***subgame E-payoff complementarity***, if it satisfies E-payoff complementarity, and in each of the four 2×2 subgames in stage 2, the payoff function of each player is quasisupermodular in own strategy and satisfies single crossing property in (own strategy; other player strategy). This coincides with the definition in Echenique (2004).

Corollary 2. *(1) In the set of two stage, 2×2 games with differential payoffs to outcomes, the set of games that satisfy subgame E-payoff complementarity is empty.*
(2) In the set of all two stage, 2×2 games, the set of games that satisfy subgame E-payoff complementarity has (Lebesgue) measure zero.

Proof. Each statement follows by noting that a game with subgame E-payoff complementarity is a game with E-payoff complementarity and then apply the corresponding statement in theorem 4. □

The robustness conclusions may be stated in terms of open and closed sets. That is, the closure of the complement of the set on which quasisupermodularity and single crossing property hold is the entire space of games, and games with strategic complements are found in open neighborhoods of games with strategic complements. Similarly, we may normalize payoffs to lie in the closed unit ball in $\mathbb{R}^{16}$ with measure normalized to 1 and use a probabilistic interpretation. Almost surely there is no game with E-payoff complementarity, and there is always a positive probability of finding games with strategic complements near each other.

As is well known, in one shot 2×2 matrix games, the role of strategic complements may be larger, because a 2×2 matrix game with strategic substitutes may be transformed into a game with strategic complements by reversing the order on the actions of one player. In other words, it is useful to know if our results remain true when we include the larger set of payoffs that include those sufficient for strategic substitutes. We show this is indeed the case. Strategic substitutes is typically formalized using dual single crossing property on payoffs (a weaker form of quasisubmodularity). We show below (lemma 5 and theorem 5) that even if we include payoffs with dual single crossing property, the set of games with these payoffs has measure zero, but the set of games in which best responses are either increasing or decreasing has infinite measure.

Player 1 payoff $u_1 : \mathcal{S} \times \mathcal{T} \to \mathbb{R}$ satisfies ***dual single crossing property in (s; t)***, if for all $t, t' \in \mathcal{T}$ such that $t \prec t'$ and for all $s, s' \in \mathcal{S}$ such that $s \prec s'$, $u_1(s', t) < (\leq) \; u_1(s, t) \implies u_1(s', t') < (\leq) \; u_1(s, t')$. Player 1 has ***payoff substitutes***, if u_1 is quasisupermodular in s and satisfies dual single crossing property in $(s; t)$. For completeness, say that player 1 has ***strategic substitutes***, if best response of player 1 is decreasing in player 2 strategies in the lattice set order. It is well known that payoff substitutes is a sufficient condition for strategic substitutes (see, for example, Roy and Sabarwal (2010)). We may define the same concepts for player 2 analogously.

Payoff substitutes restricts the strategic role of player actions in particular subgames as follows.

Lemma 5. *In a two stage, 2×2 game, if player 1 has payoff substitutes, then*
(1) A_1^1 weakly dominates A_2^1, and A_2^1 weakly dominates A_1^1, and
(2) A_1^4 weakly dominates A_2^4, and A_2^4 weakly dominates A_1^4.

Proof. For the first statement, consider the following strategies: $\hat{s} = (A_1^0, A_2^1, A_1^2, A_1^3, A_1^4)$, $\tilde{s} = (A_2^0, A_1^1, A_1^2, A_1^3, A_1^4)$, $\hat{t} = (B_1^0, B_2^1, B_1^2, B_1^3, B_1^4)$, and $\tilde{t} = (B_1^0, B_1^1, B_1^2, B_1^3, B_1^4)$. In this case, $u_1(\hat{s} \vee \tilde{s}, \hat{t}) = a_1^3 = u_1(\tilde{s}, \hat{t})$, and therefore, quasisupermodularity implies $a_4^1 = u_1(\hat{s}, \hat{t}) \leq u_1(\hat{s} \wedge \tilde{s}, \hat{t}) = a_2^1$. Moreover, $u_1(\hat{s} \vee \tilde{s}, \tilde{t}) = a_1^3 = u_1(\tilde{s}, \tilde{t})$, and therefore, quasisupermodularity implies $a_3^1 = u_1(\hat{s}, \tilde{t}) \leq u_1(\hat{s} \wedge \tilde{s}, \tilde{t}) = a_1^1$. This shows that A_1^1 weakly dominates A_2^1.

For the other part, consider the following strategies: $\hat{s} = (A_1^0, A_1^1, A_1^2, A_1^3, A_1^4)$, $\tilde{s} = (A_1^0, A_2^1, A_1^2, A_1^3, A_1^4)$, $\hat{t} = (B_1^0, B_1^1, B_1^2, B_1^3, B_1^4)$, and $\tilde{t} = (B_2^0, B_1^1, B_1^2, B_1^3, B_1^4)$. Notice that $\hat{s} \prec \tilde{s}$ and $\hat{t} \prec \tilde{t}$. In this case, $u_1(\hat{s}, \tilde{t}) = a_1^2 = u_1(\tilde{s}, \tilde{t})$, and therefore, dual single crossing property implies $a_1^1 = u_1(\hat{s}, \hat{t}) \leq u_1(\tilde{s}, \hat{t}) = a_3^1$. Now consider the same $\hat{s}$ and $\tilde{s}$, and the following $\hat{t} = (B_1^0, B_2^1, B_1^2, B_1^3, B_1^4)$ and $\tilde{t} = (B_2^0, B_2^1, B_1^2, B_1^3, B_1^4)$. Again, notice that $\hat{s} \prec \tilde{s}$ and $\hat{t} \prec \tilde{t}$. In this case, $u_1(\hat{s}, \tilde{t}) = a_1^2 = u_1(\tilde{s}, \tilde{t})$, and therefore, dual single crossing property implies $a_2^1 = u_1(\hat{s}, \hat{t}) \leq u_1(\tilde{s}, \hat{t}) = a_4^1$. This shows that A_2^1 weakly dominates A_1^1.

The second statement is proved similarly. □

Statement 1 shows that $a_1^1 = a_3^1$ and $a_2^1 = a_4^1$, and therefore, in every two stage, 2×2 game, quasisupermodular and dual single crossing property require that player 1 must be indifferent between actions A_1^1 and A_2^1 in subgame 1, essentially eliminating any strategic role for player 1 actions in subgame 1. Statement 2 shows that player 1 must be indifferent between actions A_1^4 and A_2^4 in subgame 4, eliminating a strategic role for player 1 actions in subgame 4.

A two stage, 2×2 game is a ***payoff monotone game***, if for each $i \in \{1, 2\}$, player i payoff satisfies either E-payoff complementarity or payoff substitutes. It is a ***monotone game***, if for each $i \in \{1, 2\}$, either player i has strategic complements or player i has strategic substitutes. We have the following theorem.

Theorem 5. *In the set of all two stage, 2×2 games,*

1. *The set of monotone games has infinite (Lebesgue) measure, and*
2. *The set of payoff monotone games has (Lebesgue) measure zero.*

Proof. For statement (1), theorem 2 shows that the condition player i has strategic complements is satisfied on a set of infinite (Lebesgue) measure, and Sabarwal and Vu (2019) show that the condition player i has strategic substitutes is satisfied on a set of infinite (Lebesgue) measure. Therefore, their union is a set of infinite (Lebesgue) measure.

For statement (2), lemma 4 shows that the set of payoffs satisfying E-payoff complementarity has (Lebesgue) measure zero and lemma 5 shows that the set of payoffs satisfying payoff substitutes has (Lebesgue) measure zero. Therefore, their union has (Lebesgue) measure zero. □

5 Three Player, Two Action, Two Stage Game

The restrictiveness of sufficient conditions for complementarity given by quasisupermodularity and single crossing property may be exacerbated with more than two players.

In this section, we consider the case of three player, two action games played in two stages. These are denoted $3 \times 2 \times 2$ games. Timing is standard. In stage one, 3 players (each with two actions) play a simultaneous game. After observing stage one actions (there are a total of eight possible outcomes in stage one), the three players (each with two actions) play another simultaneous game in stage two. Payoffs are realized at terminal nodes after the second stage.

The game tree for this game is given in figure 6. We use the following notation and terminology. For notational convenience, in each stage, the actions of each player are labeled L and H, with $L \prec H$. For player $i \in \{1, 2, 3\}$, let I_i^0 denote the initial information set for player i in stage 1, represented by dotted lines when needed. For each of the 8 possible outcomes in stage 1, there is a subgame in stage 2. These are termed sugbames 1 through 8. For player $i \in \{1, 2, 3\}$ and subgame $n \in \{1, \ldots, 8\}$, I_i^n denotes the information set of player i in subgame n in stage 2, represented by dotted lines when needed. When there is no confusion, some labels are suppressed due to space constraints.

As usual, a strategy for player i, denoted s_i, is a function from information sets of player i to actions available at the particular information set. The set of strategies of player i is denoted $\mathcal{S}_i$. When convenient, we use the notation s_{-i} to denote the profile of strategies for opponents of player i and $\mathcal{S}_{-i}$ for the set of all such profiles.

Player i payoff $u_i : \mathcal{S}_i \times \mathcal{S}_{-i} \to \mathbb{R}$ is ***quasisupermodular (in*** s_i***)***, if for every $s_{-i} \in \mathcal{S}_{-i}$ and for every $s_i, s_i' \in \mathcal{S}_i$, $u_i(s_i \wedge s_i', s_{-i}) < (\leq)\ u_i(s_i, s_{-i}) \implies u_i(s_i', s_{-i}) < (\leq)\ u_i(s_i \vee s_i', s_{-i})$. Player i payoff $u_i : \mathcal{S}_i \times \mathcal{S}_{-i} \to \mathbb{R}$ satisfies ***single crossing property in*** $(s_i; s_{-i})$, if for all $s_{-i}, s_{-i}' \in \mathcal{S}_{-i}$ such that $s_{-i} \prec s_{-i}'$ and for all $s_i, s_i' \in \mathcal{S}_i$ such that $s_i \prec s_i'$, $u_i(s_i, s_{-i}) < (\leq)\ u_i(s_i', s_{-i}) \implies u_i(s_i, s_{-i}') < (\leq)\ u_i(s_i', s_{-i}')$. Player i satisfies ***E-payoff complementarity***, if u_i is quasisupermodular (in s_i) and satisfies single crossing property in $(s_i; s_{-i})$. A $3 \times 2 \times 2$ game satisfies ***E-payoff complementarity***, if every player i satisfies E-payoff complementarity.

An important result here is that the restriction imposed by E-payoff complementarity

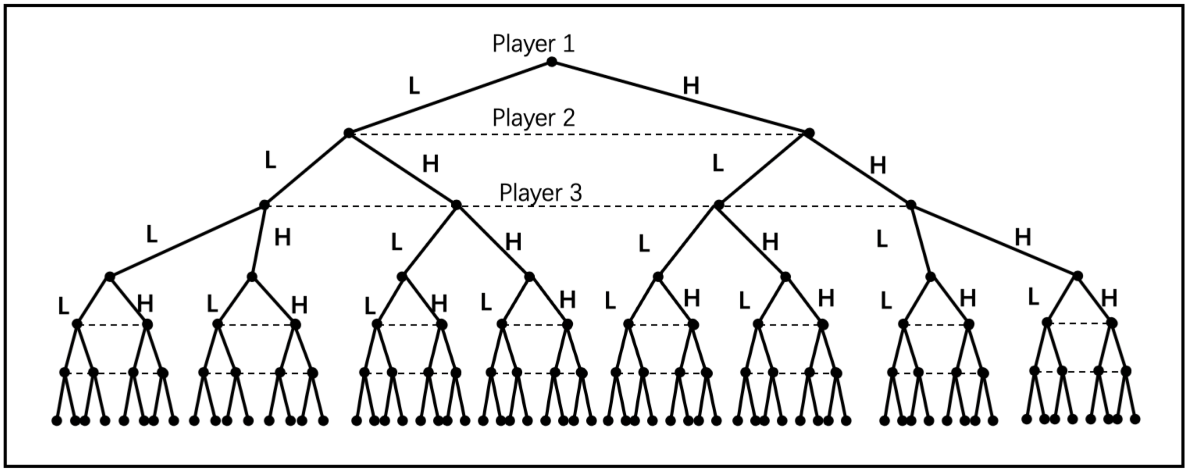


Figure 6: Game Tree For $3 \times 2 \times 2$ Game

is even more severe than earlier. It rules out any strategic role for a player's actions in six of the eight subgames in stage 2.

Theorem 6. *(1) In each $3 \times 2 \times 2$ game, if player 1 satisfies E-payoff complementarity, then in every subgame 2 through 7, player 1 is indifferent between L and H.*
(2) The set of all $3 \times 2 \times 2$ games that satisfy E-payoff complementarity lies in $\mathbb{R}^{40} \times \mathbb{R}^{40} \times \mathbb{R}^{40}$, or equivalently, in $\mathbb{R}^{120}$.

Proof. For statement (1), suppose player 1 satisfies E-payoff complementarity. To prove the result for subgame 2, let $\hat{s}_1$ be player 1 strategy given by L at I_1^0, H in subgame 2, and an arbitrarily chosen actions in subgame 1 and in every subgame 3 to 8. Let $\tilde{s}_1$ be given by H at I_1^0, L in subgame 2, and the same action as $\hat{s}_1$ in subgame 1 and in every subgame 3 to 8. For players 2 and 3, let s_{-1} be given by (L, H) at I_{-1}^0 and arbitrarily chosen actions at each information set in each subgame 1 to 8. Subgame 6 is reached on path of play for $(\hat{s}_1 \vee \tilde{s}_1, s_{-1})$ and for $(\tilde{s}_1, s_{-1})$, and as both $\hat{s}_1$ and $\tilde{s}_1$ play the same action in subgame 6, it follows that $u_1(\hat{s}_1 \vee \tilde{s}_1, s_{-1}) = u_1(\tilde{s}_1, s_{-1})$, and therefore, quasisupermodularity implies $u_1(\hat{s}_1, s_{-1}) \leq u_1(\hat{s}_1 \wedge \tilde{s}_1, s_{-1})$. As subgame 2 is reached on the path of play for $(\hat{s}_1, s_{-1})$ and for $(\hat{s}_1 \wedge \tilde{s}_1, s_{-1})$, the inequality $u_1(\hat{s}_1, s_{-1}) \leq u_1(\hat{s}_1 \wedge \tilde{s}_1, s_{-1})$ combined with arbitrary choices in s_{-1} for subgame 2 imply that for player 1, L weakly dominates H in subgame 2.

Let $\tilde{s}_1'$ be given by L at I_1^0 and the same action as $\tilde{s}_1$ at every information set for player 1 in subgames 1 to 8. Let s_{-1}' be given by (L, L) at I_{-1}^0 and the same actions as s_{-1} at all information sets for players 2 and 3 in subgames 1 to 8. It is easy to see that $\tilde{s}_1' \prec \hat{s}_1$ and $s_{-1}' \prec s_{-1}$. Subgame 1 is reached on path of play for $(\hat{s}_1, s_{-1}')$ and for $(\tilde{s}_1', s_{-1}')$ and as both $\hat{s}_1$ and $\tilde{s}_1'$ play the same action in subgame 1, it follows that $u_1(\hat{s}_1, s_{-1}') = u_1(\tilde{s}_1', s_{-1}')$, and therefore, single crossing property implies $u_1(\hat{s}_1, s_{-1}) \geq u_1(\tilde{s}_1', s_{-1})$. As subgame 2 is reached on the path of play for $(\hat{s}_1, s_{-1})$ and for $(\tilde{s}_1', s_{-1})$, the inequality $u_1(\hat{s}_1, s_{-1}) \geq u_1(\tilde{s}_1', s_{-1})$ combined with arbitrary choices in s_{-1} for subgame 2 imply that for player 1, H weakly dominates L in subgame 2.

This shows that player 1 is indifferent between L and H in subgame 2. A similar construction shows that player 1 is indifferent between L and H in subgames 3 and 4 as well.

To prove the result for subgame 5, let $\hat{s}_1$ be player 1 strategy given by H at I_1^0, L in subgame 5, and an arbitrarily chosen action in every subgame 1 to 4 and in every subgame 6 to 8. Let $\tilde{s}_1$ be given by L at I_1^0, H in subgame 5, and the same action as $\hat{s}_1$ in every subgame 1 to 4 and in every subgame 6 to 8. For players 2 and 3, let s_{-1} be given by (L, L) at I_{-1}^0, and arbitrarily chosen actions at each information set in each subgame 1 to 8. Subgame 1 is reached on path of play for $(\tilde{s}_1, s_{-1})$ and for $(\hat{s}_1 \wedge \tilde{s}_1, s_{-1})$, and as both $\hat{s}_1$ and $\tilde{s}_1$ play the same action in subgame 1, it follows that $u_1(\hat{s}_1 \wedge \tilde{s}_1, s_{-1}) = u_1(\tilde{s}_1, s_{-1})$, and therefore, quasisupermodularity implies $u_1(\hat{s}_1, s_{-1}) \leq u_1(\hat{s}_1 \vee \tilde{s}_1, s_{-1})$. As subgame 5 is reached on the path of play for $(\hat{s}_1, s_{-1})$ and for $(\hat{s}_1 \vee \tilde{s}_1, s_{-1})$, the inequality $u_1(\hat{s}_1, s_{-1}) \leq u_1(\hat{s}_1 \vee \tilde{s}_1, s_{-1})$ combined with arbitrary choices in s_{-1} for subgame 5 imply that for player 1, H weakly dominates L in subgame 5.

Let $\tilde{s}_1'$ be given by H at I_1^0 and the same action as $\tilde{s}_1$ at every subgame 1 to 8. Let s_{-1}' be given by (H, H) at I_{-1}^0 and the same actions as s_{-1} at all information sets for players

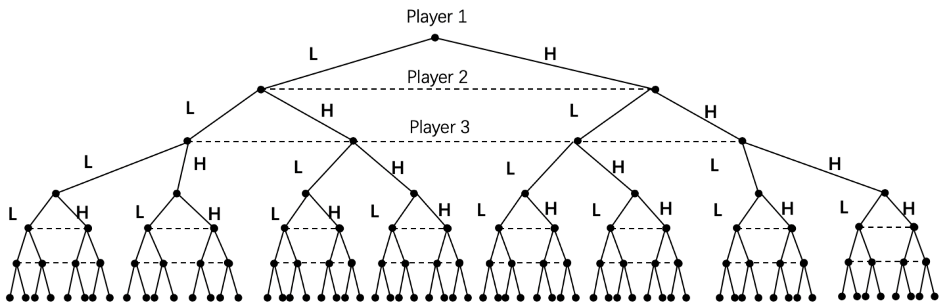
2 and 3 in subgames 1 to 8. It is easy to see that $\hat{s}_1 \prec \tilde{s}'_1$ and $s_{-1} \prec s'_{-1}$. Subgame 8 is reached on path of play for $(\hat{s}_1, s'_{-1})$ and for $(\tilde{s}'_1, s'_{-1})$ and as both $\hat{s}_1$ and $\tilde{s}'_1$ play the same action in subgame 8, it follows that $u_1(\hat{s}_1, s'_{-1}) = u_1(\tilde{s}'_1, s'_{-1})$, and therefore, single crossing property implies $u_1(\hat{s}_1, s_{-1}) \geq u_1(\tilde{s}'_1, s_{-1})$. As subgame 5 is reached on the path of play for $(\hat{s}_1, s_{-1})$ and for $(\tilde{s}'_1, s_{-1})$, the inequality $u_1(\hat{s}_1, s_{-1}) \geq u_1(\tilde{s}'_1, s_{-1})$ combined with arbitrary choices in s_{-1} for subgame 5 imply that for player 1, L weakly dominates H in subgame 5.

This shows that player 1 is indifferent between L and H in subgame 5. A similar construction shows that player 1 is indifferent between H and L in subgames 6 and 7 as well. This proves statement (1).

Consider statement (2). Statement (1) implies that player 1 payoff cannot have more than 40 distinct components (4 for each of six subgames in which player 1 is indifferent between L and H and 8 in each of the two remaining subgames). Changing the index for a player and following the same logic as in statement (1) shows that each of player 2 and player 3 must also be indifferent between L and H in six of the eight subgames, and therefore, player 2 and player 3 payoff cannot have more than 40 distinct components each. This shows that the set of games that satisfy E-payoff complementarity lies in $\mathbb{R}^{40} \times \mathbb{R}^{40} \times \mathbb{R}^{40}$, or equivalently, in $\mathbb{R}^{120}$. □

E-payoff complementarity is not necessary to have increasing best responses in these games, as shown in the following example. As earlier, player i has ***strategic complements***, if best response of player i is increasing in opponent strategies in the lattice set order. A game is a ***game with strategic complements (GSC)***, if for every $i \in \{1, 2, 3\}$, player i has strategic complements.

Figure 7 presents a $3 \times 2 \times 2$ GSC that does not satisfy E-payoff complementarity for any player. Payoffs presented at the terminal nodes are for player 1 and player 2. Player 3 payoffs are symmetric to those of player 2. Notice that player 1 is not indifferent between L and H in subgames 4, 5, 6, and 7, and therefore, player 1 payoff does not



Player 1

	Subgame 1 L	Subgame 1 H	Subgame 2 L	Subgame 2 H	Subgame 3 L	Subgame 3 H	Subgame 4 L	Subgame 4 H
LL	91	17	72	72	48	48	27	54
LH	82	14	83	83	62	62	38	81
HL	79	16	65	65	66	66	40	86
HH	63	11	102	102	74	74	43	77

	Subgame 5 L	Subgame 5 H	Subgame 6 L	Subgame 6 H	Subgame 7 L	Subgame 7 H	Subgame 8 L	Subgame 8 H
LL	41	4	12	44	1	36	22	15
LH	37	9	14	21	5	24	25	50
HL	11	23	28	7	8	45	53	46
HH	13	26	39	10	24	33	29	49

Player 2

	Subgame 1 L	Subgame 1 H	Subgame 2 L	Subgame 2 H	Subgame 3 L	Subgame 3 H	Subgame 4 L	Subgame 4 H
LL	71	37	25	18	32	14	51	51
LH	62	34	12	32	19	27	55	55
HL	59	36	16	36	21	33	41	41
HH	43	31	14	24	23	26	45	45

	Subgame 5 L	Subgame 5 H	Subgame 6 L	Subgame 6 H	Subgame 7 L	Subgame 7 H	Subgame 8 L	Subgame 8 H
LL	26	22	21	12	47	47	27	34
LH	27	31	11	20	53	53	18	38
HL	32	34	13	16	45	45	20	36
HH	15	28	15	19	38	38	23	32

Figure 7: A 3x2x2 GSC

satisfy E-payoff complementarity. Nevertheless, player 1 has strategic complements, as shown by the lemma and theorem below. A similar statement is true for players 2 and 3.

The game in figure 7 may be motivated as follows. Consider an industry with three firms, firm 1 is large and firms 2 and 3 are small (and symmetric). In each stage, each firm may compete less aggressively (play L) or more aggressively (play H). Less aggressive competition may occur, perhaps, by setting high prices and more aggressive competition may occur, perhaps, by setting low prices. Each firm chooses how to compete in each of two stages. Outcomes are observed after each stage.

Payoffs for firm 1 in figure 7 may be motivated as follows. Suppose profit of firm 1 is impacted less with one aggressive competitor and more with two aggressive competitors and firm 1 is large enough to withstand two aggressive competitors in one stage but not in both stages. Firm 1 has an incentive to compete less aggressively in the first stage. If both competitors compete less aggressively as well, it is in firm 1's best interest to compete less aggressively in the second stage. If one competitor competes less aggressively and the other competes more aggressively, then firm 1 is indifferent between competing less or more agressively in the second stage. If both competitors compete more aggressively in the first stage, it is in firm 1's best interest to compete aggressively in the second stage. It is not in firm 1's best interest to compete more aggressively in the first stage. It does so in the second stage, when forced by widespread competition from its competitors.

Payoffs for firm 2 in figure 7 may be motivated as follows. Firm 2 is small and its profit is impacted with one aggressive competitor in either stage. Firm 2 has an incentive to compete less aggressively in the first stage, if both competitors compete less aggressively in stage one, and in this case, it is in firm 1's best interest to compete less aggressively in the second stage as well. If one competitor competes less aggressively in the first stage and the other competes more aggressively, then firm 2 competes aggressively in stage one and is indifferent between competing less or more agressively in the second stage. If both competitors compete more aggressively in the first stage, it is in firm 2's best interest to compete aggressively in both stages. Firm 3 is symmetric to firm 2 and its payoff is motivated similarly.

Another motivation may be given in terms of coordination games. Consider three friends, and each friend may decide to go to a movie (L) or go to the opera (H). Each friend makes a decision about which event to go to for each of two weekends. Friend 1 prefers very much to go to a movie (at least once) but would also like the company of her friends. Friends 2 and 3 like movies but also like opera and would similarly like the company of others.

Friend 1 has an incentive to go to the movie the first weekend. If the other friends go to the movie as well, it is in friend 1's best interest to go to the movie the second weekend. If one other friend (2 or 3) goes to the movie the first weekend (with friend 1) and the third friend goes to the opera the first weekend, then friend 1 is indifferent between going to the movie or opera the second weekend. If both other friends go to the opera the first weekend, friend 1 is not so well off going alone to the movie the first weekend and chooses to go to the opera second weekend. It is not in friend 1's best interest to go to opera the first weekend. She does so the second weekend, if her friends decide to go to opera without her the first weekend.

Similar motivations may be made in terms of other coordination games. For example, one large depositor may not make a run on a bank (L) on a given day. If it observes both small depositors make a run on the bank (play (H, H)) on a given day, it is in its interest to make a run next day. The same thing with runs on groceries or gasoline in a pandemic. A similar motivation may be made with technology adoption. An entrenched firm (or department in a firm, or a person) may like to continue using an inferior technology (L) unless forced by a coordinated move by other firms (or departments within a firm, or other persons) to use a superior technology (H). These types of situations can exhibit strategic complements but cannot be modeled with E-payoff complementarity.

The following results show that each player in this game has strategic complements.

Lemma 6. *In the game given in figure 7, player 1 best response has the following structure.*

1. *For every* $s_{-1} \in \{s_{-1} \in S_{-1} \mid s_{-1}(I_{-1}^0) = (L, L)\}$,

$$BR_1(s_{-1}) = \{s_1 \in S_1 \mid s_1(I_1^0) = L, s_1(I_1^1) = L\}.$$

2. *For every* $s_{-1} \in \{s_{-1} \in S_{-1} \mid s_{-1}(I_{-1}^0) = (L, H)\}$,

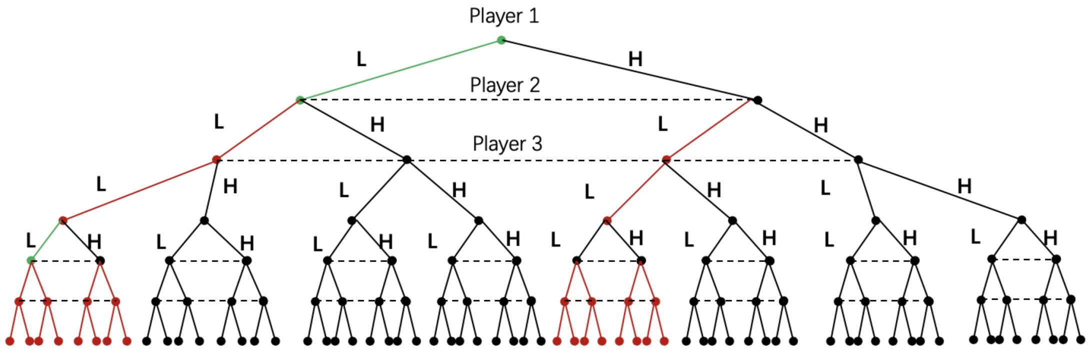
$$BR_1(s_{-1}) = \{s_1 \in S_1 \mid s_1(I_1^0) = L\}.$$

3. *For every* $s_{-1} \in \{s_{-1} \in S_{-1} \mid s_{-1}(I_{-1}^0) = (H, L)\}$,

$$BR_1(s_{-1}) = \{s_1 \in S_1 \mid s_1(I_1^0) = L\}.$$

4. *For every* $s_{-1} \in \{s_{-1} \in S_{-1} \mid s_{-1}(I_{-1}^0) = (H, H)\}$,

$$BR_1(s_{-1}) = \{s_1 \in S_1 \mid s_1(I_1^0) = L, s_1(I_1^4) = H\}.$$



	L	H
LL	91	17
LH	82	14
HL	79	16
HH	63	11

	L	H
LL	41	4
LH	39	18
HL	23	11
HH	18	26

Figure 8: Player 1 Incentives, Case 1

Proof. See Appendix. □

Statement (1) is represented in figure 8, statement (2) in figure 9, statement (3) in figure 10, and statement (4) in figure 11. This lemma is useful to prove that player 1 has strategic complements.

Theorem 7. *In the game given in figure 7, player 1 has strategic complements.*

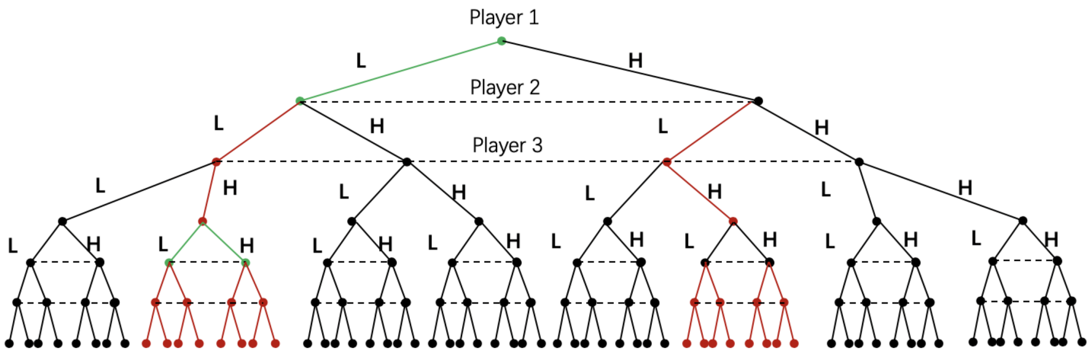
Proof. See Appendix. □

A similar argument holds for players 2 and 3.

Lemma 7. *In the game given in figure 7, player 2 best response has the following structure.*

1. *For every* $s_{-2} \in \{s_{-2} \in S_{-2} \mid s_{-2}(I^0_{-2}) = (L, L)\}$,

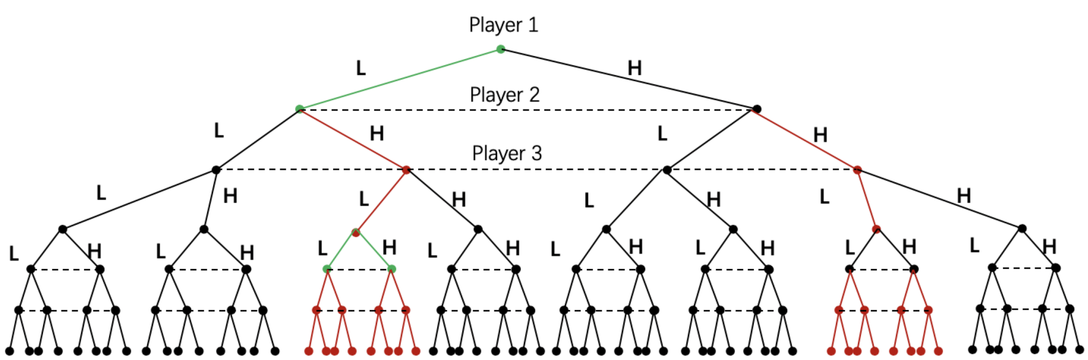
$$BR_2(s_{-2}) = \{s_2 \in S_2 | s_2(I^0_2) = L, s_2(I^1_2) = L\}.$$



	L	H
LL	72	72
LH	83	83
HL	65	65
HH	102	102

	L	H
LL	44	12
LH	14	21
HL	28	17
HH	10	39

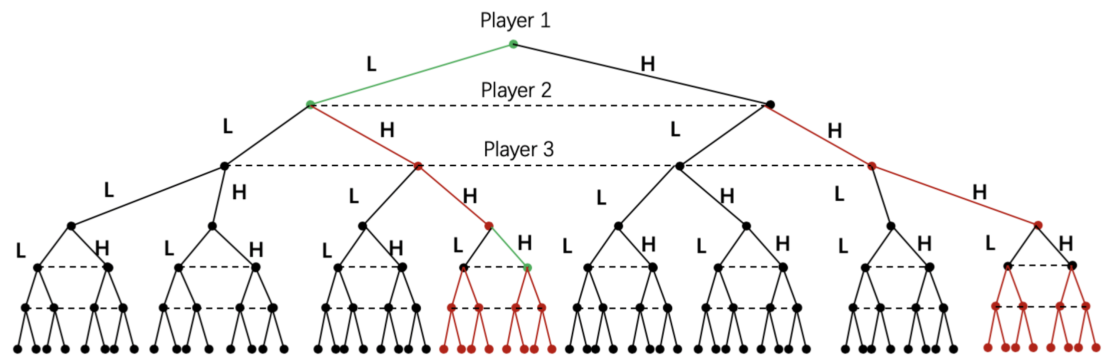
Figure 9: Player 1 Incentives, Case 2



	L	H
LL	48	48
LH	62	62
HL	66	66
HH	74	74

	L	H
LL	36	1
LH	24	5
HL	45	8
HH	24	33

Figure 10: Player 1 Incentives, Case 3



	L	H
LL	27	54
LH	38	81
HL	40	86
HH	43	77

	L	H
LL	22	15
LH	50	25
HL	53	46
HH	29	49

Figure 11: Player 1 Incentives, Case 4

2. *For every* $s_{-2} \in \{s_{-2} \in S_{-2} \mid s_{-2}(I^0_{-2}) = (H, L)\}$*,*

$$BR_2(s_{-2}) = \{s_2 \in S_2 | s_2(I^0_2) = H\}.$$

3. *For every* $s_{-2} \in \{s_{-2} \in S_{-2} \mid s_{-2}(I^0_{-2}) = (L, H)\}$*,*

$$BR_2(s_{-2}) = \{s_2 \in S_2 | s_2(I^0_2) = H\}.$$

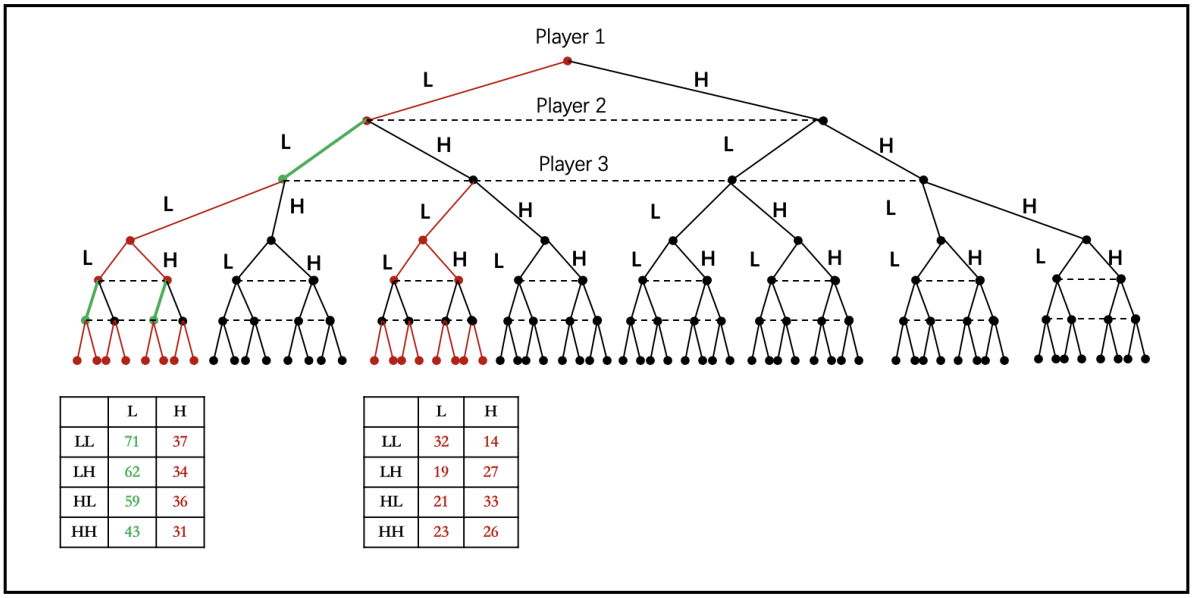
4. *For every* $s_{-2} \in \{s_{-2} \in S_{-2} \mid s_{-2}(I^0_{-2}) = (H, H)\}$*,*

$$BR_2(s_{-2}) = \{s_2 \in S_2 | s_2(I^0_2) = H, s_2(I^8_2) = H\}.$$

Proof. Similar to that of lemma 6. □

For reference, statement (1) is presented in figure 12, statement (2) in figure 13, statement (3) in figure 14, and statement (4) in figure 15. This lemma helps to prove that player 2 has strategic complements.

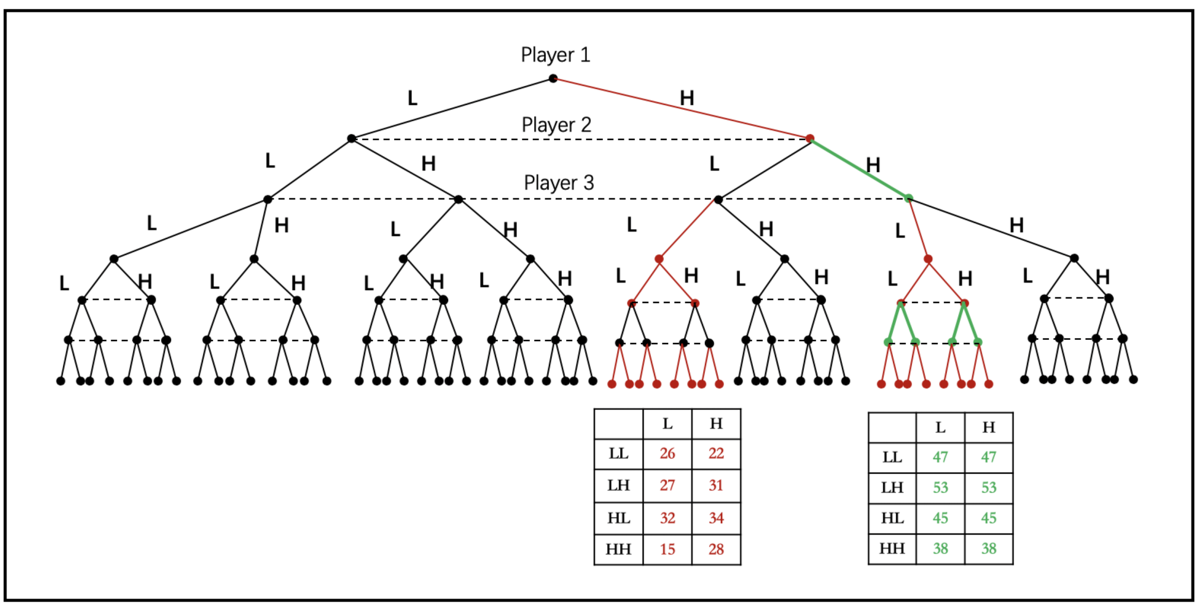
Theorem 8. *In the game given in figure 7, player 2 has strategic complements.*



	L	H
LL	71	37
LH	62	34
HL	59	36
HH	43	31

	L	H
LL	32	14
LH	19	27
HL	21	33
HH	23	26

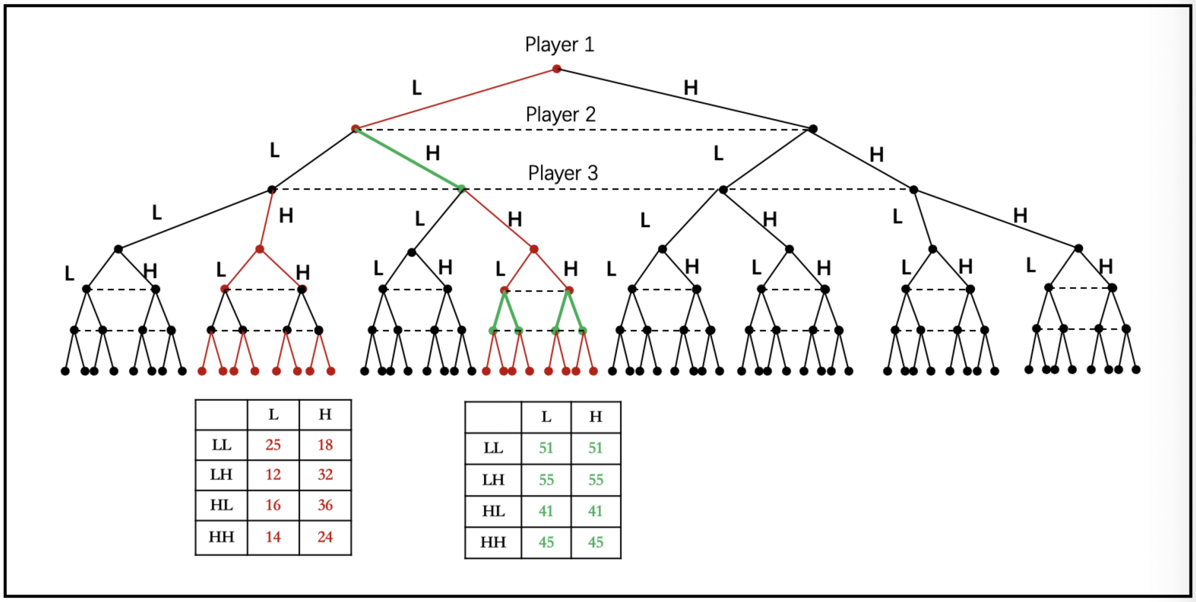
Figure 12: Player 2 Incentives, Case 1



	L	H
LL	26	22
LH	27	31
HL	32	34
HH	15	28

	L	H
LL	47	47
LH	53	53
HL	45	45
HH	38	38

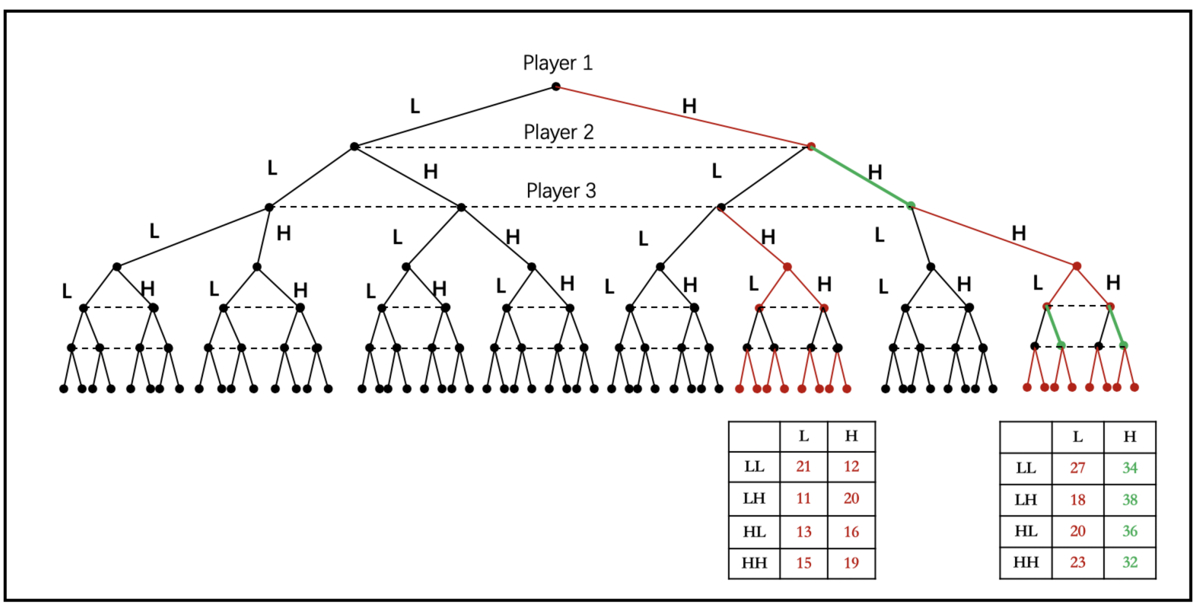
Figure 13: Player 2 Incentives, Case 2



	L	H
LL	25	18
LH	12	32
HL	16	36
HH	14	24

	L	H
LL	51	51
LH	55	55
HL	41	41
HH	45	45

Figure 14: Player 2 Incentives, Case 3



	L	H
LL	21	12
LH	11	20
HL	13	16
HH	15	19

	L	H
LL	27	34
LH	18	38
HL	20	36
HH	23	32

Figure 15: Player 2 Incentives, Case 4

Proof. Similar to that of theorem 7. □

Player 3 is symmetric to player 2 and a similar argument shows that player 3 has strategic complements. Therefore, this is a game with strategic complements.

More generally, the conditions in six of the eight subgames are open conditions which may be extended to yield sets of infinite (Lebesgue) measure in the corresponding space.

Theorem 9. *The set of* $3 \times 2 \times 2$ *games with strategic complements has infinite (Lebesgue) measure in* $\mathbb{R}^{56} \times \mathbb{R}^{56} \times \mathbb{R}^{56}$*, or equivalently, in* $\mathbb{R}^{168}$.

Proof. For player 1, denote their eight payoffs after each second stage subgame $n \in \{1, \ldots, 8\}$ by

$$\begin{aligned}(\pi_1^n(L,(L,L)), \pi_1^n(L,(L,H)), \pi_1^n(L,(H,L)), \pi_1^n(L,(H,H))) &= (a_1^n, a_2^n, a_3^n, a_4^n), \text{ and} \\ (\pi_1^n(H,(L,L)), \pi_1^n(H,(L,H)), \pi_1^n(H,(H,L)), \pi_1^n(H,(H,H))) &= (a_5^n, a_6^n, a_7^n, a_8^n),\end{aligned}$$

and consider payoffs in the following set of infinite (Lebesgue) measure in $\mathbb{R}^{56}$.

$$(a^1_1, a^1_2, a^1_3, a^1_4) \in (90,\infty) \times (80,\infty) \times (70,\infty) \times (60,\infty) \subseteq \mathbb{R}^4$$
$$(a^1_5, a^1_6, a^1_7, a^1_8) \in (40,45) \times (30,35) \times (20,25) \times (10,15) \subseteq \mathbb{R}^4$$
$$(a^2_1, a^2_2, a^2_3, a^2_4) \in (90,\infty) \times (80,\infty) \times (70,\infty) \times (60,\infty) \subseteq \mathbb{R}^4$$
$$(a^2_5, a^2_6, a^2_7, a^2_8) = (a^2_1, a^2_2, a^2_3, a^2_4)$$
$$(a^3_1, a^3_2, a^3_3, a^3_4) \in (90,\infty) \times (80,\infty) \times (70,\infty) \times (60,\infty) \subseteq \mathbb{R}^4$$
$$(a^3_5, a^3_6, a^3_7, a^3_8) = (a^3_1, a^3_2, a^3_3, a^3_4)$$
$$(a^4_1, a^4_2, a^4_3, a^4_4) \in (40,45) \times (30,35) \times (20,25) \times (10,15) \subseteq \mathbb{R}^4$$
$$(a^4_5, a^4_6, a^4_7, a^4_8) \in (90,\infty) \times (80,\infty) \times (70,\infty) \times (60,\infty) \subseteq \mathbb{R}^4$$
$$(a^5_1, a^5_2, a^5_3, a^5_4) \in (40,45) \times (30,35) \times (20,25) \times (10,15) \subseteq \mathbb{R}^4$$
$$(a^5_5, a^5_6, a^5_7, a^5_8) \in (45,50) \times (35,40) \times (25,30) \times (15,20) \subseteq \mathbb{R}^4$$
$$(a^6_1, a^6_2, a^6_3, a^6_4) \in (45,50) \times (35,40) \times (25,30) \times (10,15) \subseteq \mathbb{R}^4$$
$$(a^6_5, a^6_6, a^6_7, a^6_8) \in (40,45) \times (30,35) \times (20,25) \times (15,20) \subseteq \mathbb{R}^4$$
$$(a^7_1, a^7_2, a^7_3, a^7_4) \in (45,50) \times (35,40) \times (25,30) \times (10,15) \subseteq \mathbb{R}^4$$
$$(a^7_5, a^7_6, a^7_7, a^7_8) \in (40,45) \times (30,35) \times (20,25) \times (15,20) \subseteq \mathbb{R}^4$$
$$(a^8_1, a^8_2, a^8_3, a^8_4) \in (45,50) \times (35,40) \times (25,30) \times (10,15) \subseteq \mathbb{R}^4$$
$$(a^8_5, a^8_6, a^8_7, a^8_8) \in (40,45) \times (30,35) \times (20,25) \times (15,20) \subseteq \mathbb{R}^4.$$

Following the logic in the proof of lemma 6 and theorem 7, it can be shown that for each collection of player 1 payoffs in the above set of infinite (Lebesgue) measure in $\mathbb{R}^{56}$, player 1 has strategic complements.

For player 2, denote their eight payoffs after each second stage subgame $n \in \{1, \dots, 8\}$ by

$$\begin{aligned}
(\pi^n_2(L,(L,L)), \pi^n_2(L,(L,H)), \pi^n_2(H,(L,L)), \pi^n_2(H,(L,H)) &= (b^n_1, b^n_2, b^n_3, b^n_4) \text{ and} \\
(\pi^n_2(L,(H,L)), \pi^n_2(L,(L,H)), \pi^n_2(H,(H,L)), \pi^n_2(H,(H,H))) &= (b^n_5, b^n_6, b^n_7, b^n_8),
\end{aligned}$$

and consider payoffs in the following set of infinite (Lebesgue) measure in $\mathbb{R}^{56}$.

$$(b^1_1, b^1_2, b^1_3, b^1_4) \in (90, \infty) \times (80, \infty) \times (40, 45) \times (30, 35) \subseteq \mathbb{R}^4$$
$$(b^1_5, b^1_6, b^1_7, b^1_8) \in (70, \infty) \times (60, \infty) \times (20, 25) \times (10, 15) \subseteq \mathbb{R}^4$$
$$(b^2_1, b^2_2, b^2_3, b^2_4) \in (45, 50) \times (30, 35) \times (40, 45) \times (35, 40) \subseteq \mathbb{R}^4$$
$$(b^2_5, b^2_6, b^2_7, b^2_8) \in (20, 25) \times (10, 15) \times (25, 30) \times (15, 20) \subseteq \mathbb{R}^4$$
$$(b^3_1, b^3_2, b^3_3, b^3_4) \in (45, 50) \times (30, 35) \times (40, 45) \times (35, 40) \subseteq \mathbb{R}^4$$
$$(b^3_5, b^3_6, b^3_7, b^3_8) \in (20, 25) \times (10, 15) \times (25, 30) \times (15, 20) \subseteq \mathbb{R}^4$$
$$(b^4_1, b^4_2, b^4_5, b^4_6) \in (90, \infty) \times (80, \infty) \times (70, \infty) \times (60, \infty) \subseteq \mathbb{R}^4$$
$$(b^4_3, b^4_4, b^4_7, b^4_8) = (b^4_1, b^4_2, b^4_5, b^4_6)$$
$$(b^5_1, b^5_2, b^5_3, b^5_4) \in (45, 50) \times (30, 35) \times (40, 45) \times (35, 40) \subseteq \mathbb{R}^4$$
$$(b^5_5, b^5_6, b^5_7, b^5_8) \in (20, 25) \times (10, 15) \times (25, 30) \times (15, 20) \subseteq \mathbb{R}^4$$
$$(b^6_1, b^6_2, b^6_3, b^6_4) \in (45, 50) \times (30, 35) \times (40, 45) \times (35, 40) \subseteq \mathbb{R}^4$$
$$(b^6_5, b^6_6, b^6_7, b^6_8) \in (20, 25) \times (10, 15) \times (25, 30) \times (15, 20) \subseteq \mathbb{R}^4$$
$$(b^7_1, b^7_2, b^7_5, b^7_6) \in (90, \infty) \times (80, \infty) \times (70, \infty) \times (60, \infty) \subseteq \mathbb{R}^4$$
$$(b^7_3, b^7_4, b^7_7, b^7_8) = (b^7_1, b^7_2, b^7_5, b^7_6)$$
$$(b^8_1, b^8_2, b^8_3, b^8_4) \in (40, 45) \times (30, 35) \times (90, \infty) \times (80, \infty) \subseteq \mathbb{R}^4$$
$$(b^8_5, b^8_6, b^8_7, b^8_8) \in (20, 25) \times (10, 15) \times (70, \infty) \times (60, \infty) \subseteq \mathbb{R}^4.$$

Following the logic in the proof of lemma 6 and theorem 7, it can be shown that for each collection of player 2 payoffs in the above set of infinite (Lebesgue) measure in $\mathbb{R}^{56}$, player 2 has strategic complements. A similar argument holds for player 3. The conclusion follows because their product is a set of infinite (Lebesgue) measure in $\mathbb{R}^{56} \times \mathbb{R}^{56} \times \mathbb{R}^{56}$, or equivalently, in $\mathbb{R}^{168}$. □

This is the highest dimensional space in which we are able to formulate such a class of examples.

6 Conclusion

We study necessary and sufficient conditions for strategic complements in two stage, 2×2 games.

We find that the restrictiveness imposed by standard sufficient conditions of quasisupermodularity and single crossing property is particularly severe, in the sense that the set of games in which payoffs satisfy these conditions has measure zero. In particular, payoffs with these conditions require the player to be indifferent between their actions in two of the four subgames in stage two, eliminating any strategic role for their actions in these two subgames.

In contrast, the set of games that exhibit strategic complements (increasing best responses) has infinite measure. Players are not required to be indifferent between their actions in stage 2 subgames. This enlarges the scope of strategic complements in two stage, 2×2 games (and provides a basis for possibly greater scope in more general games). The conditions identified here are easy to verify directly from payoffs, and examples with several economic motivations show the additional types of interactions that are made possible with the more general results. The set of subgame perfect Nash equilibria in the larger class of games continues to remain a nonempty, complete lattice.

The results are easy to apply, and are robust to including dual payoff conditions and adding a third player.

The problem of characterizing strategic complements in general extensive form games is difficult and remains intractable. Some of the difficulties can be seen in our work. Research addressing more general cases (more players, more stages, more actions, continuum of actions, and so on) would be helpful. As two stage, 2×2 games are a basic building block for multi-stage games and infinitely repeated games, the results here may provide insight for other researchers to solve more general cases. In particular, our results show the need to go beyond a direct adaptation of quasisupermodularity and single crossing property in extensive form games.

References

Amir, R. (1996): "Continuous stochastic games of capital accumulation with convex transitions," Games and Economic Behavior, 15, 111–131.

Balbus, Ł., K. Reffett, and Ł. Woźny (2014): "A constructive study of Markov equilibria in stochastic games with strategic complementarities," Journal of Economic Theory, 150, 815–840.

Curtat, L. (1996): "Markov equilibria of stochastic games with complementarities," Games and Economic Behavior, 17, 177–199.

Echenique, F. (2004): "Extensive-form games and strategic complementarities," Games and Economic Behavior, 46(2), 348–364.

Milgrom, P., and C. Shannon (1994): "Monotone Comparative Statics," Econometrica, 62(1), 157–180.

Roy, S., and T. Sabarwal (2010): "Monotone comparative statics for games with strategic substitutes," Journal of Mathematical Economics, 46(5), 793–806.

Sabarwal, T., and H. Vu (2019): "Two stage 2x2 games with strategic substitutes and strategic heterogeneity," Working paper, Department of Economics, University of Kansas.

Topkis, D. (1998): Supermodularity and Complementarity. Princeton University Press.

Vives, X. (2009): "Strategic complementarities in multi-stage games," Economic Theory, 40, 151–171.

Walker, M., J. Wooders, and R. Amir (2011): "Equilibrium play in matches: Binary Markov games," Games and Economic Behavior, 71(2), 487–502.

7 Appendix

Proof of lemma 1

Proof. Notice first that the assumption of differential payoffs to outcomes has the following implications for the structure of best responses. For every $t \in T$, and for every $\hat{s}, \tilde{s} \in BR^1(t)$, the subgame reached on the path of play for profile $(\hat{s}, t)$ is the same as the subgame reached on the path of play for profile $(\tilde{s}, t)$. Moreover, the actions played by each player in the subgame reached on the path of play for profile $(\hat{s}, t)$ are the same as the actions played by each player in the subgame reached on the path of play for profile $(\tilde{s}, t)$. Furthermore, every $s \in S$ that has the same actions as $\hat{s}$ on the path of play for profile $(\hat{s}, t)$ is also a member of $BR^1(t)$.

To prove the lemma, fix $\hat{t}, \tilde{t} \in T$, $\hat{s} \in BR^1(\hat{t})$, and $\tilde{s} \in BR^1(\tilde{t})$.

Suppose first that $\hat{t}^0 = \tilde{t}^0 = B_1^0$, and suppose that $\hat{s}^0 = A_1^0$ and $\tilde{s}^0 = A_2^0$. Notice that the structure of the best response of player 1 implies that $\tilde{s}' = (A_2^0, A_2^1, A_2^2, \tilde{s}^3, A_2^4) \in BR^1(\tilde{t})$. Form $\overline{t} = (B_2^0, \tilde{t}^1, \tilde{t}^2, \tilde{t}^3, \tilde{t}^4)$ and consider $\overline{s} \in BR^1(\overline{t})$. Then $\tilde{t} \preceq \overline{t}$, and using strategic complements for player 1, it follows that $\tilde{s}' \vee \overline{s} \in BR^1(\overline{t})$. In particular, subgame 4 is reached with profile $(\tilde{s}' \vee \overline{s}, \overline{t})$, and therefore, $\overline{s}' = (A_2^0, A_1^1, A_1^2, A_1^3, A_2^4) \in BR^1(\overline{t})$. Moreover, $\tilde{t} \preceq \overline{t}$ implies $\overline{s}' = \overline{s}' \wedge \tilde{s}' \in BR^1(\tilde{t})$. Notice that on path of play for profile $(\overline{s}', \tilde{t})$, subgame 3 is reached and the action played by player 1 in subgame 3 is A_1^3.

Consider $\hat{s} \in BR^1(\hat{t})$ and notice that the structure of best response of player 1 implies that $\hat{s}' = (A_1^0, \hat{s}^1, A_1^2, A_1^3, A_1^4) \in BR^1(\hat{t})$. Let $\underline{t} = \hat{t} \wedge \tilde{t}$ and consider $\underline{s} \in BR^1(\underline{t})$. As $\underline{t} \preceq \hat{t}$, strategic complements for player 1 implies that $\underline{s} \wedge \hat{s}' \in BR^1(\underline{t})$. Notice that on path of play for profile $(\underline{s} \wedge \hat{s}', \underline{t})$, subgame 1 is reached, and therefore, the structure of best response for player 1 implies that $\underline{s}' = (A_1^0, \underline{s}^1 \wedge \hat{s}^1, A_2^2, A_2^3, A_2^4) \in BR^1(\underline{t})$. Using $\underline{t} \preceq \tilde{t}$ and strategic complements for player 1 implies that $\underline{s}' \vee \tilde{s}' \in BR^1(\tilde{t})$. Notice that on path of play for profile $(\underline{s}' \vee \tilde{s}', \tilde{t})$, subgame 3 is reached and the action played by player 1 in subgame 3 is A_2^3. As shown above, this is different from the action played by player 1 on path of play for profile $(\overline{s}', \tilde{t})$, contradicting that both $\overline{s}'$ and $\underline{s}' \vee \tilde{s}'$ are best responses of player 1 to $\tilde{t}$. The case where $\hat{s}^0 = A_2^0$ and $\tilde{s}^0 = A_1^0$ is proved similarly.

Now suppose $\hat{t}^0 = \tilde{t}^0 = B_2^0$, and suppose that $\hat{s}^0 = A_1^0$ and $\tilde{s}^0 = A_2^0$. As subgame 2 is reached on path of play for profile $(\hat{s}, \hat{t})$, it follows that $\hat{s}' = (A_1^0, A_1^1, \hat{s}^2, A_1^3, A_1^4) \in BR^1(\hat{t})$. Form $\underline{t} = (B_1^0, \hat{t}^1, \hat{t}^2, \hat{t}^3, \hat{t}^4)$ and consider $\underline{s} \in BR^1(\underline{t})$. Then $\underline{t} \preceq \hat{t}$, and using strategic complements for player 1, it follows that $\hat{s}' \wedge \underline{s} \in BR^1(\underline{t})$. In particular, subgame 1 is reached with profile $(\underline{s} \wedge \hat{s}', \underline{t})$, and therefore, $\underline{s}' = (A_1^0, A_1^1, A_2^2, A_2^3, A_1^4) \in BR^1(\underline{t})$. Moreover, $\underline{t} \preceq \hat{t}$ implies $\underline{s}' = \underline{s}' \vee \hat{s}' \in BR^1(\hat{t})$. Notice that on path of play for profile $(\underline{s}', \hat{t})$, subgame 2 is reached and the action played by player 1 in subgame 2 is A_2^2.

Consider $\tilde{s} \in BR^1(\tilde{t})$ and notice that the structure of best response of player 1 implies that $\tilde{s}' = (A_2^0, A_2^1, A_2^2, A_2^3, \tilde{s}^4) \in BR^1(\tilde{t})$. Let $\overline{t} = \hat{t} \vee \tilde{t}$ and consider $\overline{s} \in BR^1(\overline{t})$. As $\tilde{t} \preceq \overline{t}$, strategic complements for player 1 implies that $\tilde{s}' \vee \overline{s} \in BR^1(\overline{t})$. Notice that on path of play for profile $(\tilde{s}' \vee \overline{s}, \overline{t})$, subgame 4 is reached, and therefore, the structure of best response for player 1 implies that $\overline{s}' = (A_2^0, A_1^1, A_1^2, A_1^3, \overline{s}^4 \vee \tilde{s}^4) \in BR^1(\overline{t})$. Using $\hat{t} \preceq \overline{t}$ and strategic complements for player 1 implies that $\hat{s}' \wedge \overline{s}' \in BR^1(\hat{t})$. Notice that on path of play for profile $(\hat{s}' \wedge \overline{s}', \hat{t})$, subgame 2 is reached and the action played by player 1 in subgame 2 is A_1^2. This is different from the action played by player 1 on path of play for profile $(\underline{s}', \hat{t})$, contradicting that both $\underline{s}'$ and $\hat{s}' \wedge \overline{s}'$ are best responses of player 1 to $\hat{t}$. The case where $\hat{s}^0 = A_2^0$ and $\tilde{s}^0 = A_1^0$ is proved similarly. □

Proof of lemma 2

Proof. Notice that the assumption of differential payoffs to outcomes implies the following about the structure of best responses: For every $t \in T$, and for every $\hat{s}, \tilde{s} \in BR^1(t)$, $\hat{s}^0 = \tilde{s}^0$. To prove statement (1), fix $\hat{t} \in T$ and $\hat{s} \in BR^1(\hat{t})$ such that $\hat{t}^0 = B_1^0$ and $\hat{s}^0 = A_2^0$. Form $\underline{t} = (B_1^0, B_1^1, B_1^2, B_1^3, B_1^4) \in T$ and let $\underline{s} \in BR^1(\underline{t})$. Then by the previous lemma, $\underline{s}^0 = \hat{s}^0 = A_2^0$. Now fix arbitrarily $t \in T$ and $s \in BR^1(t)$. As $\underline{t} \preceq t$, strategic complements implies that $\underline{s} \vee s \in BR^1(t)$. As $\underline{s}^0 = A_2^0$, it follows that $(\underline{s} \vee s)^0 = A_2^0$. Finally, as noted above, differential payoffs implies that $s^0 = (\underline{s} \vee s)^0 = A_2^0$, as desired. Statement (2) is proved similarly. □

Proof of lemma 3

Proof. To prove statement (1), fix $\hat{t} \in T$ and $\hat{s} \in BR^1(\hat{t})$ such that $\hat{t}^0 = B_1^0$ and $\hat{s}^0 = A_1^0$. Fix arbitrarily $t \in T$, $s \in BR^1(t)$ such that $t^0 = B_1^0$. By lemma 1, $s^0 = A_1^0$, and therefore, $s' = (A_1^0, s^1, A_1^2, A_1^3, A_1^4) \in BR^1(t)$. Let $\overline{t} = (B_2^0, t^1, t^2, t^3, t^4) \in T$ and $\overline{s} \in BR^1(\overline{t})$. Structure of best responses implies that $\overline{s}' = (\overline{s}^0, A_1^1, \overline{s}^2, A_1^3, \overline{s}^4) \in BR^1(\overline{t})$. Moreover, $t \preceq \overline{t}$ and strategic complements implies that $s' \wedge \overline{s}' \in BR^1(t)$ and consequently, structure of best responses implies that $s^1 = (s' \wedge \overline{s}')^1 = A_1^1$.

To prove statement (2), fix $\hat{t} \in T$ and $\hat{s} \in BR^1(\hat{t})$ such that $\hat{t}^0 = B_1^0$ and $\hat{s}^0 = A_2^0$. Fix arbitrarily $t \in T$, $s \in BR^1(t)$ such that $t^0 = B_1^0$. By lemma 1, $s^0 = A_2^0$, and therefore, $s' = (A_2^0, A_2^1, A_2^2, s^3, A_2^4) \in BR^1(t)$. Let $\overline{t} = (B_2^0, t^1, t^2, t^3, t^4) \in T$ and $\overline{s} \in BR^1(\overline{t})$. By previous lemma, $\overline{s}^0 = A_2^0$, and therefore, $\overline{s}' = (A_2^0, A_1^1, A_1^2, A_1^3, \overline{s}^4) \in BR^1(\overline{t})$. Moreover, $t \preceq \overline{t}$ and strategic complements imply that $(A_2^0, A_1^1, A_1^2, A_1^3, A_2^4 \wedge \overline{s}^4) = s' \wedge \overline{s}' \in BR^1(t)$ and consequently, structure of best responses implies that $s^3 = (s' \wedge \overline{s}')^3 = A_1^3$.

Statements (3) and (4) are proved similarly. □

Proof of lemma 6

Proof. For statement (1), consider profile s_{-1} in which players 2 and 3 play (L, L) in stage 1, that is, $s_{-1}(I^0_{-1}) = (L, L)$. If player 1 chooses L in first stage, then subgame 1 is reached, and playing L in subgame 1 is a dominant strategy for player 1. If player 1 chooses H in first stage, then subgame 5 is reached, and player 1 payoffs after subgame 5 are dominated by player 1 payoffs after subgame 1. Therefore, L dominates H in first stage. Similar arguments may be made for statements (2), (3), and (4). □

Proof of theorem 7

Proof. We want to show that for every $s_{-1}, s'_{-1} \in S_{-1}$ with $s_{-1} \prec s'_{-1}$, $BR_1(s_{-1}) \sqsubseteq BR_1(s'_{-1})$. Pick arbitrarily $s_{-1} \in S_{-1}$.

As case 1, suppose $s_{-1} \in \{s_{-1} \mid s_{-1}(I^0_{-1}) = (L, L)\}$. Then statement (1) of lemma 6 shows that $BR_1(s_{-1}) = \{s_1 \in S_1 \mid s_1(I^0_1) = L, s_1(I^1_1) = L\}$. Pick arbitrarily $s'_{-1} \in S_{-1}$ such that $s_{-1} \prec s'_{-1}$. If $s'_{-1} \in \{s_{-1} \mid s_{-1}(I^0_{-1}) = (L, L)\}$, then statement (1) of lemma 6 shows that $BR_1(s_{-1}) = BR_1(s'_{-1})$, and therefore, $BR_1(s_{-1}) \sqsubseteq BR_1(s'_{-1})$. If $s'_{-1} \in \{s_{-1} \mid s_{-1}(I^0_{-1}) = (L, H)\}$, then statement (2) of lemma 6 shows that $BR_1(s'_{-1}) = \{s_1 \in S_1 \mid s_1(I^0_1) = L\}$, and it is easy to check that $BR_1(s_{-1}) \sqsubseteq BR_1(s'_{-1})$. If $s'_{-1} \in \{s_{-1} \mid s_{-1}(I^0_{-1}) = (H, L)\}$, then statement (3) of lemma 6 shows that $BR_1(s'_{-1}) = \{s_1 \in S_1 \mid s_1(I^0_1) = L\}$, and it is easy to check that $BR_1(s_{-1}) \sqsubseteq BR_1(s'_{-1})$. If $s'_{-1} \in \{s_{-1} \mid s_{-1}(I^0_{-1}) = (H, H)\}$, then statement (4) of lemma 6 shows that $BR_1(s'_{-1}) = \{s_1 \in S_1 \mid s_1(I^0_1) = L, s_1(I^4_1) = H\}$, and it is easy to check that $BR_1(s_{-1}) \sqsubseteq BR_1(s'_{-1})$.

As case 2, suppose $s_{-1} \in \{s_{-1} \mid s_{-1}(I^0_{-1}) = (L, H)\}$. Then statement (1) of lemma 6 shows that $BR_1(s_{-1}) = \{s_1 \in S_1 \mid s_1(I^0_1) = L\}$. Pick arbitrarily $s'_{-1} \in S_{-1}$ such that $s_{-1} \prec s'_{-1}$. Then it cannot be that $s'_{-1}(I^0_{-1}) = (L, L)$ or $s'_{-1}(I^0_{-1}) = (H, L)$. If $s'_{-1} \in \{s_{-1} \mid s_{-1}(I^0_{-1}) = (L, H)\}$, then $BR_1(s_{-1}) = BR_1(s'_{-1})$, and if $s'_{-1} \in \{s_{-1} \mid s_{-1}(I^0_{-1}) = (H, H)\}$, then it is easy to check that $BR_1(s_{-1}) \sqsubseteq BR_1(s'_{-1})$.

The cases where $s_{-1} \in \{s_{-1} \mid s_{-1}(I^0_{-1}) = (H, L)\}$ and $s_{-1} \in \{s_{-1} \mid s_{-1}(I^0_{-1}) = (H, H)\}$ are analyzed similarly. □