Two Stage 2×2 Games With Strategic Substitutes and Strategic Heterogeneity

By

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Abstract

Feng and Sabarwal (2018) show that there is additional scope to study strategic complements in extensive form games, by investigating in detail the case of two stage, 2×2 games. We show the same for two stage, 2×2 games with strategic substitutes and with strategic heterogeneity. We characterize strategic substitutes and strategic heterogeneity in such games, and show that the set of each class of games has infinite Lebesgue measure. Our conditions are easy to apply and yield uncountably many examples of such games, indicating greater possibilities for the manifestation and study of these types of interactions. In contrast to the case for strategic complements, we show that generically, the set of subgame perfect Nash equilibria in both classes of games is totally unordered (no two equilibria are comparable). Consequently, with multiple equilibria, some nice features of strategic complements that depend on the complete lattice structure of the equilibrium set may not transfer to the case of strategic substitutes or strategic heterogeneity.

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1 Introduction

Extensive form games with strategic complements have been viewed as a restrictive class of games, as shown by Echenique (2004). In a recent paper, Feng and Sabarwal (2018) investigate two stage, 2×2 games and show that there is additional scope to study strategic complements in such games. In particular, they show that the set of two stage, 2×2 games with subgame strategic complements has infinite Lebesgue measure (in the space of payoffs), as compared to previous definitions that yielded strategic complements on a set of measure zero. Moreover, they show that Echenique (2004)'s result on the complete lattice structure of subgame perfect Nash equlibria continues to hold for this larger class of games. Their work has expanded the scope of strategic complements in extensive form games.

This raises the question whether cases with strategic substitutes and with strategic heterogeneity occur more generally than what was believed previously. We show this to be true for two stage, 2×2 games. In particular, we characterize strategic substitutes and strategic heterogeneity in such games, and show that the set of each class of such games has infinite Lebesgue measure. Our conditions are easy to apply and yield infinitely many examples of such games. This expands the class of games in which such strategic interactions may arise.

In contrast to Feng and Sabarwal (2018), we show that generically (on an open, dense, and full Lebesgue measure set), the set of subgame perfect Nash equilibria in both classes of games (strategic substitutes and strategic heterogeneity) considered here is totally unordered (no two equilibria are comparable). This contrasts the case for strategic complements where the set of SPNE is always a nonempty complete lattice, as shown in Echenique (2004).

Our results highlight some similarities and differences when considering strategic complements and strategic substitutes in extensive form games. There are some similarities in what drives the monotone behavior of best responses in both cases. There is a considerable difference in the structure of the equilibrium set. Consequently, some of the nice features of strategic complements may not transfer to the case of strategic substitutes or strategic heterogeneity.

The problem of characterizing strategic complements, substitutes, and heterogeneity in general extensive form games remains intractable. The results here, combined with those in Feng and Sabarwal (2018), may provide other researchers with some resources in this regard.

The paper proceeds as follows. The next section presents a motivating example. The section after that lays out the general framework and results. The last section concludes.

2 Motivating Example

Consider the following two stage, 2×2 game. In the first stage, a 2×2 game (denoted game 0) is played in which player 1 can take actions in $\{A_1^0, A_2^0\}$ and player 2 can take actions in $\{B_1^0, B_2^0\}$. For each player, assume action 1 is lower than action 2, that is $A_1^0 \prec A_2^0$ and $B_1^0 \prec B_2^0$. The normal form is given in figure 1.

	B_1^0	B_2^0
A_1^0	$3, 3$	$1, -2$
A_2^0	$-2, 1$	$-4, -4$

Figure 1: Stage One Game

In the second stage, another 2×2 game is played depending on first stage outcome. If first state outcome is (A_1^0, B_1^0), then game 1 (top left game in figure 2) is played, if outcome is (A_1^0, B_2^0), (A_2^0, B_1^0), or (A_2^0, B_2^0) then game 2, 3, or 4 is played, respectively. In each game $n = 1, 2, 3, 4$, suppose action 1 is lower than action 2, that is $A_1^n \prec A_2^n$ and $B_1^n \prec B_2^n$.

	B_1^1	B_2^1
A_1^1	$4,8$	$0,0$
A_2^1	$12,12$	$8,4$

(a) Game 1

	B_1^2	B_2^2
A_1^2	$12,0$	$8,8$
A_2^2	$4,4$	$0,12$

(b) Game 2

	B_1^3	B_2^3
A_1^3	$0,0$	$4,12$
A_2^3	$12,4$	$8,8$

(c) Game 3

	B_1^4	B_2^4
A_1^4	$8,8$	$4,12$
A_2^4	$12,4$	$0,0$

(d) Game 4

Figure 2: Stage Two Games

The extensive form of the overall two stage game is depicted in figure 3 (assuming a discount factor of $\delta = \frac{3}{4}$)

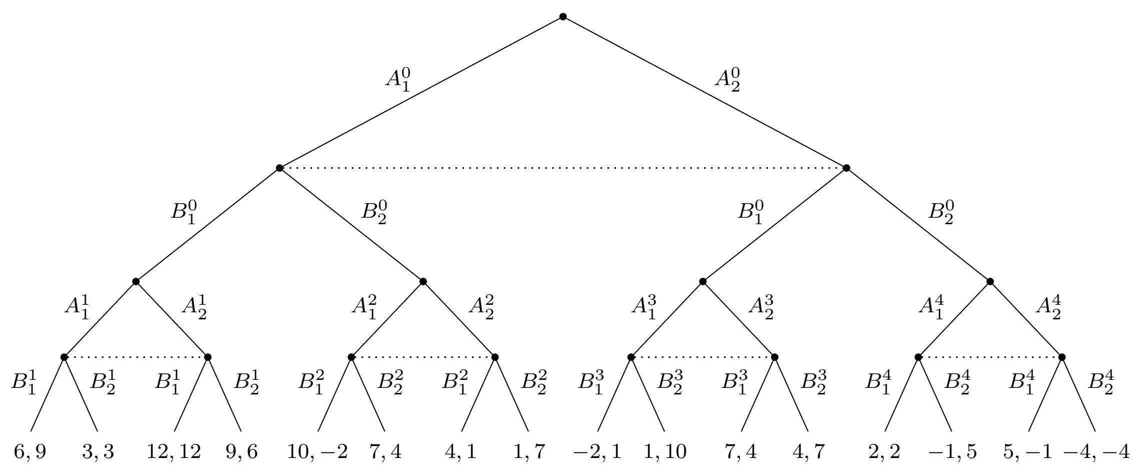

Figure 3: Extensive Form of Two Stage Game

In this game, a strategy for player 1 is a 5-tuple $s = (s^0, s^1, s^2, s^3, s^4)$, where for each $n = 0, 1, 2, 3, 4$, $s^n \in \{A_1^n, A_2^n\}$. The strategy space for player 1 is the collection of all strategies, denoted $\mathcal{S}$, and is endowed with the product order. Notice that $\mathcal{S}$ is a complete lattice in the product order.[3] Similarly, a strategy for player 2 is a 5-tuple $t = (t^0, t^1, t^2, t^3, t^4)$, where for each $n = 0, 1, 2, 3, 4$, $t^n \in \{B_1^n, B_2^n\}$. The strategy space for player 2 is the collection of all strategies, denoted $\mathcal{T}$, and is endowed with the product order. Notice that $\mathcal{T}$ is also a complete lattice in the product order as well.

This makes the game into a lattice game (each player's strategy space is a lattice), and we can inquire if this is a game with strategic substitutes. In other words, is the best response of one player decreasing (in the lattice set order)[4] in the strategy of the other player?

Notice that the component games are very well behaved in terms of best responses. Each of the games 0, 1, 2, and 3 has a strictly dominant action for each player, and game 4 is a classic Dove-Hawk game with two strict Nash equilibria. Therefore, we may expect that this is a game with strategic substitutes.

Indeed, as shown below in more generality, this is an example of a large class of two stage, 2×2 games with strategic substitutes. Moreover, it is straightforward to check that this game has two subgame perfect Nash equilibria, one given by $\hat{s}^* = (A_1^0, A_2^1, A_1^2, A_2^3, A_1^4)$ and $\hat{t}^* = (B_1^0, B_1^1, B_2^2, B_2^3, B_2^4)$, and the other given by $\tilde{s}^* = (A_1^0, A_2^1, A_1^2, A_2^3, A_2^4)$ and $\tilde{t}^* = (B_1^0, B_1^1, B_2^2, B_2^3, B_1^4)$. The set of SPNE is totally unordered.

3 General Framework and Results

Consider a general two stage, 2×2 game. In the first stage, a 2×2 game (denoted game 0) is played in which player 1 can take actions in $\{A_1^0, A_2^0\}$ and player 2 can take actions in $\{B_1^0, B_2^0\}$. In the second stage, another 2×2 game is played depending on first stage outcome. If first stage outcome is (A_1^0, B_1^0), then game 1 is played, in which player 1 can take actions in $\{A_1^1, A_2^1\}$ and player 2 can take actions

[3]We use standard lattice theoretic concepts. Useful reference are Milgrom and Shannon (1994) and Topkis (1998).

[4]See next section for the (standard) definition.

in $\{B_1^1, B_2^1\}$. Similarly, if first stage outcome is (A_1^0, B_2^0), (A_2^0, B_1^0), or (A_2^0, B_2^0) then game 2, 3, or 4 is played, respectively, in which player 1 can take actions in $\{A_1^n, A_2^n\}$ and player 2 can take actions in $\{B_1^n, B_2^n\}$, respectively, for $n = 2, 3, 4$.

The extensive form is depicted in figure 4, with general payoffs at terminal nodes. When there is no confusion, we use the term *game* for such a two stage, 2×2 game. The set of all such games is identified naturally with $\mathbb{R}^{16} \times \mathbb{R}^{16}$. Throughout the paper, we view Euclidean space as a standard measure space with the Borel sigma-algebra and Lebesgue measure.

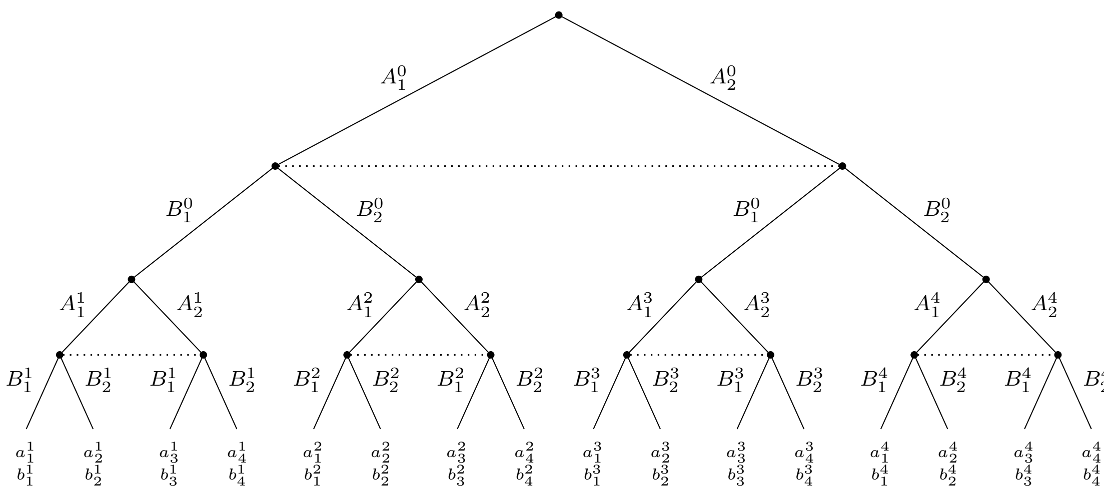

Figure 4: General Two Stage, 2×2 Game

In each component game of a two stage, 2×2 game, suppose action 1 is lower than action 2, that is, for $n = 0, 1, 2, 3, 4$, $A_1^n \prec A_2^n$ and $B_1^n \prec B_2^n$. A strategy for player 1 is a 5-tuple $s = (s^0, s^1, s^2, s^3, s^4)$, where for each $n = 0, 1, 2, 3, 4$, $s^n \in \{A_1^n, A_2^n\}$. The strategy space for player 1 is the collection of all strategies, denoted $\mathcal{S}$, and is endowed with the product order. Notice that $\mathcal{S}$ is a complete lattice in the product order.

Similarly, a strategy for player 2 is a 5-tuple $t = (t^0, t^1, t^2, t^3, t^4)$, where for each $n = 0, 1, 2, 3, 4$, $t^n \in \{B_1^n, B_2^n\}$. The strategy space for player 2 is the collection of all strategies, denoted $\mathcal{T}$, and is endowed with the product order. Notice that $\mathcal{T}$ is also a complete lattice in the product order as well. This makes Γ into a lattice game (each player's strategy space is a lattice).

Strategic substitutes and complements are defined in terms of decreasing or increasing best responses, as usual. ***Player 1 exhibits strategic substitutes***, if best response of player 1, denoted $BR^1(t)$, is (weakly) decreasing in t in the lattice set order (denoted $\sqsubseteq$).[5] That is, $\forall \hat{t}, \tilde{t} \in \mathcal{T},\ \hat{t} \preceq \tilde{t} \implies BR^1(\tilde{t}) \sqsubseteq BR^1(\hat{t})$. Similarly, ***player 1 exhibits strategic complements***, if best response of player 1, denoted $BR^1(t)$, is (weakly) increasing in t in the lattice set order. That is, $\forall \hat{t}, \tilde{t} \in \mathcal{T},\ \hat{t} \preceq \tilde{t} \implies BR^1(\hat{t}) \sqsubseteq BR^1(\tilde{t})$. Similarly, we may define ***player 2 exhibits strategic substitutes*** and ***player 2 exhibits strategic complements***. The game is a ***game with strategic substitutes***, if both players exhibit strategic substitutes. It is a ***game with strategic heterogeneity***, if player 1 exhibits strategic substitutes and player 2 exhibits strategic complements.[6]

As in Feng and Sabarwal (2018), it is useful to assume that payoffs to different final outcomes are different. Such a two stage, 2×2 game is termed a ***game with differential payoffs to outcomes***. This assumption is sufficient to prove the results in this paper. Theoretically, the set of two stage, 2×2 games with differential payoffs to outcomes is open, dense, and has full (Lebesgue) measure in $\mathbb{R}^{16} \times \mathbb{R}^{16}$ (the set of all such games).

The next three lemmas are strategic substitutes analogues of related lemmas in Feng and Sabarwal (2018) for strategic complements. For completeness, proofs are included in the appendix.

Lemma 1 *Consider a game with differential payoffs to outcomes and suppose player 1 exhibits strategic substitutes.*

[5]The lattice set order is the standard set order on lattices: $A \sqsubseteq B$ means that $\forall a \in A, \forall b \in B$, $a \wedge b \in A$ and $a \vee b \in B$. It is sometimes termed the Veinott set order, or the strong set order.

[6]The results do not depend on which player has strategic substitutes or complements. Moreover, notice that strategic substitutes and complements are defined for best responses in the overall game. We shall add subgame strategic substitutes and complements later.

$\forall \hat{t}, \tilde{t} \in T$, $\forall \hat{s} \in BR^1(\hat{t})$, *and* $\forall \tilde{s} \in BR^1(\tilde{t})$, *if* $\hat{t}^0 = \tilde{t}^0$, *then* $\hat{s}^0 = \tilde{s}^0$

Lemma 2 *Consider a game with differential payoffs to outcomes and suppose player 1 exhibits strategic substitutes.*

(1) If there exists $\hat{t} \in T$ *and* $\hat{s} \in BR^1(\hat{t})$ *such that* $\hat{t}^0 = B_1^0$ *and* $\hat{s}^0 = A_1^0$, *then* $\forall t \in T$, $\forall s \in BR^1(t)$, $s^0 = A_1^0$

(2) If there exists $\hat{t} \in T$ *and* $\hat{s} \in BR^1(\hat{t})$ *such that* $\hat{t}^0 = B_2^0$ *and* $\hat{s}^0 = A_2^0$, *then* $\forall t \in T$, $\forall s \in BR^1(t)$, $s^0 = A_2^0$

Lemma 3 *Consider a game with differential payoffs to outcomes and suppose player 1 exhibits strategic substitutes.*

(1) If there exists $\hat{t} \in T$ *and* $\hat{s} \in BR^1(\hat{t})$ *such that* $\hat{t}^0 = B_1^0$ *and* $\hat{s}^0 = A_1^0$ *then* $\forall t \in T$, $\forall s \in BR^1(t)$, *if* $t^0 = B_1^0$ *then* $s^1 = A_2^1$.

(2) If there exists $\hat{t} \in T$ *and* $\hat{s} \in BR^1(\hat{t})$ *such that* $\hat{t}^0 = B_1^0$ *and* $\hat{s}^0 = A_2^0$ *then* $\forall t \in T$, $\forall s \in BR^1(t)$, *if* $t^0 = B_1^0$ *then* $s^3 = A_2^3$.

(3) If there exists $\hat{t} \in T$ *and* $\hat{s} \in BR^1(\hat{t})$ *such that* $\hat{t}^0 = B_2^0$ *and* $\hat{s}^0 = A_2^0$ *then* $\forall t \in T$, $\forall s \in BR^1(t)$, *if* $t^0 = B_2^0$ *then* $s^4 = A_1^4$.

(4) If there exists $\hat{t} \in T$ *and* $\hat{s} \in BR^1(\hat{t})$ *such that* $\hat{t}^0 = B_2^0$ *and* $\hat{s}^0 = A_1^0$ *then* $\forall t \in T$, $\forall s \in BR^1(t)$, *if* $t^0 = B_2^0$ *then* $s^2 = A_1^2$.

These lemmas have the same flavor as in Feng and Sabarwal (2018) but adjusted for the case of strategic substitutes. For example, lemma 1 shows that strategic substitutes for player 1 implies that for a fixed first stage action of player 2, every best of player 1 has the same first stage action, and therefore, leads to the same subgame in stage two. Lemma 3 shows that whenever a particular subgame is reached on the best response path, there is a unique action chosen in that subgame. For example, statement (1) says that if subgame 1 is ever on the best response path, then whenever there is a chance to reach subgame 1 (that is, $t^0 = B_1^0$), player 1 must play A_2^1 in subgame 1. This helps to characterize strategic substitues in theorem 4 below.

In order to characterize strategic substitutes, it is useful to define when an action dominates another action, not just in a given subgame, but across subgames as well. For $m, n \in \{1, 2, 3, 4\}$, and for $k, \ell \in \{1, 2\}$, ***action*** A_k^m ***dominates action*** A_ℓ^n, if subgames m and n can be reached under the same stage one action for player 2, and regardless of which action player 2 plays in subgame n, action A_k^m in subgame m gives player 1 a higher payoff than A_ℓ^n.

This definition allows comparison of actions within the same subgame, or between subgames 1 and 3, or between subgames 2 and 4. It does not apply to comparisons between subgames 1 and 4, or subgames 2 and 3, because these cannot be reached under the same stage one action by player 2, and therefore, those comparisons are irrelevant. In particular, a statement of the form A_2^1 *dominates* A_1^1 means that player 1 payoffs satisfy $a_3^1 > a_1^1$ and $a_4^1 > a_2^1$, a statement of the form A_2^1 *dominates* A_1^3 means that $\min\{a_3^1, a_4^1\} > \max\{a_1^3, a_2^3\}$, and a statement of the form A_2^1 *dominates* A_2^3 means that $\min\{a_3^1, a_4^1\} > \max\{a_3^3, a_4^3\}$. Consequently, the statement A_2^1 *dominates* A_1^1*,* A_1^3*, and* A_2^3 is equivalent to $a_3^1 > a_1^1$, $a_4^1 > a_2^1$, and $\min\{a_3^1, a_4^1\} > \max\{a_1^3, a_2^3, a_3^3, a_4^3\}$.

Theorem 4 *Consider a game with differential payoffs to outcomes. The following are equivalent.*

1. *Player 1 has strategic substitutes*
2. *Exactly one of the following holds*
   (a) A_2^1 *dominates* A_1^1, A_1^3, A_2^3*; and* A_1^2 *dominates* A_2^2, A_1^4, A_2^4
   (b) A_2^3 *dominates* A_1^3, A_1^1, A_2^1*; and* A_1^4 *dominates* A_2^4, A_1^2, A_2^2
   (c) A_2^3 *dominates* A_1^3, A_1^1, A_2^1*; and* A_1^2 *dominates* A_2^2, A_1^4, A_2^4

Proof. Let $\underline{T} = \{t \in \mathcal{T} : t^0 = B_1^0\}$ and $\bar{T} = \{t \in \mathcal{T} : t^0 = B_2^0\}$

(⇒): Suppose player 1 has strategic substitutes

Case 1: suppose $\exists \hat{t} \in \underline{T}$, $\exists \hat{s} \in BR^1(\hat{t})$ such that $\hat{s}^0 = A_1^0$. Then lemma 3(1) implies that A_2^1 dominates A_1^1 for player 1 in subgame 1. Moreover, by lemma 1 and lemma 3(1), whenever player 2 plays B_1^0 in the first-stage game, player 1 chooses to reach subgame 1 over subgame 3 and then to play A_2^1 in subgame 1, regardless of player 2 choice in the second-stage game. Therefore, A_2^1 dominates A_1^3, A_2^3.

Now consider $\bar{T}$. Suppose $\exists \tilde{t} \in \bar{T}$, $\exists \tilde{s} \in BR^1(\tilde{t})$, $\tilde{s} = A_1^0$. Then lemma 3(4) implies that A_1^2 dominates A_2^2, and that dominates A_1^4 and A_2^4 in subgame 4. Therefore, statement 2(a) holds.

Case 2: Suppose $\forall \hat{t} \in \underline{T}$, $\forall \hat{s} \in BR^1(\hat{t})$, $\hat{s} = A_2^0$ Then lemma 3(2) implies that A_2^3 dominates A_1^3 for player 1 in subgame 3. And as reasoning as above, it follows that A_1^3 dominates A_1^1, A_2^1 in subgame 1. Now consider $\bar{T}$

Subcase 1: Suppose $\exists \tilde{t} \in \bar{T}$, $\exists \tilde{s} \in BR^1(\tilde{t})$ such that $\tilde{s}^0 = A_2^0$. Then lemma 3(3) implies that A_1^4 dominates A_2^4 for player 1 subgame 4 and that A_1^4 dominates A_1^2, A_2^2. Therefore, statement 2(b) holds.

Subcase 2: Suppose $\forall \tilde{t} \in \bar{T}$, $\forall \tilde{s} \in BR^1(\tilde{t})$ such that $\tilde{s}^0 = A_1^0$. Then lemma 3(4) implies that A_1^2 dominates A_2^2 for player 1 subgame 2 and that A_1^2 dominates A_1^4, A_2^4. Therefore, statement 2(c) holds.

The reasoning above shows that one of the statement 2(a), 2(b), or 2(c) holds. It is easy to check that no more than one statement holds, because statements are mutually exclusive. In particular, A_2^1 dominates $A_2^1 \Rightarrow A_1^1$ does not dominate A_2^1. A_1^2 dominates $A_2^2 \Rightarrow A_2^2$ does not dominate A_1^2.

($\Leftarrow$): Suppose exactly one of 2(a), 2(b), or 2(c) holds . Suppose 2(a) holds. In this case, A_2^1 dominates A_1^1, A_1^3, A_2^3 implies that $\forall t \in \underline{T}$, player 1 chooses to reach subgame 1 over subgame 3 and to play A_2^1 in subgame 1. In other words, $\forall t \in \underline{T}$, player 1's best response is given by

$$BR^1(t) = \left\{(A_1^0, A_2^1, s^2, s^3, s^4) \in \mathcal{S} : s^n \in \{A_1^n, A_2^n\}, n = 2, 3, 4\right\}$$

Notice that this is a sublattice of $\mathcal{S}$. Similarly, A_1^2 dominates A_2^2, A_1^4, A_2^4 implies that $\forall t \in \bar{T}$, player 1's best response is given by

$$BR^1(t) = \left\{(A_1^0, s^1, A_1^2, s^3, s^4) \in \mathcal{S} : s^n \in \{A_1^n, A_2^n\}, n = 1, 3, 4\right\}$$

Notice that this is a sublattice of $\mathcal{S}$ as well.

Now consider arbitrary $\hat{t}, \tilde{t} \in T$ such that $\hat{t} \preceq \tilde{t}$. If $\hat{t}^0 = \tilde{t}^0$, then $BR^1(\hat{t}) = BR^1(\tilde{t})$, and therefore, $BR^1(\tilde{t}) \sqsubseteq BR^1(\hat{t})$. And if $\hat{t}^0 = B_1^0$ and $\tilde{t}^0 = B_2^0$, then it is easy to check that $BR^1(\tilde{t}) \sqsubseteq BR^1(\hat{t})$. Thus player 1 exhibits strategic substitutes.

The cases where statement 2(b) or 2(c) holds are proved similarly. ■

The characterization in theorem 4 is useful to show that the strategic substitutes property holds on sets of infinite measure, as shown in Theorem 5. Indeed, this holds for subgame strategic substitutes and subgame strategic heterogeneity as well, as follows. ***A player exhibits subgame strategic substitutes***, if in every subgame, best response of the player is (weakly) decreasing (in the lattice set order) in the other player's strategy. ***A player exhibits subgame strategic complements***, if in every subgame, best response of the player is (weakly) increasing (in the lattice set order) in the other player's strategy. A game is a ***game with subgame strategic substitutes***, denoted GSSS, if both players exhibit subgame strategic substitutes. It is a ***game with subgame strategic heterogeneity***, denoted GSSH, if player 1 exhibits subgame strategic substitutes and player 2 exhibits subgame strategic complements.

Theorem 5 *In the set of all two stage,* 2×2 *games,*

1. *The set of GSSS has infinite (Lebesgue) measure.*

2. *The set of GSSH has infinite (Lebesgue) measure.*

Proof. To prove the first statement, we first show that the strategic substitutes property holds on a set of infinite measure. Indeed, each of the conditions 2(a), 2(b), and 2(c) in theorem 4 holds on a set of infinite (Lebesgue) measure. For example, in condition 2(a), A_2^1 dominates A_1^1, A_1^3, A_2^3 is equivalent to $a_3^1 > a_1^1$, $a_4^1 > a_2^1$, and $\min\{a_3^1, a_4^1\} > \max\{a_1^3, a_2^3, a_3^3, a_4^3\}$, and A_1^2 dominates A_2^2, A_1^4, A_2^4 is equivalent to $a_1^2 > a_3^2$, $a_2^2 > a_4^2$, and $\min\{a_1^2, a_2^2\} > \max\{a_1^4, a_2^4, a_3^4, a_4^4\}$. Therefore, the set of payoffs satisfying condition 2(a) includes the following set:

$$
\begin{aligned}
(a_1^1, a_2^1, a_3^1, a_4^1) &\in (13, 14) \times (10, 11) \times (20, +\infty) \times (15, 16) && \subset \mathbb{R}^4 \\
(a_1^2, a_2^2, a_3^2, a_4^2) &\in (15, 16) \times (20, +\infty) \times (10, 11) \times (13, 14) && \subset \mathbb{R}^4 \\
(a_1^3, a_2^3, a_3^3, a_4^3) &\in (0, 1) \times (2, 3) \times (5, 6) \times (8, 9) && \subset \mathbb{R}^4 \\
(a_1^4, a_2^4, a_3^4, a_4^4) &\in (0, 1) \times (2, 3) \times (5, 6) \times (8, 9) && \subset \mathbb{R}^4
\end{aligned}
$$

The product of these sets has infinite Lebesgue measure in $\mathbb{R}^{16}$. Therefore, the property *player 1 exhibits strategic substitutes* holds on a set of infinite measure. Observe that the set above is constructed to also satisfy subgame strategic substitutes. In particular, games with payoffs in that set have the property that for player 1, action A_2^1, A_1^2, A_2^3, and A_2^4 are dominant in stage two subgames 1, 2, 3, and 4, respectively. Therefore, the property *player 1 exhibits subgame strategic substitutes* holds on a set of infinite measure. Similarly, the property *player 2 exhibits subgame strategic substitutes* holds on a set of infinite measure, and consequently, the set of GSSS has infinite measure.

The second statement follows immediately from statement 1 for player 1 and applying corollary 1(3) in Feng and Sabarwal (2018) for player 2. ■

Notice that the measurable rectangle constructed in the proof above has a minimum side length of one unit. It is easy to see that the proof can be modified to construct a measurable rectangle with minimum side length that is arbitrarily large.

Finally, we show that the set of SPNE in GSSS and GSSH are totally unordered (that is, no two equilibria are comparable).

Theorem 6 *Consider two stage 2×2 games with differential payoffs to outcomes.*

1. *The set of SPNE in every GSSS is totally unordered*

2. *The set of SPNE in every GSSH is totally unordered*

Proof. For statement (1), suppose $(\hat{s}, \hat{t})$ and $(\tilde{s}, \tilde{t})$ are two distinct SPNE and suppose $(\hat{s}, \hat{t}) < (\tilde{s}, \tilde{t})$. As case 1, suppose $\hat{s} < \tilde{s}$. As subcase 1, suppose there is $n \in \{1, 2, 3, 4\}$ such that $\hat{s}^n < \tilde{s}^n$. Differential payoffs to outcomes implies that $\hat{t}^n \neq \tilde{t}^n$, and then subgame strategic substitutes implies that $\hat{t}^n = B_2^n$ and $\tilde{t}^n = B_1^n$, contradicting $\hat{t} \leq \tilde{t}$. As subcase 2, suppose $\hat{s}^0 < \tilde{s}^0$. Then $\hat{s}^0 = A_1^0$ and $\tilde{s}^0 = A_2^0$. Moreover, differential payoffs to outcomes implies that $\hat{t}^0 \neq \tilde{t}^0$. Notice that if $\hat{t}^0 = B_1^0$, then lemma 2 implies that $\tilde{s}^0 = A_1^0$, a contradiction. Therefore, $\hat{t}^0 = B_2^0$ and $\tilde{t}^0 = B_1^0$, contradicting $\hat{t} \leq \tilde{t}$. The case where $\hat{t} < \tilde{t}$ is proved similarly, because both players exhibit subgame strategic substitutes.

For statement (2), again consider distinct SPNE $(\hat{s}, \hat{t})$ and $(\tilde{s}, \tilde{t})$ with $(\hat{s}, \hat{t}) < (\tilde{s}, \tilde{t})$. The case where $\hat{s} < \tilde{s}$ is the same as above. So suppose $\hat{s} = \tilde{s}$ and $\hat{t} < \tilde{t}$. As subcase 1, suppose there is $n \in \{1, 2, 3, 4\}$ such that $\hat{t}^n < \tilde{t}^n$. Differential payoffs to outcomes implies that $\hat{s}^n \neq \tilde{s}^n$, contradicting $\hat{s} = \tilde{s}$. As subcase 2, suppose $\hat{t}^0 < \tilde{t}^0$. Then $\hat{t}^0 = B_1^0$ and $\tilde{t}^0 = B_2^0$. By lemma 2 in Feng and Sabarwal (2018), $\hat{s}^0 \neq \tilde{s}^0$, contradicting $\hat{s} = \tilde{s}$. ∎

This theorem contrasts the case for games with subgame strategic complements, in which the set of SPNE is always a nonempty, complete lattice, applying a result due to Echenique (2004). On the other hand, it is similar to results for simultaneous move games with strategic substitutes and strategic heterogeneity, as in Roy and Sabarwal (2008) and in Monaco and Sabarwal (2016).

Notice that we cannot apply the result in Roy and Sabarwal (2008), because the best response correspondence here does not satisfy their never-increasing property, and we cannot apply the result in Monaco and Sabarwal (2016), because the best response correspondence here does not satisfy their strictly decreasing property. Intuitively, both properties are violated because off the best response path, all actions are admissible in the best response correspondence.

4 Conclusion

For some time now, extensive form games with strategic complements have been viewed as a restrictive class of games. Feng and Sabarwal (2018) show that there is additional scope to study strategic complements in such games, at least in the basic and foundational case of two stage, 2×2 games, which are a building block for multi-stage games. We show the same for two stage, 2×2 games with strategic substitutes and strategic heterogeneity. Taken together, these results indicate greater possibilities for the manifestation and study of these types of interactions in extensive form games.

The results here highlight some similarities and differences between strategic complements and strategic substitutes in extensive form games. There are some similarities in what drives the monotone behavior of best responses in both cases. There is a considerable difference in the structure of the equilibrium set. Consequently, with multiple equilibria, some of nice features of strategic complements that depend on the complete lattice structure of the equilbrium set may not transfer to the case of strategic substitutes or strategic heterogeneity.

The problem of characterizing strategic complements, substitutes, and heterogeneity in general extensive form games remains open. Hopefully, the results here, combined with those in Feng and Sabarwal (2018), may spur additional research in this regard.

References

Echenique, F. (2004): "Extensive-form games and strategic complementarities," *Games and Economic Behavior*, 46, 348–364.

Feng, Y., and T. Sabarwal (2018): "Strategic Complements in Two Stage, 2 × 2 Games," Working Paper Series in Theoretical and Applied Economics 201801, University of Kansas, Department of Economics.

Milgrom, P., and C. Shannon (1994): "Monotone Comparative Statics," *Econometrica*, 62, 157–80.

Monaco, A. J., and T. Sabarwal (2016): "Games with Strategic Complements and Substitutes," *Economic Theory*, 62(1), 65–91.

Roy, S., and T. Sabarwal (2008): "On the (Non-)Lattice Structure of the Equilibrium Set in Games With Strategic Substitutes," *Economic Theory*, 37(1), 161–169.

Topkis, D. M. (1998): *Supermodularity and Complementarity*. Princeton University Press.

Appendix

Proof. (of lemma 1) Pick $\hat{t}, \tilde{t} \in T$, $\hat{s} \in BR^1(\hat{t})$, and $\tilde{s} \in BR^1(\tilde{t})$ arbitrary. Suppose $\hat{t}^0 = \tilde{t}^0 = B_1^0$, suppose $\hat{s}^0 = A_1^0$, $\tilde{s}^0 = A_2^0$. As subgame 1 is reached on the path of play for profile $(\hat{s}, \hat{t})$, it follows that $\hat{s}' = (A_1^0, \hat{s}^1, A_1^2, A_1^3, A_1^4) \in BR^1(\hat{t})$. Form $\bar{t} = (B_2^0, \hat{t}^1, \hat{t}^2, \hat{t}^3, \hat{t}^4)$ and consider $\bar{s} \in BR^1(\bar{t})$. Then $\hat{t} \preceq \bar{t}$, and using strategic substitutes of player 1, it follows that $\hat{s}' \wedge \bar{s} \in BR^1(\bar{t})$. In particular, subgame 2 is reached with profile $(\hat{s}' \wedge \bar{s}, \bar{t})$. Therefore, $\bar{s}' = (A_1^0, A_2^1, A_1^2, A_2^3, A_2^4) \in BR^1(\bar{t})$. Moreover, $\hat{t} \preceq \bar{t}$ implies $\bar{s}' \vee \hat{s}' \in BR^1(\hat{t})$. Notice that on path of play for profile $(\bar{s}' \vee \hat{s}', \bar{t})$, subgame 1 is reached and the action played by player 1 in subgame 1 is A_2^1.

Consider $\tilde{s} \in BR^1(\tilde{t})$ and notice that the structure of best response of player 1 implies $\tilde{s}' = (A_2^0, A_1^1, A_1^2, \tilde{s}^3, A_1^4) \in BR^1(\tilde{t})$. Let $\underline{t} = \hat{t} \wedge \tilde{t}$ and consider $\underline{s} \in BR^1(\underline{t})$. As $\underline{t} \preceq \tilde{t}$, strategic substitutes of player 1 implies that $\underline{s} \vee \tilde{s}' \in BR^1(\underline{t})$. Notice that on path of play for profile $(\underline{s} \vee \tilde{s}', \underline{t})$, subgame 3 is reached, therefore, the structure of best response for player 1 implies $\underline{s}' = (A_2^0, A_1^1, A_1^2, \underline{s}^3 \vee \tilde{s}'^3, A_1^4) \in BR^1(\underline{t})$. Using $\underline{t} \preceq \hat{t}$ and strategic substitutes for player 1 imply $\underline{s}' \wedge \hat{s}' \in BR^1(\hat{t})$. Notice that on path of play for profile $(\underline{s}' \wedge \hat{s}', \hat{t})$, subgame 1 is reached, and the action played by player 1 in subgame 1 is A_1^1. This is different from the action played by player 1 on path of play for profile $(\bar{s}' \vee \hat{s}', \hat{t})$, contradicting that both $(\underline{s}' \wedge \hat{s}')$ and $(\bar{s}' \vee \hat{s}')$ are best responses of player 1 to $\hat{t}$.

The case where $\hat{s}^0 = A_2^0$ and $\tilde{s}^0 = A_1^0$ is proved similarly.

Now suppose $\hat{t}^0 = \tilde{t}^0 = B_2^0$, suppose $\hat{s}^0 = A_1^0$, $\tilde{s}^0 = A_2^0$. As subgame 4 is reached on the path of play for profile $(\tilde{s}, \tilde{t})$, it follows that $\tilde{s}' = (A_2^0, A_2^1, A_2^2, A_1^3, \tilde{s}^4) \in BR^1(\tilde{t})$. Form $\underline{t} = (B_1^0, \tilde{t}^1, \tilde{t}^2, \tilde{t}^3, \tilde{t}^4)$ and consider $\underline{s} \in BR^1(\underline{t})$. Then $\underline{t} \preceq \tilde{t}$, and using strategic substitutes of player 1, it follows that $\tilde{s}' \vee \underline{s} \in BR^1(\underline{t})$. In particular, subgame 3 is reached with profile $(\tilde{s}' \vee \underline{s}, \underline{t})$. Therefore, $\underline{s}' = (A_2^0, A_2^1, A_2^2, \tilde{s}'^3 \vee \underline{s}^3, A_1^4) \in BR^1(\underline{t})$. Moreover, $\underline{t} \preceq \tilde{t}$ implies $\underline{s}' \wedge \tilde{s}' \in BR^1(\tilde{t})$. Notice that on path of play for profile $(\underline{s}' \wedge \tilde{s}', \tilde{t})$, subgame 4 is reached and the action played by player 1 in subgame 4 is A_1^4.

Consider $\hat{s} \in BR^1(\hat{t})$ and notice that the structure of best response of player 1 implies $\hat{s}' = (A_1^0, A_1^1, \hat{s}^2, A_1^3, A_2^4) \in BR^1(\hat{t})$. Let $\bar{t} = \hat{t} \vee \tilde{t}$ and consider $\bar{s} \in BR^1(\bar{t})$. As $\hat{t} \preceq \bar{t}$, strategic substitutes of player 1 implies that $\bar{s} \wedge \hat{s}' \in BR^1(\bar{t})$. Notice that on path of play for profile $(\bar{s} \wedge \hat{s}', \bar{t})$, subgame 2 is reached, therefore, the structure of best response for player 1 implies $\bar{s}' = (A_1^0, A_1^1, \bar{s}^2 \wedge \hat{s}'^2, A_1^3, A_2^4) \in BR^1(\bar{t})$. Using $\tilde{t} \preceq \bar{t}$ and strategic substitutes for player 1 imply $\tilde{s}' \vee \bar{s}' \in BR^1(\tilde{t})$. Notice that on path of play for profile $(\tilde{s}' \vee \bar{s}', \tilde{t})$, subgame 4 is reached, and the action played by player 1 in subgame 4 is A_2^4. This is different from the action played by player 1 on path of play for profile $(\underline{s}' \wedge \tilde{s}', \tilde{t})$, contradicting that both $(\tilde{s}' \vee \bar{s}')$ and $(\underline{s}' \wedge \tilde{s}')$ are best responses of player 1 to $\tilde{t}$.

The case where $\hat{s}^0 = A_2^0$ and $\tilde{s}^0 = A_1^0$ is proved similarly. ∎

Proof. (of lemma 2) To prove statement (1), fix $\hat{t} \in T$ and $\hat{s} \in BR^1(\hat{t})$ such that $\hat{t}^0 = B_1^0$ and $\hat{s}^0 = A_1^0$. Form $\underline{t} = (B_1^0, B_1^1, B_1^2, B_1^3, B_1^4) \in T$ and let $\underline{s} \in BR^1(\underline{t})$. Then by lemma 1, $\underline{s}^0 = \hat{s}^0 = A_1^0$. Fix $t \in T$, $s \in BR^1(t)$. As $\underline{t} \preceq t$, strategic substitutes implies that $\underline{s} \wedge s \in BR^1(t)$. As $\underline{s}^0 = A_1^0$, it follows that $(\underline{s} \wedge s)^0 = A_1^0$. Notice that the assumption of differential payoffs to outcomes implies: $\forall t \in T$, $\forall \hat{s}, \tilde{s} \in BR^1(t)$, $\hat{s}^0 = \tilde{s}^0$. Thus, differential payoffs implies that $s^0 = (\underline{s} \wedge s)^0 = A_1^0$ as desired. Statement (2) is proved similarly. ■

Proof. (of lemma 3) To prove statement (1), fix $\hat{t} \in T$ and $\hat{s} \in BR^1(\hat{t})$ such that $\hat{t}^0 = B_1^0$ and $\hat{s}^0 = A_1^0$. Fix $t \in T$ and $s \in BR^1(t)$ such that $t^0 = B_1^0$. [Want to show: $s^1 = A_2^1$] By lemma 1, $s^0 = \hat{s}^0 = A_1^0$. Therefore, $s' = (A_1^0, s^1, A_1^2, A_1^3, A_1^4) \in BR^1(t)$. Let $\bar{t} = (B_2^0, t^1, t^2, t^3, t^4) \in T$ and $\bar{s} \in BR^1(\bar{t})$. By lemma 2(1), $\bar{s}^0 = A_1^0$. Structure of best responses implies $\bar{s}' = (A_1^0, A_2^1, \bar{s}^2, A_2^3, A_2^4) \in BR^1(\bar{t})$. Moreover, $t \preceq \bar{t}$ and strategic substitutes imply $s' \vee \bar{s}' \in BR^1(t)$. Structure of best responses implies $s^1 = (s' \vee \bar{s}')^1 = A_2^1$. To prove statement (2), fix $\hat{t} \in T$ and $\hat{s} \in BR^1(\hat{t})$ such that $\hat{t}^0 = B_1^0$ and $\hat{s}^0 = A_2^0$. Fix $t \in T$ and $s \in BR^1(t)$ such that $t^0 = B_1^0$. [Want to show: $s^3 = A_2^3$] By lemma 1, $s^0 = \hat{s}^0 = A_2^0$. Therefore, $s' = (A_2^0, A_2^1, A_2^2, s^3, A_2^4) \in BR^1(t)$. Let $\bar{t} = (B_2^0, t^1, t^2, t^3, t^4) \in T$ and $\bar{s} \in BR^1(\bar{t})$. Structure of best responses implies that $\bar{s}' = (\bar{s}^0, A_2^1, \bar{s}^2, A_2^3, \bar{s}^4) \in BR^1(\bar{t})$. Moreover, $t \preceq \bar{t}$ and strategic substitutes imply that $s' \vee \bar{s}' \in BR^1(t)$. Finally, structure of best responses implies $s^3 = (s' \vee \bar{s}')^3 = A_2^3$ as desired. (since both s and $(s' \vee \bar{s}') \in BR^1(t)$)

Statement (3) and (4) are proved similarly. ■