Strategic Complements in Two Stage, 2×2 Games

By

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Abstract

Echenique (2004) concludes that extensive form games with strategic complementarities are a very restrictive class of games. In the context of two stage, 2×2 games, we find that the restrictiveness imposed by quasisupermodularity and single crossing property is particularly severe, in the sense that the set of games in which payoffs satisfy these conditions has measure zero. In contrast, the set of such games that exhibit strategic complements (in the sense of increasing best responses) has infinite measure. Our characterization allows one to write uncountably many examples of two stage, 2×2 games with strategic complements. The results show a need to go beyond a direct application of quasisupermodularity and single crossing property to define strategic complements in extensive form games.

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1 Introduction

Echenique (2004) identifies a potential concern about the possibility of strategic complements in extensive form games. Using a natural definition of strategic complements for such games (each player's payoff function satisfies standard notions of quasisupermodularity and single crossing property in every subgame), he gives several examples showing that games that should intuitively exhibit strategic complementarities do not satisfy this definition. He also gives examples of simple extensive form games that cannot be made into extensive form games with strategic complements. He concludes that extensive form games with strategic complementarities are a very restrictive class of games.

We explore the extent of this restrictiveness in the context of two stage, 2×2 games. In particular, we inquire if this restrictiveness is due to the assumption of quasisupermodularity and single crossing property (which are typical sufficient conditions for strategic complements in games)? Or, is it related to the more fundamental notion of strategic complements (in terms of increasing best responses)?

We find that the restrictiveness imposed by quasisupermodularity and single crossing properties is particularly severe, in the sense that the set of two stage, 2×2 games in which payoffs satisfy these conditions has measure zero. We also explore the more general question of when such games exhibit strategic complements (in the sense of increasing best responses) and find that the set of such games has infinite measure.[1]

Our results are based on a detailed study of the notion of strategic complements in

[1] Such a distinction does not hold for normal form games in general, as can be shown readily for the case of 2×2 games, where the set of games in which payoffs satisfy quasisupermodularity and single crossing properties has infinite measure.

two stage, 2×2 games, and a characterization of when a player exhibits strategic complements in such games. This yields conditions on payoff functions that allow us to write (uncountably) many such extensive form games with strategic complements. Moreover, as steps in the development of the main results, we show that strategic complements implies a particular structure for best choices in the first and second stage games. This is important to characterize strategic complements.

The notion of subgame strategic complements used here is consistent with the notion of increasing extended best responses in Echenique (2004), and therefore, his result that the set of subgame perfect Nash equilibria is a nonempty, complete lattice continues to hold in the class of games considered here.

As is well-known, the problem of characterizing strategic complements in general extensive form games remains intractable.[2] As two stage, 2×2 games are a basic building block for multi-stage games and infinitely repeated games, our results may help other researchers to explore more general cases. In particular, our results show the need to go beyond a direct adaptation of quasisupermodularity and single crossing property as used in Echenique (2004). In this regard, the lemmas below shed useful light on the structure of best responses that are consistent with strategic complements.

In order to present ideas more concretely, we consider an explicit example in the next section. The section after that defines the general framework and presents the main result characterizing strategic complements. The section after that formalizes the connection to

[2]The reader may get a flavor of additional complexities related to Markov strategies and transition probabilities in infinite horizon models, as discussed in Amir (1996), Curtat (1996), and Balbus, Reffett, and Woźny (2014).

Echenique (2004).

2 Example

Consider the following two stage, 2×2 game. In the first stage, a 2×2 game (denoted game 0) is played in which player 1 can take actions in $\{A_1^0, A_2^0\}$ and player 2 can take actions in $\{B_1^0, B_2^0\}$. For each player, assume that action 1 is lower than action 2, that is, $A_1^0 \prec A_2^0$ and $B_1^0 \prec B_2^0$. The normal form is given in figure 1.

	B_1^0	B_2^0
A_1^0	3, 3	1, -2
A_2^0	-2, 1	-4, -4

Figure 1: Stage One Game

In the second stage, another 2×2 game is played depending on first stage outcome. If first stage outcome is (A_1^0, B_1^0), then game 1 (top left game in the figure 2) is played, if outcome is (A_1^0, B_2^0), then game 2 (top right) is played, if outcome is (A_2^0, B_1^0), then game 3 (bottom left) is played, and if outcome is (A_2^0, B_2^0), then game 4 (bottom right) is played. In each game $n = 1, 2, 3, 4$, suppose action 1 is lower than action 2, that is, $A_1^n \prec A_2^n$ and $B_1^n \prec B_2^n$.

The extensive form of the overall two stage game is depicted in figure 3 (assuming a discount factor of $\delta = 0.8$).

In this two stage game, a strategy for player 1 is a 5-tuple $s = (s^0, s^1, s^2, s^3, s^4)$, where

	B_1^1	B_2^1
A_1^1	15, 15	10, 5
A_2^1	5, 10	0, 0

	B_1^2	B_2^2
A_1^2	5, 5	0, 15
A_2^2	15, 0	10, 10

	B_1^3	B_2^3
A_1^3	0, 0	5, 15
A_2^3	15, 5	10, 10

	B_1^4	B_2^4
A_1^4	15, 15	5, 0
A_2^4	0, 5	10, 10

Figure 2: Stage Two Games

for each $n = 0, 1, 2, 3, 4$, $s^n \in \{A_1^n, A_2^n\}$. The strategy space for player 1 is the collection of all strategies, denoted $\mathcal{S}$, and is endowed with the product order. It is a (complete) lattice in the product order.[3] Similarly, a strategy for player 2 is a 5-tuple $t = (t^0, t^1, t^2, t^3, t^4)$, where for each $n = 0, 1, 2, 3, 4$, $t^n \in \{B_1^n, B_2^n\}$. The strategy space for player 1 is the collection of all strategies, denoted $\mathcal{T}$, and is endowed with the product order. It is also a (complete) lattice in the product order. We denote payoffs from a strategy profile (s, t) by $u_1(s, t)$ for player 1 and $u_2(s, t)$ for player 2, as usual.

This makes the game into a lattice game (each player's strategy space is a lattice), and we can inquire if this game exhibits strategic complements. In other words, is the best response of one player increasing (in the lattice set order)[4] in the strategy of the other player?

Notice that the component games are very well behaved in terms of monotone com-

[3]We use standard lattice theoretic concepts. Useful references are Milgrom and Shannon (1994) and Topkis (1998).

[4]See next section for the (standard) definition.

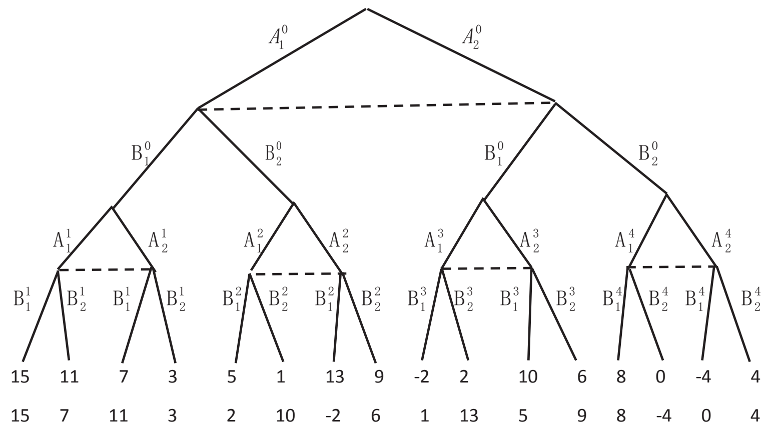

Figure 3: Extensive Form of Two Stage Game

parative statics. Each of the games 0, 1, 2, and 3 has a strictly dominant action for each player, and game 4 is a classic coordination game with two strict Nash equilibria. Therefore, it is natural to expect that there are strategic complements in the two stage game.

Indeed, as shown below in more generality, this game does exhibit strategic complements. Moreover, it is straightforward to check that this game has two subgame perfect Nash equilibria, one given by $\hat{s}^* = (A_1^0, A_1^1, A_2^2, A_2^3, A_1^4)$ and $\hat{t}^* = (B_1^0, B_1^1, B_2^2, B_2^3, B_1^4)$, and the other given by $\tilde{s}^* = (A_1^0, A_1^1, A_2^2, A_2^3, A_2^4)$ and $\tilde{t}^* = (B_1^0, B_1^1, B_2^2, B_2^3, B_2^4)$, and the set of subgame perfect Nash equilibria is a complete lattice.

Nevertheless, this game does not satisfy the definition of an extensive form game with strategic complementarities used in Echenique (2004). For example, the payoff function of player 1 is not quasisupermodular.[5] Consider $\hat{s} = (A_1^0, A_1^1, A_2^2, A_1^3, A_1^4)$, $\tilde{s} = (A_2^0, A_1^1, A_1^2, A_1^3, A_1^4)$, and $\hat{t} = (B_2^0, B_1^1, B_1^2, B_1^3, B_1^4)$. In this case, player 1 payoff is $u_1(\hat{s} \vee$

[5]See section 3 for the (standard) definition of a quasisupermodular function.

$\tilde{s}, \hat{t}) = 8 = u_1(\tilde{s}, \hat{t})$, and therefore, quasisupermodularity implies $13 = u_1(\hat{s}, \hat{t}) \leq u_1(\hat{s} \wedge \tilde{s}, \hat{t}) = 5$, a contradiction.

As shown below in theorems 1 and 2, this example is one of a large class of two stage, 2×2 games that exhibit strategic complements but do not satisfy Echenique's definition. Indeed, we show that the set of such games that satisfy Echenique's definition has measure zero, whereas the set of such games that exhibit strategic complements has infinite measure.

3 General Framework

Consider a general two stage, 2×2 game (denoted Γ). In the first stage, a 2×2 game (denoted game 0) is played in which player 1 can take actions in $\{A_1^0, A_2^0\}$ and player 2 can take actions in $\{B_1^0, B_2^0\}$. In the second stage, another 2×2 game is played depending on first stage outcome. If first stage outcome is (A_1^0, B_1^0), then game 1 is played, in which player 1 can take actions in $\{A_1^1, A_2^1\}$ and player 2 can take actions in $\{B_1^1, B_2^1\}$. If outcome is (A_1^0, B_2^0), then game 2 is played, in which player 1 can take actions in $\{A_1^2, A_2^2\}$ and player 2 can take actions in $\{B_1^2, B_2^2\}$. If outcome is (A_2^0, B_1^0), then game 3 is played, in which player 1 can take actions in $\{A_1^3, A_2^3\}$ and player 2 can take actions in $\{B_1^3, B_2^3\}$. If outcome is (A_2^0, B_2^0), then game 4 is played, in which player 1 can take actions in $\{A_1^4, A_2^4\}$ and player 2 can take actions in $\{B_1^4, B_2^4\}$. The extensive form of Γ is depicted in figure 4, with general payoffs at terminal nodes. When there is no confusion, we use the term *game* for such a two stage, 2×2 game. The set of all such games is identified naturally with $\mathbb{R}^{16} \times \mathbb{R}^{16}$. Throughout the paper, we view Euclidean space as a standard measure space with the Borel sigma-algebra and Lebesgue measure.

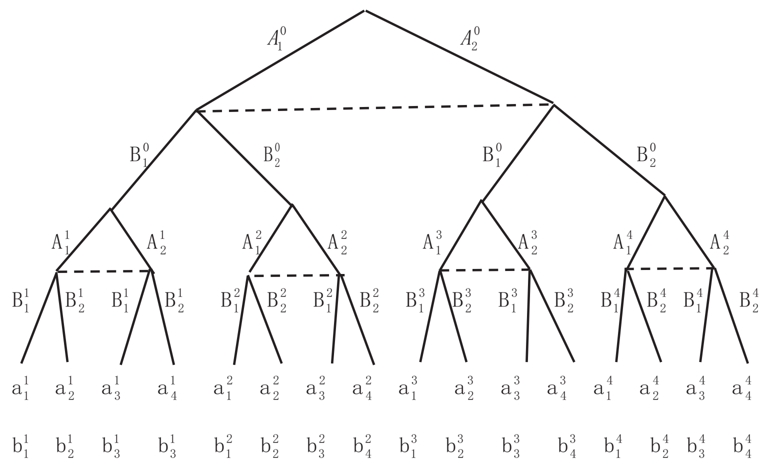

Figure 4: General Two Stage, 2×2 Game

In each component game of a two stage, 2×2 game, suppose action 1 is lower than action 2, that is, for $n = 0, 1, 2, 3, 4$, $A_1^n \prec A_2^n$ and $B_1^n \prec B_2^n$. A strategy for player 1 is a 5-tuple $s = (s^0, s^1, s^2, s^3, s^4)$, where for each $n = 0, 1, 2, 3, 4$, $s^n \in \{A_1^n, A_2^n\}$. The strategy space for player 1 is the collection of all strategies, denoted $\mathcal{S}$, and is endowed with the product order. Notice that $\mathcal{S}$ is a complete lattice in the product order. Similarly, a strategy for player 2 is a 5-tuple $t = (t^0, t^1, t^2, t^3, t^4)$, where for each $n = 0, 1, 2, 3, 4$, $t^n \in \{B_1^n, B_2^n\}$. The strategy space for player 2 is the collection of all strategies, denoted $\mathcal{T}$, and is endowed with the product order. The strategy space $\mathcal{T}$ is a complete lattice in the product order. This makes Γ into a lattice game (each player's strategy space is a lattice). We denote payoffs from a strategy profile (s, t) as $u_1(s, t)$ for player 1 and $u_2(s, t)$ for player 2, as usual.

We shall formulate conditions under which such games exhibit strategic complements, defined in terms of increasing best responses, as usual. ***Player 1 exhibits strategic***

complements, if best response of player 1, denoted $BR^1(t)$, is increasing in t in the lattice set order (denoted $\sqsubseteq$).[6] That is, $\forall \hat{t}, \tilde{t} \in \mathcal{T},\ \hat{t} \preceq \tilde{t} \implies BR^1(\hat{t}) \sqsubseteq BR^1(\tilde{t})$. Similarly, we may define when ***player 2 exhibits strategic complements***. The game Γ ***is a game with strategic complements***, if both players exhibit strategic complements.

Notice that strategic complements is defined for best response sets in the overall game. As shown by a closer analysis of examples in Echenique (2004) and in more detail here, this is the hard case. When we want to include strategic complements in subgames, we shall assume that second-stage subgames exhibit strategic complements. As those are standard 2×2 games, conditions under which they exhibit strategic complements are well known.

In the remainder of this section, we make the assumption that payoffs to different final outcomes are different. Such a two stage, 2×2 game is termed a ***game with differential payoffs to outcomes***. This assumption is sufficient to prove the results in this paper. Theoretically, the set of two stage, 2×2 games with differential payoffs to outcomes is open, dense, and has full (Lebesgue) measure in $\mathbb{R}^{16} \times \mathbb{R}^{16}$ (the set of all such games).

The next three lemmas are important because they show implications of strategic complements in the class of games studied here. In addition to their contribution to prove theorem 1, these lemmas may help researchers to explore more general cases.

Lemma 1. *Consider a game with differential payoffs to outcomes and suppose player 1 exhibits strategic complements.*

For every $\hat{t}, \tilde{t} \in T$, for every $\hat{s} \in BR^1(\hat{t})$, and for every $\tilde{s} \in BR^1(\tilde{t})$, if $\hat{t}^0 = \tilde{t}^0$, then

[6] The lattice set order is the standard set order on lattices: $A \sqsubseteq B$ means that $\forall a \in A, \forall b \in B$, $a \wedge b \in A$ and $a \vee b \in B$. It is sometimes termed the Veinott set order, or the strong set order.

$\hat{s}^0 = \tilde{s}^0$.

Proof. Notice first that the assumption of differential payoffs to outcomes has the following implications for the structure of best responses. For every $t \in T$, and for every $\hat{s}, \tilde{s} \in BR^1(t)$, the subgame reached on the path of play for profile $(\hat{s}, t)$ is the same as the subgame reached on the path of play for profile $(\tilde{s}, t)$. Moreover, the actions played by each player in the subgame reached on the path of play for profile $(\hat{s}, t)$ are the same as the actions played by each player in the subgame reached on the path of play for profile $(\tilde{s}, t)$. Furthermore, every $s \in S$ that has the same actions as $\hat{s}$ on the path of play for profile $(\hat{s}, t)$ is also a member of $BR^1(t)$.

To prove the lemma, fix $\hat{t}, \tilde{t} \in T$, $\hat{s} \in BR^1(\hat{t})$, and $\tilde{s} \in BR^1(\tilde{t})$.

Suppose first that $\hat{t}^0 = \tilde{t}^0 = B_1^0$, and suppose that $\hat{s}^0 = A_1^0$ and $\tilde{s}^0 = A_2^0$. Notice that the structure of the best response of player 1 implies that $\tilde{s}' = (A_2^0, A_2^1, A_2^2, \tilde{s}^3, A_2^4) \in BR^1(\tilde{t})$. Form $\overline{t} = (B_2^0, \tilde{t}^1, \tilde{t}^2, \tilde{t}^3, \tilde{t}^4)$ and consider $\overline{s} \in BR^1(\overline{t})$. Then $\tilde{t} \preceq \overline{t}$, and using strategic complements for player 1, it follows that $\tilde{s}' \vee \overline{s} \in BR^1(\overline{t})$. In particular, subgame 4 is reached with profile $(\tilde{s}' \vee \overline{s}, \overline{t})$, and therefore, $\overline{s}' = (A_2^0, A_1^1, A_1^2, A_1^3, A_2^4) \in BR^1(\overline{t})$. Moreover, $\tilde{t} \preceq \overline{t}$ implies $\overline{s}' = \overline{s}' \wedge \tilde{s}' \in BR^1(\tilde{t})$. Notice that on path of play for profile $(\overline{s}', \tilde{t})$, subgame 3 is reached and the action played by player 1 in subgame 3 is A_1^3.

Consider $\hat{s} \in BR^1(\hat{t})$ and notice that the structure of best response of player 1 implies that $\hat{s}' = (A_1^0, \hat{s}^1, A_1^2, A_1^3, A_1^4) \in BR^1(\hat{t})$. Let $\underline{t} = \hat{t} \wedge \tilde{t}$ and consider $\underline{s} \in BR^1(\underline{t})$. As $\underline{t} \preceq \hat{t}$, strategic complements for player 1 implies that $\underline{s} \wedge \hat{s}' \in BR^1(\underline{t})$. Notice that on path of play for profile $(\underline{s} \wedge \hat{s}', \underline{t})$, subgame 1 is reached, and therefore, the structure of best response for player 1 implies that $\underline{s}' = (A_1^0, \underline{s}^1 \wedge \hat{s}^1, A_2^2, A_2^3, A_2^4) \in BR^1(\underline{t})$. Using $\underline{t} \preceq \tilde{t}$

and strategic complements for player 1 implies that $\underline{s}' \vee \tilde{s}' \in BR^1(\tilde{t})$. Notice that on path of play for profile $(\underline{s}' \vee \tilde{s}', \tilde{t})$, subgame 3 is reached and the action played by player 1 in subgame 3 is A_2^3. As shown above, this is different from the action played by player 1 on path of play for profile $(\overline{s}', \tilde{t})$, contradicting that both $\overline{s}'$ and $\underline{s}' \vee \tilde{s}'$ are best responses of player 1 to $\tilde{t}$. The case where $\hat{s}^0 = A_2^0$ and $\tilde{s}^0 = A_1^0$ is proved similarly.

Now suppose $\hat{t}^0 = \tilde{t}^0 = B_2^0$, and suppose that $\hat{s}^0 = A_1^0$ and $\tilde{s}^0 = A_2^0$. As subgame 2 is reached on path of play for profile $(\hat{s}, \hat{t})$, it follows that $\hat{s}' = (A_1^0, A_1^1, \hat{s}^2, A_1^3, A_1^4) \in BR^1(\hat{t})$. Form $\underline{t} = (B_1^0, \hat{t}^1, \hat{t}^2, \hat{t}^3, \hat{t}^4)$ and consider $\underline{s} \in BR^1(\underline{t})$. Then $\underline{t} \preceq \hat{t}$, and using strategic complements for player 1, it follows that $\hat{s}' \wedge \underline{s} \in BR^1(\underline{t})$. In particular, subgame 1 is reached with profile $(\underline{s} \wedge \hat{s}', \underline{t})$, and therefore, $\underline{s}' = (A_1^0, A_1^1, A_2^2, A_2^3, A_1^4) \in BR^1(\underline{t})$. Moreover, $\underline{t} \preceq \hat{t}$ implies $\underline{s}' = \underline{s}' \vee \hat{s}' \in BR^1(\hat{t})$. Notice that on path of play for profile $(\underline{s}', \hat{t})$, subgame 2 is reached and the action played by player 1 in subgame 2 is A_2^2.

Consider $\tilde{s} \in BR^1(\tilde{t})$ and notice that the structure of best response of player 1 implies that $\tilde{s}' = (A_2^0, A_2^1, A_2^2, A_2^3, \tilde{s}^4) \in BR^1(\tilde{t})$. Let $\overline{t} = \hat{t} \vee \tilde{t}$ and consider $\overline{s} \in BR^1(\overline{t})$. As $\tilde{t} \preceq \overline{t}$, strategic complements for player 1 implies that $\tilde{s}' \vee \overline{s} \in BR^1(\overline{t})$. Notice that on path of play for profile $(\tilde{s}' \vee \overline{s}, \overline{t})$, subgame 4 is reached, and therefore, the structure of best response for player 1 implies that $\overline{s}' = (A_2^0, A_1^1, A_1^2, A_1^3, \overline{s}^4 \vee \tilde{s}^4) \in BR^1(\overline{t})$. Using $\hat{t} \preceq \overline{t}$ and strategic complements for player 1 implies that $\hat{s}' \wedge \overline{s}' \in BR^1(\hat{t})$. Notice that on path of play for profile $(\hat{s}' \wedge \overline{s}', \hat{t})$, subgame 2 is reached and the action played by player 1 in subgame 2 is A_1^2. This is different from the action played by player 1 on path of play for profile $(\underline{s}', \hat{t})$, contradicting that both $\underline{s}'$ and $\hat{s}' \wedge \overline{s}'$ are best responses of player 1 to $\hat{t}$. The case where $\hat{s}^0 = A_2^0$ and $\tilde{s}^0 = A_1^0$ is proved similarly. ∎

Lemma 1 shows that in the class of games considered here, strategic complements for player 1 implies that if a fixed first stage action is part of player 1's best response to $\hat{t}$, then for every player 2 strategy $\tilde{t}$ that has the same first stage action as $\hat{t}$, every best response of player 1 must play the same fixed first stage action, and therefore, lead to the same subgame in stage two.

Lemma 2. *Consider a game with differential payoffs to outcomes and suppose player 1 exhibits strategic complements.*
(1) If there exists $\hat{t} \in T$ and $\hat{s} \in BR^1(\hat{t})$ such that $\hat{t}^0 = B_1^0$ and $\hat{s}^0 = A_2^0$, then for every $t \in T$ and for every $s \in BR^1(t)$, $s^0 = A_2^0$.
(2) If there exists $\hat{t} \in T$ and $\hat{s} \in BR^1(\hat{t})$ such that $\hat{t}^0 = B_2^0$ and $\hat{s}^0 = A_1^0$, then for every $t \in T$ and for every $s \in BR^1(t)$, $s^0 = A_1^0$.

Proof. Notice that the assumption of differential payoffs to outcomes implies the following about the structure of best responses: For every $t \in T$, and for every $\hat{s}, \tilde{s} \in BR^1(t)$, $\hat{s}^0 = \tilde{s}^0$. To prove statement (1), fix $\hat{t} \in T$ and $\hat{s} \in BR^1(\hat{t})$ such that $\hat{t}^0 = B_1^0$ and $\hat{s}^0 = A_2^0$. Form $\underline{t} = (B_1^0, B_1^1, B_1^2, B_1^3, B_1^4) \in T$ and let $\underline{s} \in BR^1(\underline{t})$. Then by the previous lemma, $\underline{s}^0 = \hat{s}^0 = A_2^0$. Now fix arbitrarily $t \in T$ and $s \in BR^1(t)$. As $\underline{t} \preceq t$, strategic complements implies that $\underline{s} \vee s \in BR^1(t)$. As $\underline{s}^0 = A_2^0$, it follows that $(\underline{s} \vee s)^0 = A_2^0$. Finally, as noted above, differential payoffs implies that $s^0 = (\underline{s} \vee s)^0 = A_2^0$, as desired. Statement (2) is proved similarly. ∎

Part (1) of this lemma shows that if playing the higher action in the first stage is ever a best response of player 1 to player 2 playing the lower action in the first stage, then for every player 2 strategy t, playing the higher action must be a best response of player 1. Similarly, part (2) of this lemma shows that if playing the lower action in the first stage is ever a best response of player 1 to player 2 playing the higher action in the first stage, then for every player 2 strategy t, playing the lower action must be a best response of player 1.

Lemma 3. *Consider a game with differential payoffs to outcomes and suppose player 1 exhibits strategic complements.*
(1) If there exists $\hat{t} \in T$ *and* $\hat{s} \in BR^1(\hat{t})$ *such that* $\hat{t}^0 = B_1^0$ *and* $\hat{s}^0 = A_1^0$*, then for every* $t \in T$ *and for every* $s \in BR^1(t)$*, if* $t^0 = B_1^0$ *then* $s^1 = A_1^1$.
(2) If there exists $\hat{t} \in T$ *and* $\hat{s} \in BR^1(\hat{t})$ *such that* $\hat{t}^0 = B_1^0$ *and* $\hat{s}^0 = A_2^0$*, then for every* $t \in T$ *and for every* $s \in BR^1(t)$*, if* $t^0 = B_1^0$ *then* $s^3 = A_1^3$.
(3) If there exists $\hat{t} \in T$ *and* $\hat{s} \in BR^1(\hat{t})$ *such that* $\hat{t}^0 = B_2^0$ *and* $\hat{s}^0 = A_2^0$*, then for every* $t \in T$ *and for every* $s \in BR^1(t)$*, if* $t^0 = B_2^0$ *then* $s^4 = A_2^4$.
(4) If there exists $\hat{t} \in T$ *and* $\hat{s} \in BR^1(\hat{t})$ *such that* $\hat{t}^0 = B_2^0$ *and* $\hat{s}^0 = A_1^0$*, then for every* $t \in T$ *and for every* $s \in BR^1(t)$*, if* $t^0 = B_2^0$ *then* $s^2 = A_2^2$.

Proof. To prove statement (1), fix $\hat{t} \in T$ and $\hat{s} \in BR^1(\hat{t})$ such that $\hat{t}^0 = B_1^0$ and $\hat{s}^0 = A_1^0$. Fix arbitrarily $t \in T$, $s \in BR^1(t)$ such that $t^0 = B_1^0$. By lemma 1, $s^0 = A_1^0$, and therefore, $s' = (A_1^0, s^1, A_1^2, A_1^3, A_1^4) \in BR^1(t)$. Let $\overline{t} = (B_2^0, t^1, t^2, t^3, t^4) \in T$ and $\overline{s} \in BR^1(\overline{t})$. Structure of best responses implies that $\overline{s}' = (\overline{s}^0, A_1^1, \overline{s}^2, A_1^3, \overline{s}^4) \in BR^1(\overline{t})$. Moreover, $t \preceq \overline{t}$ and strategic complements implies that $s' \wedge \overline{s}' \in BR^1(t)$ and consequently, structure of best responses implies that $s^1 = (s' \wedge \overline{s}')^1 = A_1^1$.

To prove statement (2), fix $\hat{t} \in T$ and $\hat{s} \in BR^1(\hat{t})$ such that $\hat{t}^0 = B_1^0$ and $\hat{s}^0 = A_2^0$. Fix arbitrarily $t \in T$, $s \in BR^1(t)$ such that $t^0 = B_1^0$. By lemma 1, $s^0 = A_2^0$, and therefore, $s' = (A_2^0, A_2^1, A_2^2, s^3, A_2^4) \in BR^1(t)$. Let $\overline{t} = (B_2^0, t^1, t^2, t^3, t^4) \in T$ and $\overline{s} \in BR^1(\overline{t})$. By previous lemma, $\overline{s}^0 = A_2^0$, and therefore, $\overline{s}' = (A_2^0, A_1^1, A_1^2, A_1^3, \overline{s}^4) \in BR^1(\overline{t})$. Moreover, $t \preceq \overline{t}$ and strategic complements imply that $(A_2^0, A_1^1, A_1^2, A_1^3, A_2^4 \wedge \overline{s}^4) = s' \wedge \overline{s}' \in BR^1(t)$

and consequently, structure of best responses implies that $s^3 = (s' \wedge \overline{s}')^3 = A_1^3$.

Statements (3) and (4) are proved similarly. ∎

Lemma 3 presents a very useful characteristic of strategic complements in this setting. Whenever a particular subgame is reached on the best response path, lemma 3 locates the unique action that must be chosen in that subgame to be consistent with strategic complements. For example, statement (1) says that if subgame 1 is ever on the best response path, then whenever there is a chance to reach subgame 1 (that is, $t^0 = B_1^0$), player 1 must play A_1^1 in subgame 1. This helps to characterize strategic complements in theorem 1 below. Notice that similar lemmas hold for player 2.

In order to make theorem 1 more accessible, it is useful to define when an action dominates another action, not just in a given subgame, but across subgames as well. For $m, n \in \{1, 2, 3, 4\}$, and for $k, \ell \in \{1, 2\}$, ***action*** A_k^m ***dominates action*** A_ℓ^n, if subgames m and n can be reached under the same stage one action for player 2, and regardless of which action player 2 plays in subgame n, action A_k^m in subgame m gives player 1 a higher payoff than A_ℓ^n.

Notice that this definition allows comparison of actions within the same subgame, or between subgames 1 and 3, or between subgames 2 and 4. It does not apply to comparisons between subgames 1 and 4, or subgames 2 and 3, because these cannot be reached under the same stage one action by player 2, and therefore, those comparisons are irrelevant. In particular, a statement of the form A_1^1 *dominates* A_2^1 means that player 1 payoffs satisfy $a_1^1 > a_3^1$ and $a_2^1 > a_4^1$, a statement of the form A_1^1 *dominates* A_1^3 means that $\min\{a_1^1, a_2^1\} > \max\{a_1^3, a_2^3\}$, and a statement of the form A_1^1 *dominates* A_2^3 means that $\min\{a_1^1, a_2^1\} > \max\{a_3^3, a_4^3\}$. Consequently, the statement A_1^1 *dominates* A_2^1*,* A_1^3*, and*

A_2^3 is equivalent to $a_1^1 > a_3^1$, $a_2^1 > a_4^1$, and $\min\{a_1^1, a_2^1\} > \max\{a_1^3, a_2^3, a_3^3, a_4^3\}$. Here is the main theorem.

Theorem 1. *Consider a game with differential payoffs to outcomes. The following are equivalent.*

1. *Player 1 has strategic complements*

2. *Exactly one of the following holds*

   (a) A_1^1 *dominates* A_2^1, A_1^3, *and* A_2^3, *and* A_2^2 *dominates* A_1^2, A_1^4, *and* A_2^4

   (b) A_1^1 *dominates* A_2^1, A_1^3, *and* A_2^3, *and* A_2^4 *dominates* A_1^4, A_1^2, *and* A_2^2

   (c) A_1^3 *dominates* A_2^3, A_1^1, *and* A_2^1, *and* A_2^4 *dominates* A_1^4, A_1^2, *and* A_2^2

Proof. For this proof, let $\underline{T} = \{t \in \mathcal{T} : t^0 = B_1^0\}$ and $\overline{T} = \{t \in \mathcal{T} : t^0 = B_2^0\}$.

For sufficiency, suppose player 1 has strategic complements.

As case 1, suppose there exists $\hat{t} \in \underline{T}$, there exists $\hat{s} \in BR^1(\hat{t})$ such that $\hat{s}^0 = A_2^0$. Then lemma 3(2) implies that action A_1^3 dominates action A_2^3 for player 1 in subgame 3. Moreover, by lemma 1 and lemma 3(2), whenever player 2 plays B_1^0 in the first-stage game, player 1 chooses to reach subgame 3 over subgame 1, and then to play A_1^3 in subgame 3, regardless of player 2 choice in the second-stage game. Therefore, A_1^3 dominates A_1^1 and A_2^1. Furthermore, lemma 2(1) implies that for every $t \in T$ and $s \in BR^1(t)$, if $t^0 = B_2^0$, then $s^0 = A_2^0$, and therefore, lemma 3(3) implies that action A_2^4 dominates action A_1^4 for player 1 in subgame 4. Reasoning as above, A_2^4 dominates A_1^2 and A_2^2, and therefore, statement 2(c) holds.

As case 2, suppose for every $\hat{t} \in \underline{T}$, for every $\hat{s} \in BR^1(\hat{t})$, $\hat{s} = A_1^0$. Then lemma 3(1) implies that action A_1^1 dominates A_2^1 for player 1 in subgame 1, and reasoning as above, it follows that A_1^1 dominates A_1^3 and A_2^3 in subgame 3. Now consider $\overline{T}$. As subcase 1, suppose there exists $\tilde{t} \in \overline{T}$, there exists $\tilde{s} \in BR^1(\tilde{t})$ such that $\tilde{s}^0 = A_1^0$. Then lemma 3(4) implies that action A_2^2 dominates A_1^2 for player 1 in subgame 2, and that A_2^2 dominates A_1^4 and A_2^4. Therefore, statement 2(a) holds. As subcase 2, suppose for every $\tilde{t} \in \overline{T}$, for every $\tilde{s} \in BR^1(\tilde{t})$, $\tilde{s}^0 = A_2^0$. Then lemma 3(3) implies that action A_2^4 dominates A_1^4, and that A_2^4 dominates A_1^2 and A_2^2 in subgame 2. Therefore, statement 2(b) holds.

The reasoning above shows that one of the statements 2(a), 2(b), or 2(c) holds. It is easy to check that no more than one statement holds, because the statements are mutually exclusive. (In particular, A_2^2 dominates A_2^4 implies A_2^4 does not dominate A_2^2, A_1^1 dominates A_1^3 implies that A_1^3 does not dominate A_1^1, and so on.)

For necessity, suppose exactly one of 2(a), 2(b), or 2(c) holds. Suppose statement 2(a) holds. In this case, A_1^1 dominates A_2^1, A_1^3, and A_2^3 implies that for every $t \in \underline{T}$, player 1 chooses to reach subgame 1 over subgame 3 and to play A_1^1 in subgame 1. In other words, for every $t \in \underline{T}$, player 1's best response is given by

$$BR^1(t) = \{(A_1^0, A_1^1, s^2, s^3, s^4) \in \mathcal{S} : s^n \in \{A_1^n, A_2^n\}, n = 2, 3, 4\}.$$

Notice that this is a sublattice of $\mathcal{S}$. Similarly, A_2^2 dominates A_1^2, A_1^4, and A_2^4 implies that for every $t \in \overline{T}$, player 1 chooses to reach subgame 2 over subgame 4, and to play A_2^2 in subgame 2. In other words, for every $t \in \overline{T}$, player 1's best response is given by

$$BR^1(t) = \{(A_1^0, s^1, A_2^2, s^3, s^4) \in S : s^n \in \{A_1^n, A_2^n\}, n = 1, 3, 4\}.$$

Notice that this is a sublattice of $\mathcal{S}$ as well.

Now consider arbitrary $\hat{t}, \tilde{t} \in T$ such that $\hat{t} \preceq \tilde{t}$. If $\hat{t}^0 = \tilde{t}^0$, then $BR^1(\hat{t}) = BR^1(\tilde{t})$, and therefore, $BR^1(\hat{t}) \sqsubseteq BR^1(\tilde{t})$. And if $\hat{t}^0 = B_1^0$ and $\tilde{t}^0 = B_2^0$, then it is easy to check that $BR^1(\hat{t}) \sqsubseteq BR^1(\tilde{t})$. Thus, player 1 exhibits strategic complements.

The cases where statement 2(b) or 2(c) holds are proved similarly. ∎

Notice that a similar characterization holds for *player 2 has strategic complements*. As shown above, it is easy to write the conditions in statements 2(a), (b), and (c) in terms of the corresponding payoffs, and these conditions are easy to satisfy (see proof of theorem 2 below). Therefore, this characterization yields uncountably many examples of two stage, 2×2 games with strategic complements.[7] For reference, the example above satisfies statement 2(a) of the theorem.

4 Comparison to Echenique (2004)

Echenique (2004) defines an extensive form game with strategic complementarities as an extensive form game in which each player's payoff function satisfies quasisupermodularity (in own strategy) and single crossing property in (own strategy; other players' strategy) in all subgames. For consistency in comparison, we shall first restrict the definition to the overall game and then include stage two subgames.

Player 1 payoff function $u_1 : \mathcal{S} \times \mathcal{T} \to \mathbb{R}$ is ***quasisupermodular (in*** s***)***, if for every $t \in \mathcal{T}$ and for every $s, s' \in \mathcal{S}$, $u_1(s \wedge s', t) < (\leq)\ u_1(s, t) \implies u_1(s', t) < (\leq)\ u_1(s \vee s', t)$. Player 1 payoff function $u_1 : \mathcal{S} \times \mathcal{T} \to \mathbb{R}$ satisfies ***single crossing property in (s; t)***, if for all $t, t' \in \mathcal{T}$ such that $t \prec t'$ and for all $s, s' \in \mathcal{S}$ such that $s \prec s'$, $u_1(s, t) < (\leq$

[7] The case with strategic complements in subgames is similar. Details are presented in the next section.

) $u_1(s', t) \implies u_1(s, t') < (\leq)\ u_1(s', t')$. These are defined similarly for player 2 payoff function u_2. A two stage, 2×2 game satisfies ***Echenique complementarity***, if the payoff function of each player is quasisupermodular in own strategy and satisfies single crossing property in (own strategy; other player strategy).

In order to state the following lemma, it is useful to recall when an action weakly dominates another action in a given subgame. For $n \in \{1, 2, 3, 4\}$ and for $k, \ell \in \{1, 2\}$, ***action*** A_k^n ***weakly dominates action*** A_ℓ^n, if regardless of which action player 2 plays in subgame n, playing action A_k^n in subgame n gives player 1 a weakly higher payoff than A_ℓ^n. This definition shows that a statement like A_1^2 *weakly dominates* A_2^2 is equivalent to $a_1^2 \geq a_3^2$ and $a_2^2 \geq a_4^2$.

Lemma 4. *Consider a two stage,* 2×2 *game that satisfies Echenique complementarity. For player 1,*

(1) A_1^2 *weakly dominates* A_2^2*, and* A_2^2 *weakly dominates* A_1^2

(2) A_1^3 *weakly dominates* A_2^3*, and* A_2^3 *weakly dominates* A_1^3

Proof. For the first statement, consider the following strategies: $\hat{s} = (A_1^0, A_1^1, A_2^2, A_1^3, A_1^4)$, $\tilde{s} = (A_2^0, A_1^1, A_1^2, A_1^3, A_1^4)$, $\hat{t} = (B_2^0, B_1^1, B_2^2, B_1^3, B_1^4)$, and $\tilde{t} = (B_2^0, B_1^1, B_1^2, B_1^3, B_1^4)$. In this case, $u_1(\hat{s} \vee \tilde{s}, \hat{t}) = a_1^4 = u_1(\tilde{s}, \hat{t})$, and therefore, quasisupermodularity implies $a_4^2 = u_1(\hat{s}, \hat{t}) \leq u_1(\hat{s} \wedge \tilde{s}, \hat{t}) = a_2^2$. Moreover, $u_1(\hat{s} \vee \tilde{s}, \tilde{t}) = a_1^4 = u_1(\tilde{s}, \tilde{t})$, and therefore, quasisupermodularity implies $a_3^2 = u_1(\hat{s}, \tilde{t}) \leq u_1(\hat{s} \wedge \tilde{s}, \tilde{t}) = a_1^2$. This shows that A_1^2 weakly dominates A_2^2.

For the other part, consider the following strategies: $\hat{s} = (A_1^0, A_1^1, A_1^2, A_1^3, A_1^4)$, $\tilde{s} = (A_1^0, A_1^1, A_2^2, A_1^3, A_1^4)$, $\hat{t} = (B_1^0, B_1^1, B_1^2, B_1^3, B_1^4)$, and $\tilde{t} = (B_2^0, B_1^1, B_1^2, B_1^3, B_1^4)$. Notice that $\hat{s} \prec \tilde{s}$ and $\hat{t} \prec \tilde{t}$. In this case, $u_1(\hat{s}, \hat{t}) = a_1^1 = u_1(\tilde{s}, \hat{t})$, and therefore, single crossing property implies $a_1^2 = u_1(\hat{s}, \tilde{t}) \leq u_1(\tilde{s}, \tilde{t}) = a_3^2$. Now consider the same $\hat{s}$ and $\tilde{s}$, and the following $\hat{t} = (B_1^0, B_1^1, B_2^2, B_1^3, B_1^4)$ and $\tilde{t} = (B_2^0, B_1^1, B_2^2, B_1^3, B_1^4)$. Again, notice that $\hat{s} \prec \tilde{s}$ and $\hat{t} \prec \tilde{t}$. In this case, $u_1(\hat{s}, \hat{t}) = a_1^1 = u_1(\tilde{s}, \hat{t})$, and therefore, single crossing property implies $a_2^2 = u_1(\hat{s}, \tilde{t}) \leq u_1(\tilde{s}, \tilde{t}) = a_4^2$. This shows that A_2^2 weakly dominates A_1^2.

The second statement is proved similarly. ∎

Statement 1 shows that $a_1^2 = a_3^2$ and $a_2^2 = a_4^2$, and therefore, in every two stage, 2×2 game, quasisupermodular and single crossing property require that player 1 must be indifferent between actions A_1^2 and A_2^2 in subgame 2, essentially eliminating any strategic role for player 1 actions in subgame 2. Statement 2 shows that player 1 must be indifferent between actions A_1^3 and A_2^3 in subgame 3, eliminating a strategic role for player 1 actions in subgame 3. A similar lemma holds for player 2. This yields the following theorem.

Theorem 2. *(1) In the set of two stage, 2×2 games with differential payoffs to outcomes, the set of games that satisfy Echenique complementarity is empty.*
(2) In the set of all two stage, 2×2 games, the set of games that satisfy Echenique complementarity has (Lebesgue) measure zero.
(3) In the set of all two stage, 2×2 games, the set of games that satisfy strategic complements has infinite (Lebesgue) measure.

Proof. For the first statement, if a game satisfies Echenique complementarity, then lemma 4(1) shows that $a_1^2 = a_3^2$ and $a_2^2 = a_4^2$, contradicting differential payoffs to outcomes.

The second statement follows, because the set of games with differential payoffs to outcomes has full (Lebesgue) measure and the first statement here shows that the set of games satisfying Echenique complementarity lies in the complement of this set.

The third statement follows because the set of games that satisfy conditions 2(a), 2(b), and 2(c) in theorem 1 has infinite (Lebesgue) measure. For example, in condition 2(a), A_1^1 dominates A_2^1, A_1^3, and A_2^3 is equivalent to $a_1^1 > a_3^1$, $a_2^1 > a_4^1$, and $\min\{a_1^1, a_2^1\} > \max\{a_1^3, a_2^3, a_3^3, a_4^3\}$, and A_2^2 dominates A_1^2, A_1^4, and A_2^4 is equivalent to $a_3^2 > a_1^2$, $a_4^2 > a_2^2$, and $\min\{a_3^2, a_4^2\} > \max\{a_1^4, a_2^4, a_3^4, a_4^4\}$. Therefore, the set of payoffs satisfying condition 2(a) includes the following set.

$$
\begin{aligned}
&(a_1^1, a_2^1, a_3^1, a_4^1) \in (20, +\infty) \times (15, 16) \times (13, 14) \times (10, 11) && \subset \mathbb{R}^4 \\
&(a_1^2, a_2^2, a_3^2, a_4^2) \in (10, 11) \times (13, 14) \times (15, 16) \times (20, +\infty) && \subset \mathbb{R}^4 \\
&(a_1^3, a_2^3, a_3^3, a_4^3) \in (0, 1) \times (2, 3) \times (5, 6) \times (8, 9) && \subset \mathbb{R}^4 \\
&(a_1^4, a_2^4, a_3^4, a_4^4) \in (0, 1) \times (2, 3) \times (5, 6) \times (8, 9) && \subset \mathbb{R}^4
\end{aligned}
$$

The product of these sets has infinite Lebesgue measure in $\mathbb{R}^{16}$. Therefore, the set of games satisfying condition 2(a) has infinite measure. Consequently, the set of games satisfying *player 1 has strategic complements* has infinite measure. Similarly, it can be shown that the set of games satisfying *player 2 has strategic complements* has infinite measure. ∎

Theorem 2 may be extended to include complementarity in subgames, as follows. A two stage, 2×2 game satisfies ***subgame Echenique complementarity***, if it satisfies Echenique complementarity, and in each of the four 2×2 subgames in stage 2, the payoff function of each player is quasisupermodular in own strategy and satisfies single crossing property in (own strategy; other player strategy). We define a notion of subgame strategic complements similarly. A two stage, 2×2 game satisfies ***subgame strategic complements***, if it exhibits strategic complements, and in each of the four 2×2 subgames in stage 2, the best response of each player is increasing (in the lattice set order) in the other player's strategy.

Corollary 1. *(1) In the set of two stage, 2×2 games with differential payoffs to outcomes, the set of games that satisfy subgame Echenique complementarity is empty.*

(2) In the set of all two stage, 2×2 games, the set of games that satisfy subgame Echenique complementarity has (Lebesgue) measure zero.

(3) In the set of all two stage, 2×2 games, the set of games that satisfy subgame strategic complements has infinite (Lebesgue) measure.

Proof. The first two statements follow immediately from the corresponding statements in theorem 2. For the third statement, notice that the infinite measure set listed in the proof of theorem 2 is constructed to also satisfy subgame strategic complements. In particular, games with payoffs in that set have the property that for player 1, action A_1^1, A_2^2, A_2^3, and A_2^4 are dominant in stage two subgames 1, 2, 3, and 4, respectively. A similar statement holds for player 2. ∎

Finally, the next theorem follows immediately by noting that subgame strategic complements implies increasing extended best response correspondences, as used in Echenique (2004), and to apply his corresponding result.

Theorem 3. *In every two stage, 2×2 game with subgame strategic complements, the set of subgame perfect Nash equilibria is a nonempty, complete lattice.*

Proof. Apply theorem 9 in Echenique (2004) by noting that its proof only requires nonempty, increasing best responses in every subgame, which is satisfied here. ∎

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