User Cost of Credit Card Services under Risk with Intertemporal Nonseparability

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Abstract

This paper derives the user cost of monetary assets and credit card services with interest rate risk under the assumption of intertemporal non-separability. Barnett and Su (2016) derived theory permitting inclusion of credit card transaction services into Divisia monetary aggregates. The risk adjustment in their theory is based on CCAPM[1] under intertemporal separability. The equity premium puzzle focusses on downward bias in the CCAPM risk adjustment to common stock returns. Despite the high risk of credit card interest rates, the risk adjustment under the CCAPM assumption of intertemporal separability might nevertheless be similarly small. While the known downward bias of CCAPM risk adjustments are of little concern with Divisia monetary aggregates containing only low risk monetary assets, that downward bias cannot be ignored, once high risk credit card services are included. We believe that extending to intertemporal non-separability could provide a non-negligible risk adjustment, as has been emphasized by Barnett and Wu (2015).

In this paper, we extend the credit-card-augmented Divisia monetary quantity aggregates to the case of risk aversion and intertemporal non-separability in consumption.

[1] CCAPM is the "consumptions capital asset pricing model," as opposed to the conventional "capital asset pricing model," CAPM.

Our results are for the “representative consumer” aggregated over all consumers. While credit-card interest-rate risk may be low for some consumers, the volatility of credit card interest rates for the representative consumer is high, as reflected by the high volatility of the Federal Reserve’s data on credit card interest rates aggregated over consumers.[2] One method of introducing intertemporal non-separability is to assume habit formation. We explore that possibility.

To implement our theory, we introduce a pricing kernel, in accordance with the approach advocated by Barnett and Wu (2015). We assume that the pricing kernel is a linear function of the rate of return on a well-diversified wealth portfolio. We find that the risk adjustment of the credit-card-services user cost to its certainty equivalence level can be measured by its beta. That beta depends upon the covariance between the interest rates on credit card services and on the wealth portfolio of the consumer, in a manner analogous to the standard CAPM adjustment. As a result, credit card services’ risk premia depend on their market portfolio risk exposure, which is measured by the beta of the credit card interest rates.

We are currently conducting research on empirical implementation of the theory proposed in this paper. We believe that under intertemporal non-separability, we will be able to generate an accurate credit-card-augmented Divisia monetary index to explain the available empirical data.

Keywords— Divisia Index, monetary aggregation, intertemporal non-separability, credit card services, risk adjustment.

JEL—C43, D81, E03, E40, E41, E44, E51, G12.

[2] The relationship between that volatility and the theory of aggregation over consumers under risk is beyond the scope of this research, but is a serious matter meriting future research.

1. Introduction

The simple sum monetary aggregate are consistent with economic aggregation theory, only if the user costs of all component assets are the same. If currency has a zero rate of interest, then all assets in the aggregate must have a zero rate of interest. The assumptions on which simple sum monetary aggregation are based have been unreasonable, since monetary assets began yielding interest. But conventional index numbers from economic index number theory, such as Divisia and Fisher ideal, do not assume that components are perfect substitutes, and hence permit different user cost prices of the component assets. The best known of the economic monetary aggregates are the Divisia monetary aggregates (Barnett (1980)).

The simple sum monetary aggregates do not and cannot include credit card transactions, because of accounting conventions. Monetary assets are assets, while credit card balances are liabilities. Assets and liabilities cannot be added in accounting. However, economic monetary aggregates, such as the Divisia monetary aggregates, measure flows of services, and are not based on accounting conventions. Economic index numbers are based on microeconomic theory. Using economic aggregation and index number theory, the transactions services of credit card and monetary assets can be aggregated jointly.

Credit card transactions play a significant role in the flow of monetary transactions services and provide a deferred payment service not available from monetary assets. Ignoring credit card services from monetary aggregates can lead to bias in the measurement of monetary services. The inability of simple sum monetary aggregates to include credit card transactions services is a fundamental defect of accounting aggregation of monetary services. However, the interest rate on credit card balances, when aggregated over credit card holders, is much more volatile than interest rates on monetary assets. As a result, the user cost of credit card transactions services is much more risky than the user cost of monetary assets.

While the need for risk adjustment of the user costs of monetary assets might be low in the Divisia monetary aggregates, the need for risk adjustment of the user cost of credit card transactions services cannot be ignored. This paper studies the risk adjustment of the user cost of credit card services under intertemporal non-separability.

Barnett (1978, 1980) first derived the user cost of monetary assets under perfect certainty. Barnett, Liu and Jensen (1997) and Barnett and Liu (2000) further introduced risk adjustment of the monetary-asset user cost in a consumption based, CCAPM, asset pricing model. Barnett and Su (2016) extended the derivation to include credit card services under the assumption of intertemporal separability of tastes. Despite the high volatility of the interest rate on credit card holdings of the representative consumer, the risk adjustment is likely to be downward biased by the assumption of intertemporal separability. The CCAPM approach under intertemporal separability implies that the entire effect of financial risk on consumption is contemporaneous without any lags. As a result, CCAPM risk adjustments ignore two sources of correlation between investment risk and consumption life style. (1) It overlooks the fact that current consumption of goods depends not only on current period investment risk, but also on future period expected investment risk. (2) In addition, CCAPM ignores the fact that current period investment risk not only affects current period consumption of goods but also future period consumption. Our extension weakens CCAPM by removing the first restriction, but not the second one, which remains a subject for possible future research.[3]

[3] Consider the consequence of stock market risk on luxury car sales, as related to the equity premium puzzle. The first restriction says that a stock market capital gain this period will be translated into a highly correlated increase in contemporaneous luxury car sales, even if common stock holders are not convinced that the contemporaneous capital gain is permanent, but might be offset by future capital losses. The second restriction says that if indeed the stock holder decides to spend the current capital gain on a luxury car, he will do so immediately, rather than waiting until next period to buy the car. Although these two restrictions are fundamentally different, neither seems realistic. Although we believe both are relevant to the equity premium puzzle, we are addressing only the first in the present research.

2. Consumer's Optimization Problem

In this section, we formulate a representative consumer's stochastic decision problem over consumer goods, monetary asset services, and credit card transaction services. The consumer's decisions are made in discrete time over an infinite planning horizon for the time intervals, $t, t+1, ..., t+s, ...$, where t is the current time period.

Our assumptions on credit card services are the same as those in Barnett and Su (2016). All transactions are made at the beginning of periods, and the payments for the transactions are either by credit card or money. Credit card purchases take place at the start of intervals, but interest on credit card transactions and payments on credit card balances occur at the end of the current and future periods. In other words, the discrete time periods are closed on the left and open on the right. After aggregation over consumers, the expected interest rate paid by the representative credit card holder can be very high, despite the fact that some consumers pay no explicit interest on credit card balances. It is important to recognize that our model is not for a single consumer, but for the representative consumer aggregated over consumers, with all quantities in the model understood to be per capita.

Definitions of the variables are as follows:

x_t = per capita (planned) aggregate consumption of goods;

p_t^* = the true cost of living index;

$\mathbf{m}_t^a = \left(m_{1t}^a, ..., m_{it}^a, ..., m_{It}^a\right)'$ = vector of planned per capita real balances of I monetary assets during period t;

$\mathbf{m}_t^c = \left(m_{1t}^c, ..., m_{it}^c, ..., m_{Lt}^c\right)'$ = vector of planned per-capita real credit-card expenditure balances on goods using L credit cards during period t. In the jargon of the credit card industry, those contemporaneous expenditure balances are called credit card "volumes."

Those volumes are not the total credit card balances, which include the rotating balances resulting from purchases in previous periods and not yet paid off;

$\mathbf{m}_t = (\mathbf{m}_t^{a\prime}, \mathbf{m}_t^{c\prime})'$ = vector of current period monetary asset balances and credit card transactions volumes;

$\mathbf{z}_t = \left(z_{1t}, ..., z_{lt}, ..., z_{Lt}\right)'$ = vector of per capita rotating real balances in the L credit cards during period t, so that the total balances in the credit cards are $\mathbf{m}_t^c + \mathbf{z}_t$;

$\mathbf{k}_t = (\mathrm{k}_{1t},...,\mathrm{k}_{jt},...\mathrm{k}_{Jt})' =$ vector of planned per capita real holdings of J nonmonetary assets during period t;

$\mathbf{R}_t = (R_{1t}, ...R_{jt}, ...R_{Jt})'$ = vector of yields on the J nonmonetary assets during period t;

$\mathbf{e}_t = \left(e_{1t}, ..., e_{lt}, ..., e_{Lt}\right)'$ = vector of interest rates on $\mathbf{m}_t^c$ during period t;

$\mathbf{e}_t^z = \left(e_{1t}^z, ..., e_{lt}^z, ..., e_{Lt}^z\right)'$ = vector of interest rates on $\mathbf{z}_t$ during period t;

$\mathbf{r}_t = \left(r_{1t}, ..., r_{it}, ..., r_{It}\right)'$ = vector rates of return on $\mathbf{m}_t^a$ during period t;

y_t = other sources of income, such as wages or transfers from the government;

$\mathbf{1}$ = unit vector, with each element equalling the number 1. The dimension of the vector is defined to conform to its application.

It should be observed that under risk, there can be multiple nonmonetary assets, j = 1, …, J, subject to different amounts of risk and different rates of return, R_{jt}. What they have in common is providing no services, other than their investment rates of return, while differing from each other only in their degree of risk. It is important to emphasize that relative to our definition, the only source of differences among rates of return on nonmonetary assets is differences in risk. Among those nonmonetary assets, the risk free one is often called the "benchmark asset" or "reference asset." Its rate of return is unique.[4]

Under our assumptions, the benchmark asset is risk free pure capital, providing no services other than its rate of return as an investment. The benchmark asset is fully secured by its ownership. The interest rate on credit card transaction volumes, $\mathbf{e}_t$, is much higher than the benchmark asset rate, because $\mathbf{e}_t$ is the interest rate on an unsecured liability, subject to substantial default and fraud risk. The value of $\mathbf{e}_t$ that we are using is the explicit interest rate on credit card service, as in Barnett and Su (2017). There also is implicit interest on credit card services, such as the annual service fees and increased price of goods commonly purchased with credit cards. Since we are using the Federal Reserve's average explicit interest rate series on credit cards, our measure of $\mathbf{e}_t$ could be biased downward. The actual $\mathbf{e}_t$, including implicit interest, could be even higher.

We use assumptions similar to those in Barnett and Wu (2005). The representative consumer has an intertemporally non-separable utility function. The value of $\mathbf{e}_t$ is averaged over consumers who pay both explicit and implicit interest on credit card volumes and consumers who pay only implicit interest on credit card volumes. But in our initial applications, implicit interest will be assumed to be zero, since implicit interest is not included in the Federal Reserve's data.

The current period utility function, U, is defined over current and past consumption, a vector of current period monetary assets, and a vector of credit card transaction volumes. The consumers' holdings of nonmonetary assets, including the benchmark asset, do not enter the utility function, since those pure capital assets are defined to produce no services other

[4] The conventional benchmark rate can be acquired by risk adjusting any of the nonmonetary assets' rates of return. The same benchmark rate would result from risk adjusting any of the risky nonmonetary assets, since they differ only in their risk. For our purpose, any pure investment asset can be used as the benchmark asset after risk adjustment, since our theory accounts for the differences in risk among those nonfinancial assets.

than their investment rate of return. That non-monetary asset holdings are solely a means of accumulating wealth to endow the next period. The carried-forward rotating balances in credit cards also do not enter the current period utility function, and hence will not enter our derived monetary services aggregate, since they are from transactions in prior periods. During the current period they appear only in constraints. We exclude from current-period utility the rotating balances used for transactions in prior periods to avoid the double counting of transaction services.

We have partitioned the vector $\mathbf{m}_t$ into $\mathbf{m}_t = (\mathbf{m}_t^{a\prime}, \mathbf{m}_t^{c\prime})'$ and have correspondingly partitioned the vector of interest rate into $(\mathbf{r}', \mathbf{e}')'$. We assume that the utility function, U, is blockwise weakly separable in $\mathbf{m}_t$. Hence, there exists a monetary aggregator function, M , and utility function, V, such that

$$U_t = U(\mathbf{m}_t, x_{t,} x_{t-1}, ..., x_{t-n}) = V(M(\mathbf{m}_t), x_{t,} x_{t-1}, ..., x_{t-n}) \tag{2.1}$$

where $M(\mathbf{m}_t)$ is the aggregate over monetary asset and over credit card transaction volumes. The function, U, is assumed to be increasing and strictly concave in $\mathbf{m}_t, x_t$, conditionally on $x_{t-1}, ..., x_{t-n}$.

The implications of this extension to CCAPM are the following. The expected rate-of-return risk on $\mathbf{m}_{t+s}$ during future period $t + s$ will not only correlate contemporaneously with expected consumption of goods in that future period, x_{t+s}, as in CCAPM under intertemporal separability, but also will correlate with x_{t+s-1}, x_{t+s-2}, … x_{t+1} , x_t. As a result, the CCAPM contemporaneous correlation would capture only part of the correlation between expected investment risk and consumption of goods. Conversely current period consumption of goods, x_t, will not only correlate with current period rate of return on holdings of risky assets, $\mathbf{m}_t$, but also on expected rates of return on all future holdings of risky monetary service assets. If risk from long term holding of assets is viewed as being less than short term risk, consumption risk will be decreased by the non-separability of utility.[5] As an extension to CCAPM, we anticipate that this model will produce more plausible risk adjustments than CCAPM, as used in the theory developed by Barnett and Su (2016).

But it should be observed that there is a possible further extension that could produce even larger risk adjustments. If the current period utility function contained future consumption of goods, then current period asset risk could correlate with future planned consumption of consumer goods. Our current extension to intertemporal non-separability does not capture that possible source of increased risk adjustment, which could be a productive source of future research.

The fact that blockwise weak separability is a necessary condition for exact aggregation is well known in the perfect-certainty case. Barnett, Liu, and Jensen (1997) have shown that $M(\mathbf{m}_t)$ is the exact quantity aggregate, even under risk, when $\mathbf{m}_t$ contains only monetary assets.

We assume that credit card transaction services are weakly separable from consumer goods. In the credit card industry, credit cards are defined to include only those cards that can be used widely to buy many kinds of goods and services and also provide a line of credit. They include only Visa, Mastercard, Discover, and American Express. They are used for the same purposes as cash, except that they defer payment. Store cards, such as Sears cards and gasoline cards, are not counted as credit cards. Since store cards are associated with specific goods, store cards are not weakly separable from consumer goods. The four "credit cards" are in the same weakly separable block as monetary assets, which also do not link to any specific consumer goods.

[55] This observation is relevant to the equity premium puzzle, since the buy and hold approach to stock market investment is widely viewed as being more conservative than high frequency trading.

Given initial net wealth, W_t, the consumer seeks to maximize expected lifetime utility. We introduce the expectations operator, E_t, to designate expectations conditional upon information available at current period *t*. The decision is to maximize $V = V(\mathbf{m}_0, x_0, \mathbf{m}_1, x_1, \mathbf{m}_2, x_2, \ldots\ldots)$ defined by

$$V = E_t \sum_{s=0}^{\infty} \beta^s U(\mathbf{m}_{t+s}, x_{t+s}, x_{t+s-1}, \ldots, x_{t+s-n}) \tag{2.2}$$

subject to the budget constraints,

$$W_t + p_t^* \mathbf{1}'\mathbf{m}_t^c + p_t^* \mathbf{1}'\mathbf{z}_t = p_t^* x_t + p_t^* \mathbf{1}'\mathbf{k}_t + p_t^* \mathbf{1}'\mathbf{m}_t^a \tag{2.3}$$

and

$$W_{t+1} = (\mathbf{1}+\mathbf{r}_{t+1})' p_t^* \mathbf{m}_t^a - (\mathbf{1}+\mathbf{e}_{t+1})' p_t^* \mathbf{m}_t^c - (\mathbf{1}+\mathbf{e}_{t+1}^z)' p_t^* \mathbf{z}_t + (\mathbf{1}+\mathbf{R}_{t+1})' p_t^* \mathbf{k}_t + y_{t+1}. \tag{2.4}$$

The intertemporal utility function, $V(\mathbf{m}_0, x_0, \mathbf{m}_1, x_1, \mathbf{m}_2, x_2, \ldots\ldots)$, is assumed to be increasing and strictly concave in all of its arguments. This assumption implicitly constrains the properties of single period utility functions, *U*, as a function of lagged quantities.[6]

The decision also is subject to the transversality condition,

$$\lim_{s\to\infty} \beta^s p_t^* \mathbf{k}_{t+s} = \mathbf{0}. \tag{2.5}$$

The consumer's subjective rate of time preference, β, is assumed to be constant.[7] The transversality condition rules out perpetual borrowing at the pure investment rates, $\mathbf{R}_t$.

Solving for the Euler equations (see appendix), we have

$$E_t \begin{pmatrix} \dfrac{\partial U_t}{\partial \mathbf{m}_t^a} + \beta\lambda_{t+1} \dfrac{p_t^*}{p_{t+1}^*}(\mathbf{1}+\mathbf{r}_{t+1}) - \lambda_t \mathbf{1} \\ \dfrac{\partial U_t}{\partial \mathbf{m}_t^c} - \beta\lambda_{t+1} \dfrac{p_t^*}{p_{t+1}^*}(\mathbf{1}+\mathbf{e}_{t+1}) + \lambda_t \mathbf{1} \end{pmatrix} = \mathbf{0}\ , \tag{2.6}$$

[6] In the literature on habit formation, the implicit constraint on those properties of *U* may merit further theoretical research.

[7] Habit formation is considered to be "myopic," if *U* is optimized separately for each period of time, conditionally upon lagged consumption. Our decision is not myopic, since we optimize the complete intertemporal utility function, *V*.

$$E_t\left(\beta\lambda_{t+1}\frac{p_t^*}{p_{t+1}^*}(\mathbf{1}+\mathbf{R}_{t+1})-\lambda_t\mathbf{1}\right)=\mathbf{0}, \tag{2.7}$$

where $\lambda_t = E_t\sum_{s=0}^{n}\beta^s\frac{\partial U_{t+s}}{\partial x_t}$ is the expected present value of the marginal utility of consumption, x_t. Observe that equation (2.7) holds independently of *j*. These are the Euler equations for consumer optimization under intertemporal non-separability.

When the utility function is intertemporally separable, $\lambda_t = \frac{\partial U_t}{\partial x_t}$. Substituting into the Euler equations, we have

$$E_t\left(\frac{\partial U_t}{\partial m_{it}^a}+\beta\frac{\partial U_{t+1}}{\partial x_{t+1}}\frac{p_t^*}{p_{t+1}^*}(1+r_{i,t+1})-\frac{\partial U_t}{\partial x_t}\right)=0, \tag{2.8}$$

$$E_t\left(\frac{\partial U_t}{\partial m_{lt}^c}-\beta\frac{\partial U_{t+1}}{\partial x_{t+1}}\frac{p_t^*}{p_{t+1}^*}(1+e_{l,t+1})+\frac{\partial U_t}{\partial x_t}\right)=0, \tag{2.9}$$

$$E_t\left(\beta\frac{\partial U_{t+1}}{\partial x_{t+1}}\frac{p_t^*}{p_{t+1}^*}(1+R_{j,t+1})-\frac{\partial U_t}{\partial x_t}\right)=0. \tag{2.10}$$

From (2.10), we have

$$\frac{\partial U_t}{\partial x_t}=E_t\left(\beta\frac{\partial U_{t+1}}{\partial x_{t+1}}\frac{p_t^*}{p_{t+1}^*}(1+R_{j,t+1})\right). \tag{2.11}$$

Substituting equation (2.11) back into (2.8), we have

$$E_t\left(\frac{\partial U_t}{\partial m_{it}^a}+\beta\frac{\partial U_{t+1}}{\partial x_{t+1}}\frac{p_t^*}{p_{t+1}^*}(1+r_{i,t+1})-\beta\frac{\partial U_{t+1}}{\partial x_{t+1}}\frac{p_t^*}{p_{t+1}^*}(1+R_{j,t+1})\right)=0, \tag{2.12}$$

while by substituting equation (2.11) into (2.9) , we have

$$E_t\left(\frac{\partial U_t}{\partial m_{lt}^c}-\beta\frac{\partial U_{t+1}}{\partial x_{t+1}}\frac{p_t^*}{p_{t+1}^*}(1+e_{l,t+1})+\beta\frac{\partial U_{t+1}}{\partial x_{t+1}}\frac{p_t^*}{p_{t+1}^*}(1+R_{j,t+1})\right)=0. \tag{2.13}$$

Rearranging the above equations, we have

$$\left\{\begin{array}{l} E_t\left(\dfrac{\partial U_t}{\partial m_{it}^a}-\beta\dfrac{\partial U_{t+1}}{\partial x_{t+1}}\dfrac{p_t^*}{p_{t+1}^*}(R_{j,t+1}-r_{i,t+1})\right)=0 \\ E_t\left(\dfrac{\partial U_t}{\partial m_{lt}^c}-\beta\dfrac{\partial U_{t+1}}{\partial x_{t+1}}\dfrac{p_t^*}{p_{t+1}^*}(e_{l,t+1}-R_{j,t+1})\right)=0 \\ E_t\left(\dfrac{\partial U_t}{\partial x_t}-\beta\dfrac{\partial U_{t+1}}{\partial x_{t+1}}\dfrac{p_t^*}{p_{t+1}^*}(1+R_{j,t+1})\right)=0 \end{array}\right\}. \tag{2.14}$$

Hence the result in Barnett and Su (2016) still holds, as a special case under intertemporally separable consumption.

For notational convenience, we sometimes convert the nominal rates of return to real gross rates of return, such that

$$\left\{\begin{array}{l} \tilde{\mathbf{r}}_{t+1}=\dfrac{p_t^*}{p_{t+1}^*}(\mathbf{1}+\mathbf{r}_{t+1}) \\ \tilde{\mathbf{e}}_{t+1}=\dfrac{p_t^*}{p_{t+1}^*}(\mathbf{1}+\mathbf{e}_{t+1}) \\ \tilde{\mathbf{R}}_{t+1}=\dfrac{p_t^*}{p_{t+1}^*}(\mathbf{1}+\mathbf{R}_{t+1}) \end{array}\right\}, \tag{2.15}$$

where $\tilde{\mathbf{r}}_{t+1}$ and $\tilde{\mathbf{R}}_{t+1}$ are the real gross rates of excess return on assets, and $\tilde{\mathbf{e}}_{t+1}$ are the real gross interest rate on credit card transaction services. Under this change of variables and observing that current-period marginal utilities are known with certainty, the Euler equations become

$$E_t\begin{pmatrix} \dfrac{\partial U_t}{\partial \mathbf{m}_t^a}+\beta\lambda_{t+1}\tilde{\mathbf{r}}_{t+1}-\lambda_t\mathbf{1} \\ \dfrac{\partial U_t}{\partial \mathbf{m}_t^c}-\beta\lambda_{t+1}\tilde{\mathbf{e}}_{t+1}+\lambda_t\mathbf{1} \end{pmatrix}=\mathbf{0}, \tag{2.16}$$

$$E_t\left(\beta\lambda_{t+1}\tilde{\mathbf{R}}_{t+1}-\lambda_t\mathbf{1}\right)=\mathbf{0}. \tag{2.17}$$

3. Generalized Augmented Divisia Index under Intertemporal Non-separability

3.1. Risk adjusted user cost of monetary assets and credit card services

We now return to the Euler equations. The equations in (2.16) are for monetary assets and credit card transactions, and equation (2.17) is for consumer goods. Money is a durable good. The single period cost of consuming the services of a durable good or asset is the user cost price or rental price. User cost price aggregates are duals to quantity aggregates. Either implies the other uniquely. In addition, user-cost aggregates of monetary services imply the corresponding interest-rate aggregates uniquely. We now derive the user cost of monetary services and the user cost of credit card transactions services under intertemporal non-separability.

As in Barnett, Liu, and Jensen (1997), we define the contemporaneous real user-cost price of the services of monetary asset *i* to be the marginal rate of substitution between monetary asset *i* and consumption of goods, so that the contemporaneous risk-adjusted real user cost price of services of monetary asset *i* is defined such that

$$\pi_{it}^{a} = \frac{\dfrac{\partial U_t}{\partial m_{it}^{a}}}{E_t \sum_{s=0}^{n} \beta^s \dfrac{\partial U_{t+s}}{\partial x_t}} = \frac{\dfrac{\partial U_t}{\partial m_{it}^{a}}}{\lambda_t} \ . \tag{3.1}$$

Similarly, we define the contemporaneous real user-cost price of the transaction services of credit card *l* to be the marginal rate of substitution between credit card *l* transaction volumes and consumption of goods, so that the contemporaneous risk-adjusted real user cost price of credit card *l* transaction volumes is defined such that

$$\pi_{lt}^{c} = \frac{\dfrac{\partial U_t}{\partial m_{lt}^{c}}}{E_t \sum_{s=0}^{n} \beta^s \dfrac{\partial U_{t+s}}{\partial x_t}} = \frac{\dfrac{\partial U_t}{\partial m_{lt}^{c}}}{\lambda_t} . \tag{3.2}$$

With the user costs defined above, we can show that the solution value of the exact monetary aggregate, $M\left(\mathbf{m}_t\right)$, can be tracked accurately in continuous time by the generalized Divisia index, as proved in the perfect certainty special case for monetary assets alone by Barnett (1980).

Proposition 1. *Let* $s_{it} = \pi_{it}^a m_{it}^a / \sum_{i=1}^{I} \pi_{it}^a m_{it}^a$ *be the user-cost-evaluated expenditure share of monetary asset i and* $s_{lt} = \pi_{lt}^c m_{lt}^c / \sum_{l=1}^{L} \pi_{lt}^c m_{lt}^c$ *be the user-cost-evaluated expenditure share of credit card l transactions. Under the weak-separability assumption, we have for any linearly homogenous monetary aggregator function,* $M(.)$, *that*

$$d\log M_t = \mathbf{s}_t' d\log \mathbf{m}_t , \tag{3.3}$$

which can also be written as

$$d\log M_t = \sum_{i=1}^{I} s_{it} d\ log\ m_{it}^a + \sum_{l=1}^{L} s_{lt} d\ log\ m_{lt}^c \tag{3.4}$$

where $M_t = M(\mathbf{m}_t)$, $\mathbf{s}_t$ *is the vector of shares,* $\mathbf{s}_t = \left(\mathbf{s}_t^{a\prime}, \mathbf{s}_t^{c\prime}\right)'$ with $\mathbf{s}_t^a = \left(s_{1t},..., s_{it},..., s_{It}\right)'$, and $\mathbf{s}_t^c = \left(s_{1t},..., s_{lt},..., s_{Lt}\right)'$.

Proof. Under our assumption of weak separability, current period utility is

$$U_t = U(\mathbf{m}_t, x_t, x_{t-1},..., x_{t-n}) = V(M(\mathbf{m}_t), x_t, x_{t-1},..., x_{t-n}) , \tag{3.5}$$

so that

$$\frac{\partial U_t}{\partial m_{it}^a} = \frac{\partial V_t}{\partial M_t}\frac{\partial M_t}{\partial m_{it}^a} , \tag{3.6}$$

$$\frac{\partial U_t}{\partial m_{lt}^c} = \frac{\partial V_t}{\partial M_t}\frac{\partial M_t}{\partial m_{lt}^c} . \tag{3.7}$$

By definitions (3.1) and (3.2), it then follows that

$$\frac{\partial M_t}{\partial m_{it}^a} = \pi_{it}^a (\lambda_t / \frac{\partial V_t}{\partial M_t}), \tag{3.8}$$

$$\frac{\partial M_t}{\partial m_{lt}^c} = \pi_{lt}^c (\lambda_t / \frac{\partial V_t}{\partial M_t}). \tag{3.9}$$

Taking the total differential of $M_t = M(\mathbf{m}_t)$ and using the above results, we obtain,

$$\left\{ \begin{aligned} & dM_t = (\lambda_t / \frac{\partial V_t}{\partial M_t}) \sum_{i=1}^{I} \pi_{it}^a dm_{it}^a + (\lambda_t / \frac{\partial V_t}{\partial M_t}) \sum_{l=1}^{L} \pi_{lt}^c dm_{lt}^c \\ & = (\lambda_t / \frac{\partial V_t}{\partial M_t}) (\sum_{i=1}^{I} \pi_{it}^a dm_{it}^a + \sum_{l=1}^{L} \pi_{lt}^c dm_{lt}^c) \\ & = (\lambda_t / \frac{\partial V_t}{\partial M_t}) (\boldsymbol{\pi}_t' d\mathbf{m}_t). \end{aligned} \right\} \tag{3.10}$$

Because of the linear homogeneity of $M_t = M(\mathbf{m}_t) = M(m_{it}^a, m_{lt}^c)$, it follows that

$$\left\{ \begin{aligned} & M_t = \sum_{i=1}^{I} \frac{\partial M_t}{\partial m_{it}^a} m_{it}^a + \sum_{l=1}^{L} \frac{\partial M_t}{\partial m_{lt}^c} m_{lt}^c \\ & = (\lambda_t / \frac{\partial V_t}{\partial M_t}) \sum_{i=1}^{I} \pi_{it}^a m_{it}^a + (\lambda_t / \frac{\partial V_t}{\partial M_t}) \sum_{l=1}^{L} \pi_{lt}^c m_{lt}^c. \end{aligned} \right\} \tag{3.11}$$

Dividing (3.10) by (3.11), the proposition follows. ■

The user-cost price aggregate, $\Pi_t = \Pi(\boldsymbol{\pi}_t)$, dual to the monetary quantity aggregator function, $M_t = M(\mathbf{m}_t) \; = M(\mathbf{m}_t^a, \mathbf{m}_t^c)$, is computed from factor reversal,

$\Pi(\boldsymbol{\pi}_t) M(\mathbf{m}_t) \; = \sum_{i=1}^{I} \pi_{it}^a m_{it}^a + \sum_{l=1}^{L} \pi_{lt}^c m_{lt}^c$, so that

$$\Pi(\boldsymbol{\pi}_t) = \frac{\sum_{i=1}^{I} \pi_{it}^a m_{it}^a + \sum_{l=1}^{L} \pi_{lt}^c m_{lt}^c}{M(\mathbf{m}_t)} \; . \tag{3.12}$$

In continuous time, the user-cost price dual can be tracked without error by the Divisia user cost price index

$$d \log \Pi_t = \sum_{i=1}^{I} s_{it} d \log \pi_{it}^a + \sum_{l=1}^{I} s_{lt} d \log \pi_{lt}^c . \tag{3.13}$$

The result shows that the tracking ability of the Divisia aggregation index holds regardless of the form of the unknown utility function, U . The result in Barnett and Su (2016) is a special case under intertemporal separability.

As a means of illustrating the nature of the risk adjustment and to acquire a more convenient expression for the user cost, π_{it}^{a} and π_{lt}^{c}, we define the pricing kernel to be

$$Q_{t+1} = \frac{\beta\lambda_{t+1}}{\lambda_t}. \tag{3.14}$$

Recall that λ_t is the present value of the marginal utility of consumption at time t. Hence, Q_{t+1} measures the marginal utility growth from t to $t+1$. For example, if the utility function is time-separable, we have that $Q_{t+1} = \beta \frac{\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}}{\partial U(\mathbf{m}_t, x_t) / \partial x_t}$, which is the subjectively discounted marginal rate of substitution between consumption this period and consumption next period. As required of marginal rates of substitution, Q_{t+1} is positive.

Since the pricing kernel is the subjectively discounted marginal rate of substitution in consumption, it reflects the trade-off among monetary services, risk, and rate of return on different assets through the first-order conditions. If we use the approximation that characterizes conventional CAPM, the pricing kernel becomes a linear function of the interest rates.

The Euler equations can be written as

$$1 = E_t(\tilde{R}_{j,t+1} Q_{t+1}), \tag{3.15}$$

$$\pi_{it}^{a} = 1 - E_t(\tilde{r}_{i,t+1} Q_{t+1}), \tag{3.16}$$

$$\pi_{lt}^{c} = E_t(\tilde{e}_{l,t+1} Q_{t+1}) - 1, \tag{3.17}$$

where $\tilde{R}_{jt}$ is the gross rate of return on a nonmonetary asset, which need not be risk free.

Recall that there can be multiple nonmonetary assets, j = 1, …, J, subject to different amounts of risk and thereby different rates of return, $\tilde{R}_{jt}$. What they have in common is providing no services, other than their investment rates of return and differing from each other only in their degrees of risk. But equation (3.15) must be equally as applicable to all of them, so long as they differ only in their risk.

Equation (3.15) imposes restrictions on the nonmonetary assets' return processes. For monetary assets, equation (3.16) implies that the deviation from the usual Euler equation measures the user cost of that monetary asset. For credit card transactions, equation (3.17) implies that the deviation from the usual Euler equation measures the user cost of credit card transaction services. The non-monetary asset pricing within the asset portfolio's pricing kernel, Q_{t+1}, should be as accurate as possible. Otherwise, we could attribute non-monetary asset pricing errors to the credit-card user costs and monetary-asset user costs. From the Euler equations, we can obtain the following proposition.

Proposition 2. *Given the real gross rate of return,* $\tilde{r}_{i,t+1}$*, on a monetary asset and the real gross rate of return,* $\tilde{R}_{j,t+1}$*, on a non-monetary asset, and* $\tilde{e}_{l,t+1}$ *the real gross interest rate on credit card transaction volumes, as defined in* (*2.15), the risk-adjusted real user-cost price of the services of monetary asset i can be obtained as*

$$\pi_{it}^{a} = \frac{(1+\omega_{it})E_t(\tilde{R}_{j,t+1}) - (1+\omega_{jt})E_t(\tilde{r}_{i,t+1})}{E_t\tilde{R}_{j,t+1}}, \tag{3.18}$$

where

$$\omega_{it} = -\mathrm{cov}_t(Q_{t+1}, \tilde{r}_{i,t+1}), \tag{3.19}$$

$$\omega_{jt} = -\mathrm{cov}_t(Q_{t+1}, \tilde{R}_{j,t+1}). \tag{3.20}$$

The risk-adjusted real user-cost price of the services of the credit card volumes can be obtained as

$$\pi_{lt}^{c} = \frac{(1+\omega_{jt})E_t(\tilde{e}_{l,t+1}) - (1+\omega_{lt})E_t(\tilde{R}_{j,t+1})}{E_t(\tilde{R}_{j,t+1})}, \tag{3.21}$$

where

$$\omega_{lt} = -\mathrm{cov}_t(Q_{t+1}, \tilde{e}_{l,t+1}) . \tag{3.22}$$

Proof. From the Euler equation in (3.15), we have

$$1 = E_t Q_{t+1} E_t(\tilde{R}_{j,t+1}) + \mathrm{cov}_t(Q_{t+1}, \tilde{R}_{j,t+1}) . \tag{3.23}$$

From the Euler equation in (3.16), we have

$$\pi_{it}^{a} = 1 - E_t Q_{t+1} E_t(\tilde{r}_{i,t+1}) - \mathrm{cov}_t(Q_{t+1}, \tilde{r}_{i,t+1}) . \tag{3.24}$$

From the Euler equation in (3.17), we have

$$\pi_{lt}^{c} = E_t Q_{t+1} E_t(\tilde{e}_{l,t+1}) + \mathrm{cov}_t(Q_{t+1}, \tilde{e}_{l,t+1}) - 1 . \tag{3.25}$$

From equation (3.23), we have

$$E_t\, Q_{t+1} = \frac{1 - \mathrm{cov}_t(Q_{t+1}, \tilde{R}_{j,t+1})}{E_t\,(\tilde{R}_{j,t+1})} . \tag{3.26}$$

Substituting equation (3.26) into equation (3.24), we have

$$\pi_{it}^{a} = 1 - \frac{1 - \mathrm{cov}_t(Q_{t+1}, \tilde{R}_{j,t+1})}{E_t(\tilde{R}_{j,t+1})} E_t(\tilde{r}_{i,t+1}) - \mathrm{cov}_t(Q_{t+1}, \tilde{r}_{i,t+1}) . \tag{3.27}$$

Rearranging the equation, we have

$$\pi_{it}^{a} = E_t(\tilde{R}_{j,t+1}) \frac{1 - \mathrm{cov}_t(Q_{t+1}, \tilde{r}_{i,t+1})}{E_t \tilde{R}_{j,t+1}} - E_t(\tilde{r}_{i,t+1}) \frac{1 - \mathrm{cov}_t(Q_{t+1}, \tilde{R}_{j,t+1})}{E_t \tilde{R}_{j,t+1}} . \tag{3.28}$$

Substituting equation (3.19) and equation (3.20) into equation (3.28), we have

$$\pi_{it}^{a} = E_t(\tilde{R}_{j,t+1})\frac{(1+\omega_{i,t+1})}{E_t\tilde{R}_{j,t+1}} - E_t(\tilde{r}_{i,t+1})\frac{(1+\omega_{j,t+1})}{E_t\tilde{R}_{j,t+1}}, \tag{3.29}$$

so that,

$$\pi_{it}^{a} = \frac{E_t(\tilde{R}_{j,t+1})(1+\omega_{i,t+1}) - E_t(\tilde{r}_{i,t+1})(1+\omega_{j,t+1})}{E_t\tilde{R}_{j,t+1}} \ . \tag{3.30}$$

Similarly, substituting equation (3.26) into equation (3.25), we have

$$\pi_{lt}^{c} = \frac{1-\mathrm{cov}_t(Q_{t+1},\tilde{R}_{j,t+1})}{E_t(\tilde{R}_{j,t+1})}E_t(\tilde{e}_{l,t+1}) + \mathrm{cov}_t(Q_{t+1},\tilde{e}_{l,t+1}) - 1 \ . \tag{3.31}$$

Rearranging the equation, we have

$$\pi_{lt}^{c} = \frac{1-\mathrm{cov}_t(Q_{t+1},\tilde{R}_{j,t+1})}{E_t(\tilde{R}_{j,t+1})}E_t(\tilde{e}_{l,t+1}) + \frac{\mathrm{cov}_t(Q_{t+1},\tilde{e}_{l,t+1})-1}{E_t(\tilde{R}_{j,t+1})}E_t(\tilde{R}_{j,t+1}), \tag{3.32}$$

so that

$$\pi_{lt}^{c} = \frac{(1-\mathrm{cov}_t(Q_{t+1},\tilde{R}_{j,t+1}))E_t(\tilde{e}_{l,t+1}) - (1-\mathrm{cov}_t(Q_{t+1},\tilde{e}_{l,t+1}))E_t(\tilde{R}_{j,t+1})}{E_t(\tilde{R}_{j,t+1})}. \tag{3.33}$$

Substituting equations (3.20) and (3.22) into equation (3.33), we have

$$\pi_{lt}^{c} = \frac{(1+\omega_{j,t+1})E_t(\tilde{e}_{l,t+1}) - (1+\omega_{l,t+1})E_t(\tilde{R}_{j,t+1})}{E_t(\tilde{R}_{j,t+1})}. \qquad \blacksquare \tag{3.34}$$

We are assuming that the sole source of differences among rates of return on different benchmark assets, *j*, is differences in risk. Since our results are risk adjusted, our results are not dependent upon the choice of the benchmark asset, and hence the subscript *j* does not appear in the results.

3.2 The perfect certainty case

To observe the intuition associated with the above proposition, assume that one of the non-monetary assets is the "benchmark asset," defined to be risk-free with gross real interest rate of r_t^f at time t. As shown by Barnett (1978), the certainty-equivalent user cost, π_{it}^{ae}, of a monetary asset m_{it}^a is

$$\pi_{it}^{ae} = \frac{r_t^f - E_t \tilde{r}_{i,t+1}}{r_t^f}. \tag{3.35}$$

From (3.15), the first order condition for r_t^f is

$$1 = E_t (Q_{t+1} r_t^f). \tag{3.36}$$

From the non-randomness of r_t^f we have that

$$E_t Q_{t+1} = \frac{1}{r_t^f}. \tag{3.37}$$

Substituting equation (3.37) into equation (3.16), we have

$$\pi_{it}^a = 1 - E_t \tilde{r}_{i,t+1} E_t Q_{t+1} - \text{cov}_t(\tilde{r}_{i,t+1}, Q_{t+1}) = 1 - E_t \tilde{r}_{i,t+1} \frac{1}{r_t^f} - \text{cov}_t(\tilde{r}_{i,t+1}, Q_{t+1}), \tag{3.38}$$

so that

$$\pi_{it}^a = \frac{r_t^f - E_t \tilde{r}_{i,t+1}}{r_t^f} - \text{cov}_t(\tilde{r}_{i,t+1}, Q_{t+1}).$$

Letting $\omega_{it} = -\text{cov}_t(\tilde{r}_{i,t+1}, Q_{t+1})$, we have

$$\pi_{it}^a = \frac{r_t^f - E_t \tilde{r}_{i,t+1}}{r_t^f} + \omega_{i,t+1} = \pi_{it}^{ae} + \omega_{i,t+1}.$$

Substituting equation (3.37) into equation (3.17) for the credit card case, we have

$$\pi_{lt}^c = E_t \tilde{e}_{l,t+1} E Q_{t+1} + \text{cov}(\tilde{e}_{l,t+1}, Q_{t+1}) - 1 = E_t \tilde{e}_{l,t+1} \frac{1}{r_t^f} + \text{cov}(\tilde{e}_{l,t+1}, Q_{t+1}) - 1, \tag{3.39}$$

so that

$$\pi_{lt}^c = \frac{E_t \tilde{e}_{l,t+1} - r_t^f}{r_t^f} + \text{cov}(\tilde{e}_{l,t+1}, Q_{t+1}).$$

Letting $\omega_{lt} = -\text{cov}_t(Q_{t+1}, \tilde{e}_{l,t+1})$, we have

$$\pi_{lt}^{c} = \pi_{lt}^{ce} - \omega_{l,t+1},$$

where the certainty equivalent credit card user cost is

$$\pi_{lt}^{ce} = \frac{E_t \tilde{e}_{l,t+1} - r_t^f}{r_t^f}.$$

This proposition relates the user cost of credit cards to the rates of return on financial assets, which need not be risk-free. The user costs of monetary assets are consistent with the result in Barnett and Wu (2005), which relates the user costs of monetary assets to the rates of return on financial assets.

Therefore, π_{it}^{a} can be larger or smaller than the certainty-equivalent user cost π_{it}^{ae}, depending on the sign of the covariance between $\tilde{r}_{i,t+1}$ and Q_{t+1}. When the return on a monetary asset is positively correlated with the pricing kernel Q_{t+1}, and thereby negatively correlated with the rate of return on the full portfolio of monetary and nonmonetary assets, the monetary asset's user cost will be adjusted downwards from the certainty-equivalent user cost. Such assets offer a hedge against aggregate risk by performing well, when the return on a monetary asset is negatively correlated with the pricing kernel Q_{t+1}, and thereby positively correlated with the rate of return on the full asset portfolio. The asset's user cost will be adjusted upwards from the certainty-equivalent user cost, since such assets tend to pay off when the asset portfolio's rate of return is high.

Similarly, π_{lt}^{c} can be larger or smaller than the certainty-equivalent user cost, π_{lt}^{ce}, depending on the sign of the covariance between $\tilde{e}_{l,t+1}$ and Q_{t+1} When the interest rate on a credit card service is positively correlated with the pricing kernel, Q_{t+1}, and thereby negatively correlated with the rate of return on the full portfolio of monetary assets, non-monetary assets, and credit card service, the credit card service's user cost will be adjusted upwards from the certainty-equivalent user cost.

To calculate the risk adjustment of credit card service, we need to compute the covariance between the interest rate on credit card service $\tilde{e}_{l,t+1}$ and the pricing kernel Q_{t+1}, which is unobservable. Consumption-based asset pricing models allow us to relate Q_{t+1} to consumption growth through a specific intertemporally separable utility function. But the empirical results from Barnett, Liu, and Jensen (1997) show that consumption risk adjustments for the user costs of monetary assets are small in most cases under the standard utility function with moderate risk aversion. We anticipate that the same problem would arise with risk adjustment of credit card services user costs, despite their higher risk than monetary assets. The reason is that such consumption based intertemporally separable risk adjustments to common stock returns, including high risk common stocks, have been shown to be small in the literature on the "equity premium puzzle."

With more general, intertemporally nonseparable utility functions, we can use the theory in this paper to extend the existing empirical studies on the user costs of risky monetary assets and credit card services, and thereby on the induced risk adjusted Divisia monetary quantity and user cost aggregates.

3.3. Simple sum aggregation special case

In the general case, the simple summation of the monetary asset components can be written tautologically as the following identity,

$$\sum_{i=1}^{I} m_{it}^{a} = \left[\sum_{i=1}^{I} \frac{r_t^f - E_t \tilde{r}_{i,t+1}}{r_t^f} m_{it}^{a} + \omega_{it} \right] + \sum_{i=1}^{I} \frac{E_t \tilde{r}_{i,t+1}}{r_t^f} m_{it}^{a} - \omega_{it} . \tag{3.40}$$

This decomposition of the simple sum can be interpreted as follows. The term $\sum_{i=1}^{I} \frac{E_t \tilde{r}_{i,t+1}}{r_t^f} m_{it}^{a}$ is the discounted investment yield part of the simple sum, while ω_{it} is its risk adjustment, and $\sum_{i=1}^{I} \frac{r_t^f - E_t \tilde{r}_{i,t+1}}{r_t^f} m_{it}^{a} + \omega_{it}$ is the discounted monetary service flow part. Thus, the simple sum monetary aggregate represents a joint product, consisting of a discounted monetary service flow, a discounted investment yield flow, and risk adjustment. The joint product exceeds the economic stock of monetary services, which does not include the discounted investment yield.[8] If investment yield were a monetary service, then money would include the discounted present value of the investment return on the entire capital stock of the country.

When central banks first began producing monetary aggregates, all of the components over which they aggregated yielded no interest. Hence, there was perfect certainty about the rate of return on each component. In addition, because that rate of return was exactly zero for each component, the user costs were known to be the same for each component. Under those circumstances, it is well known in aggregation theory that the exact monetary quantity index becomes the simple summation. Under those assumptions, it follows from (3.40) that

$$\sum_{i=1}^{I} \frac{r_t^f - E_t \tilde{r}_{i,t+1}}{r_t^f} m_{it}^a + \omega_{it} = \sum_{i=1}^{I} m_{it}^a \,. \tag{3.41}$$

The simple sum index is a special case of the generalized Divisia index. As the financial innovation and deregulation of financial intermediation have progressed, the assumption that all monetary assets yield zero interest rates has become increasingly unrealistic. The introduction of credit card transaction services into the generalized Divisia index renders the simple sum index not only unreasonable but also impossible. Credit card volumes have high interest costs for the representative consumer and can have high risk. But even if that were not the case, monetary assets cannot be added to credit card volumes, since accounting procedures do not permit adding assets to liabilities.

The need for the generalized Divisia Index can be interpreted as follows. The consumer has to make a three-dimensional decision involving trade-off among investment return, risk adjustment, and liquidity service consumption. Monetary assets can produce

[8] Regarding measurement of the economic stock of money, defined to be the discounted service flow, see Barnett, Keating, and Chae (2006).

investment returns, contribute to risk, and provide liquidity services. Credit card transactions can produce payment deferred liquidity services involving future interest payment and can also contribute to risk. Our objective is to track the exact aggregator function, $M\left(\mathbf{m}_t\right)$, which measures only liquidity. To do so, we must remove the investment yield and adjust for risk.

3.4. Intertemporally separable special case

In this section, we show that the result in Barnett and Su (2016) is a special cases of our result, by demonstrating that the Barnett and Su (2016) result becomes ours under the assumption of intertemporal separability.

Rewriting equation (3.27), we have

$$\begin{aligned}\pi_{it}^{a} &= \frac{E_t(\tilde{R}_{j,t+1}) - E_t(\tilde{r}_{i,t+1}) + \operatorname{cov}_t(Q_{t+1}, \tilde{R}_{j,t+1})E_t(\tilde{r}_{i,t+1}) - \operatorname{cov}_t(Q_{t+1}, \tilde{r}_{i,t+1})E_t(\tilde{R}_{j,t+1})}{E_t(\tilde{R}_{j,t+1})} \\ &= \frac{E_t(\tilde{R}_{j,t+1}) - E_t(\tilde{r}_{i,t+1})}{E_t(\tilde{R}_{j,t+1})} + \frac{\operatorname{cov}_t(Q_{t+1}, \tilde{R}_{j,t+1})E_t(\tilde{r}_{i,t+1}) - \operatorname{cov}_t(Q_{t+1}, \tilde{r}_{i,t+1})E_t(\tilde{R}_{j,t+1})}{E_t(\tilde{R}_{j,t+1})}.\end{aligned} \tag{3.42}$$

Now define the risk neutral monetary asset user cost to be

$$\pi_{it}^{em} = \frac{E_t(\tilde{R}_{j,t+1}) - E_t(\tilde{r}_{i,t+1})}{E_t(\tilde{R}_{j,t+1})}, \tag{3.43}$$

and let the adjustment for risk aversion be

$$\eta_i = \frac{\operatorname{cov}_t(Q_{t+1}, \tilde{R}_{j,t+1})E_t(\tilde{r}_{i,t+1}) - \operatorname{cov}_t(Q_{t+1}, \tilde{r}_{i,t+1})E_t(\tilde{R}_{j,t+1})}{E_t(\tilde{R}_{j,t+1})}. \tag{3.44}$$

When consumption is intertemporal separable, the price kernel is

$$Q_{t+1} = \beta \frac{\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}}{\partial U(\mathbf{m}_t, x_t) / \partial x_t} \ ,$$

which we can substitute back into the equation (3.44) to acquire the following

$$\eta_i = \frac{\operatorname{cov}_t\left(\beta \frac{\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}}{\partial U(\mathbf{m}_t, x_t) / \partial x_t}, \tilde{R}_{j,t+1}\right)E_t(\tilde{r}_{i,t+1}) - \operatorname{cov}_t\left(\beta \frac{\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}}{\partial U(\mathbf{m}_t, x_t) / \partial x_t}, \tilde{r}_{i,t+1}\right)E_t(\tilde{R}_{j,t+1})}{E_t(\tilde{R}_{j,t+1})}. \tag{3.45}$$

Rearranging the equation, we have

$$\eta_i = \frac{\beta \operatorname{cov}_t(\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}, \tilde{R}_{j,t+1}) E_t(\tilde{r}_{i,t+1}) - \beta \operatorname{cov}_t(\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}, \tilde{r}_{i,t+1}) E_t(\tilde{R}_{j,t+1})}{\left[E_t(\tilde{R}_{j,t+1})\right] \partial U(\mathbf{m}_t, x_t) / \partial x_t}. \tag{3.46}$$

Further rearranging the equation, we have

$$\eta_i = \frac{\beta \operatorname{cov}_t(\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}, \tilde{R}_{j,t+1}) E_t(\tilde{r}_{i,t+1})}{\left[E_t(\tilde{R}_{j,t+1})\right] \partial U(\mathbf{m}_t, x_t) / \partial x_t} - \frac{\beta \operatorname{cov}_t(\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}, \tilde{r}_{i,t+1})}{\partial U(\mathbf{m}_t, x_t) / \partial x_t}. \tag{3.47}$$

But recalling that,

$$\pi_{it}^{em} = \frac{E_t(\tilde{R}_{j,t+1}) - E_t(\tilde{r}_{i,t+1})}{E_t(\tilde{R}_{j,t+1})}, \tag{3.48}$$

we have

$$\frac{E_t(\tilde{r}_{i,t+1})}{E_t(\tilde{R}_{j,t+1})} = 1 - \pi_{it}^{em}. \tag{3.49}$$

Substituting equation (3.49) back into equation (3.47), we have

$$\eta_i = \frac{\beta(1-\pi_{it}^{em}) \operatorname{cov}_t(\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}, \tilde{R}_{j,t+1})}{\partial U(\mathbf{m}_t, x_t) / \partial x_t} - \frac{\beta \operatorname{cov}_t(\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}, \tilde{r}_{i,t+1})}{\partial U(\mathbf{m}_t, x_t) / \partial x_t}, \tag{3.50}$$

while substituting equation (3.50) back into equation (3.42), we have

$$\begin{aligned} \pi_{it}^{a} &= \pi_{it}^{em} \\ &+ \frac{\beta(1-\pi_{i,t}^{e}) \operatorname{cov}_t(\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}, \tilde{R}_{j,t+1})}{\partial U(\mathbf{m}_t, x_t) / \partial x_t} - \frac{\beta \operatorname{cov}_t(\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}, \tilde{r}_{i,t+1})}{\partial U(\mathbf{m}_t, x_t) / \partial x_t}, \end{aligned} \tag{3.51}$$

where the first term on the right side of the equation represents the risk neutral user cost of the monetary asset, and the rest is the risk adjustment.

Similarly, we can derive the intertemporally separable credit card user cost from the general form of credit card user cost, which is

$$\pi_{lt}^c = \frac{1-\mathrm{cov}_t(Q_{t+1}, \tilde{R}_{j,t+1})}{E_t(\tilde{R}_{j,t+1})} E_t(\tilde{e}_{l,t+1}) + \mathrm{cov}_t(Q_{t+1}, \tilde{e}_{l,t+1}) - 1$$

$$= \frac{E_t(\tilde{e}_{l,t+1}) - \mathrm{cov}_t(Q_{t+1}, \tilde{R}_{j,t+1}) E_t(\tilde{e}_{l,t+1}) + \mathrm{cov}_t(Q_{t+1}, \tilde{e}_{l,t+1}) E_t(\tilde{R}_{j,t+1}) - E_t(\tilde{R}_{j,t+1})}{E_t(\tilde{R}_{j,t+1})} \quad (3.52)$$

$$= \frac{E_t(\tilde{e}_{l,t+1}) - E_t(\tilde{R}_{j,t+1})}{E_t(\tilde{R}_{j,t+1})} + \frac{-\mathrm{cov}_t(Q_{t+1}, \tilde{R}_{j,t+1}) E_t(\tilde{e}_{l,t+1}) + \mathrm{cov}_t(Q_{t+1}, \tilde{e}_{l,t+1}) E_t(\tilde{R}_{j,t+1})}{E_t(\tilde{R}_{j,t+1})}.$$

When consumption is intertemporally separable, we can substitute the price kernel

$$Q_{t+1} = \beta \frac{\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}}{\partial U(\mathbf{m}_t, x_t) / \partial x_t},$$

back into equation (3.52) to acquire

$$\pi_{lt}^c = \frac{E_t(\tilde{e}_{l,t+1}) - E_t(\tilde{R}_{j,t+1})}{E_t(\tilde{R}_{j,t+1})}$$
$$+ \frac{-\mathrm{cov}_t(\beta \frac{\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}}{\partial U(\mathbf{m}_t, x_t) / \partial x_t}, \tilde{R}_{j,t+1}) E_t(\tilde{e}_{l,t+1}) + \mathrm{cov}_t(\beta \frac{\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}}{\partial U(\mathbf{m}_t, x_t) / \partial x_t}, \tilde{e}_{l,t+1}) E_t(\tilde{R}_{j,t+1})}{E_t(\tilde{R}_{j,t+1})}$$
$$= \frac{E_t(\tilde{e}_{l,t+1}) - E_t(\tilde{R}_{j,t+1})}{E_t(\tilde{R}_{j,t+1})}$$
$$+ \frac{-\beta\, \mathrm{cov}_t(\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}, \tilde{R}_{j,t+1}) E_t(\tilde{e}_{l,t+1}) + \beta\, \mathrm{cov}_t(\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}, \tilde{e}_{l,t+1}) E_t(\tilde{R}_{j,t+1})}{\partial U(\mathbf{m}_t, x_t) / \partial x_t E_t(\tilde{R}_{j,t+1})}.$$

(3.53)

Letting

$$\eta_l = \frac{-\beta\, \mathrm{cov}_t(\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}, \tilde{R}_{j,t+1}) E_t(\tilde{e}_{l,t+1}) + \beta\, \mathrm{cov}_t(\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}, \tilde{e}_{l,t+1}) E_t(\tilde{R}_{j,t+1})}{\partial U(\mathbf{m}_t, x_t) / \partial x_t E_t(\tilde{R}_{j,t+1})},$$

(3.54)

we have

$$\pi_{lt}^c = \frac{E_t(\tilde{e}_{l,t+1}) - E_t(\tilde{R}_{j,t+1})}{E_t(\tilde{R}_{j,t+1})} + \eta_l. \quad (3.55)$$

Letting the risk neutral user cost of credit card volumes be

$$\pi_{lt}^{ec} = \frac{E_t(\tilde{e}_{l,t+1}) - E_t(\tilde{R}_{j,t+1})}{E_t(\tilde{R}_{j,t+1})},$$

we have

$$\frac{E_t(\tilde{e}_{l,t+1})}{E_t(\tilde{R}_{j,t+1})} = \pi_{lt}^{ec} + 1. \tag{3.56}$$

Substituting equation (3.56) into equation (3.54), we have

$$\begin{aligned}\eta_l &= (\pi_{lt}^{ec}+1)\frac{-\beta \operatorname{cov}_t(\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}, \tilde{R}_{j,t+1})}{\partial U(\mathbf{m}_t, x_t) / \partial x_t} + \frac{\beta \operatorname{cov}_t(\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}, \tilde{e}_{l,t+1})}{\partial U(\mathbf{m}_t, x_t) / \partial x_t} \\ &= \frac{\beta \operatorname{cov}_t(\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}, \tilde{e}_{l,t+1})}{\partial U(\mathbf{m}_t, x_t) / \partial x_t} - \beta(\pi_{lt}^{ec}+1)\frac{\operatorname{cov}_t(\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}, \tilde{R}_{j,t+1})}{\partial U(\mathbf{m}_t, x_t) / \partial x_t}.\end{aligned} \tag{3.57}$$

Then substituting equation (3.57) into equation (3.53), we acquire

$$\begin{aligned}\pi_{lt}^{c} &= \pi_{lt}^{ec} \\ &+ \frac{\beta \operatorname{cov}_t(\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}, \tilde{e}_{l,t+1})}{\partial U(\mathbf{m}_t, x_t) / \partial x_t} - \beta(\pi_{lt}^{ec}+1)\frac{\operatorname{cov}_t(\partial U(\mathbf{m}_{t+1}, x_{t+1}) / \partial x_{t+1}, \tilde{R}_{j,t+1})}{\partial U(\mathbf{m}_t, x_t) / \partial x_t},\end{aligned} \tag{3.58}$$

which is the same as those in Barnett and Su (2016).

3.5. Approximation to the risk adjustment

Consumption-based asset pricing models, CCAPM, require explicit assumptions about investors' utility functions. An alternative approach, CAPM, which is commonly used in finance, is to approximate Q_{t+1} by a simple function of observable macroeconomic factors that are believed to be closely related to investor's marginal utility growth. Sharpe (1964) and Lintner (1965) approximate Q_{t+1} by a linear function of the rate of return on the market portfolio. Then the rate of return on each individual asset is linked to its covariance with the market rate of return.

Barnett and Wu (2005) showed that there exists a similar CAPM relationship among user costs of risky monetary assets, under the assumption that Q_{t+1} is a linear function of the rate of return on a well-diversified wealth portfolio. In this paper, we accept that assumption. Specifically, define $r_{A,t+1}$ to be the share-weighted real gross rate of return on the consumer's asset portfolio, including the monetary assets, $m_{it}^a\ (i = 1,\ldots, I)$, and the non-monetary assets, $k_{jt}\left(j = 1,\ldots, J \right)$. Then the traditional CAPM approximation to Q_{t+1} mentioned above is of the form $Q_{t+1} = a_t - b_t r_{A,t+1}$, where a_t, and b_t can be time dependent. The monetary aggregate, $M\left(\mathbf{m}_t\right)$, measures the flow of monetary services rather than the stock of financial portfolio wealth.

Let A_t be the real value of the portfolio's stock, and let ϕ_{it} and φ_{jt} denote the share of m_{it}^a and k_{jt} in the portfolio's stock value, respectively, so that

$$\begin{aligned} \phi_{it} &= \frac{m_{it}^a}{\sum_{i=1}^{I} m_{it}^a + \sum_{j=1}^{J} k_{jt}} = \frac{m_{it}^a}{A_t}, \\ \varphi_{jt} &= \frac{k_{jt}}{\sum_{i=1}^{I} m_{it}^a + \sum_{j=1}^{J} k_{jt}} = \frac{k_{jt}}{A_t}. \end{aligned} \tag{3.59}$$

Then, by construction, $r_{A,t+1} = \sum_{i=1}^{I} \phi_{it} \tilde{r}_{i,t+1} + \sum_{j=1}^{J} \varphi_{jt} \tilde{R}_{j,t+1}$, where

$$\sum_{i=1}^{I} \phi_{it} + \sum_{j=1}^{J} \varphi_{jt} = 1.$$

Multiplying (3.15) by φ_{jt} and (3.16) by ϕ_{it}, we acquire

$$\begin{aligned} &0 = \varphi_{jt} - E_t(\tilde{R}_{j,t+1} Q_{t+1} \varphi_{jt}),\ j = (1,\ldots, J), \\ &\phi_{it} \pi_{it}^a = \phi_{it} - E_t(\tilde{r}_{i,t+1} Q_{t+1} \phi_{it}),\ i = (1,\ldots, I). \end{aligned} \tag{3.60}$$

Summing the above equations over *i* and *j*, then adding the two summed equations together, and using the definition of $r_{A,t+1}$, we get

$$\sum_{i=1}^{I} \phi_{it} \pi_{it}^{a} = 1 - E_t (r_{A,t+1} Q_{t+1}) . \tag{3.61}$$

Let $\Pi_{At} = \sum_{i=1}^{I} \phi_{it} \pi_{it}^{a} + \sum_{j=1}^{J} \varphi_{jt} \pi_{jt}^{k}$, where π_{it}^{k} is the user cost of non-monetary asset *j*. We define Π_{At} to be the user cost of the consumer's asset wealth portfolio. But the user cost, π_{jt}^{k}, of every non-monetary asset is 0, so equivalently $\Pi_{At} = \sum_{i=1}^{I} \phi_{it} \pi_{it}^{a}$. The reason is that consumers do not pay a price, in terms of foregone interest, for the monetary services of non-monetary assets, since they provide no monetary services and provide only their investment rate of return. Barnett and Wu (2005) showed that our definition of Π_{At} is consistent with Fisher's factor reversal test, in the following sense:

$$\Pi_{At} A_t = \sum_{i=1}^{I} \pi_{it}^{a} m_{it}^{a} + \sum_{j=1}^{J} \pi_{jt}^{k} k_{jt} . \tag{3.62}$$

Since we know that $\pi_{it}^{k} = 0$ for all *j*, portfolio factor reversal equivalently can be written as

$$\Pi_{At} A_t = \sum_{i=1}^{I} \pi_{it}^{a} m_{it}^{a} . \tag{3.63}$$

Recall that $M(\mathbf{m}_t) = M(\mathbf{m}_t^a, \mathbf{m}_t^c)$. Suppose that *M* is weakly separable in monetary assets, so there exists *f* such that $M(\mathbf{m}_t^a, \mathbf{m}_t^c) = M(f(\mathbf{m}_t^a), \mathbf{m}_t^c)$. Then $f(\mathbf{m}_t^a)$ is the aggregation theoretic quantity aggregate over the services of monetary assets alone. By factor reversal, there exists a user cost aggregate, Π_{at}, dual to $f(\mathbf{m}_t^a)$, such that $\Pi_{ct} f(\mathbf{m}_t^a) = \sum_{i=1}^{I} \pi_{i,t}^{a} m_{it}^{a}$. It follows from (3.68) that $\Pi_{At} A_t = \Pi_{ct} f(\mathbf{m}_t^a)$, and hence $\Pi_{At} = \Pi_{ct} \left[f(\mathbf{m}_t^a) / A_t \right]$.

Suppose one of the non-monetary assets is risk-free with gross real interest rate r_t^f .

By substituting $\pi_{it}^{ae} = \frac{r_t^f - E_t \tilde{r}_{i,t+1}}{r_t^f}$ for π_{it}^{a} into the definition of $\mathit{\Pi}_{At}$, using the definition of $r_{A,t+1}$, and letting $r_t^f = E\tilde{r}_{i,t+1}$ for all j , it follows that the certainty equivalent user cost of the asset wealth portfolio is $\Pi_{A,t}^{e} = \frac{r_t^f - E_t r_{A,t+1}}{r_t^f}$. We now can prove the following proposition.

Proposition 4. *If one of the non-monetary assets is (locally) risk-free with gross real interest rate* r_t^f *, and if* $Q_{t+1} = a_t - b_t r_{A,t+1}$*, where* $r_{A,t+1}$ *is the gross real rate of return on the consumer's wealth portfolio, then the user cost of any monetary asset i or credit card volume* l *is given by*

$$\pi_{it}^{a} - \pi_{it}^{ae} = \beta_{it}(\mathit{\Pi}_{At} - \mathit{\Pi}_{At}^{e}), \tag{3.64}$$

$$\pi_{lt}^{c} - \pi_{lt}^{ce} = \beta_{lt}(\mathit{\Pi}_{At} - \mathit{\Pi}_{At}^{e}), \tag{3.65}$$

where π_{it}^{a} *,* π_{lt}^{c}*, and* $\mathit{\Pi}_{At}$ *are the user costs of monetary asset i, credit card transaction volume l, and asset wealth portfolio, respectively; and*

$$\pi_{it}^{ae} = \frac{r_t^f - E_t(\tilde{r}_{i,t+1})}{r_t^f} \; , \quad \pi_{lt}^{ce} = \frac{E_t(\tilde{e}_{l,t+1}) - r_t^f}{r_t^f} \; \textit{and} \; \mathit{\Pi}_{At}^{e} = \frac{r_t^f - E_t(r_{A,t+1})}{r_t^f}$$

are the certainty-equivalent user costs of monetary asset i, credit card transaction volume l, and asset wealth portfolio, respectively. The “betas" of monetary asset i and credit card transaction volume l in equation (3.64) *and* (3.65) *are given by*

$$\begin{aligned} \beta_{it} &= \frac{\operatorname{cov}_t(r_{A,t+1}, \tilde{r}_{i,t+1})}{\operatorname{var}_t(r_{A,t+1})}, \\ \beta_{lt} &= -\frac{\operatorname{cov}_t(r_{A,t+1}\tilde{e}_{l,t+1})}{\operatorname{var}_t(r_{A,t+1})}. \end{aligned} \tag{3.66}$$

Proof. Equation (3.64) follows from Barnett and Wu (2005). The rest of this proof is about (3.65).

From equation (3.61) and the definition of Π_{At}, we have for the wealth portfolio that

$$\Pi_{At} = 1 - E_t Q_{t+1} E_t r_{A,t+1} - \mathrm{cov}_t(Q_{t+1}, r_{A,t+1}) . \tag{3.67}$$

Given the risk-free rate r_t^f , we have $E_t Q_{t+1} = \dfrac{1}{r_t^f}$. Hence

$$\Pi_{At} = 1 - \frac{E_t r_{A,t+1}}{r_t^r} - \mathrm{cov}_t(Q_{t+1}, r_{A,t+1}) = \Pi_{At}^e - \mathrm{cov}_t(Q_{t+1}, r_{A,t+1}) . \tag{3.68}$$

Using the assumption that $Q_{t+1} = a_t - b_t r_{A,t+1}$, so that $\mathrm{cov}_t(Q_{t+1}, r_{A,t+1}) = -b_t \,\mathrm{var}_t(r_{A,t+1})$, it follows that

$$\Pi_{At} = \Pi_{At}^e + b_t \,\mathrm{var}_t(r_{A,t+1}) . \tag{3.69}$$

For the monetary asset case, Barnett and Wu (2005) have shown that

$$\pi_{it}^a = \pi_{it}^{ae} - \mathrm{cov}_t(Q_{t+1}, \tilde{r}_{i,t+1}) = \pi_{it}^{ae} + b_t \,\mathrm{cov}_t(r_{A,t+1}, \tilde{r}_{i,t+1}) . \tag{3.70}$$

Similarly, for credit card volume *l*, we have $\pi_{lt}^c = \dfrac{E_t \tilde{e}_{l,t+1} - r_t^f}{r_t^f} - \omega_{lt} = \pi_{lt}^{ce} - \omega_{lt}$ where $\omega_{lt} = -\mathrm{cov}_t(Q_{t+1}, \tilde{e}_{l,t+1})$ and $Q_{t+1} = a_t - b_t r_{A,t+1}$, so that

$$\pi_{lt}^c = \pi_{lt}^{ce} - \mathrm{cov}_t(Q_{t+1}, \tilde{e}_{l,t+1}) = \pi_{lt}^{ce} - b_t \,\mathrm{cov}_t(r_{A,t+1}, \tilde{e}_{l,t+1}) . \tag{3.71}$$

From equations (3.69) and (3.70), Barnett and Wu (2005) conclude that

$$\frac{\pi_{it}^a - \pi_{it}^{ae}}{\Pi_{At} - \Pi_{At}^e} = \frac{\mathrm{cov}_t(r_{A,t+1}, \tilde{r}_{i,t+1})}{\mathrm{var}_t(r_{A,t+1})} . \tag{3.72}$$

Similarly from equation (3.69) and equation (3.71), we can conclude that

$$\frac{\pi_{lt}^c - \pi_{lt}^{ce}}{\Pi_{At} - \Pi_{At}^e} = -\frac{\mathrm{cov}_t(r_{A,t+1}, \tilde{e}_{l,t+1})}{\mathrm{var}_t(r_{A,t+1})} . \qquad \blacksquare \tag{3.73}$$

In the approximation, $Q_{t+1} = a_t - b_t r_{A,t+1}$, to the theoretical pricing kernel, Q_{t+1}, the reason for the minus sign is similar to the reason for the minus signs before the own rates of return within monetary asset user costs: the intent in the finance literature is to measure a “price”, not a rate of return. With the minus sign in front of b_t, and with b_t positive, we can interpret b_t in equations (3.69), (3.70) and (3.71) as a “price” of risk. Then b_t measures the amount of risk premium added to the left-hand side per unit of covariance in equation (3.70) and (3.71), or variance in equation (3.69). Also recall that the pricing kernel itself, as a subjectively discounted marginal rate of substitution, should be positive. The signs of a_t and b_t must both be positive, and a_t must be sufficiently large, so that the pricing kernel is positive for all observed values of $r_{A,t+1}$.

Proposition 4 is very similar to the standard CAPM formula for asset returns. In CAPM theory, the expected excess rate of return on an individual asset is determined by its covariance with the excess rate of return on market portfolio, $r_{M,t+1}$, in accordance with

$$E_t\tilde{r}_{i,t+1} - r_t^f = \beta_{it}(E_t r_{M,t+1} - r_t^f), \tag{3.74}$$

where $\beta_{it} = \mathrm{cov}_t(\tilde{r}_{i,t+1} - r_t^f, r_{M,t+1} - r_t^f) / \mathrm{var}_t(r_{M,t+1} - r_t^f)$.

The result from Proposition 4 implies that credit card transaction volume l’s risk premium depends upon its market portfolio risk exposure, which is measured by the beta of this exposure. The larger the beta, through risk exposure to the wealth portfolio, the larger the risk adjustment. Credit card user costs will be adjusted upwards for those credit card transactions whose rates of return are positively correlated with the interest rate on the market portfolio, and conversely for those credit card transactions whose rates of return are negatively correlated with the asset portfolio.

While CCAPM adjusts for risk relative to its correlation with only current period consumption of goods, our CAPM result adjusts for risk relative its correlation with asset portfolio wealth value. Compared to our CAPM adjustment, the CCAPM adjustment is “myopic.”

4. Empirical Study

We are currently working on implementing this paper's theory for a future empirical paper. The data source for credit card services are documented in Barnett and Su (2017). Transaction volumes of credit card services are from four sources: Visa, MasterCard, American Express, and Discover. The credit card interest rates imputed to the representative consumer are based on the Federal Reserve Board's data on all commercial bank credit card accounts. All the other quantities and interest rates, including the benchmark rate on monetary assets, are from the Center for Financial Stability's monthly releases on current Divisia monetary aggregates.

An extension of the current paper could be to introduce heterogeneous agents. This extension would disaggregate the consumers who fully repay their credit card transaction volumes each period from those consumers with rotating balances.

5. Conclusion

Simple sum monetary aggregates treat monetary assets as perfect substitutes. That assumption has not been justifiable since monetary assets began paying interest. Barnett (1978, 1980) showed that the Divisia monetary quantity index, with user cost prices, is directly derivable from aggregation theory in the absence of uncertainty. Barnett, Liu, and Jensen. (1997) extended to include risk under intertemporal separability in accordance with CCAPM conventions. Barnett and Su (2016) first included credit cards transaction services into monetary aggregates. They further extend it to the case of uncertain returns and risk aversion using CCAPM assumptions. Despite the fact that credit card interest rates are high and volatile, the CCAPM risk adjustment could be small, for the same reason causing the equity premium puzzle in the asset pricing literature. Extension to include intertemporal non-separability could resolve this problem. Barnett and Wu (2004) developed the user cost of monetary assets under intertemporal non-separability, but without inclusion of credit card transactions services. This paper extends to the inclusion of credit card transactions services under intertemporally non-separable utility and risk aversion.

The risk adjustment of monetary asset user costs to their certainty equivalent user costs can be measured by the adjustment's beta, which depends on the covariance between the consumer's wealth portfolio and the rate of return on the monetary asset. Similarly, for any credit card service, the risk adjustment of its user cost to its certainty equivalent user cost can be measured by the adjustment's beta, which depends on the covariance between the consumer's wealth portfolio and the interest rate on credit card transactions. This is analogous to the standard Capital Asset Pricing Model (CAPM) and is more likely to capture the effects of risk on consumer behaviour than the covariance only with current period consumption of goods, as in CCAPM.

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Appendix

In this appendix, we derive the Euler equations (2.6) and (2.7) from the decision problem consisting of maximizing equations (2.2) subject to equations (2.3) and (2.4). Given initial net wealth, W_t , the consumer seeks to maximize his expected lifetime utility function. The expectations operator, E_t , is designated to be expectations conditional upon all information available at current period t.

The decision is to maximize $V = V(\mathbf{m}_0, x_0, \mathbf{m}_1, x_1, \mathbf{m}_2, x_2, \ldots\ldots)$ defined by

$$V = E_t \sum_{s=0}^{\infty} \beta^s U(\mathbf{m}_{t+s}, x_{t+s}, x_{t+s-1}, \ldots, x_{t+s-n}), \tag{A.1}$$

subject to the budget constraints,

$$W_t + p_t^* \mathbf{1}'\mathbf{m}_t^c + p_t^* \mathbf{1}'\mathbf{z}_t = p_t^* x_t + p_t^* \mathbf{1}'\mathbf{k}_t + p_t^* \mathbf{1}'\mathbf{m}_t^a \tag{A.2}$$

and

$$W_{t+1} = (\mathbf{1} + \mathbf{r}_{t+1})' p_t^* \mathbf{m}_t^a - (\mathbf{1} + \mathbf{e}_{t+1})' p_t^* \mathbf{m}_t^c - (\mathbf{1} + \mathbf{e}_{t+1}^z)' p_t^* \mathbf{z}_t + (\mathbf{1} + \mathbf{R}_{t+1})' p_t^* \mathbf{k}_t + y_{t+1}. \tag{A.3}$$

The intertemporal utility function, $V(\mathbf{m}_0, x_0, \mathbf{m}_1, x_1, \mathbf{m}_2, x_2, \ldots\ldots)$, is assumed to be increasing and strictly concave in all of its arguments. The decision also is subject to the transversality condition,

$$\lim_{s \to \infty} \beta^s p_t^* \mathbf{k}_{t+s} = \mathbf{0}. \tag{A.4}$$

The consumer's subjective rate of time preference, β , is assumed to be constant. The transversality condition rules out perpetual borrowing at the nonmonetary pure investment rates, $\mathbf{R}_t$.

Define the current period value function of the consumer's optimization problem as follows:

$$H_t = H(W_t, x_{t-1}, \ldots, x_{t-n}). \tag{A.5}$$

Given the price, wage, and interest rate processes, the consumer selects the deterministic point and stochastic process $(x_t, \mathbf{m}_t, \mathbf{k}_t)$ to maximize the expected value of U over the planning horizon, subject to the sequence of constraints. Assuming the solution to the decision problem exists, we then have the Bellman equation

$$H_t = \sup_{(x_t, \mathbf{m}_t, \mathbf{k}_t)} E_t(U(\mathbf{m}_t, x_t, x_{t-1}, ..., x_{t-n}) + \beta H_{t+1}). \tag{A.6}$$

With any given $\mathbf{m}_t$ and $\mathbf{k}_t$, the value of x_t is determined from (A.2). At each period t, the consumer's decision problem is therefore to choose $\mathbf{m}_t$ and $\mathbf{k}_t$ to maximize utility subject to the constraints. Therefore, we can reformulate the optimization problem as the following. From equation (A.2) , we have

$$x_t = \frac{W_t}{p_t^*} - \mathbf{1}'\mathbf{m}_t^a - \mathbf{1}'\mathbf{k}_t + \mathbf{1}'\mathbf{m}_t^c + \mathbf{1}'\mathbf{z}_t. \tag{A.7}$$

Moving it forward one period, we have

$$x_{t+1} = \frac{W_{t+1}}{p_{t+1}^*} - \mathbf{1}'\mathbf{m}_{t+1}^a - \mathbf{1}'\mathbf{k}_{t+1} + \mathbf{1}'\mathbf{m}_{t+1}^c + \mathbf{1}'\mathbf{z}_{t+1}. \tag{A.8}$$

Substituting equation (A.3) into (A.8), we have

$$\begin{aligned} x_{t+1} &= \left(\frac{p_t^*}{p_{t+1}^*}(\mathbf{1} + \mathbf{r}_{t+1})'\mathbf{m}_t^a - \mathbf{1}'\mathbf{m}_{t+1}^a\right) + \left(\mathbf{1}'\mathbf{m}_{t+1}^c - \frac{p_t^*}{p_{t+1}^*}(\mathbf{1} + \mathbf{e}_{t+1})'\mathbf{m}_t^c\right) \\ &+ \left(\mathbf{1}'\mathbf{z}_{t+1} - \frac{p_t^*}{p_{t+1}^*}(\mathbf{1} + \mathbf{e}_{t+1}^z)'\mathbf{z}_t\right) + \left(\frac{p_t^*}{p_{t+1}^*}(\mathbf{1} + \mathbf{R}_{t+1})'\mathbf{k}_t - \mathbf{1}'\mathbf{k}_{t+1}\right) + \frac{1}{p_{t+1}^*} y_{t+1}. \end{aligned} \tag{A.9}$$

On the right hand side of the equation, the first term is the released or absorbed funds from rolling over the portfolio of monetary assets. The second term is the net change in credit card debt from purchases of consumption goods. The third term is the changes in credit card rotating balances. The fourth term is the released or absorbed funds from rolling over non-monetary assets. The last term is the income from other sources, such as labor income.

Since rotating balances, $\mathbf{z}_t$, reflect the credit card transactions during previous periods, their transactions services were valued in the utility function of previous periods and should not again be in the utility function of the current period. In addition, y_{t+1} consists of income sources exogenous to the decision, such as wage income. Thus equation (A.9) is of the following form:

$$x_{t+1} = x_{_{t+1}}(\mathbf{m}_t, \mathbf{m}_{t+1}, \mathbf{k}_t, \mathbf{k}_{t+1}, \mathbf{z}_t, y_{t+1}, \mathbf{r}_{t+1}, \mathbf{e}_{t+1}, \mathbf{e}^z_{t+1}, p^*_t, p^*_{t+1}). \tag{A.10}$$

The consumption next period is a function of monetary assets, credit card services, and non-monetary assets along with variables exogenous to the current period, including $\mathbf{z}_t$ and y_{t+1}. Taking total differentials on both sides of equation (A.7) conditionally upon fixed values of the exogenous variables, we have

$$dx_t = -\mathbf{1}' d\mathbf{m}^a_t + \mathbf{1}' d\mathbf{m}^c_t - d\mathbf{1}'\mathbf{k}_t\ . \tag{A.11}$$

Taking total differentials on both sides of equation (A.9) conditionally upon fixed values of the exogenous variables, we have

$$dx_{t+1} = \frac{p^*_t}{p^*_{t+1}}(\mathbf{1}+\mathbf{r}_{_{t+1}})' d\mathbf{m}^a_t - \frac{p^*_t}{p^*_{t+1}}(\mathbf{1}+\mathbf{e}_{_{t+1}})' d\mathbf{m}^c_t + \frac{p^*_t}{p^*_{t+1}}(\mathbf{1}+\mathbf{R}_{_{t+1}})' d\mathbf{k}_t - \mathbf{1}' d\mathbf{m}^a_{t+1} + \mathbf{1}' d\mathbf{m}^c_{t+1} - \mathbf{1}' d\mathbf{k}_{t+1}. \tag{A.12}$$

Differentiating equation (A.6) with respect to x_t, we have

$$\begin{aligned}
\frac{\partial H_t}{\partial x_t} &= E_t\left(\frac{\partial U_t}{\partial x_t} + \beta\frac{\partial H_{t+1}}{\partial x_t}\right) \\
&= E_t\left(\frac{\partial U_t}{\partial x_t} + \beta\left(\frac{\partial U_{t+1}}{\partial x_t} + \beta\frac{\partial H_{t+2}}{\partial x_t}\right)\right) \\
&= E_t\left(\frac{\partial U_t}{\partial x_t} + \beta\frac{\partial U_{t+1}}{\partial x_t} + \beta^2\left(\frac{\partial U_{t+2}}{\partial x_t} + \beta\frac{\partial H_{t+3}}{\partial x_t}\right)\right) \\
&= E_t\left(\frac{\partial U_t}{\partial x_t} + \beta\frac{\partial U_{t+1}}{\partial x_t} + \beta^2\frac{\partial U_{t+2}}{\partial x_t} + \ldots + \beta^s\left(\frac{\partial U_{t+s}}{\partial x_t} + \beta\frac{\partial H_{t+1+s}}{\partial x_t}\right)\right) \\
&= E_t\sum_{s=0}^{n}\beta^s\frac{\partial U_{t+s}}{\partial x_t}.
\end{aligned} \tag{A.13}$$

Moving equation (A.6) forward one period and taking the derivative of H_{t+1} with respect to x_t, we get

$$\begin{aligned}
\frac{\partial H_{t+1}}{\partial x_t} &= E_t\left(\frac{\partial U_{t+1}}{\partial x_t}+\beta\frac{\partial H_{t+2}}{\partial x_t}\right)\\
&= E_t\left(\frac{\partial U_{t+1}}{\partial x_t}+\beta\left(\frac{\partial U_{t+2}}{\partial x_t}+\beta\frac{\partial H_{t+3}}{\partial x_t}\right)\right)\\
&= E_t\left(\frac{\partial U_{t+1}}{\partial x_t}+\beta\frac{\partial U_{t+2}}{\partial x_t}+\beta^2\left(\frac{\partial U_{t+3}}{\partial x_t}+\beta\frac{\partial H_{t+4}}{\partial x_t}\right)\right) \qquad \text{(A.14)}\\
&= E_t\left(\frac{\partial U_{t+1}}{\partial x_t}+\beta\frac{\partial U_{t+2}}{\partial x_t}+\beta^2\frac{\partial U_{t+3}}{\partial x_t}+\ldots+\beta^s\left(\frac{\partial U_{t+1+s}}{\partial x_t}+\beta\frac{\partial H_{t+2+s}}{\partial x_t}\right)\right)\\
&= E_t\sum_{s=0}^{n}\beta^s\frac{\partial U_{t+1+s}}{\partial x_t}.
\end{aligned}$$

Define the expected present value of the marginal utility of consumption by

$$\lambda_t = E_t\sum_{s=0}^{n}\beta^s\frac{\partial U_{t+s}}{\partial x_t}. \qquad \text{(A.15)}$$

Substituting equation (A.15) into equation (A.13), we have

$$\frac{\partial H_t}{\partial x_t} = E_t\sum_{s=0}^{n}\beta^s\frac{\partial U_{t+s}}{\partial x_t} = \lambda_t. \qquad \text{(A.16)}$$

Moving (A.16) forward one period, we have

$$\frac{\partial H_{t+1}}{\partial x_{t+1}} = \lambda_{t+1}. \qquad \text{(A.17)}$$

Differentiating equation (A.6) with respect to $\mathbf{m}_t^a$, we have the first order conditions for $\mathbf{m}_t^a$ as follows.

$$E_t\left(\frac{\partial U_t}{\partial \mathbf{m}_t^a}+\frac{\partial U_t}{\partial x_t}\frac{\partial x_t}{\partial \mathbf{m}_t^a}+\beta\frac{\partial H_{t+1}}{\partial x_t}\frac{\partial x_t}{\partial \mathbf{m}_t^a}+\beta\frac{\partial H_{t+1}}{\partial x_{t+1}}\frac{\partial x_{t+1}}{\partial \mathbf{m}_t^a}\right)=\mathbf{0}. \qquad \text{(A.18)}$$

Similarly differentiating equation (A.6) with respect to $\mathbf{m}_t^c$, we have the first order conditions for $\mathbf{m}_t^c$ as follows.

$$E_t\left(\frac{\partial U_t}{\partial \mathbf{m}_t^c}+\frac{\partial U_t}{\partial x_t}\frac{\partial x_t}{\partial \mathbf{m}_t^c}+\beta\frac{\partial H_{t+1}}{\partial x_t}\frac{\partial x_t}{\partial \mathbf{m}_t^c}+\beta\frac{\partial H_{t+1}}{\partial x_{t+1}}\frac{\partial x_{t+1}}{\partial \mathbf{m}_t^c}\right)=\mathbf{0}. \qquad \text{(A.19)}$$

Differentiating equation (A.6) with respect to $\mathbf{k}_t$, we have the first order conditions for $\mathbf{k}_t$ as follows.

$$E_t\left(\frac{\partial U_t}{\partial x_t}\frac{\partial x_t}{\partial \mathbf{k}_t}+\beta\frac{\partial H_{t+1}}{\partial x_t}\frac{\partial x_t}{\partial \mathbf{k}_t}+\beta\frac{\partial H_{t+1}}{\partial x_{t+1}}\frac{\partial x_{t+1}}{\partial \mathbf{k}_t}\right)=\mathbf{0}. \tag{A.20}$$

From equation (A.11), we have

$$\frac{\partial x_t}{\partial \mathbf{m}_t^a}=-\mathbf{1}, \tag{A.21}$$

while from equation (A.12), we have

$$\frac{\partial x_{t+1}}{\partial \mathbf{m}_t^a}=\frac{p_t^*}{p_{t+1}^*}(\mathbf{1}+\mathbf{r}_{t+1}). \tag{A.22}$$

Substituting (A.14), (A.17), (A.21), and (A.22) into equation (A.18), we have

$$E_t\left(\frac{\partial U_t}{\partial \mathbf{m}_t^a}-\frac{\partial U_t}{\partial x_t}\mathbf{1}-\beta E_t\sum_{s=0}^{n}\beta^s\frac{\partial U_{t+1+s}}{\partial x_t}\mathbf{1}+\beta\lambda_{t+1}\frac{p_t^*}{p_{t+1}^*}(\mathbf{1}+\mathbf{r}_{t+1})\right)=\mathbf{0}. \tag{A.23}$$

Rearranging equation (A.23), we have

$$E_t\left(\frac{\partial U_t}{\partial \mathbf{m}_t^a}+\beta\lambda_{t+1}\frac{p_t^*}{p_{t+1}^*}(\mathbf{1}+\mathbf{r}_{t+1})-(\frac{\partial U_t}{\partial x_t}+\beta E_t\sum_{s=0}^{n}\beta^s\frac{\partial U_{t+1+s}}{\partial x_t})\mathbf{1}\right)=\mathbf{0}. \tag{A.24}$$

The last term in equation (A.24) could be simplified as the follows.

$$\begin{aligned}
&\frac{\partial U_t}{\partial x_t}+\beta E_t\sum_{s=0}^{n}\beta^s\frac{\partial U_{t+1+s}}{\partial x_t}\\
&=\frac{\partial U_t}{\partial x_t}+\beta E_t(\frac{\partial U_{t+1}}{\partial x_t}+\beta^1\frac{\partial U_{t+2}}{\partial x_t}+\beta^2\frac{\partial U_{t+3}}{\partial x_t}+...)\\
&=E_t(\frac{\partial U_t}{\partial x_t}+\beta\frac{\partial U_{t+1}}{\partial x_t}+\beta^2\frac{\partial U_{t+2}}{\partial x_t}+\beta^3\frac{\partial U_{t+3}}{\partial x_t}+...)\\
&=E_t\sum_{s=0}^{n}\beta^s\frac{\partial U_{t+s}}{\partial x_t}\\
&=\lambda_t.
\end{aligned} \tag{A.25}$$

Substituting equation (A.25) into equation (A.24), we have

$$E_t\left(\frac{\partial U_t}{\partial \mathbf{m}_t^a}+\beta\lambda_{t+1}\frac{p_t^*}{p_{t+1}^*}(\mathbf{1}+\mathbf{r}_{t+1})-\lambda_t\mathbf{1}\right)=\mathbf{0}\,. \tag{A.26}$$

From equation (A.11), we have

$$\frac{\partial x_t}{\partial \mathbf{m}_t^c}=\mathbf{1}\,, \tag{A.27}$$

while from equation (A.12), we have

$$\frac{\partial x_{t+1}}{\partial \mathbf{m}_t^c}=-\frac{p_t^*}{p_{t+1}^*}(\mathbf{1}+\mathbf{e}_{t+1})\,. \tag{A.28}$$

Substituting (A.14), (A.17), (A.27), and (A.28) into equation (A.19), we have

$$E_t\left(\frac{\partial U_t}{\partial \mathbf{m}_t^c}+\frac{\partial U_t}{\partial x_t}\mathbf{1}+\beta E_t\sum_{s=0}^{n}\beta^s\frac{\partial U_{t+1+s}}{\partial x_t}\mathbf{1}-\beta\lambda_{t+1}\frac{p_t^*}{p_{t+1}^*}(\mathbf{1}+\mathbf{e}_{t+1})\right)=\mathbf{0}\,. \tag{A.29}$$

Rearranging equation (A.29), we have

$$E_t\left(\frac{\partial U_t}{\partial \mathbf{m}_t^c}-\beta\lambda_{t+1}\frac{p_t^*}{p_{t+1}^*}(\mathbf{1}+\mathbf{e}_{t+1})+(\frac{\partial U_t}{\partial x_t}+\beta E_t\sum_{s=0}^{n}\beta^s\frac{\partial U_{t+1+s}}{\partial x_t})\mathbf{1}\right)=\mathbf{0}\,. \tag{A.30}$$

Substituting equation (A.25) into equation (A.30), we have

$$E_t\left(\frac{\partial U_t}{\partial \mathbf{m}_t^c}-\beta\lambda_{t+1}\frac{p_t^*}{p_{t+1}^*}(\mathbf{1}+\mathbf{e}_{t+1})+\lambda_t\mathbf{1}\right)=\mathbf{0}\,. \tag{A.31}$$

From equation (A.11), we have

$$\frac{\partial x_t}{\partial \mathbf{k}_t}=-\mathbf{1}\,, \tag{A.32}$$

From equation (A.12), we have

$$\frac{\partial x_{t+1}}{\partial \mathbf{k}_t}=\frac{p_t^*}{p_{t+1}^*}(\mathbf{1}+\mathbf{R}_{t+1})\,. \tag{A.33}$$

Substituting (A.14), (A.17), (A.32) and (A.33) into equation (A.20), we have

$$E_t\left(-\frac{\partial U_t}{\partial x_t}\mathbf{1}-\beta E_t\sum_{s=0}^{n}\beta^s\frac{\partial U_{t+1+s}}{\partial x_t}\mathbf{1}+\beta\lambda_{t+1}\frac{p_t^*}{p_{t+1}^*}(\mathbf{1}+\mathbf{R}_{t+1})\right)=\mathbf{0}\,. \tag{A.34}$$

Rearranging equation (A.34), we have

$$E_t\left(\beta\lambda_{t+1}\frac{p_t^*}{p_{t+1}^*}(\mathbf{1}+\mathbf{R}_{t+1})-(\frac{\partial U_t}{\partial x_t}+\beta E_t\sum_{s=0}^{n}\beta^s\frac{\partial U_{t+1+s}}{\partial x_t})\mathbf{1}\right)=\mathbf{0}. \tag{A.35}$$

Substituting equation (A.25) into equation (A.35), we have

$$E_t\left(\beta\lambda_{t+1}\frac{p_t^*}{p_{t+1}^*}(\mathbf{1}+\mathbf{R}_{t+1})-\lambda_t\mathbf{1}\right)=\mathbf{0}. \tag{A.36}$$

Combining equations (A.26), (A.31), (A.36), the first order conditions can be written as

$$\begin{pmatrix} E_t\left(\dfrac{\partial U_t}{\partial \mathbf{m}_t^a}+\beta\lambda_{t+1}\dfrac{p_t^*}{p_{t+1}^*}(\mathbf{1}+\mathbf{r}_{t+1})-\lambda_t\mathbf{1}\right)=\mathbf{0} \\ E_t\left(\dfrac{\partial U_t}{\partial \mathbf{m}_t^c}-\beta\lambda_{t+1}\dfrac{p_t^*}{p_{t+1}^*}(\mathbf{1}+\mathbf{e}_{t+1})+\lambda_t\mathbf{1}\right)=\mathbf{0} \\ E_t\left(\beta\lambda_{t+1}\dfrac{p_t^*}{p_{t+1}^*}(\mathbf{1}+\mathbf{R}_{t+1})-\lambda_t\mathbf{1}\right)=\mathbf{0} \end{pmatrix}. \tag{A.37}$$

Substituting (2.15) into (A.37), we have

$$E_t\begin{pmatrix} \dfrac{\partial U_t}{\partial \mathbf{m}_t^a}+\beta\lambda_{t+1}\tilde{\mathbf{r}}_{t+1}-\lambda_t\mathbf{1} \\ \dfrac{\partial U_t}{\partial \mathbf{m}_t^c}-\beta\lambda_{t+1}\tilde{\mathbf{e}}_{t+1}+\lambda_t\mathbf{1} \end{pmatrix}=\mathbf{0}, \tag{A.38}$$

$$E_t\left(\beta\lambda_{t+1}\tilde{\mathbf{R}}_{t+1}-\lambda_t\mathbf{1}\right)=\mathbf{0}. \qquad \blacksquare \tag{A.39}$$