REGULARITY OF THE GENERALIZED QUADRATIC PRODUCTION MODEL: A COUNTEREXAMPLE

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ABSTRACT

Recently there has been a growing tendency to impose curvature, but not monotonicity, on specifications of technology. But regularity requires satisfaction of both curvature and monotonicity conditions. Without both satisfied, the second order conditions for optimizing behavior fail and duality theory fails. When neither curvature nor monotonicity are imposed, estimated flexible specifications of technology are much more likely to violate curvature than monotonicity. Hence it has been argued that there is no need to impose or check for monotonicity, when curvature has been imposed globally. But imposition of curvature may induce violations of monotonicity that otherwise would not have occurred.

We explore the regularity properties of our earlier results with a multiproduct financial technology specified to be generalized quadratic. In our earlier work, we used the usual approach and accepted the usual view. We now find that imposition of curvature globally and monotonicity locally does not assure monotonicity within the region of the data. Our purpose is to alert researchers to the kinds of problems that we encountered and which we believe are largely being overlooked in the production modelling literature, as we had been overlooking them.

Key Words: Technology; Regularity; Curvature; Production; Flexibility

JEL Classification Codes: C51, C3, C13.

1. INTRODUCTION

1.1. The Issue

Recently there has been a growing tendency to impose curvature, but not monotonicity, on specifications of technology.[1] This practice is especially

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[1] See, e.g., Moschini (1998,1999), Hernandez (1994), Ryan and Wales (1998,1999), Diewert and Wales (1987,1995), Cooper, McLaren, and Parameswran (1994), Kohli (1992), Diewert and Ostensoe (1988), and Morey (1986).

common with the currently popular generalized quadratic model. We believe that this practice of overlooking monotonicity should not be taken lightly. Regularity requires satisfaction of both the curvature and the monotonicity conditions. Without both satisfied, the second order conditions for optimizing behavior fail, duality theory fails, and the specification should be viewed as compromised in a serious manner. The damage done to inferences when regularity conditions fail has been emphasized by Basmann, Molina, and Slottje (1983), Basmann, Diamond, Frentrup, and White (1985), Basmann, Fawson, and Shumway (1990), and Basmann, Hayes, and Slottje (1994).

An earlier practice with "flexible functional forms" was to impose neither monotonicity nor curvature, but check those conditions at each data point *ex post*. Experience in that tradition has suggested that when violations of regularity occur, they are much more likely to occur through violations of curvature conditions than through violations of monotonicity conditions. Based upon those results, the more recent approach of imposing curvature alone seems constructive and reasonable. But once curvature is imposed without the imposition of monotonicity, the earlier observation may no longer apply. Permitting a highly parameterized function to depart from the neoclassical function space is usually fit-improving, regardless of whether or not the neoclassical null would pass a hypothesis test. With curvature imposed, the only way that an estimator’s fit can be improved spuriously in that manner is through violations of monotonicity. This problem is likely to be especially common with quadratic models, which can have bliss points.

In short, violations of monotonicity that had not occurred prior to imposition of curvature might be induced by imposition of curvature. This possibility motivated us to investigate two cases in which the quadratic model had been estimated with curvature imposed globally and monotonicity imposed at a central point. In both of those cases, the authors had accepted the usual view that global monotonicity need not be checked, when global curvature and local monotonicity had been imposed, and in fact we ourselves were among the three coauthors who published the results in those two cases. But we now have gone back to check monotonicity by generating graphs of isoquants. We had not attempted to graph those isoquants, when we published the results is Barnett, Kirova, and Pasupathy (1995), and hence our more cursory checks at that time had not alerted us to the seriousness of the problem. Having now attempted to produce those isoquants and having observed the nature of the problem, we now view our results as counterexamples to the usual view that we had accepted at the time that we published our two estimated models. Because of space constraints, we report primarily on the model that produced the most disturbing regularity problems, but we do mention in this paper that we found similar, but less extreme, violations of monotonicity in the other case.

In addition, a further complication can be produced by imposing curvature. In those studies that follow this recent approach, the technology usually is specified as a composite function, including an outer function and inner aggregator subfunctions. Curvature is imposed both on the outer and the inner subfunctions globally. But if the outer function violates monotonicity in the level of the inner aggregate, then curvature of the composite function can fail. This paradoxical possibility is analogous to the fact that changing the sign of a concave function can produce a convex function. Hence the currently growing approach to imposing curvature may not only result in violations of monotonicity, but even in violations of curvature itself.

We believe that the recent literature that takes curvature seriously is to be welcomed as a constructive step in a positive direction. But the potential damage done by violations of monotonicity should not be underestimated, when curvature is imposed. We explore that overlooked issue in this paper. Based upon the results with our two counterexamples, we believe that monotonicity needs to be checked at all data points in applications of the generalized quadratic production technology as well as with all other models that do not satisfy monotonicity globally, regardless of whether or not curvature has been imposed globally. In addition, the needed checks may be more complicated than previously considered in this literature, since we are not aware of any prior publications that had considered such disturbing possibilities as curvature reversal of composite functions, nonuniqueness of isoquants, or complex valued solutions with production models.

We also believe that this phenomenon should be explored with parameters estimated from other models and with other data, including simulated data produced from technologies having various elasticity properties. With the generalized quadratic model, we believe that it also would be constructive to set its parameters to attain various elasticities at the point at which monotonicity is imposed, while curvature is imposed globally, and produce the model's regular region to see under what circumstances data reasonably can be expected to lie within its regular region. The regular region is defined to be the subset of the data space within which the technology satisfies both the neoclassical monotonicity and curvature conditions.

It is tempting to dismiss one or two counterexamples as perhaps resulting from odd data or some other coincidence that need not be taken seriously. But since users of the generalized quadratic model, and other models in which only curvature is imposed, have not been reporting the result of checks of monotonicity at their data points---or even mentioning monotonicity---, our two counterexamples comprise 100% of the available published evidence on this subject.

The burden of proof of the contrary hypothesis should be on those who are failing to provide their relevant evidence. In addition, it is important to recognize that since we imposed curvature globally on all inner and outer functions of the technology and monotonicity at a central point, our estimated model must produce elasticities that are consistent with theory at that central point. With the generalized quadratic model, the parameterization is parsimonious in the sense that flexibility at a point cannot be attained, if curvature and monotonicity are both imposed globally. By imposing curvature globally and monotonicity at a point, the remaining degrees of freedom in Barnett, Kirova, and Pasupathy's (1995) estimated models are exactly equal to the number needed to permit elasticities to be set arbitrarily at a point. No degrees of freedom remain to control global monotonicity properties. Other data exhibiting similar elasticities at the central point plus globally imposed curvature could easily produce the same phenomena on which we report in this paper. With earlier models, including the translog and generalized Leontief, the effects of these limited degrees of freedom were investigated in depth by Caves and Christensen (1980), Barnett, Lee, and Wolfe (1987), Guilkey, Lovell, and Sickles (1983), and Barnett and Lee (1985). Since the elasticities reported by Barnett, Kirova, and Pasupathy (1995) were not unusual, the disturbing results in the current paper are likely to be representative of what often would likely be revealed by similar isoquant plots from other published models estimated in this tradition.

We could have used new data for these comparisons, but then there would have been obvious concerns about whether or not we sought out adverse cases to make our point. We decided instead to investigate regularity with two models on which we ourselves had previously reported favorably. We obviously had not generated and published those results for the purpose of permitting our own published research to be discredited at a later date by us or by anyone else. We did not become aware of the nature of the problem until we recently produced graphs of the isoquants. Under these circumstances it is tempting to overlook subsequently discovered negative properties of one's own previously published research. But we felt that it was important for us to reveal the unpleasant realities of what we have discovered about our previous work. These realities provide counterexamples that we feel should be known to others who might similarly be inclined to accept globaly monotonicity without testing, after having imposed curvature globally and monotonicity locally.

1.2. The Approach

Since we condition our investigation upon estimation of the parameters, rather than upon a large range of possible settings of the parameters, we believe that estimation in a manner consistent with the state of the art is appropriate, and the state of the art currently could be viewed as generalized method of moments (GMM) estimation of Euler equations for production under risk. Barnett, Kirova, and Pasupathy (1995) apply an approach to the estimation of technology parameters when financial assets are included among outputs of the technology. Using their parametric estimates, we explore both the monotonicity and curvature properties of the resulting technology. The relevant technologies are those of financial intermediaries that produce inside money as output services. The need for GMM estimation results from the inclusion of monetary assets in the model, with interest paid at the end of the period and thereby unknown at the start of the period.

The specification of technology that was used by Barnett, Kirova, and Pasupathy (1995) was the currently popular generalized quadratic model. That model can be estimated subject to imposition of global curvature, but monotonicity can be imposed only at one point, since flexibility is damaged if global monotonicity also is imposed. They imposed curvature globally on all inner and outer subfunctions of the technology, but imposed monotonicity only at a point. Unlike prior users of this model, we do check for monotonicity at each data point, and we explore the implications of the of regularity violations for the implied properties of the model's isoquants, including induced nonuniqueness of isoquants, complex valued solutions, and other such unanticipated problems not previously reported in the production modeling literature.

2. FINANCIAL INTERMEDIARIES

One of the recent approaches to modeling financial intermediaries is to model them as profit maximizing neoclassical multiproduct firms, which produce financial services, such as demand deposits and time deposits, as outputs by employing financial and non financial factors as inputs.[2]

In our current investigation of the regularity properties of the model, we use Barnett, Kirova, and Pasupathy's (1995) model and GMM parameter estimates as an illustration of the implications of the generalized quadratic model under state-of-the-art application and estimation. Since details of the model and estimation are available in Barnett, Kirova, and Pasupathy (1995), we discuss the model and estimation only to the degree necessary to understand our current investigation of

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[2]Early work that used this approach was based on the assumption of perfect certainty. See Hancock (1985, 1987, 1991), Barnett (1987), and Barnett and Hahm (1994). This approach is based upon a tradition that has been highly developed on the demand side. See Barnett, Fisher, and Serletis (1992) and Barnett (1997). Barnett and Zhou (1994) extended the supply side approach to the case of uncertainty. Barnett, Kirova, and Pasupathy (1995) introduced capital accumulation and relaxed the assumption of "no retained earnings." They also rigorously nested exact monetary output aggregates within the transformation function of the financial intermediary and report the behavior of the resulting exact monetary aggregate.

The resulting model can be viewed as a step in the direction of exploring technological change and economies of scale and scope in financial intermediation in a manner that is invariant to central bank policy intervention, and in a manner that can produce inside money aggregates that are consistent with the theory that produced the policy invariant Euler equations.

the estimated model's regularity properties and to put our current findings about the estimated isoquants into context.

2.1. Financial Firm's Production Model

The financial firm uses real resources such as labor, capital, and other material inputs, plus a monetary input in the form of cash, in the production of the services of the produced liabilities. The output of the firm in Barnett, Kirova, and Pasupathy's (1995) application consists of demand deposits and time deposits, which are liabilities to the firm.

Let Y_t be the real balances of the asset (loan) portfolio, $y_{i,t}$ the real balances of the *i*th produced account (liability) type, C_t the real balances of cash holdings, $z_{j,t}$ the quantity of *j*th real input (including labor), and K_t the quantity of capital stock of the financial firm at time *t*. In the model, $y_{i,t}$ constitute the outputs of the financial firm, while C_t, $z_{j,t}$, and K_t are the inputs. Let R_t be the portfolio rate of return, which is unknown at the beginning of period *t*, and let $h_{i,t}$ be the holding cost per dollar of the *i*th liability. All financial transactions are contracted at the beginning of the period. Interests on the deposits are paid at the end of the period. The cost per unit of the *j*th real input, $w_{j,t}$, is incurred at the beginning of the period. Let $P_{K,t}$ be the cost of capital and P_t be the general price index, which is used to deflate nominal to real terms.

Variable profits (net of investment expenditure), π_t, at the beginning of period *t*, can be represented by

$$\begin{aligned}\pi_t &= (1+R_{t-1})Y_{t-1}P_{t-1} - Y_tP_t + C_{t-1}P_{t-1} - C_tP_t \\ &\quad + \sum_{i=1}^{I}[y_{i,t}P_t - (1+h_{i,t-1})y_{i,t-1}P_{t-1}] - \sum_{j=1}^{J} w_{j,t}z_{j,t} - P_{K,t}I_t\end{aligned} \qquad (2.1)$$

The first two terms in the above equation represent the change in variable profits from rolling over the loan portfolio during period *t*. The third and fourth terms represent the change in the nominal value of excess reserves. The fifth term represents the change in the firm's variable profits from the change in the issuance of produced financial liabilities. The sixth term constitutes payments for real inputs, and the last term is the expenditure on investments.

Portfolio investment , Y_t , is constrained by total available funds. The constraint is given by

$$Y_t P_t = \sum_{i=1}^{I} [(1-k_{i,t}) y_{i,t} P_t] - C_t P_t - \sum_{j=1}^{J} w_{j,t} z_{j,t} - P_{K,t} I_t \,, \qquad (2.2)$$

where $\mathrm{k}_{\mathrm{i,t}}$ is the required reserves ratio on the *i*th produced liability. Equation (2.2) implies that the total deposits $\sum_{i=1}^{I} y_{i,t} P_t$ are allocated to required reserves, excess reserves, payment for all real inputs used in production, investment in capital, and investment in loans.

The time to build approach is adopted to model capital dynamics. Capital accumulation based on this approach is given by:

$$K_t = I_{t-1} + (1-\delta) K_{t-1} \qquad (2.3)$$

where the depreciation rate δ is a constant and is assumed to be given. Gross investment at time *t-1*, I_{t-1}, becomes productive only in period *t*. Substituting equations (2.2) and (2.3) into equation (2.1) to eliminate investment in loans and investment in capital goods, we get the variable profits at time *t* to be

$$\begin{aligned}\pi_t = & \sum_{i=1}^{I} [(1+R_{t-1})(1-k_{i,t-1}) - (1+h_{i,t-1})] y_{i,t-1} P_{t-1} + k_{i,t} y_{i,t} P_t \\ & - R_{t-1} C_{t-1} P_{t-1} - (1+R_{t-1}) \sum_{j=1}^{J} w_{j,t-1} z_{j,t-1} \\ & + (1-\delta)(1+R_{t-1}) K_{t-1} P_{K,t-1} - (1+R_{t-1}) K_t P_{K,t-1}\end{aligned} \qquad (2.4)$$

The financial firm maximizes the expected value of the discounted intertemporal utility of its variable profits stream, subject to the firm's technological constraint. The firm's optimization problem is then given by:

$$Max \quad E_t\left[\sum_{s=t}^{\infty}\left(\frac{1}{1+\mu}\right)^{s-t} U(\pi_s)\right]$$

$$s.t. \quad \Omega(y_{1,s},\ldots,y_{I,s},C_s,z_{1,s},\ldots,z_{J,s},K_s)=0 \quad \forall s\geq t \tag{2.5}$$

where E_t is the expectation at time *t*, μ is the subjective rate of time preference, U is the utility function, π_s is the variable profit at time *s*, and Ω is the transformation function.

The transformation function, Ω, is convex in its arguments. The derivatives of Ω with respect to the inputs and outputs are respectively given by:

$$\frac{\partial\Omega}{\partial C_s}\leq 0, \quad \frac{\partial\Omega}{\partial K_s}\leq 0, \quad \frac{\partial\Omega}{\partial z_{j,s}}\leq 0 \quad \forall\ j=1,\ldots,J \tag{2.6}$$

and

$$\frac{\partial\Omega}{\partial y_{i,s}}\geq 0 \quad \forall\ i=1,\ldots,I. \tag{2.7}$$

Barnett, Kirova, and Pasupathy (1995) specify the utility function, *U*, to be in the class of functions exhibiting Hyperbolic Absolute Risk Aversion (HARA):

$$U(\pi_t)=\frac{1-\rho}{\rho}\left(\frac{h}{1-\rho}\pi_t+d\right)^{\rho} \tag{2.8}$$

where ρ, *h* and *d* are parameters to be estimated.

Using Bellman's method and the Benveniste and Scheinkman equation, Barnett, Kirova, and Pasupathy (1995) obtain the following set of Euler equations:

$$E_t\left\{\frac{\partial U}{\partial \pi_t}(\pi_t)k_{i,t}P_t\right\}$$
$$+E_t\left\{\frac{1}{1+\mu}\frac{\partial U}{\partial \pi_t}(\pi_{t+1})P_t\left[R_t\frac{\partial\Omega/\partial y_{i,t}}{\partial\Omega/\partial C_t}+\left[(1+R_t)(1-k_{i,t})-(1+h_{i,t})\right]\right]\right\}$$
$$=0 \qquad (2.9)$$
$$\forall y_{i,t},\ i=1,\ldots,I.$$

$$E_t\left\{\frac{\partial U}{\partial \pi_t}(\pi_{t+1})\left[R_tP_t\frac{\partial\Omega/\partial z_{j,t}}{\partial\Omega/\partial C_t}-(1+R_t)w_{j,t}\right]\right\}=0 \qquad (2.10)$$
$$\forall z_{j,t},\ j=1,\ldots,J.$$

$$E_t\left\{\frac{\partial U}{\partial \pi_t}(\pi_t)\left[(1+R_{t-1})P_{K,t-1}\right]\right.$$
$$\left.-\frac{1}{1+\mu}\frac{\partial U}{\partial \pi_t}(\pi_{t+1})\left[R_tP_t\frac{\partial\Omega/\partial K_t}{\partial\Omega/\partial C_t}+(1-\delta)(1+R_t)P_{K,t}\right]\right\} \qquad (2.11)$$
$$=0,$$

where

$$\frac{\partial U}{\partial \pi_t}=h\left(\frac{h}{1-\rho}\pi_t+d\right)^{\rho-1}. \qquad (2.12)$$

2.2 Output Aggregation

The financial firm's outputs consist of demand deposits and time deposits. The financial firm's outputs of demand and time deposits are important in determining the level of inside money in the economy. In this section, we find the aggregation-theoretic exact quantity output aggregate that measures the firm's produced service flow. Relative to the money markets, our aggregation is on the supply side.

Generating the exact quantity aggregate consists of first identifying the components over which aggregation is admissible and then determining the aggregator function defined over the identified components.[3] The first step determines the existence of an exact aggregate, and the second step produces that aggregate in the manner that is consistent with microeconomic theory. Let $\mathbf{y} = (y_{1t}, \ldots, y_{It})'$ be the firm's output vector, and let $\mathbf{x} = (x_{1t}, \ldots, x_{Jt})'$ be the input vector, so the transformation function can be written as $\Omega(\mathbf{y}, \mathbf{x}) = 0$ An exact supply side aggregate exists over all of the firm's outputs if and only if **y** is weakly separable from **x** within the function Ω. In accordance with the definition of weak separability, there then exist two functions H and y_0 such that

$$\Omega(\mathbf{y}, \mathbf{x}) = H(y_0(\mathbf{y}), \mathbf{x}),$$

where the output aggregator function, $y_0(\mathbf{y})$, is a convex function of **y**. The econometric estimate of the aggregator function y_0 is obtained by estimating the Euler equations using GMM.

2.3 Testing for Weak Separability

The conventional parametric approach to testing weak separability was adopted in Barnett, Kirova, and Pasupathy (1995), since weak separability is a strictly nested null hypothesis within our parametric specification of technology. To minimize the biases that can be produced from specification error, they used a flexible functional form for technology.[4] Unfortunately flexible functional forms need not satisfy the regularity conditions imposed by economic theory, including the monotonicity and curvature conditions. Hence we must consider methods for testing and imposing those conditions, at least locally, as well as methods for testing and imposing global blockwise weak separability of the technology in its

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[3]For details, refer to Barnett (1980).

[4]The form of flexibility that we use is called Diewert-flexibility or second-order flexibility. See Barnett (1983) for the definition and its connection with other definitions of parametric flexibility. The newer concept of global or Sobolev flexibility (see Barnett, Geweke, and Wolfe (1991a,b)) is beyond the scope of this paper.

outputs. For existence of aggregator functions, the weak separability must be global. Barnett, Kirova, and Pasupathy (1995) used the generalized quadratic (previously called the Generalized McFadden) functional form to specify the technology of the firm. That specification was originated by Diewert and Wales (1987), who also originated the Generalized Barnett functional form.[5]

We assume that the transformation function, Ω, is linearly homogeneous. Instead of specifying the form of the full transformation function Ω, and then imposing weak separability in **y**, Barnett, Kirova, and Pasupathy (1995) directly impose weak separability by specifying $H(y_0, \mathbf{x})$ and $y_0(\mathbf{y})$ separately[6]. The specification for Ω is then obtained by substituting $y_0(\mathbf{y})$ into $H(y_0, \mathbf{x})$. Since $y_0(\mathbf{y})$ and $H(y_0, \mathbf{x})$ are both specified to be flexible, the full technology Ω is flexible, subject to the separability restriction.

The function H is specified to be the symmetric generalized quadratic functional form

$$H(y_0, \mathbf{x}) = a_0 y_0 + \mathbf{a}'\mathbf{x} + \frac{1}{2}[y_0, \mathbf{x}']\overline{\mathbf{A}}\begin{bmatrix} y_0 \\ \mathbf{x} \end{bmatrix} / \boldsymbol{\alpha}'\mathbf{x}, \qquad (2.13)$$

where $\boldsymbol{\alpha}'\mathbf{x} \neq 0$, and a_0, $\mathbf{a}' = (a_1, \ldots, a_n)$, and $\overline{\mathbf{A}}$ are parameters to be estimated. The matrix $\overline{\mathbf{A}}$ is $(n+1)\times(n+1)$ and symmetric. The vector $\boldsymbol{\alpha}' = (\alpha_1, \ldots, \alpha_n)$ contains all fixed nonnegative constants, which are chosen by the researcher. The matrix $\overline{\mathbf{A}}$ is partitioned as follows:

$$\overline{\mathbf{A}} = \begin{bmatrix} A_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A} \end{bmatrix},$$

[5] That latter model was applied by Barnett and Hahm (1994) in the perfect certainty case, but has not yet been adapted to the case of stochastic choice.
[6] For the more general form of the model which does not include the imposition of weak separability, refer to Barnett and Zhou (1994).

where A_{11} is a scalar, $\mathbf{A_{12}}$ is a $1\times n$ row vector, $\mathbf{A_{21}}$ is an $n\times 1$ column vector, and $\mathbf{A}$ is an $n \times n$ symmetric matrix. Since $\overline{\mathbf{A}}$ is symmetric, we have $\mathbf{A_{12}} = \mathbf{A'_{21}}$.

Let $(y_0^*, \mathbf{x}^*) \neq 0$ be the chosen point about which the functional form is locally flexible. Within the class of linearly homogeneous transformation functions, the specification given above is not parsimonious, and hence we can impose further restrictions on the model without losing local flexibility. We impose the following restrictions, *which reduce the number of free parameters in our specification to the minimum required number to maintain local flexibility*.

$$\boldsymbol{\alpha}'\mathbf{x}^* = 1, \tag{2.14}$$

$$A_{11}y_0^* + \mathbf{A_{12}}\mathbf{x}^* = 0, \tag{2.15}$$

$$\mathbf{A'_{12}}\mathbf{y_0^*} + \mathbf{A}\mathbf{x}^* = \mathbf{0_n}, \tag{2.16}$$

Solving (2.18) and (2.19) for A_{11} and $\mathbf{A_{12}}$, and then substituting into (2.19) results in

$$\begin{aligned} H(y_0,\mathbf{x}) = a_0 y_0 + \mathbf{a}'\mathbf{x} + \frac{1}{2}(\boldsymbol{\alpha}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{A}\mathbf{x} \\ -(\boldsymbol{\alpha}'\mathbf{x})^{-1}\mathbf{x}^{*\prime}\mathbf{A}\mathbf{x}(y_0/y_0^*) + \frac{1}{2}(\boldsymbol{\alpha}'\mathbf{x})^{-1}\mathbf{x}^{*\prime}\mathbf{A}\mathbf{x}^*(y_0/y_0^*)^2, \end{aligned} \tag{2.17}$$

which is flexible at $(y_0^*, \mathbf{x}^*)$.[7]

The aggregator function $y_0(\mathbf{y})$ is specified as:

$$y_0(\mathbf{y}) = \mathbf{b}'\mathbf{y} + \frac{1}{2}\mathbf{y}'\mathbf{B}\mathbf{y}/\boldsymbol{\beta}'\mathbf{y}, \tag{2.18}$$

where $\mathbf{b}' = (b_1, \ldots, b_m)$ and the $m \times m$ symmetric matrix $\mathbf{B}$ contain the parameters to be estimated, while $\boldsymbol{\beta}' = (\beta_1, \ldots, \beta_m)$ is the vector of fixed

[7]See Diewert and Wales (1987) for the proof of flexibility.

nonnegative constants chosen by the researcher. As similarly done above with H, we can impose the following restrictions without losing local flexibility:

$$\boldsymbol{\beta}'\mathbf{y}^{*} = 1, \tag{2.19}$$

$$y_0^{*} = \mathbf{b}'\mathbf{y}^{*} \tag{2.20}$$

$$\mathbf{B}\mathbf{y}^{*} = \mathbf{0_m} \tag{2.21}$$

Substituting (2.18) into (2.17), we get the following flexible functional form for $\Omega(\mathbf{y},\mathbf{x})$, which satisfies the weak separability condition

$$\begin{aligned} \Omega(\mathbf{y},\mathbf{x}) &= H(y_0(\mathbf{y}),\mathbf{x}) \\ &= a_0\left(\mathbf{b}'\mathbf{y} + \frac{1}{2}(\boldsymbol{\beta}'\mathbf{y})^{-1}\mathbf{y}'\mathbf{B}\mathbf{y}\right) + \mathbf{a}'\mathbf{x} + \frac{1}{2}(\boldsymbol{\alpha}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{A}\mathbf{x} \\ &\quad - (y_0^{*}\boldsymbol{\alpha}'\mathbf{x})^{-1}\mathbf{x}^{*\prime}\mathbf{A}\mathbf{x}\left(\mathbf{b}'\mathbf{y} + \frac{1}{2}(\boldsymbol{\beta}'\mathbf{y})^{-1}\mathbf{y}'\mathbf{B}\mathbf{y}\right) \\ &\quad + \frac{1}{2}(y_0^{*2}\boldsymbol{\alpha}'\mathbf{x})^{-1}\mathbf{x}^{*\prime}\mathbf{A}\mathbf{x}^{*}\left(\mathbf{b}'\mathbf{y} + \frac{1}{2}(\boldsymbol{\beta}'\mathbf{y})^{-1}\mathbf{y}'\mathbf{B}\mathbf{y}\right)^{2}. \end{aligned} \tag{2.22}$$

The neoclassical curvature conditions require $\Omega(\mathbf{y},\mathbf{x})$ and $y_0(\mathbf{y})$ to be convex functions. Monotonicity requires that $\partial\Omega/\partial\mathbf{y} \geq 0$ and $\partial\Omega/\partial\mathbf{x} \leq 0$. Convexity of $H(y_0,\mathbf{x})$ and $y_0(\mathbf{y})$ requires the matrices $\mathbf{A}$ and $\mathbf{B}$ to be positive semidefinite. Global convexity of $\Omega(\mathbf{y},\mathbf{x})$ further requires the condition

$$\frac{\partial H(y_0,\mathbf{x})}{\partial y_0} \geq 0. \tag{2.23}$$

Positive semidefiniteness of the matrices $\mathbf{A}$ and $\mathbf{B}$ can be imposed without loss of flexibility by substituting

$$\mathbf{A} = \mathbf{q}\mathbf{q}' \tag{2.24}$$

and

$$\mathbf{B} = \mathbf{u}\mathbf{u}', \tag{2.25}$$

where **q** is an $n \times n$ lower triangular matrix and **u** is an $m \times m$ lower triangular matrix. Although global convexity can be imposed on the functions $H(y_0,\mathbf{x})$ and

$y_0(\mathbf{y})$ separately, *global convexity of the composite function* $\Omega(\mathbf{y},\mathbf{x})$ *cannot be imposed without damaging the flexibility of the function. Similarly, global monotonocity with respect to the inputs and outputs cannot be imposed on the transformation function* $\Omega(\mathbf{y},\mathbf{x})$ *without destroying the flexibility of the functional form*.

The first derivatives of $\Omega(\mathbf{y},\mathbf{x})$ are

$$\begin{aligned}\frac{\partial\Omega}{\partial\mathbf{y}} &= a_0\left[\mathbf{b}+\frac{1}{2}\left(2(\boldsymbol{\beta}'\mathbf{y})^{-1}\mathbf{B}\mathbf{y}-(\boldsymbol{\beta}'\mathbf{y})^{-2}\boldsymbol{\beta}\mathbf{y}'\mathbf{B}\mathbf{y}\right)\right]\\ &-(y_0^{*}\boldsymbol{\alpha}'\mathbf{x})^{-1}\mathbf{x}^{*\prime}\mathbf{A}\mathbf{x}\left[\mathbf{b}+\frac{1}{2}\left(2(\boldsymbol{\beta}'\mathbf{y})^{-1}\mathbf{B}\mathbf{y}-(\boldsymbol{\beta}'\mathbf{y})^{-2}\boldsymbol{\beta}\mathbf{y}'\mathbf{B}\mathbf{y}\right)\right]\\ &+(y_0^{*2}\boldsymbol{\alpha}'\mathbf{x})^{-1}\mathbf{x}^{*\prime}\mathbf{A}\mathbf{x}^{*}\left[\mathbf{b}+\frac{1}{2}\left(2(\boldsymbol{\beta}'\mathbf{y})^{-1}\mathbf{B}\mathbf{y}-(\boldsymbol{\beta}'\mathbf{y})^{-2}\boldsymbol{\beta}\mathbf{y}'\mathbf{B}\mathbf{y}\right)\right]\left(\mathbf{b}'\mathbf{y}+\frac{1}{2}(\boldsymbol{\beta}'\mathbf{y})^{-1}\mathbf{y}'\mathbf{B}\mathbf{y}\right)\end{aligned} \tag{2.26}$$

$$\begin{aligned}\frac{\partial\Omega}{\partial\mathbf{x}} &= \mathbf{a}+\frac{1}{2}\left(2(\boldsymbol{\alpha}'\mathbf{x})^{-1}\mathbf{A}\mathbf{x}-(\boldsymbol{\alpha}'\mathbf{x})^{-2}\boldsymbol{\alpha}\mathbf{x}'\mathbf{A}\mathbf{x}\right)-\\ &-\left[(y_0^{*}\boldsymbol{\alpha}'\mathbf{x})^{-1}\mathbf{A}\mathbf{x}^{*}-(y_0^{*}\boldsymbol{\alpha}'\mathbf{x})^{-2}\,y_0^{*}\boldsymbol{\alpha}\mathbf{x}^{*\prime}\mathbf{A}\mathbf{x}\right]\left(\mathbf{b}'\mathbf{y}+\frac{1}{2}(\boldsymbol{\beta}'\mathbf{y})^{-1}\mathbf{y}'\mathbf{B}\mathbf{y}\right)\\ &-\frac{1}{2}(y_0^{*2}\boldsymbol{\alpha}'\mathbf{x})^{-2}\,y_0^{*2}\boldsymbol{\alpha}\mathbf{x}^{*\prime}\mathbf{A}\mathbf{x}^{*}\left(\mathbf{b}'\mathbf{y}+\frac{1}{2}(\boldsymbol{\beta}'\mathbf{y})^{-1}\mathbf{y}'\mathbf{B}\mathbf{y}\right)^{2}\end{aligned} \tag{2.27}$$

At $(\mathbf{y}^{*},\mathbf{x}^{*})$ the value of the derivatives reduces to

$$\frac{\partial\Omega}{\partial\mathbf{y}} = a_0\mathbf{b} \tag{2.28}$$

and

$$\frac{\partial\Omega}{\partial\mathbf{x}} = \mathbf{a}\,. \tag{2.29}$$

Imposing monotonicity on (2.31) and (2.32) results in

$$\frac{\partial\Omega}{\partial\mathbf{y}}(\mathbf{y}^{*},\mathbf{x}^{*}) = a_0\mathbf{b}\geq 0 \text{ and } \frac{\partial\Omega}{\partial\mathbf{x}}(\mathbf{y}^{*},\mathbf{x}^{*}) = \mathbf{a}\leq 0 \tag{2.30}$$

The transformation function $\Omega(\mathbf{y},\mathbf{x})$ defined by (2.22) and restricted to satisfy equations (2.14), (2.19)-(2.21), (2.23)-(2.25), and (2.30) is flexible and locally monotone at $(\mathbf{y}^*,\mathbf{x}^*)$, subject to the weak separability condition. Monotonicity is verified empirically at each point within the data. Although *h*, *H*, and y_0 are globally convex, the composite function Ω still can fail convexity locally at points at which Ω fails monotonicity.

Substituting the functional form given by equation (2.22) into the system of Euler equations, we obtain the structural model, which is a system of integral equations. Barnett, Kirova, and Pasupathy (1995) tested the separability hypothesis and accepted the null hypotheses. Since that test is not central to purposes of this current paper, we do not present or discuss the results of that test in this paper, but the fact that the null was accepted does suggest that the separability hypothesis is not the source of the regularity problems that we more recently have found in this estimated model and report below.

2.4. Empirical Application

Barnett, Kirova, and Pasupathy (1995) apply the approach to estimating the technology of commercial banks. The outputs of that aggregated financial firm in our application consist of demand deposits and time deposits.[8] Demand deposits and time deposits account for the major portion of the fund-providing functions of the bank's balance sheet. The inputs used in the production process include both financial and nonfinancial inputs. The financial input in the form of cash is excess reserves. The nonfinancial inputs includes labor, materials, and physical capital. The output vector is given by $\mathbf{y}' = (D_t, T_t)$ and the input vector is

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[8]As shown in Debreu (1959), perfect competition alone is a sufficient condition for the existence of a representative firm. Since we are assuming perfect competition in this paper, no additional assumptions are implicit in our use of a representative neoclassical firm to model aggregate banking behavior.

$\mathbf{x}' = (C_t, L_t, M_t, K_t)$, where D_t is demand deposits, T_t is time deposits, C_t is excess reserves, L_t is labor input, M_t is material inputs, and K_t is capital.

In our empirical application we use the power utility function, which is a nested special case of the general class of HARA utility functions, $U(\pi_t)$, given by equation (2.8). We use this simplification, since the available sample size does not permit the use of the more general form. The power utility function is obtained by setting $d = 0,$ and by imposing the restriction $0 < \rho < 1,$ in equation (2.8).

Using equations (2.9) - (2.11), the Euler equations are

$$E_t\{(\pi_t)^{\rho-1}k_{1,t}P_t\} + E_t\left\{\frac{1}{1+\mu}P_t(\pi_{t+1})^{\rho-1}\left[R_t\frac{\partial\Omega/\partial D_t}{\partial\Omega/\partial C_t} + (1+R_t)(1-k_{1,t}) - (1+h_{1,t})\right]\right\} = 0 \quad (2.31)$$

$$E_t\{(\pi_t)^{\rho-1}k_{2,t}P_t\} + E_t\left\{\frac{1}{1+\mu}P_t(\pi_{t+1})^{\rho-1}\left[R_t\frac{\partial\Omega/\partial T_t}{\partial\Omega/\partial C_t} + (1+R_t)(1-k_{2,t}) - (1+h_{2,t})\right]\right\} = 0 \quad (2.32)$$

$$E_t\left\{(\pi_{t+1})^{\rho-1}\left[R_tP_t\frac{\partial\Omega/\partial L_t}{\partial\Omega/\partial C_t} - (1+R_t)w_{1,t}\right]\right\} = 0. \quad (2.33)$$

$$E_t\left\{(\pi_{t+1})^{\rho-1}\left[R_tP_t\frac{\partial\Omega/\partial M_t}{\partial\Omega/\partial C_t} - (1+R_t)w_{2,t}\right]\right\} = 0. \quad (2.34)$$

$$E_t\left\{(\pi_t)^{\rho-1}\left[(1+R_{t-1})P_{K,t-1}\right] - \frac{1}{1+\mu}(\pi_{t+1})^{\rho-1}\left[R_tP_t\frac{\partial\Omega/\partial K_t}{\partial\Omega/\partial C_t} + (1-\delta)(1+R_t)P_{K,t}\right]\right\} = 0 \quad (2.35)$$

where $h_{1,t}$ and $h_{2,t}$ are respectively the holding costs of demand deposits and time deposits, $k_{1,t}$ and $k_{2,t}$ are respectively the required reserves ratio on demand and time deposits, and $w_{1,t}$ and $w_{2,t}$ are respectively the prices of labor and material inputs. The derivatives of Ω with respect to the various inputs and outputs are given by equations (2.26) and (2.27).

Since we impose monotonicity only at one point, we need to select the "center of the approximation." We choose $y_0^* = 1$, $\mathbf{y}^{*\prime} = (1,1)$, and $\mathbf{x}^{*\prime} = (1,1,1,1)$ as the center of approximation. The fixed nonnegative constants are chosen such that

$$\alpha_i = \frac{|\tilde{\tilde{x}}_i|}{\sum_{j=1}^{4} |\tilde{\tilde{x}}_j|} \quad \forall \text{i} = 1,2,3,4 \tag{2.36}$$

and

$$\beta_i = \frac{|\tilde{\tilde{y}}_i|}{\sum_{j=1}^{2} |\tilde{\tilde{y}}_j|} \quad \forall \text{i} = 1,2\,, \tag{2.37}$$

where $\tilde{\tilde{x}}_i$ and $\tilde{\tilde{y}}_i$ are the sample means of $\tilde{x}_i$ and $\tilde{y}_i$ respectively. The α_i and β_i are thus chosen to satisfy restrictions (2.14) and (2.19) respectively.

Equation (2.20) implies $b_1 + b_2 = 1$, which is imposed through the substitution $b_1 = 1 - b_2$. There are also inequality restrictions to be imposed. The monotonicity condition (2.30) implies $b_i \geq 0$ for $i = 1,2,$ and hence from equation (2.20) we also have $b_i \leq 1\ \forall i = 1,2.$ Combining these two conditions and the mathematical identity $\sin^2\theta + \cos^2\theta = 1$, we have the substitutions $b_1 = \sin^2\theta$ and $\text{b}_2 = \cos^2\theta$, where the parameter θ must now be estimated.

We further normalize $a_0 = 1$, since $\Omega(\mathbf{y}, \mathbf{x}) = 0.$ The monotonicity condition (2.30) implies that $a_i \leq 0$ *f*or $i = 1,2,3,4$. We impose that restriction by replacing a_i by $-\tilde{a}_i^2$ for $\text{i} = 1,2,3,4$, and estimating $\tilde{a}_i$. The convexity conditions are imposed by replacing the matrices $\mathbf{A}$ and $\mathbf{B}$ by the matrices $\mathbf{qq}'$ and $\mathbf{uu}'$ respectively, where the lower triangular matrices $\mathbf{q}$ and $\mathbf{u}$ are given by

$$\mathbf{q} = \begin{bmatrix} q_{11} & 0 & 0 & 0 \\ q_{21} & q_{22} & 0 & 0 \\ q_{31} & q_{32} & q_{33} & 0 \\ q_{41} & q_{42} & q_{43} & q_{44} \end{bmatrix} \text{ and } \mathbf{u} = \begin{bmatrix} u_{11} & 0 \\ u_{21} & u_{22} \end{bmatrix}. \tag{2.38}$$

Equation (2.21) implies

$$\begin{bmatrix} u_{11} & 0 \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{21} \\ 0 & u_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2.39}$$

which, when solved, produces the restrictions $u_{22} = 0$ and $u_{21} = -u_{11}$.

These relationships reduce the matrix $\mathbf{B}$ to

$$\mathbf{B} = u_{11}^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Following these substitutions, the parameters that remain to be estimated within technology are θ, u_{11}, $\mathbf{q}$, and $\tilde{\mathbf{a}} = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4)$. In addition the subjective rate of time discount μ and the risk aversion parameter ρ must be estimated.

The data used for estimating the model was mainly obtained from the Federal Reserve Bank Functional Cost Analysis (FCA) Program.[9]

2.5. Results

Barnett, Kirova, and Pasupathy (1995) estimated the Euler equations ((2.33)-(2.37)) using the GMM estimation procedure on heteroskedasticity and autocorrelation in the disturbance terms. They specified a second order moving average serial correlation. Bartlett kernels were specified for the kernel density. Discount window rate, federal funds rate, composite bond rate, lagged value of excess reserves, lagged value of interest paid on time deposits, and a constant

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[9]For details of the data used in this mainframe TSP (version 7.02). This estimation process allows for project, see Barnett, Kirova, and Pasupathy (1995). The sample period was 1966-1992, and the primary source of data was the Federal Reserve's Functional Cost Analysis for National Average Banks.

were chosen as instruments. To ensure that $0 < \rho < 1$, they replace ρ by $\sin^2(\hat{\rho})$ and estimate $\hat{\rho}$. Similarly, to rule out the possiblity of negative values for the subjective rate of time preference, μ, they replace μ by $\tilde{\mu}^2$ and estimate $\tilde{\mu}$.

The test for weak separability of monetary assets, using Hansen's χ^2 test for no over-identifying restrictions, was conducted by Barnett, Kirova, and Pasupathy (1995) and showed that weak separability could not be rejected. This implies the existence of an output aggregator function. Using the Barnett, Kirova, and Pasupathy (1995) parameter estimates, we now check the functional form to see if the regularity conditions, namely monotonicity and curvature conditions, are satisfied at all data points. This exercise reveals that monotonicity conditions were satisfied at all points within the output aggregator function: i.e., the condition $\frac{\partial y_0}{\partial \mathbf{y}} \geq 0$ was satisfied at all data points. Within the transformation function, Ω, the condition $\frac{\partial \Omega}{\partial y_i} \geq 0,\ \forall i$ was satisfied at all points, so there were no violations of the monotonicity conditions with respect to the outputs. For inputs, the condition $\frac{\partial \Omega}{\partial \mathbf{x}} \leq 0$ was tested for each of the four inputs at all data points. It was found that this condition was satisfied everywhere in the case of cash and labor, while it was violated at two data points in the case of materials and at four data points in the case of capital. The curvature condition for the composite function, $H(y_0(\mathbf{y}), \mathbf{x})$, was satisfied globally, since it was found that the condition $\frac{\partial H(y_0, \mathbf{x})}{\partial y_0} \geq 0$ was satisfied at all data points, so that reversals of curvature of the composite function could not be induced by violations of monotonicity of H in y_0.

2.6 Consequences of Regularity Violations

As stated in the concluding portion of the previous section, the functional form failed to satisfy some of the regularity conditions that are required by economic theory of a valid transformation function. In this section, we provide further details of this problem.

The weak separability of outputs from inputs within the transformation function implies that the equation $H(y_0(\mathbf{y}),\mathbf{x}) = 0$ can be solved to represent $y_0(\mathbf{y})$ as a function of the inputs $\mathbf{x}$, if the conditions imposed by the implicit function theorem are satisfied. In our case this condition reduces to verifying that $\frac{\partial H(y_0,\mathbf{x})}{\partial y_0} \neq 0$ at all data points. We found that this condition was indeed satisfied at all data points. Hence, we proceeded to solve the equation for $y_0(\mathbf{y})$, so that the model can be used to obtain the elasticities of substitution between the inputs and outputs of the production technology.

We were in fact able to find an explicit solution to the equation. The function H, specified as an adaptation of the symmetric generalized quadratic functional form, is given (from equation 2.17) by:

$$\begin{aligned} H(y_0,\mathbf{x}) = a_0 y_0 + \mathbf{a}'\mathbf{x} + \frac{1}{2}(\boldsymbol{\alpha}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{A}\mathbf{x} \\ - (\boldsymbol{\alpha}'\mathbf{x})^{-1}\mathbf{x}^{*\prime}\mathbf{A}\mathbf{x}(y_0/y_0^*) + \frac{1}{2}(\boldsymbol{\alpha}'\mathbf{x})^{-1}\mathbf{x}^{*\prime}\mathbf{A}\mathbf{x}^*(y_0/y_0^*)^2 \end{aligned} \tag{2.40}$$

where $(y_0^*,\mathbf{x}^*)$ is the point of local flexibility.

These points were chosen as $y_0^* = 1$ and $\mathbf{x}^* = (1,1,1,1)$ during estimation. At this point using the representation given in equation (2.40), equation $H(y_0(\mathbf{y}),\mathbf{x}) = 0$, can be rewritten as a quadratic equation in y_0. The quadratic equation is given by:

$$\left[(\boldsymbol{\alpha}'\mathbf{x})^{-1}\mathbf{x}^{*\prime}\mathbf{A}\mathbf{x}^*\right]y_0^2 + 2\left[a_0 - (\boldsymbol{\alpha}'\mathbf{x})^{-1}\mathbf{x}^{*\prime}\mathbf{A}\mathbf{x}\right]y_0 + 2\mathbf{a}'\mathbf{x} + (\boldsymbol{\alpha}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{A}\mathbf{x} = 0 \tag{2.41}$$

We found that both roots to the equation (2.41) existed. Hence, there were two values for the production possibility surface y_0, which implies that y_0 was in fact a correspondence and not a function, as it should be. We believe that the problem of multiple solutions has arisen due to the violations of regularity conditions reported earlier. We further found that for part of the sample (1976 - 1992), the term

$$\left(a_0 - (\boldsymbol{\alpha}'\mathbf{x})^{-1}\mathbf{x}^{*\prime}\mathbf{A}\mathbf{x}\right)^2 - (\boldsymbol{\alpha}'\mathbf{x})^{-1}\mathbf{x}^{*\prime}\mathbf{A}\mathbf{x}^*\left(2\mathbf{a}'\mathbf{x} + (\boldsymbol{\alpha}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{A}\mathbf{x}\right)$$

had negative values. Since, this term appears under the square root in the solution to equation (2.43), the roots to the equation had complex values at these sample points.

To study this rather strange behavior of the functional form further, we tried to plot the isoquants. We once again encountered the phenomenon of multiple solutions. We found that there were two sets of isoquants for every pair of inputs. The isoquants between labor and capital and between materials and capital entered the complex plane.

3. THE OTHER COUNTEREXAMPLE

Barnett, Kirova, and Pasupathy (1995) not only used the generalized quadratic model to estimate the technology of financial intermediaries, but also to estimate the technology of manufacturing firms. In this paper, we report primarily on the results with the financial intermediary, since the regularity violations were the most disturbing in that case. But we also ran some similar checks on the manufacturing firm model. The manufacturing firm model is of a particularly common form, with financial services and other factors treated as inputs to the technology of a single product firm, as opposed to the financial intermediary model, in which financial services are outputs of a multiproduct firm. Some of the strangest regularity violations that we found in our financial intermediary model results did not arise, when we ran similar checks with the manufacturing firm model. But when we plotted the isoquants for the estimated manufacturing firm model, we did find that the slopes of many isoquants, instead of intersecting axes or approaching them asymptotically, eventually curved back by acquiring positive slopes, and some of the data points were found to lie in the regions of positive isoquant slopes. This demonstrates violations of monotonicity, even with the more conventional manufacturing firm model estimated in Barnett, Kirova, and Pasupathy (1995).

At the time that we estimated and published the results with that model, we had found all results to be plausible and had encountered no reason to suspect that anything had gone wrong. Only more recently when we acquired the isoquants of the financial intermediary model and then decided to plot the isoquants of the manufacturing firm model did we find that regularity had again been violated in very troubling ways, despite the fact that in both models we had imposed curvature globally and monotonicity locally and had found plausible elaticities at the center of the approximation.[10]

4. ARE THESE TWO COUNTEREXAMPLES REPRESENTATIVE?

The financial intermediary example that produced the most severe regularity problems is particularly sophisticated in its use of econometric methodology. But as we have observed, regularity problems also arose in our less exotic

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[10] The plots of the isoquants for the manufacturing firm model are available upon request and are very revealing. Some of those plots are available in Barnett (2002). The sample period was 1949-1988, and the data are on U. S. manufacturing. The two primary sources of the data were the Division of Multifactor Productivity of the Bureau of Labor Statistics and the *Quarterly Financial Report for Manufacturing, Mining, and Trade Corporations*. The details of the data can be found in Barnett, Kirova, and Pasupathy (1995).

manufacturing firm application. Applied econometricians should not take comfort in the sophisticated nature of either of these counterexamples, since the relevant criteria for judging whether or not these results should be viewed as representative indicate very clearly that they are representative.

The reason is the parsimoneous nature of the generalized quadratic model. As was demonstrated in the original Diewert and Wales (1987) paper that proposed the model (then called generalized McFadden), the model has just enough parameters to be able to attain arbitrary elasticities at a point, when curvature is imposed globally and monotonicity at the point. That is the reason that monotonicity is not normally imposed globally. If monotonicity were imposed globally, the model's flexibility would be compromised, so that it would not be possible to attain arbitrary elasticities at a point.

Hence whatever adverse regularity properties this model may exhibit are a consequence of the elasticities at the "point of approximation." When Barnett, Kirova, and Pasupathy (1995) published the point estimates from these two applications, they did not suspect severe regularity problems since the elasticities at the point of approximation were very reasonable, especially in the manufacturing firm case. The only way that our results could be viewed as unlikely special cases would be if the elasticities at the center of the approximation were in some way unusual or odd---and they were not!

In addition the isoquant plots that we mention in footnote 9, and that we would be happy to supply on request, display very dramatically the manner in which the isoquants curve back such that their slopes become positive. In each of the cases of those two dimensional plots, the elasticities are entirely normal and plausible at the center of the approximation, with curvature satisfied globally.

5. CONCLUSIONS

We find that the generalized quadratic specification of technology produces violations of monotonicity in our application, when curvature is imposed globally. In addition, we find that the violations of monotonicity can create induced violations of curvature in composite functions defined by weakly separable nesting of aggregator functions within the firm's technology---even when curvature has been imposed globally for all of the inner and outer component functions defining the composite function. Although we did not encounter that local reversal problem, we did encounter considerably more troubling problems involving nonuniqueness of isoquants and complex valued solutions induced by violations of monotonicity. In short, imposing curvature without monotonicity, while perhaps to be preferred to the prior common practice of imposing neither, is not adequate without at least reporting data points at which violations of monotonicity occur. Monotonicity is too important to be overlooked.

In fact, we believe that imposition of curvature can produce spurious violations of monotonicity that would not otherwise have occurred. Without imposition of curvature, an estimator can most readily improve fit spuriously through violating curvature. But when curvature is imposed, the only spurious means remaining to improve fit by leaving the neoclassical function space is through violation of monotonicity. This problem is likely to be especially common with quadratic models, such as the generalized quadratic model, since quadratic functions can have a bliss point.

While our counterexample is produced by a particularly sophisticated application of the generalized quadratic technology for a multiproduct financial firm, we have confirmed that the same problem arises in a more elementary model of a single product manufacturing firm, also estimated using generalized method of moments. In addition, because of the parsimonious nature of the model's specification after imposition of global curvature, and because of the plausible elasticities estimates at the center of the approximation, we believe that our disturbing findings are representative of those that would be found by similar explorations of isoquants from other models estimated within this tradition.

When we first estimated those two models and published results from the two models in Barnett, Kirova, and Pasupathy (1995), we had not anticipated the regularity problems that we recently found in those results, when we went back and tried to plot the isoquants. In short, we were not looking for examples to make the point that regularity is a problem with the generalized quadratic model. But now that we have become aware of the problem in that work, we believe it is important for us to reveal the issue, which we believe has potentially widespread relevancy in the production modeling literature.

We have no doubt that replication of this experiment with other data sets estimated under perfect certainty by maximum likelihood will further confirm our conclusions. We do not expect that induced nonuniqueness of isoquants will be a common problem, and we doubt that induced violations of curvature of the composite technology will be common. But on the other hand, we strongly doubt that global satisfaction of monotonicity will often be attained by this model, when estimated subject to imposed curvature conditions, and in fact in our ongoing research on this subject, we have not yet found a single such case. We believe that a search and display of the model's regular regions, using the procedures of Caves and Christensen (1980), is warranted and will demonstrate that the now common procedure of ignoring monotonicity when curvature is imposed is not justifiable. We plan to provide those replications and to display those function properties in future research, if no one else does it.

Research on models permitting imposition of both curvature and monotonicity remains at an early stage and has so far had little impact on the literature on production modeling. While a difficult literature, we believe that research on models permitting flexible imposition of true regularity---i.e. both monotonicity and curvature---should expand.[11] In short, the recent advances in imposing curvature alone are an important step in a positive direction, but do not yet produce the ultimate result sought from this direction of research: the ability to impose full regularity without loss of flexibility.

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