# The Theoretical Regularity Properties of the Normalized Quadratic Consumer Demand Model

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**Abstract**

We conduct a Monte Carlo study of the global regularity properties of the Normalized Quadratic model. We particularly investigate monotonicity violations, as well as the performance of methods of locally and globally imposing curvature. We find that monotonicity violations are especially likely to occur, when elasticities of substitution are greater than unity. We also find that imposing curvature locally produces difficulty in the estimation, smaller regular regions, and the poor elasticity estimates in many cases considered in the paper. Imposition of curvature alone does not assure regularity, and imposing local curvature alone can have very adverse consequences.

JEL Codes: C14, C22, E37, E32.

## 1. Introduction

Uzawa (1962) proved that the constant elasticity of substitution (CES) model cannot attain arbitrary elasticities with more than two goods. As a result, the development of locally flexible functional forms evolved as a new approach to modeling specifications of tastes and technology. Flexible functional forms were defined by Diewert (1971) to be the class of functions that have enough free parameters to provide a local second-order approximation to any twice continuously differentiable function. If a flexible functional form has no more parametric freedom than needed to satisfy that definition, then the flexible functional form is called "parsimonious." Barnett (1983) proved that a functional form satisfies Diewert's definition if and only if it can attain any arbitrary elasticities at any one predetermined point in data space. Most of the available flexible functional forms are based on quadratic forms derived from second-order series expansions. The translog model of Christensen, Jorgenson, and Lau (1971) and the AIDS (almost ideal demand system) model of Deaton and Muellbauer (1980) use Taylor series expansions in logarithms, the generalized Leontief model of Diewert (1971) uses a Taylor series expansion in square roots, and the Laurent models of Barnett (1983) use the Laurent series expansion.

As these flexible functional form models became available, applied researchers tended to overlook the maintained regularity conditions required by microeconomic theory. Regularity requires satisfaction of both curvature and monotonicity conditions. Simultaneous imposition of both of these conditions on a parsimonious flexible functional form destroys the model's local flexibility property. For instance, Lau (1978) showed that imposition of global regularity reduces the translog model to Cobb-Douglas, which is not a flexible functional form and has no estimable elasticities. When regularity is not imposed, most of the estimated flexible functional forms in empirical applications exhibit frequent violations of regularity conditions at many data points.[1] Since that fact became evident, information about violations of regularity conditions in empirical applications have become hard to find.[2]

An exception to the common neglect of regularity conditions was Diewert and Wales' (1987) work on the Normalized Quadratic model. That model permits imposition of curvature globally, while remaining flexible. Since violations of curvature have more often been reported than violations of monotonicity, the imposition of curvature alone seems to merit consideration. In subsequent papers of Diewert and Wales (1992,1993,1995) and others, imposition of curvature globally, without imposition of monotonicity, has become a common practice with the Normalized Quadratic functional form.

But once curvature is imposed without the imposition of monotonicity, the earlier observation may no longer apply. When global curvature is imposed, the loss of model-fit may induce spurious improvements in fit through violations of monotonicity. This problem could be especially common with quadratic models, which can have bliss points. It is possible that violations of monotonicity could be induced by imposition of curvature.

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[1] See, e.g., Manser (1974) and Humphrey and Moroney (1975).
[2] A noteworthy exception is Moroney and Trapani (1981), who confirmed the earlier findings of frequent violations of maintained regularity conditions.

With this model, it has become common not to check for monotonicity, after imposing global curvature. Diewert and Wales (1995) and Ryan and Wales (1998) have expected that monotonicity will be satisfied, as a result of the non-negativity of the dependent variables. But non-negativity of observed dependent variables does not assure non-negativity of fitted dependent variables. In Kohli (1993) and Diewert and Fox (1999), the curvature condition is treated as the sole regularity condition. But without satisfaction of both curvature and monotonicity, the second-order condition for optimizing behavior fails, duality theory fails, and inferences resulting from derived estimating equations become invalid.[3] Hence the common practice of equating regularity solely with curvature is not justified.

Barnett (2002), and Barnett and Pasupathy (2003) confirmed the potential problem and found further troublesome consequences, when they checked regularity violations in their own previously published estimation of technology in Barnett, Kirova and Pasupathy (1995). Initially, they imposed curvature globally, but monotonicity only at a central data point with the Normalized Quadratic production model. In addition to violations of monotonicity, they encountered induced curvature reversals of composite functions, along with nonunique isoquants and complex valued solutions. Even if curvature is imposed on both inner (category) production functions and weakly separable outer functions, the composite technology still can violate curvature, if monotonicity is violated. The evidence suggested the need for a thorough investigation of the global regularity property of the Normalized Quadratic model. We undertake this task in this paper.

A well established approach to exploring regularity properties of a neoclassical function is to set the parameters of the model to produce various plausible elasticities, and then plot the regular regions within which the model satisfies monotonicity and curvature. We do so by setting the parameters at various levels to produce elasticities that span the plausible range, and then plot the regular region of the model when curvature is imposed but monotonicity not imposed. The intent is to explore the common practice with the Normalized Quadratic model. Such experiments have been conducted with the translog and the generalized Leontief by Caves and Christensen (1980) and with newer models by Barnett, Lee and Wolfe (1985,1987) and Barnett and Lee (1985).[4]

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[3] The damage done to the inference has been pointed out by Basmann, Molina and Slottje (1983), Basmann, Diamond, Frentrup and White (1985), Basmann, Fawson, and Shumway (1990), and Basmann, Hayes and Slottje (1994).

[4] Other relevant papers include Wales (1977), Blackorby, Primont, and Russell (1977), Guilkey and Lovell (1980), White (1980), Guilkey, Lovell and Sickles (1983), Barnett and Choi (1989).

In our experiment, we obtain the parameter values of the Normalized Quadratic model by estimation of those parameters with data produced by another model at various settings of the elasticities. Jensen (1997) devised the experimental design, which closely follows that of Caves and Christensen, but Jensen applied the approach to estimate the coefficients of the Asymptotically Ideal Model (AIM) of Barnett and Jonas (1983).[5] We adopt a similar experimental design, in which (1) artificial data is generated, (2) the Normalized Quadratic model then is estimated with that data, and (3) finally its regular regions are displayed at the various elasticities used with the generating model.

The functional form model that we investigate is the Normalized Quadratic reciprocal indirect utility function (Diewert and Wales, 1988b, Ryan and Wales, 1998). Globally correct curvature can be imposed on the model by imposing negative semi-definiteness of a particular coefficient matrix and non-negativity of a particular coefficient vector, but at the added cost of losing flexibility. It has been argued that global curvature imposition forces the Slutsky matrix to be "too negative semi-definite." In that sense, the method imposes too much concavity, and thereby damages the flexibility. Since concavity is required by economic theory, the model's inability to impose full concavity without loss of flexibility is a serious defect of the model. If instead of the indirect utility function, the Normalized Quadratic is used to model the expenditure function, global concavity can be imposed without loss of flexibility. But the underlying preferences then are quasihomothetic and thereby produce linear Engel curves. Because of that serious restriction on tastes, we exclude that model from our experiment.

Ryan and Wales (1998) suggested a procedure for imposing negative semi-definiteness on the Slutsky matrix, as is necessary and sufficient for the curvature requirement of economic theory. But to avoid the loss of flexibility, Ryan and Wales apply the condition only at a single point of approximation.[6] With their data, they successfully found a data point such that imposition of curvature at that point results in full regularity (both curvature and monotonicity) at every sample point. They also applied the procedure to other earlier consumer demand systems.[7] By imposing correct curvature at a point, the intent with this procedure is to attain, without imposition, the curvature and monotonicity conditions at all data points. We explore the regular regions of the models with these two methods of curvature imposition. The objective is to

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[5] The AIM is a seminonparametric model produced from a class of globally flexible series expansions. See also Barnett, Geweke and Wolfe (1991a,1991b). Gallant's Fourier flexible functional form (1981) is also globally flexible.

[6] Moschini (1996) independently developed the identical procedure to impose local curvature on the semiflexible AIDS model. See Diewert and Wales (1988a) for the definition of the semiflexibility.

[7] Moreover, Ryan and Wales (2000) showed effectiveness of the procedure when estimating the translog and generalized Leontief cost functions with the data utilized by Berndt and Khaled (1979).

determine the extent to which imposition of global, local or no curvature results in regularity violations. Imposing curvature locally may induce violations of curvature at other points, in addition to violations of monotonicity.

We find monotonicity violations to be common. With these models, the violations exist widely within the region of the data, even when neither global curvature nor local curvature is imposed. We believe that this problem is common with many non-globally-regular flexible functional forms, and is not a problem specific to the Normalized Quadratic model. For example, one of the graphs for the AIM cost function in Jensen (1997), without regularity imposition, look similar to the one obtained below.

Imposing curvature globally corrected the monotonicity violations globally in a case with complimentarity among two of the goods. But that imposition produced some overestimation of cross elasticities of substitutions in absolute values. A pair of complementary goods became more complementary and a pair of substitute goods became stronger substitutes. Diewert and Wales (1993) similarly found that some method of imposing regularity can produce upper bounds on certain elasticities for the AIM and translog models.[8]

The paper is organized as follows. Section 2 presents the model using the two methods of curvature imposition. Section 3 illustrates our experimental design, by which the artificial data is simulated, the model is estimated, and the regular region is displayed. Section 4 provides our results and discussion. We conclude in the final section.

## 2. The Model

Central to the imposition of curvature is a quadratic term of the form $\mathbf{v}'\mathbf{B}\mathbf{v}$, where $\mathbf{v}$ is a vector of the variables, and $\mathbf{B}$ is a symmetric matrix containing unknown parameters. With the Normalized Quadratic model, the quadratic term is normalized by a linear function of the form $\boldsymbol{\alpha}'\mathbf{v}$, so that the quadratic term can be written in the form $\mathbf{v}'\mathbf{B}\mathbf{v}/\boldsymbol{\alpha}'\mathbf{v}$, where $\boldsymbol{\alpha}$ is a non-negative predetermined vector. According to Diewert and Wales ((1987), theorem 10), the Hessian matrix of the quadratic term is negative semi-definite, if the matrix $\mathbf{B}$ is negative semi-definite. Imposition of that matrix constraint ensures concavity of the normalized quadratic term. As a result, imposition of global curvature starts with imposition of negative semi-definiteness on the matrix $\mathbf{B}$. We reparameterize the matrix $\mathbf{B}$ by replacing it by minus the product of a lower

[8] See also Terrell (1995). But it should be observed that more sophisticated methods of imposing regularity on AIM do not create that problem. In fact it is provable that imposition of global regularity on seminonparametric models, such as AIM, cannot reduce the span, if imposition is by the most general methods.

triangular matrix, **K**, multiplied by its transpose, so that $\mathbf{B} = -\mathbf{KK}'$. Diewert and Wales (1987,1988a,1988b) used this technique, developed by Wiley, Schmidt, and Bramble (1973) and generalized by Lau (1978).

The Normalized Quadratic reciprocal indirect utility function of Diewert and Wales (1988b) and Ryan and Wales (1998) is defined as,

$$h(\mathbf{v}) = \mathbf{b}'\mathbf{v} + \frac{1}{2}\left[\frac{\mathbf{v}'\mathbf{B}\mathbf{v}}{\boldsymbol{\alpha}'\mathbf{v}}\right] + \mathbf{a}'log(\mathbf{v}), \tag{1}$$

where **b** is a vector containing unknown parameters, and **v** is a vector of prices, **p**, normalized by a scalar of total expenditure $y$, so that $\mathbf{v} \equiv \mathbf{p}/y$.[9] A fixed reference point $\mathbf{v}_0$ is chosen, such that the matrix **B** satisfies

$$\mathbf{B}\mathbf{v}_0 = \mathbf{0}, \tag{2}$$

and the predetermined vector $\boldsymbol{\alpha}$ satisfies

$$\boldsymbol{\alpha}'\mathbf{v}_0 = 1. \tag{3}$$

Using Diewert's (1974) modification of Roy's Identity, the system of share equations is derived as,

$$\mathbf{s}(\mathbf{v}) = \frac{\hat{\mathbf{V}}\mathbf{b} + \hat{\mathbf{V}}\dfrac{\mathbf{B}\mathbf{v}}{(\boldsymbol{\alpha}'\mathbf{v})} - \dfrac{1}{2}\hat{\mathbf{V}}\left[\dfrac{\mathbf{v}'\mathbf{B}\mathbf{v}}{(\boldsymbol{\alpha}'\mathbf{v})^2}\right]\mathbf{a} + \mathbf{a}}{\mathbf{v}'\mathbf{b} + \dfrac{1}{2}\left[\dfrac{\mathbf{v}'\mathbf{B}\mathbf{v}}{\boldsymbol{\alpha}'\mathbf{v}}\right] + \boldsymbol{\iota}'\mathbf{a}}, \tag{4}$$

where $\mathbf{s} = \mathbf{s}(\mathbf{v})$ is a vector of budget shares, $\boldsymbol{\iota}$ is a unit vector with 1 as each element, and $\hat{\mathbf{V}}$ is a diagonal matrix with normalized prices on the main diagonal and zeros on the off-diagonal. Homogeneity of degree zero in all parameters of the share equations (4) requires use of an identifying normalization. The normalization usually used is

9 Diewert and Wales (1988b) include a level parameter $b_0$ additively in equation (1). But it is nonidentifiable and not estimatable, since it vanishes during the derivation of the estimating equations.

$$\mathbf{b}'\mathbf{v}_0 = 1. \tag{5}$$

The functional form (1) subject to restriction (2), (3), and (5) will be globally concave over the positive orthant, if the matrix **B** is negative semi-definite and all elements of the parameter vector **a** are non-negative. Global concavity can be imposed during the estimation by setting $\mathbf{B} = -\mathbf{KK}'$ with **K** lower triangular, while setting $a_i = c_i^2$ for each $i$, where **c** is a vector of the same dimension as **a**. We then estimate the elements of **K** and **c** instead of those of **B** and **a**. As mentioned above, this procedure for imposing global concavity damages flexibility.

Imposition of curvature locally is at the point of approximation. Without loss of generality, we choose the $\mathbf{v}_0 = \mathbf{1}$ to be that point. For ease of estimation we impose the following additional restriction:

$$\mathbf{a}'\mathbf{v}_0 = 0. \tag{6}$$

Using the other restrictions along with the restriction (6), the Slutsky matrix at the point $\mathbf{v}_0$ can be written as,

$$\mathbf{S} = \mathbf{B} - \hat{\mathbf{A}} + \mathbf{ab}' + \mathbf{ba}' + 2\mathbf{aa}', \tag{7}$$

where $\hat{\mathbf{A}} = diag\,(\mathbf{a})$. Hence $\hat{\mathbf{A}}$ is a diagonal matrix defined similarly to $\hat{\mathbf{V}}$. Imposing curvature locally is attained by setting $\mathbf{S} = -\mathbf{KK}'$ with **K** lower triangular, and solving for **B** as

$$\mathbf{B} = -(\mathbf{KK}') + \hat{\mathbf{A}} - \mathbf{ab}' - \mathbf{ba}' - 2\mathbf{aa}'. \tag{8}$$

Ryan and Wales (1998) showed that the demand system described above is flexible. Moreover, the regular regions of this model and the unconstrained model with equation (6) imposed will be exactly identical, when Caves and Christensen's method is used.

During estimation, the matrix **B** is replaced by the right hand side of (8) to guarantee that the Slutsky matrix is negative semi-definite at the point of approximation. To see why imposing curvature globally damages the flexibility while imposing curvature locally does not, recall that the Slutsky matrix, **S**, is symmetric and satisfies $\mathbf{Sp} = \mathbf{0}$ or equivalently $\mathbf{Sv} = \mathbf{0}$. As a result, the rank of **S** is reduced by one, so that the number of the independent elements of **S** becomes equal to that of **B**. Therefore **S** in equation (7) can be arbitrary determined by **B,** independently of **a** and **b**. But the Hessian matrix of the indirect utility function is usually full rank, unless linear homogeneity is imposed or attained empirically. In Diewert and Wales' (1988b) approach to proof of local flexibility, the second partial derivatives at the point of approximation depend on both **B** and **a**. However imposition of non-negativity on **a** to attain global curvature reduces the number of independent parameters, and limits the span of **B** and **a**. As a result, imposing curvature globally on this model damages flexibility. Regarding local curvature imposition, the condition that the Slutsky matrix be negative semi-definite is both necessary and sufficient for correct curvature at the point of approximation.

## 3. Experimental Design

Our Monte Carlo experiment is conducted with a model of three goods demand to permit different pairwise complementarities and substitutabilities. The design of the experiment is described below.

### 3.1. Data Generation

The data set employed in the actual estimation process includes data for normalized prices and budget shares, defined as $\mathbf{v} \equiv \mathbf{p}/y$ and $\mathbf{s} \equiv \hat{\mathbf{V}}\,\mathbf{q}$ where $\mathbf{q}$ is a vector of demand quantities and $\hat{\mathbf{V}}$ is as defined previously. The data for demand quantities are produced from the demand functions induced by two globally regular utility functions: the CES functional form and the linearly-homogeneous Constant-Differences of Elasticities-of-Substitution (CDE) functional form.[10]

The CES indirect utility function with three goods is

$$U(\mathbf{p}, y) = y\left[\sum_{k=1}^{3} p_k^r\right]^{-1/r} \qquad \text{where } r = \rho/(\rho - 1). \tag{9}$$

By applying Roy's identity to (8), Marshallian demand functions are derived as

$$q_i(\mathbf{p}, y) = \; y\, p_i^{r\text{-}1} \Big/ \sum_{k=1}^{3} p_k^r \tag{10}$$

[10] See Hanoch (1975) and Jensen (1997) for details of the model.

for $i$ = 1, 2, and 3. The CES utility function is globally regular if $\rho \leq 1$. The values of $\rho$ are chosen so that the elasticity of substitution $\sigma = 1/(1-\rho)$ covers a sufficiently wider range. Fleissig, Kastens, and Terrell (2000) also used this data generation model in comparing the performance of the Fourier flexible form, the AIM form, and a neural network in estimating technologies.

The CDE indirect reciprocal utility function $1/u = g(\mathbf{p}, y)$ is defined implicitly by an identity of the form:

$$G(\mathbf{p}, y, u) = \sum_{k=1}^{3} G^k(p_k, y, u) \equiv 1, \tag{11}$$

with $G^k = \phi_k u^{\theta_k} (p_k/y)^{\theta_k}$. Parametric restrictions required for the implicit utility function (11) to be globally regular are $\phi_k > 0$ and $\theta_k < 1$ for all $k$, and either $\theta_k \leq 0$ for all $k$ or $0 < \theta_k < 1$ for all $k$. In all cases, the $\phi_k$'s equal the corresponding budget shares at $(\mathbf{p}_0, y_0) = (\mathbf{1}, 1)$. Applying Roy's Identity, we derive the demand functions,

Table 1
True underlying preferences at a point, $(\mathbf{p}_0, y_0) = (\mathbf{1}, 1)$ in terms of budget shares and elasticities of substitution.

| | Budget shares | | | Elasticities of substitution | | |
|---|---|---|---|---|---|---|
| | $s_1$ | $s_2$ | $s_3$ | $\sigma_{12}$ | $\sigma_{13}$ | $\sigma_{23}$ |
| Case 1 | 0.333 | 0.333 | 0.333 | 0.200 | 0.200 | 0.200 |
| Case 2 | 0.333 | 0.333 | 0.333 | 0.700 | 0.700 | 0.700 |
| Case 3 | 0.333 | 0.333 | 0.333 | 2.000 | 2.000 | 2.000 |
| Case 4 | 0.333 | 0.333 | 0.333 | 4.000 | 4.000 | 4.000 |
| Case 5 | 0.300 | 0.300 | 0.400 | 1.500 | 1.500 | 1.500 |
| Case 6 | 0.300 | 0.300 | 0.400 | 0.500 | 0.500 | 0.500 |
| Case 7 | 0.395 | 0.395 | 0.211 | 0.061 | 0.561 | 0.561 |
| Case 8 | 0.409 | 0.409 | 0.182 | -0.010 | 0.591 | 0.591 |

$$q_i(\mathbf{p}, y) = \frac{\theta_i \phi_i u^{\theta_i} (p_i/y)^{\theta_i - 1}}{\sum_{k=1}^{3} \theta_k \phi_k u^{\theta_k} (p_k/y)^{\theta_k}} \tag{12}$$

for $i = 1, 2$, and 3. The utility level $u$ is set to unity without loss of generality, when generating simulated data.

Our test bed consists of eight cases. Cases 1 to 4 use data simulated from demand functions of the equation (10) CES form, and cases 5 to 8 use data from demand functions of the equation (12) CDE form. Table 1 describes each case in terms of the elasticity of substitution and budget share settings at the reference point.[11]

It is convenient to construct the data such that the mean of the normalized prices is 1. We draw from a continuous uniform distribution over the interval [0.5, 1.5] for price data and [0.8, 1.2] for total expenditure data. The sample size is 100, as would be a typical sample size with annual data. [12]

The stochastic data, adding noise to the model's solved series, are constructed in the following manner. The noise vector $\boldsymbol{\varepsilon}$ is generated from a multivariate normal distribution of mean zero and covariance matrix $\mu\text{cov}(\mathbf{p})$, where $\mu$ is a constant and $\text{cov}(\mathbf{p})$ is the covariance matrix of a generated price series. We arbitrarily set $\mu \in [0.0, 1.0]$ to adjust the influence of noise on the estimation. The price series incorporating noise is constructed as $\mathbf{p}^* = \mathbf{p} + \boldsymbol{\varepsilon}$, while making sure that the resulting prices are strictly positive with each setting of $\mu$. We then use equations (10) and (12), along with total expenditure $y$, to generate the data for quantities demanded, $\mathbf{q}^*(\mathbf{p}^*, y)$. Using the noise-added data, we compute total expenditure $\mathbf{y}^* = (\mathbf{q}^*)'(\mathbf{p}^*)$, normalized prices $\mathbf{v}^* = \mathbf{p}^*/y^*$, and budget shares $\mathbf{s}^* = \hat{\mathbf{V}}^{*\prime}\mathbf{q}^*$, where $\hat{\mathbf{v}}^* = diag(\mathbf{v}^*)$. We then have the data for the dependent variable, $\mathbf{s}^*$ and the noise-free independent variable, $\mathbf{v} = \mathbf{p}/\text{y}$. It is easier to ensure strictly positive noise-added data by this procedure, than by adding directly to the budget shares.[13]

### 3.2. Estimation

Using our simulated data, we estimate the system of budget share equations (4) with a vector of added disturbances $\mathbf{e}$. We assume that the $\mathbf{e}$'s are independently multivariate normally distributed with $E(\mathbf{e}) = \mathbf{0}$ and $E(\mathbf{ee}') = \boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is constant across observations. Since the

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[11] The values of those elasticities are computed as Allen-Uzawa elasticities of substitution. The Allen-Uzawa elasticity of substitution is the commonly used traditional measure. More complicated substitutability can be captured by the Morishima elasticity of substitution. Its superiority is maintained in Blackorby and Russell (1989).

[12] Unlike our design, Jensen's (1997) design used price series of length 1000 with a factorial design at discrete points in the interval [0.5, 2.0]. Terrell (1995) used a grid of equally spaced data in evaluating the performance of the AIM production model.

[13] Gallant and Golub (1984) used the same procedure for stochastic data generation with a production model.

budget constraint causes $\mathbf{\Omega}$ to be singular, we drop one equation, impose all restrictions by substitution, and compute the maximum likelihood estimates of the reparameterized model. Barten (1969) proved that consistent estimates can be obtained in this manner, with the estimates being invariant to the equation omitted. The unconstrained optimization is computed by MATLAB's Quasi-Newton algorithm. A complete set of parameters is recovered using the associated restrictions.

*A priori* there is no known optimal method for choosing the vector $\boldsymbol{\alpha}$, so we choose all elements of the vector to be equal. Some authors have experimented with alternative settings, such as setting $\boldsymbol{\alpha}$ as weights to form a Laspeyres-like price index, but with no clear gain over our choice.[14] Hence all elements of the vector $\boldsymbol{\alpha}$ are set at 1/3, as a result of equation (3) with $\mathbf{v}_0 = \mathbf{1}$.

The number of Monte Carlo repetitions is 1000 for each case. We use boxplots to summarize the distribution of the estimated elasticities across the 1000 replicates. We follow the standard procedure for drawing boxplots. The box has lines at the lower quartile, median, and upper quartile values. The "whisker" is a line going through each end of the box above and below the box. The length of the whisker above and below the box equals 1.5 × (the upper quartile value – the lower quartile value). Estimates above or below the whisker are considered outliers. Since we find these distributions of estimates to be asymmetric, standard errors alone cannot capture what is displayed in the boxplots. The average values of each parameter across replications are used to produce the regular regions of the models.

To begin our iterations, we start as follows. We compute the gradient vector and the Hessian matrix of the data-generating function at the point $(\mathbf{p}_0, y_0) = (\mathbf{1}, 1)$. We set that vector and that Hessian of the Normalized Quadratic model to be the same as those of the data-generating model at that point. We then solve for corresponding parameter values of the Normalized Quadratic and use the solution as the starting values for the optimization procedure. Those starting parameter values produce a local second order approximation of the Normalized Quadratic to the generating function. Our starting values facilitate convergence to the global maximum of the likelihood function, since the global maximum is likely to be near the starting point.

### 3.3. Regular Region

Following Jensen (1997), we plot two dimensional sections of the regular region in the Cartesian plane. The $x$-axis represents the natural logarithm of $(p_2/p_1)$, and the $y$-axis the natural

[14] See Kohli (1993) and Diewert and Wales (1992).

logarithm of $(p_3/p_1)$. Each axis ranges between $log(0.2/5.0) \approx -3.2189$ and $log(5.0/0.2) \approx 3.2189$. The sample range is defined as the convex hull of possible prices within the above intervals of relative values and is displayed in our figures as a rectangle in the center of the graph. Accordingly, each of the four sides of the rectangle range from $log(0.5/1.5) \approx -1.0986$ to $log(1.5/0.5) \approx 1.0986$, since our data is generated from the interval of [0.5, 1.5]. The entire section is divided into 150 × 150 grid points, at which the monotonicity and curvature conditions are evaluated. Each plot can be viewed as a display of a 2-dimensional hyperplane through the 3-dimensional space having dimensions $log(p_2/p_1)$, $log(p_3/p_1)$, and $log(y)$. We section the regular region perpendicular to the $log(y)$ axis at the reference point of $y = 1.0$. It is desirable to plot several hyperplanes at different settings of $y$, as done by Barnett, Lee and Wolfe (1985,1987) to investigate the full 3-dimensional properties of the model's regular region. But since regularity is usually satisfied at the reference point and violations increase as data points move away from the reference point, the emergence of regularity violations on the single hyperplane with $y$ fixed at the reference setting is sufficient to illustrate deficiencies of the Normalized Quadratic Model.

The monotonicity condition is evaluated using the gradient vector of the estimated equation (1), $\nabla h(\mathbf{v})$. The model is required by theory to be strictly increasing in **v**. For each grid point at which the gradient is evaluated, the monotonicity condition is satisfied, if $\nabla h > 0$.

Our approach to evaluation of the curvature condition differs from that used in most studies. In those other studies, the curvature condition is judged to be satisfied, if the Allen elasticity of substitution matrix or the Slutsky substitution matrix is negative semi-definite.[15] The problem is that satisfaction of the monotonicity condition is required for those matrix conditions to be necessary and sufficient for satisfaction of the curvature condition. Hence, we do not use substitution matrices to evaluate the curvature condition.

We evaluate the quasiconcavity of the equation (1) directly using the method proposed by Arrow and Enthoven (1961). Quasiconcavity is checked by confirming alternating signs of the principal minors of the bordered Hessian matrix, which contains the second partial derivatives bordered by first derivatives. This approach is general, regardless of whether there are any monotonicity violations. The appendix formalizes the procedure for checking quasiconcave.

Each grid is filled with different gradations of black and white, designating the evaluation results for the regularity conditions. The completely black grid designates violations of both curvature and monotonicity; the very dark grey gird designates violation of only curvature; and

[15] For example, Serletis and Shahmoradi (2005) computed the Cholesky values of the Slutsky matrix to evaluate its negative semi-definiteness. A matrix is negative semi-definite, if its Cholesky factors are non-positive (Lau, 1978).

the very light grey grid specifies violation of only monotonicity. The completely white regions are fully regular. There are 8 cases with 3 models each, resulting in 24 plots of regular regions.

## 4. Results and Discussion

We confirm convergence to a global maximum of the likelihood function by comparing the estimated elasticity values with the true ones. If the discrepancy is large, we discard that run, and rerun the program. We do not seek to explain the cases of large discrepancies, other than to conclude that under such circumstances, an unresolved problem exists. Based on this criterion, we encounter substantial difficulty in the estimation of the model with local curvature imposed. When data are generated with elasticities of substitution greater than unity (cases 3, 4, and 5), the estimates converge to values far from the true ones. But when we try the somewhat lower elasticities of $\sigma_{12} = \sigma_{13} = \sigma_{23} = 1.10$ or 1.20, convergence to reasonable estimates is more successful, although some replications still often yield unreasonable estimates.

The boxplots of case 4 in figure 1 describe the distributions of 1000 estimates of the model's elasticities of substitution with no curvature imposed (left), local curvature imposed

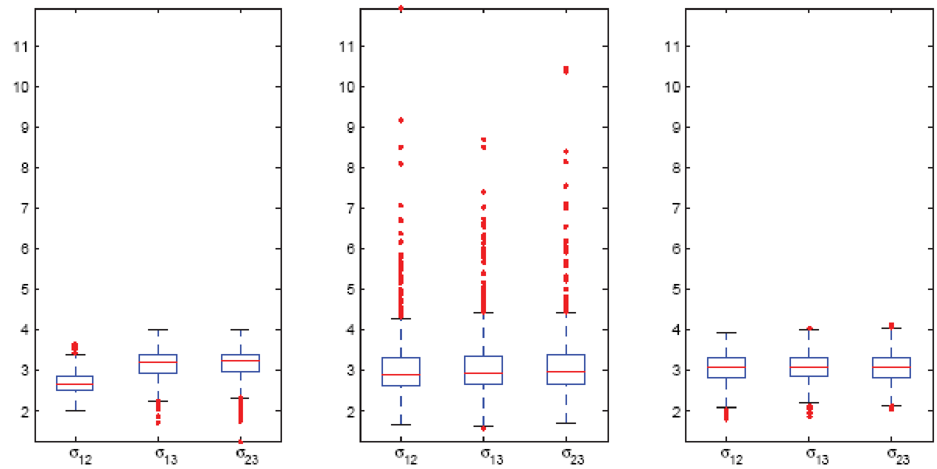


Figure 1: Boxplots of the distributions of estimates for case 4 with the model with no curvature imposed model (left), with local curvature imposed (middle), and with global curvature imposed (right).

(middle), and global curvature imposed (right).[16] The estimates of the local-curvature-imposed elasticities not only have larger variations (described as longer whiskers) than those of the no curvature imposed and global-curvature-imposed elasticities, but also include a number of severe

[16] All data used for boxplots in this paper are generated with $\mu = 0.20$ setting of the noise adjustment constant.

outliers especially in the positive direction. Even when estimates of cross elasticities are reasonable, estimates of own elasticities (not included in this paper) were often found to be far from the true values. In fact, the unconstrained and global-curvature-imposed models both require very high values of **a** (around 400,000's to 600,000's for all elements**)** to attain maximization of the likelihood function. We conclude that constraint (6) is too restrictive. For the local-curvature-imposed model to approximate a symmetric function, as in cases 3 and 4, the optimal values of **a** should all be zeros, while satisfying the restriction (6). However, any statistical tests would reject the hypothesis that the restriction is valid.

Moreover, as Diewert and Wales (1988b) observed, equation (6) often renders global concavity to be impossible. The only way to achieve global curvature with (6) is to set **a** = **0,** since globally concave requires non-negativity of all elements of **a**. In addition, since **B** is a function of **a** as well as of **b** and **K**, any poor estimate of **a** will produce poor estimation of **B,** which are important parameters in attaining concavity. With all such problems and nonlinearity embedded in the likelihood function, the optimization procedure can search for wrong local maxima. We tried a few global optimization techniques,

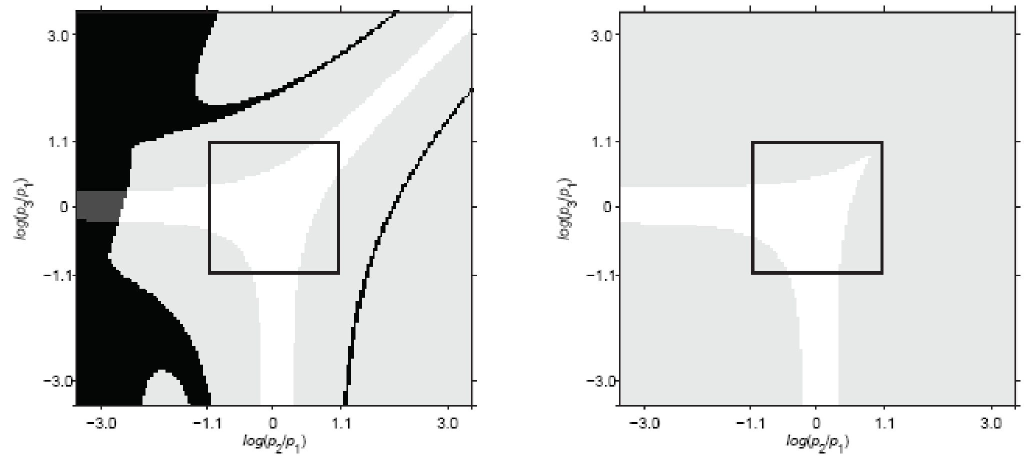


Figure 2: Section through regular regions of the model at $y$ = 1.0 with local curvature imposed and with global curvature imposed for case 4.

but without success.[17] When the point estimates of elasticities are themselves poor, we view regularity violations to be a "higher order problem" of lesser concern. For reference purposes, we plot in figure 2 the regular region of the local-curvature imposed model (left) along with that of

[17] The genetic algorithm and the pattern search method are implemented. But the lack of convergence of the former and the very slow convergence of the latter produce substantial computational burden for any study that requires a large number of repeated simulations. Dorsey and Mayer (1995) provide empirical evaluation in econometric applications of the performance of genetic algorithms versus other global optimization techniques.

the global-curvature imposed model (right). However, the validity of conclusions drawn from that figure is questionable.

In the figure 2 display of the regular region of the model with global curvature imposed, the regularity violations occupy a large part of the area. Within the very light grey areas of the plots, the regularity violations are attributed entirely to monotonicity violations. Severe regularity violations resulted from using data produced with high elasticities of substitution, and therefore the most severe violations occurred in case 4. The plot of case 3 (omitted to conserve on space) displays similar shape of regularity violation regions to case 4, but with a somewhat wider regular region. In case 4, we obtained an almost identical figure with the unconstrained model. The phenomenon of monotonicity-induced regularity violations may be common with many non-globally-regular flexible functional forms and should not be viewed as exclusive to the Normalized Quadratic model.

In case 4 the model substantially underestimates true elasticities. The boxplot (right) in figure 1 shows that the median of the 1000 sample estimates is near 3.0, while the true elasticity

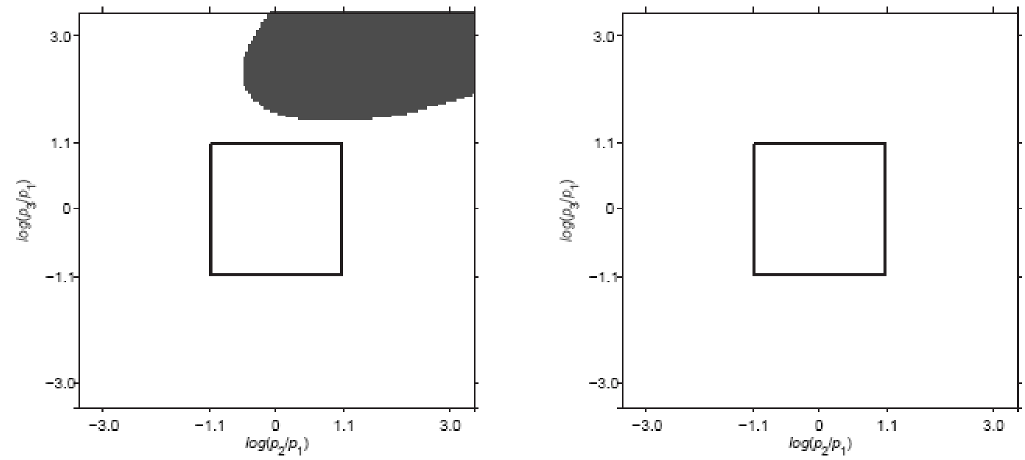


Figure 3: Section through regular regions of the model at $y$ = 1.0 with no curvature imposed and with local and global curvature imposed for case 7.

is 4.0. This finding is similar to those of Guilkey, Lovell and Sickles (1983) and Barnett and Lee (1985), who found that the generalized Leontief model performs poorly when approximating the function with high elasticities of substitution. We find that the Normalized Quadratic model performs poorly both in estimating elasticities and in maintaining regularity conditions, when the data used was produced with high elasticities of substitution.

For cases 1, 2, and 6, all models perform very well with no regularity violations. The Normalized Quadratic model performs well, when the data is characterized by low elasticities of substitution (below unity) and pairwise elasticities are relatively close to each other. The plot with case 5 is omitted to conserve on space, since the result is similar to case 4.[18]

With case 7, the plot on the left in figure 3 displays a very dark grey cloud on the top of the plot, designating curvature violations for the unconstrained model. In this case, imposing curvature locally as well as globally eliminates all of the curvature violations within the region of the data, as shown by the entirely white region in the right plot. Figure 4 describes the distributions of estimates in case 7. For all three models, the median estimates are satisfactorily close to true elasticities.

With global curvature imposed, elasticity estimates are severely downward biased when the true elasticities are high, and upward biased when the true elasticities are low as outliers. It

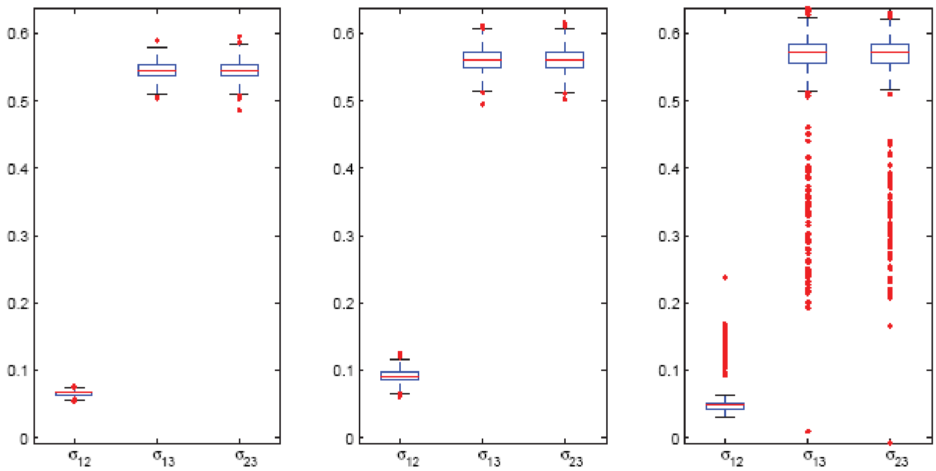


Figure 4: Boxplots of the distributions of estimates for case 7 with the unconstrained model (left), local-curvature-imposed model (middle), and global-curvature-imposed model (right).

suggests that all pairwise elasticities become close to each other. As should be expected by outliers, the cause is not easy to determine, and we do not impute much importance to results with outliers. These problems do not arise in case 7, when we impose curvature locally, which succeeds in producing global regularity within the region of our data.

Figures 5 displays case 8 plots of the regular regions for the unconstrained (left) and the local-curvature constrained models (right). The top of the left plot has a wide thick cloud of curvature violations. That region intersects a small monotonicity-violation area on the right side. The resulting small intersection region designates the set within which both violations occur.

[18] With case 5, there are regularity violations inside the sample range, as in case 4, but to a milder extent, as in case 3, since both case 3 and case 5 use data with lower elasticities of substitutions than case 4. The plot of case 5 is slightly shifted from the center, as a result of the fact that the budget shares at the center point are slightly asymmetric. This plot is very similar to the case I of Jensen (1997). Our case 5 and his case I use the same data-generating setting.

Imposing local curvature does not shrink those regions, but rather expands them. On the right plot, the region of curvature violations now covers much of the 2-dimensional section, with the exception of the white convex regular region and a wide thick pillar of monotonicity violations on the left side. In the intersection of the two irregular regions, both violations occur. Notice that regularity is satisfied at the center point, at which correct curvature is imposed. This pattern of expansion and change of regularity-violation regions is hard to explain.

Another disadvantage of the model is its failure to represent complementarity among goods. A middle boxplot in figure 6 shows that the lower whisker for $\sigma_{12}$ is strictly above zero. A typical estimate of **a** for the unconstrained model was **a** = (0.014, 0.014, 0.262).[19] Although all elements are strictly positive values, they are not substantially different from **a** = **0**, in contrast with case 4. Hence, we do not believe that the inability to characterize complementarity was caused by the restrictiveness of equation (6).

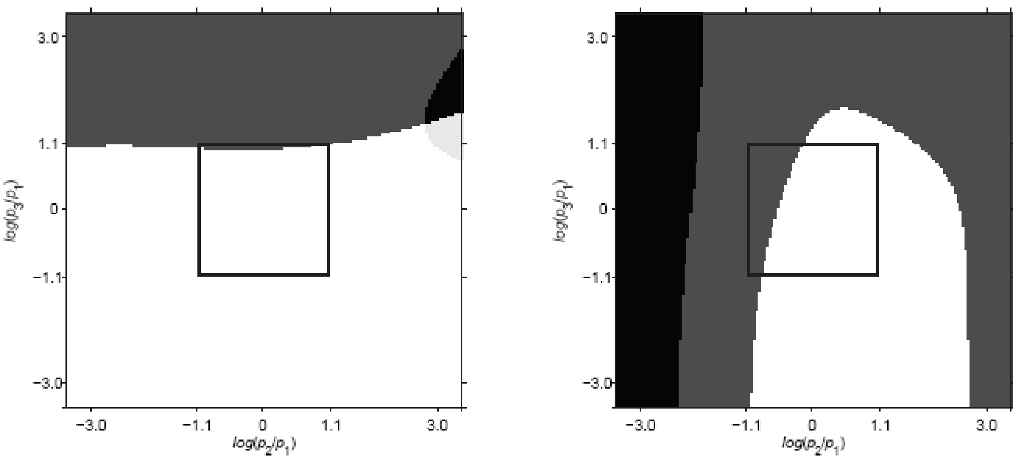


Figure 5: Section through regular regions of the model at $y$ = 1.0 with no curvature imposed and with local curvature imposed for case 8.

[19] The estimate was obtained using noise-free data with sample size of 500 instead of 100.

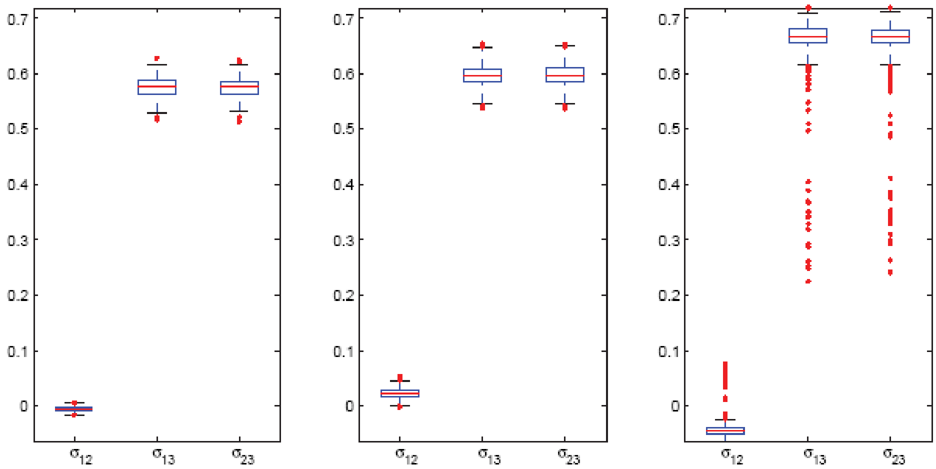


Figure 6: Boxplots of the distributions of estimates for case 8 with the unconstrained model (left), local-curvature-imposed model (middle), and global-curvature-imposed model (right).

In case 8, the global-curvature-imposed model's plot is identical to the right plot in figure 3. However, imposing global regularity can decrease approximation accuracy (Diewert and Wales 1987, Terrell 1996). Comparing the figure 6 boxplot of the unconstrained model's elasticity estimates (left) with those of the global-curvature imposed model (right), relative to the true elasticity values, we see that global-curvature-imposed model overestimates the elasticity of substitution, $\sigma_{12}$, for complementarity and the elasticities of substitution, $\sigma_{13}$ and $\sigma_{23}$ for substitutes. A possible cause is the imposition of negative semi-definiteness on the Hessian matrix. As in figure 4 the outlier estimates cause the pairwise elasticities of substitution estimates to become closer to each other.

## 5. Conclusion

We conduct a Monte Carlo study of the global regularity properties of the Normalized Quadratic model. We particularly investigate monotonicity violations, as well as the performance of methods of locally and globally imposing curvature. We find that monotonicity violations are especially likely to occur, when elasticities of substitution are greater than unity. We also find that imposing curvature locally produces difficulty in the estimation, smaller regular regions, and the poor elasticity estimates in many cases considered in the paper.

When imposing curvature globally, our results are better. Although violations of monotonicity remain common in some of our cases, those violations do not appear to be induced solely by the global curvature imposition, but rather by the nature of the Normalized Quadratic model itself. However, imposition of global curvature does induce a problem with complementary goods by biasing the estimates towards over complementarity and substitute goods towards over substitutability.

With the Normalized Quadratic model, we find that both curvature and monotonicity must be checked with the estimated model, as has previously been shown to be the case with many other flexible functional forms. Imposition of curvature alone does not assure regularity, and imposing local curvature alone can have very adverse consequences.

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## Appendix: Theorem (Arrow and Enthoven, 1961)

Let $f$ be a twice differentiable function on the open convex set $C \subset \mathrm{R}_{+}^{n}$. Define the determinants $D_k(\mathbf{x})$, $k = 1, \ldots, n$, by

$$D_k(\mathbf{x}) = \begin{vmatrix} 0 & \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_k} \\ \frac{\partial f}{\partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial x_k} & \frac{\partial^2 f}{\partial x_k \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_k \partial x_k} \end{vmatrix}. \tag{A.1}$$

A sufficient condition for $f(\mathbf{x})$ to be quasi-concave for $\mathbf{x} \geq 0$ is that the sign of $D_k$ be the same sign of $(-1)^k$ for all $\mathbf{x}$ and all $k = 1,\ldots, n$. A necessary condition for $f$ to be quasi-concave is that $(-1)^k D_k \geq 0$, for $k = 1,\ldots,n$, for all $\mathbf{x}$.