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EXISTENCE OF EQUILIBRIA FOR ECONOMIES WITH EXTERNALITIES AND A MEASURE SPACE OF CONSUMERS

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Existence of equilibria for economies with externalities and a measure space of consumers

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Abstract

This paper considers an exchange economy with a measure space of agents and consumption externalities, which take into account two possible external effects in consumers' preferences: the dependence upon prices and other agents' consumptions, respectively, as in Greenberg et al. [12] and Khan and Vohra [15] (see also Balder [4] for a general discussion). This allows to cover a general model of reference coalitions externalities, in which the agents' preferences are influenced by the global (or the mean) consumption of the agents in the finitely many reference coalitions. Our paper provides a general existence theorem of equilibria that extends previous results by Schmeidler [21], in the case of fixed reference coalitions and Noguchi [17], for a more particular concept of reference coalitions.

Key Words: Externalities, Reference coalitions, Measure space of agents, Equilibrium.
JEL Classification Numbers: D62, D51, H23

1 Introduction[1]

This paper considers consumption externalities in an exchange economy with a measure space of agents and takes into account two external effects in consumers' preferences: the dependence upon prices and upon other agents' consumptions. This question has been extensively studied in the case of finitely many agents (see, for example the book by Laffont [16]) and the case of a measure space of consumers had a recent revival of interest since the article by Balder [4] pointing out the inherent difficulties in this framework.

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The price dependence externality is a long recognized problem, which found recently new applications in the study of financial markets, where a two-period temporary equilibrium model has a reduced form as a Walrasian model with price dependent preferences. For the existence of equilibria in economies with a measure space of agents and price externalities we refer to Greenberg et al. [12], who use the concept of "abstract economies" introduced by Arrow and Debreu [1].

The dependence upon other agents' consumptions has also been considered in the last years, with attempts to extend the equilibrium existence result with interdependent preferences by Shafer and Sonnenschein [18]. We mention the paper of Khan and Vohra [15], which uses also the same concept of "abstract economies" as in [18] and we refer to Balder [4] for a discussion on the limitations of the assumptions made in [15].

In the present paper we propose a model with "finitely many externality effects", i.e., formally, the *externality space* E is assumed to be a subset of a finite dimensional Euclidean space and the externality $e \in E$ summarizes the externality effects coming both from the dependence upon prices and other agents' consumptions. Thus, the preferences of each agent a may depend upon the externality $e \in E$ and we denote by $\prec_{a,e}$ her preference relation. Formally, an *externality mapping* Φ is given, and this mapping associates to each agent a, each price p and each (integrable) consumption allocation f, the externality $e = \Phi(a,p,f) \in E$, which influences agent a's preferences, in the sense that the equilibrium choice of agent a will be made with the preference relation $\prec_{a,\Phi(a,p,f)}$. The consideration of finitely many externality effects makes an explicit restriction on the couples (p, f) of prices and (integrable) consumption allocation that can influence agents' preferences via the externality mapping Φ. The previous model contains, in particular, the case of reference coalitions externalities that we will now present. Let $(A, \mathcal{A}, \nu)$ be the measure space of consumers, then the *reference coalitions model* associates to each agent a and each price p, finitely many *reference coalitions* $C_k(a,p) \in \mathcal{A}$ $(k = 1, \ldots, K)$, which may influence the tastes of agents a in one of the two following ways. Each coalition $C_k(a,p) \in \mathcal{A}$ can be considered as the reference class of agent a for a particular group of commodities, say clothes, music, housing, travels... The externality dependence operates via *reference consumption vectors* (for the particular group of commodities) which can be obtained either as the total or as the mean consumption of agents in the reference coalition of agent a. With a single reference coalition (i.e. $K = 1$), the externality mapping can be written as follows in the cases of total (resp. mean) consumption dependence:

$$\Phi_1(a,p,f) = \int_{C(a,p)} f(\alpha) d\nu(\alpha)$$

$$\Phi_2(a,p,f) := \begin{cases} \frac{1}{\nu[C(a,p)]} \int_{C(a,p)} f(\alpha) d\nu(\alpha) & \text{if } \nu[C(a,p)] > 0 \\ 0 & \text{if } \nu[C(a,p)] = 0 \end{cases}$$

Both models consider finitely many external effects, with the externality space $E = \mathbb{R}^H_+$, the closed positive orthant of the commodity space $\mathbb{R}^H$, denoting by H the number of commodities in the economy. With the first externality mapping, Φ_1, the preferences of agent a are influenced by the total consumption of the agents in the reference coalition, whereas, with the second externality mapping, Φ_2, the preferences of agent a are influenced by the mean consumption of the agents in the reference coalition. In the first model, the quantity of the commodities and the number of persons consuming the particular (group of) commodities is important as, for example, in the case of network effects: the number of persons connected to a network (internet or mobile phone...) in the reference coalition is important for an agent to decide to connect herself, i.e., buy this particular commodity, whereas, in the second model, only the average consumption is important to define the "reference trend".

The main aim of this paper is to provide a general existence result of equilibria in the case of finitely many external effects and, then, to deduce from it an existence result in the reference coalitions model with the two externality mappings (global and mean dependence) as defined above. Our result encompasses the result by Schmeidler [19] in the case of constant reference coalitions (hence do not depend on the price system). We also generalize the existence result by Noguchi [17] who considers a particular reference coalition, which consists of all the agents who belong to a certain income range associated with agent a (see Section 3.3). Finally, we mention also the similar model considered by Balder [5], which is not comparable to the present model in terms of existence result of equilibria.

The paper is organized as follows. In Section 2, we present the model and we state the main existence result; the model of exchange economies with general externality mappings and the concept of equilibrium are presented [Section 2.1], we state our first existence result [Section 2.2] and we weaken the convexity assumption of preferences [Section 2.3] to be able to emcompass Aumann-Hildenbrand existence result. In Section 3, we present the reference coalitions model [Section 3.1], and we deduce from our main theorem the existence of equilibria in this model [Section 3.2]; finally, we present the particular case of reference coalitions model considered by Noguchi [17]. The proof of the main existence result [Theorem 2.2] is given in Section 4. We first prove an existence result [Theorem 4.1] under the additional assumption that the consumption sets correspondence is integrably bounded [Section 4.1]. We then deduce from it the main result in the general case [Section 4.2]. Finally, the Appendix presents the main properties of the individual quasi-demand that are used in the proof of the existence theorem.

2 The model and the existence result

2.1 The model and the equilibrium notion

We consider an exchange economy with a finite set H of commodities. The commodity space[2] is represented by the vector space $\mathbb{R}^H$.

[2]For a finite set H we denote by $\mathbb{R}^H$ the set of all mappings from H to $\mathbb{R}$. An element x of $\mathbb{R}^H$ will be denoted by $(x_h)_{h\in H}$ or simply by (x_h) when no confusion is possible. For two elements $x = (x_h), x' = (x'_h)$ in $\mathbb{R}^H$, we denote by $x \cdot x' = \sum_{h\in H} x_h x'_h$ the scalar product, by $\|x\| = \sqrt{x \cdot x}$ the Euclidian norm and by $B(x_0, r) = \{x \in \mathbb{R}^H \mid \|x - x_0\| \leq r\}$ the closed ball. For $X \subset \mathbb{R}^H$, we denote by $\mathrm{int}X$, $\overline{X}$ and $\mathrm{co}X$, respectively, the interior, the closure and the convex hull of X. The

The set of consumers is defined by a positive, finite, complete measure space $(A, \mathcal{A}, \nu)$, where $\mathcal{A}$ is a σ-algebra of subsets in A and ν is a positive, finite measure on $\mathcal{A}$. An element $C \in \mathcal{A}$ is a possible group of consumers, also called a coalition.

Each consumer a is endowed with a consumption set $X(a) \subset \mathbb{R}^H$, an initial endowment $\omega(a) \in \mathbb{R}^H$ and a strict preference relation $\prec_{a,e}$ on $X(a)$, which allows the dependence on externalities $e \in E$ (called the *externality space*), in a way which will be specified hereafter.

The set $X(a)$ represents the possible consumptions of consumer a. A consumption allocation of the economy specifies the possible consumptions of each consumer, hence is a selection of the correspondence $a \mapsto X(a)$, which is additionally assumed to be integrable. The set of consumption allocations is denoted by $\mathcal{L}_X$.

Specific to this economy is the fact that externalities can influence the preference relation of each agent a. Externalities summarize the effects of the price and the choices made by the other consumers on the preference relation of agent a. Thus, given the price $p \in \mathbb{R}^H$ and the allocation $f \in \mathcal{L}_X$, the choices of agent a will be made with the strict preference relation $\prec_{a,\Phi(a,p,f)}$, where $\Phi : A \times \mathbb{R}^H \times \mathcal{L}_X \to E$ is a given mapping, called the *externality mapping.*

We assume also that the initial endowment mapping $\omega : A \to \mathbb{R}^H$ is integrable and thus the total initial endowment of the economy is $\omega := \int_A \omega(a) d\nu(a)$.

In the presence of externalities, the exchange economy is completely summarized by the couple $(\mathcal{E}, \Phi)$, where $\mathcal{E}$ specifies the characteristics of the consumers

$$\mathcal{E} = \{\mathbb{R}^H, E, (A, \mathcal{A}, \nu), (X(a), (\prec_{a,e})_{e \in E}, \omega(a))_{a \in A}\},$$

E describes the externality space and the externality mapping $\Phi : A \times \mathbb{R}^H \times \mathcal{L}_X \to E$, specifies the way externalities are acting on individual preferences.

We now give the definition of an equilibrium in this economy.

Definition 2.1 *An equilibrium of the economy $(\mathcal{E}, \Phi)$ is an element $(f^*, p^*) \in \mathcal{L}_X \times \mathbb{R}^H$ such that $p^* \neq 0$ and*
(a) *[Preference Maximization] for a.e. $a \in A$, $f^*(a)$ is a maximal element for $\prec_{a,{e^*}_a}$ in the budget set $B(a, p^*) := \{x \in X(a) \mid p^* \cdot x \leq p^* \cdot \omega(a)\}$, where ${e^*}_a := \Phi(a, p^*, f^*)$, that is, $f^*(a) \in B(a, p^*)$ and there is no $x \in B(a, p^*)$ such that $f^*(a) \prec_{a,e^*_a} x$;*

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notations: $x \leq x'$, $x < x'$, $x << x'$ mean, respectively, that for all $h \in H$, $x_h \leq x'_h$, $[x \leq x'$ and $x \neq x']$, and $x_h < x'_h$. We denote by $\mathbb{R}^H_+ := \{x \in \mathbb{R}^H \mid 0 \leq x\}$ and by $\mathbb{R}^H_{++} := \{x \in \mathbb{R}^H \mid 0 << x\}$. We let $\mathbf{1} := (1, \ldots, 1) \in \mathbb{R}^H$ and the canonical basis $\{\mathbf{e^i} \mid i \in H\}$ of $\mathbb{R}^H$, defined by $e^i_h = 1$, if $h = i$ and $e^i_h = 0$, if $h \neq i$.

For a measure space $(A, \mathcal{A}, \nu)$, we recall that a measurable set $\bar{A} \in \mathcal{A}$ is called an atom if $\nu(\bar{A}) > 0$ and for every $C \in \mathcal{A}$ such that $C \subset \bar{A}$, one has $[\nu(C) = 0$ or $\nu(\bar{A} \setminus C) = 0]$ and we denote by A_{na} the nonatomic part of A, that is, the complementary in A of the union of all the atoms of A. We denote also by $L^1(A, \mathbb{R}^H)$ the space of equivalence classes of integrable mappings from A to $\mathbb{R}^H$ and we let $\|f\|_1 := \int_A \|f(a)\| d\nu(a)$, which defines a norm on $L^1(A, \mathbb{R}^H)$. The space $L^1(A, \mathbb{R}^H)$ will be endowed with two different topologies, the norm topology defined by the norm $\|f\|_1$ and the weak topology $\sigma(L^1, L^\infty)$; we recall that a sequence $\{f^n\}$ converges weakly to f if and only if $\sup_n \|f^n\|_1 < \infty$ and $\int_C f^n(a) d\nu(a) \to \int_C f(a) d\nu(a)$, for every $C \in \mathcal{A}$. For C_1, C_2 in $\mathcal{A}$, we let $C_1 \Delta C_2 := (C_1 \setminus C_2) \cup (C_2 \setminus C_1)$ and we define the characteristic function $\chi_{C_1} : A \to \mathbb{R}$ by $\chi_{C_1}(a) = 1$ if $a \in C_1$ and $\chi_{C_1}(a) = 0$ if $a \notin C_1$.

(b) [*Market Clearing*] $\int_A f^*(a)d\nu(a) = \int_A \omega(a)d\nu(a)$.

2.2 The existence result for general externality mappings

We present the list of assumptions that the economy $(\mathcal{E}, \Phi)$ will be required to satisfy.
Assumption A *The measure space $(A, \mathcal{A}, \nu)$ is positive, finite, complete and $L^1(A, \mathbb{R}^H)$ is separable for the norm topology;*

Assumption C [Consumption Side] *for a.e. $a \in A$ and every $(e, x) \in E \times X(a)$:*
(i) *E is a subset of a Euclidian space $\mathbb{R}^K$ and $X(a)$ is a closed, convex subset of $\mathbb{R}^H_+$;*
(ii)*[Irreflexivity and transitivity] $\prec_{a,e}$ is irreflexive*[3] *and transitive*[4]*;*
(iii)*[Convexity of preferences on atoms] if a belongs to some atom $\bar{A}$ of A, then the set $\{x' \in X(a) \mid not[x' \prec_{a,e} x]\}$ is convex;*
(iv)*[Continuity] the sets $\{x' \in X(a) \mid x \prec_{a,e} x'\}$ and $\{(x', e') \in X(a) \times E \mid x' \prec_{a,e'} x\}$ are open, respectively, in $X(a)$ and in $X(a) \times E$ (for their relative topologies);*
(v)*[Measurability] the consumption set correspondence $a' \mapsto X(a')$ and the preference correspondence $(a', e') \mapsto \prec_{a',e'}$ are measurable*[5]*;*
(vi) *$\omega \in \mathcal{L}_X$, i.e., $\omega : A \to \mathbb{R}^H$ is integrable and $\omega(a') \in X(a')$ for a.e. $a' \in A$;*
Assumption M(i)*[Monotonicity] for a.e. $a \in A$, $X(a) := \mathbb{R}^H_+$ and for every $e \in E$ and every x, x' in $X(a)$, $x < x'$ implies $x \prec_{a,e} x'$;*
(ii)*[Strong survival] $\int_A \omega(a)d\nu(a) >> 0$.*

The above assumptions are standard and need no special comments. In a model without externalities (say $E = \{0\}$), they coincide with Aumann-Schmeidler's assumptions, as discussed in the next section.

The next assumption concerns the externality mapping.

Assumption E [Externality Side] *For every $(a, p) \in A \times \mathbb{R}^H$, $\Phi(a, p, f) = \Phi(a, p, g)$ if $f = g$ almost everywhere on A. Without any risk of confusion, this allows us to consider Φ as a mapping $\Phi : A \times \mathbb{R}^H \times L_X \to E$, where*

$$L_X := \{f \in L^1(A, \mathbb{R}^H) \mid f(a) \in X(a) \text{ a.e. } a \in A\};$$

EC*[Caratheodory] E is a subset of a Euclidean space $\mathbb{R}^K$ and the mapping Φ is Caratheodory-type, i.e., (i) for every $(p, f) \in \mathbb{R}^H_+ \times L_X$, the mapping $a \mapsto \Phi(a, p, f)$ is measurable on A and (ii) for a.e. $a \in A$ and for every sequence $\{p^n\} \subset \mathbb{R}^H_+$ converging to p and every integrably bounded*[6] *sequence $\{f^n\} \subset L_X$ converging weakly to f, the sequence $\{\Phi(a, p^n, f^n)\}$ converges to $\Phi(a, p, f)$;*

[3] for every $x \in X(a)$, not$[x \prec_{a,e} x]$.

[4] for every $x, x', x'' \in X(a)$, $x \prec_{a,e} x'$ and $x' \prec_{a,e} x''$ imply $x \prec_{a,e} x''$.

[5] We recall that a correspondence F, from a measurable space $(A, \mathcal{A})$ to $\mathbb{R}^n$, is said to be $\mathcal{A}-$measurable if its graph is a measurable set, i.e., $G_F := \{(a, x) \in A \times \mathbb{R}^H \mid x \in F(a)\}$ belongs to $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^n)$, where $\mathcal{B}(\mathbb{R}^n)$ denotes the $\sigma-$algebra of Borel subsets of $\mathbb{R}^n$ and $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^n)$ the $\sigma-$algebra product. The preference correspondence $(a, e) \mapsto \prec_{a,e}$ is measurable in the sense that the correspondence $(a, e) \mapsto \{(x, x') \in X(a) \times X(a) \mid x \prec_{a,e} x'\}$ is $\mathcal{A} \otimes \mathcal{B}(E)-$measurable.

[6] that is, for some integrable function $\rho : A \to \mathbb{R}_+$, $\sup_n \|f^n(a)\| \leq \rho(a)$ for a.e. $a \in A$.

EB[*Boundedness*] *if the sequence* $\{(p^n, f^n)\} \subset \mathbb{R}^H_+ \times L_X$ *is (norm-)bounded, then, for every* $a \in A$, *there exists a subsequence of* $\{\Phi(a, p^n, f^n)\}$ *which is bounded in* E.

The above Caratheodory assumption is a standard regularity assumption. The boundedness assumption, which is the key assumption of the model, will be satisfied in the reference coalitions model presented hereafter. We point out that **EB** is also satisfied when the correspondence $a \mapsto X(a)$ is integrably bounded (see Assumption **IB** hereafter) and **C** and **EC** hold.

The last assumption strengthen the convexity of preferences, which needs to be assumed also on the nonatomic part in the general case (see Section 2.3 for a weakening of this assumption and the meaning of its notation).

Assumption $\mathbf{ECL}_0$ *for a.e.* $a \in A_{na}$ *and every* $(e, x) \in E \times X(a)$, *the set* $\{x' \in X(a) \mid not[x' \prec_{a,e} x]\}$ *is convex.*

We can now state our first existence result.

Theorem 2.1 *The exchange economy with externalities* $(\mathcal{E}, \Phi)$ *admits an equilibrium* (f^*, p^*) *with* $p^* >> 0$, *if it satisfies Assumptions* **A**, **C**, **M**, **E** *and* $\mathbf{ECL}_0$.

The above Theorem 2.1 is a direct consequence of a more general result [Theorem 2.2] that will be stated in the following section, which is devoted to the weakening of the convexity assumption $\mathbf{ECL}_0$.

2.3 Weakening the convexity assumption $\mathbf{ECL}_0$

Since Aumann [2], most of the existence results in models without externalities (say $E = \{0\}$) do not assume the convexity of the preferences on the nonatomic part A_{na} of the measure space of consumers (i.e., Assumption $\mathbf{ECL}_0$). Without Assumption $\mathbf{ECL}_0$, the above existence result may not be true in a general model with externalities.

The aim of this section is to show, however, that we can weaken the convexity assumption $\mathbf{ECL}_0$ to encompass the known results of the literature in the three following important cases.

$\mathbf{E}_1$: **No-externalities** [Aumann [2], Schmeidler [19], Hildenbrand [13]] $E_1 = \{0\}$ *and the mapping* $\Phi_1 : A \times \mathbb{R}^H_+ \times L_X \to E_1$ *is defined by* $\Phi_1(a, p, f) = 0$.

$\mathbf{E}_2$: **Price dependent preferences** [Greenberg et al. [12]] $E_2 = \mathbb{R}^H_+$ *and the mapping* $\Phi_2 : A \times \mathbb{R}^H_+ \times L_X \to E_2$ *is defined by* $\Phi_2(a, p, f) = p$.

$\mathbf{E}_3$: **Constant reference coalitions** [Schmeidler [21]] $E_3 = (\mathbb{R}^H_+)^K$ *and the mapping* $\Phi_3 : A \times \mathbb{R}^H_+ \times L_X \to E_3$ *is defined by*

$$\Phi_3(a, p, f) := (\int_{C_1} f(a) d\nu(a), \ldots, \int_{C_K} f(a) d\nu(a)),$$

where the sets C_k $(k = 1, \ldots, K)$ *are nonempty measurable subsets of* A_{na}, *which are pairwise disjoint, i.e.,* $C_j \cap C_k = \emptyset$ *for every* $j \neq k$.

In the three above cases, the externality mappings Φ_i $(i = 1, 2, 3)$ satisfy the *Lyapunov property* on A_{na}, in the sense that, for $C = A_{na}$ and $\Phi = \Phi_i$ the following holds (see Proposition 2.1 for the proof):

Lyapunov property of Φ on C: *for a.e.* $a \in A$ *and every* $p \in \mathbb{R}^H_+$, *for every* $\{f_i\}_{i\in I} \subset L_X$ *(I finite) and every* $f \in L_X$ *such that, for a.e.* $\alpha \in C$, $f(\alpha) \in co\{f_i(\alpha) \mid i \in I\}$, *there exists* $f^* \in L_X$ *such that, for a.e.* $\alpha \in C$, $f^*(\alpha) \in \{f_i(\alpha) \mid i \in I\}$, *for a.e.* $\alpha \in A \setminus C$, $f^*(\alpha) = f(\alpha)$, $\Phi(a,p,f) = \Phi(a,p,f^*)$ *and* $\int_A f(\alpha)d\nu(\alpha) = \int_A f^*(\alpha)d\nu(\alpha)$.

We now can state our main existence result, which extends Theorem 2.1 and allows also to cover the three above cases $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$. For this, we need to introduce a new Convexity-Lyapunov Assumption, which is clearly satisfied in the two important cases: (i) convexity of the preferences on A_{na} (i.e., Assumption $\mathbf{ECL}_0$ of Theorem 2.1), and (ii) Lyapunov property of Φ on A_{na}.

Theorem 2.2 *The exchange economy with externalities* $(\mathcal{E},\Phi)$ *admits an equilibrium* (f^*, p^*) *with* $p^* >> 0$, *if it satisfies Assumptions* **A**, **C**, **M**, **E**, *together with the following one:*
Assumption ECL *There exists a measurable set* $C \subset A_{na}$ *such that: (i) the preferences are convex*[7] *on* $A_{na} \setminus C$ *and (ii) the externality mapping* Φ *satisfies the Lyapunov property on* C.

We end this section with a proposition showing that, in the three above cases $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$, the externality mappings Φ_i $(i = 1, 2, 3)$ satisfy Assumption **ECL**, and also a stronger Assumption $\mathbf{ECL'}$ (in which no convexity assumption on preferences is made).

Proposition 2.1 *(a) In the three above cases* $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$, *the externality mappings* $\Phi = \Phi_i$ $(i = 1, 2, 3)$ *satisfy the following assumption:*
$\mathbf{ECL'}$ *There exists a measurable set* $C \subset A_{na}$ *such that:* (i) *the externality mapping* Φ *only depends on* $f_{|C}$, *in the sense that,* $\Phi(a,p,f) = \Phi(a,p,g)$, *if* $f_{|C} = g_{|C}$, *and* (ii) *the externality mapping* Φ *satisfies the Lyapunov property on* C.
(b) If Assumption $\mathbf{ECL'}$ *holds, then* Φ *satisfies the Lyapunov property on* A_{na}, *hence Assumption* **ECL** *holds.*

Proof. (a) Assumption $\mathbf{ECL'}$ is satisfied for $C = A_{na}$ for the cases $\mathbf{E}_1$ and $\mathbf{E}_2$ and for $C = \cup_{i=1}^N C_i$ for $\mathbf{E}_3$. This is a consequence of Lyapunov's theorem, applied to A_{na} in the first two cases and applied successively to each C_i $(i = 1, \dots, N)$ in the latter case.

(b) We show that the externality mapping Φ satisfies the Lyapunov property on A_{na}. Indeed, for a.e. $a \in A$ and every $p \in \mathbb{R}^H_+$, let $\{f_i\}_{i\in I} \subset L_X$ (I finite) and $f \in L_X$ such that, for a.e. $\alpha \in A_{na}$, $f(\alpha) \in \text{co}\{f_i(\alpha) \mid i \in I\}$. Then, for a.e. $\alpha \in C$, $f(\alpha) \in \text{co}\{f_i(\alpha) \mid i \in I\}$ and, since Φ satisfies the Lyapunov property on C (by $\mathbf{ECL'}$), there exists an integrable mapping $f' : A \to \mathbb{R}^H$ such that, for a.e. $\alpha \in C$, $f'(\alpha) \in \{f_i(\alpha) \mid i \in I\}$, $\Phi(a,p,f) = \Phi(a,p,f')$ and $\int_C f(\alpha)d\nu(\alpha) = \int_C f'(\alpha)d\nu(\alpha)$. From above, for a.e. $\alpha \in A_{na} \setminus C$, $f(\alpha) \in \text{co}\{f_i(\alpha) \mid i \in I\}$, hence, from Lyapunov's theorem, there exists an integrable mapping $f'' : A_{na} \setminus C \to \mathbb{R}^H$ such that $f''(\alpha) \in \{f_i(\alpha) \mid i \in I\}$ and $\int_{A_{na}\setminus C} f(\alpha)d\nu(\alpha) = \int_{A_{na}\setminus C} f''(\alpha)d\nu(\alpha)$. We consider now the mapping $f^* : A \to \mathbb{R}^H$ defined by $f^*(\alpha) = f'(\alpha)$ for every $\alpha \in C$, $f^*(\alpha) = f''(\alpha)$ for every $\alpha \in A_{na} \setminus C$ and $f^*(\alpha) = f(\alpha)$ for every

[7] that is, for a.e. $a \in A_{na} \setminus C$ and every $(e,x) \in E \times X(a)$, the set $\{x' \in X(a) \mid \text{not}[x' \prec_{a,e} x]\}$ is convex.

$\alpha \in A \setminus A_{na}$ and we note that, for a.e. $\alpha \in A_{na}$, $f^*(\alpha) \in \{f_i(\alpha) \mid i \in I\}$ and for a.e. $\alpha \in A \setminus A_{na}$, $f^*(\alpha) = f(\alpha)$. Moreover, from above, $\Phi(a,p,f) = \Phi(a,p,f') = \Phi(a,p,f^*)$ (since $f'_{|C} = f^*_{|C}$) and $\int_A f(\alpha)d\nu(\alpha) = \int_A f^*(\alpha)d\nu(\alpha)$. □

3 The reference coalitions model

3.1 The model and the existence result

The general model of an exchange economy with externalities $(\mathcal{E}, \Phi)$ allows us to consider the reference coalitions model that we now present as an extension of the Schmeidler's model.

We suppose that, given a price $p \in \mathbb{R}^H_+$, each agent a has finitely many reference coalitions of agents, $C_k(a,p) \in \mathcal{A}$ $(k = 1 \dots K)$, whose consumption choices influence the preferences of agent a in a way defined precisely hereafter. Hence, the reference coalitions may depend upon the agent and also on the price that prevails; this differs from Schmeidler's model, in which the reference coalitions are constant. We will assume that each agent a is influenced either by the global consumption or by the mean consumption of agents in the coalition $C_k(a,p)$.

The "global dependence" case is characterized by the externality space $E := (\mathbb{R}^H_+)^K$ and the externality mapping $\Phi_1^{\mathcal{C}} : A \times \mathbb{R}^H_+ \times L_X \to E$ defined by

$$\Phi_1^{\mathcal{C}}(a,p,f) = (\int_{C_1(a,p)} f(\alpha)d\nu(\alpha), \dots, \int_{C_K(a,p)} f(\alpha)d\nu(\alpha)).$$

The "mean dependence" case, is characterized by the externality space $E := (\mathbb{R}^H_+)^K$ and the externality mapping $\Phi_2^{\mathcal{C}} : A \times \mathbb{R}^H_+ \times L_X \to E$, defined by

$$\Phi_2^{\mathcal{C}}(a,p,f) = (\Phi_{21}^{\mathcal{C}}(a,p,f), \dots, \Phi_{2K}^{\mathcal{C}}(a,p,f)),$$

$$\Phi_{2k}^{\mathcal{C}}(a,p,f) := \begin{cases} \frac{1}{\nu[C_k(a,p)]} \int_{C_k(a,p)} f(\alpha)d\nu(\alpha) & \text{if } \nu[C_k(a,p)] > 0 \\ 0 & \text{if } \nu[C_k(a,p)] = 0. \end{cases}$$

The reference coalitions model can thus be summarized by the exchange economies with externalities $(\mathcal{E}, \Phi_1^{\mathcal{C}})$ and $(\mathcal{E}, \Phi_2^{\mathcal{C}})$, where

$$\mathcal{E} = \{\mathbb{R}^H, (\mathbb{R}^H_+)^K, (A, \mathcal{A}, \nu), (X(a), (\prec_{a,e})_{e \in (\mathbb{R}^H_+)^K}, \omega(a))_{a \in A}\},$$

$$\mathcal{C} := (C_1(a,p), \dots, C_K(a,p))_{(a,p) \in A \times \mathbb{R}^H_+},$$

and the externality mappings $\Phi_1^{\mathcal{C}}$ and $\Phi_2^{\mathcal{C}}$ are defined as above (and correspond, respectively, to the global and the mean dependence).

We state the following existence result.

Theorem 3.1 *The exchange economy with reference coalitions externalities* $(\mathcal{E},\mathcal{C})$ *admits an equilibrium* (p_1^*, f_1^*) *with* $p_1^* >> 0$ *for global dependence and an equilibrium* (p_2^*, f_2^*) *with* $p_2^* >> 0$ *for mean dependence (i.e.,* $(\mathcal{E}, \Phi_i^{\mathcal{C}})$ *admits an equilibrium* (p_i^*, f_i^*) $(i = 1, 2)$*), if it satisfies Assumptions* **A**, **C**, **M**, $\mathbf{ECL}_0$*, together with:*

Assumption R [Reference Coalition Side]
For every $k = 1, \dots, K$ *and for every* $(a, p) \in A \times \mathbb{R}_+^H$*:*
(i) $\nu[C_k(a, p)] > 0$*;* **(ii)** *for every* $\lambda > 0$, $C_k(a, \lambda p) = C_k(a, p)$*;*
(iii) *for every sequence* $p^n \to p$ *in* $\mathbb{R}_+^H$, $\nu[C_k(a, p^n)\Delta C_k(a, p)] \to 0$*;*
(iv) *the set* $\{(a', a'') \in A \times A \mid a'' \in C_k(a', p)\} \in \mathcal{A} \otimes \mathcal{A}$.

The proof of Theorem 3.1 is given in Section 3.2.

3.2 Proof of Theorem 3.1

[$\mathbf{EC}(i)$]**:** *For every* $(p, f) \in \mathbb{R}_+^H \times L_X$*, the mapping* $a \mapsto \Phi_i^{\mathcal{C}}(a, p, f)$ $(i = 1, 2)$ *is measurable on* A.

Let $(p, f) \in \mathbb{R}_+^H \times L_X$. We first show that the mapping

$$a \mapsto \Phi_1^{\mathcal{C}}(a, p, f) := \int_{C(a,p)} f(\alpha) d\nu(\alpha)$$

is measurable. We notice that the mappings $(a, \alpha) \to f(\alpha)$ and $(a, \alpha) \to \chi_{C(a,p)}(\alpha)$ are both measurable on $A \times A$ (endowed with the product $\sigma-$algebra $\mathcal{A} \otimes \mathcal{A}$), from the fact that $f \in L^1(A, \mathbb{R}_+^H)$ and Assumption **R(iv)** respectively. Hence, the mapping $(a, \alpha) \to \chi_{C(a,p)}(\alpha) f(\alpha)$ is measurable on $A \times A$.

Since $\chi_{C(a,p)}(\alpha) f(\alpha) \le f(\alpha)$ for a.e. $(a, \alpha) \in A \times A$ and $f \in L^1(A, \mathbb{R}_+^H)$, applying Fubini's theorem, the mapping

$$a \to \int_A \chi_{C(a,p)}(\alpha) f(\alpha) d\nu(\alpha) = \int_{C(a,p)} f(\alpha) d\nu(\alpha)$$

is correctly defined and is measurable on A. Hence, the mapping $\Phi_1^{\mathcal{C}}$ satisfies Assumption $\mathbf{EC}(i)$.

We now show that the mapping

$$a \mapsto \Phi_2^{\mathcal{C}}(a, p, f) := \frac{1}{\nu[C(a, p)]} \Phi_1^{\mathcal{C}}(a, p, f)$$

is measurable on A. Using the above argument for $f = 1$, we deduce that the mapping $a \to \nu[C(a, p)]$ is measurable on A, hence the mapping $a \to \frac{1}{\nu[C(a,p)]}$ is also measurable on A, since $\nu[C(a, p)] > 0$ for every $a \in A$ (by **R(i)**). Then, in view of the measurability property of $\Phi_1^{\mathcal{C}}$, the mapping $\Phi_2^{\mathcal{C}}$ satisfies Assumption $\mathbf{EC}(i)$. □

[$\mathbf{EC}(ii)$]**:** *For every* $a \in A$ *and for every sequence* $\{p^n\}$ *converging to* p *in* $\mathbb{R}_+^H$ *and every integrably bounded sequence* $\{f^n\}$ *converging weakly to* f *in* L_X*, the sequence* $\{\Phi_i^{\mathcal{C}}(a, p^n, f^n)\}$ *converges to* $\Phi_i^{\mathcal{C}}(a, p, f)$ $(i = 1, 2)$.

Let $a \in A$ and let $\{(p^n, f^n)\}$ as above. We first prove that $\Phi_1^{\mathcal{C}}$ satisfies [**EC**(ii)], i.e.,

$$\Phi_1^{\mathcal{C}}(a, p^n, f^n) = \int_{C(a,p^n)} f^n(\alpha) d\nu(\alpha) \to \int_{C(a,p)} f(\alpha) d\nu(\alpha) = \Phi_1^{\mathcal{C}}(a, p, f).$$

For this, one notices that

$$\| \int_{C(a,p^n)} f^n(\alpha) d\nu(\alpha) - \int_{C(a,p)} f(\alpha) d\nu(\alpha) \| \leq$$

$$\| \int_{C(a,p^n)} f^n(\alpha) d\nu(\alpha) - \int_{C(a,p)} f^n(\alpha) d\nu(\alpha) \| + \| \int_{C(a,p)} [f^n(\alpha) - f(\alpha)] d\nu(\alpha) \|.$$

For the second term, since $\{f^n\}$ converges weakly to f, one has

$$\| \int_{C(a,p)} [f^n(\alpha) - f(\alpha)] d\nu(\alpha) \| \to 0.$$

For the first term we have

$$\begin{aligned} &\| \int_A \chi_{C(a,p^n)}(\alpha) f^n(\alpha) d\nu(\alpha) - \int_A \chi_{C(a,p)}(\alpha) f^n(\alpha) d\nu(\alpha) \| \leq \\ &\int_A |\chi_{C(a,p^n)}(\alpha) - \chi_{C(a,p)}(\alpha)| \|f^n(\alpha)\| d\nu(\alpha) \leq \\ &\int_A |\chi_{C(a,p^n)}(\alpha) - \chi_{C(a,p)}(\alpha)| \rho(\alpha) d\nu(\alpha) = \int_{C(a,p^n) \Delta C(a,p)} \rho(\alpha) d\nu(\alpha), \end{aligned}$$

recalling that the sequence $\{f^n\}$ is integrably bounded, hence, for some integrably function $\rho : A \to \mathbb{R}_+$, one has $\sup_n \|f^n(a)\| \leq \rho(a)$ for a.e. $a \in A$. Moreover, $\nu[C(a, p^n) \Delta C(a, p)] \to 0$ when $p^n \to p$ (by **R(iii)**), hence

$$\int_{C(a,p^n) \Delta C(a,p)} \rho(\alpha) d\nu(\alpha) \to 0,$$

since the mapping $C \mapsto \int_C \rho(\alpha) d\nu(\alpha)$, from $\mathcal{A}$ to $\mathbb{R}_+$, is a positive measure, absolutely continuous with respect to ν. This implies that the first term converges to zero[8] and ends the proof that $\Phi_1^{\mathcal{C}}$ satisfies [**EC**(ii)].

We now prove that $\Phi_2^{\mathcal{C}}$ satisfies [**EC**(ii)] and we recall that

$$\Phi_2^{\mathcal{C}}(a, p, f) = \frac{1}{\nu[C(a, p)]} \Phi_1^{\mathcal{C}}(a, p, f).$$

[8]Note: We don't need to use the fact that the sequence $\{f^n\}$ is integrably bounded. Indeed, if $\{f^n\}$ converges weakly to f and $\nu[C(a, p^n) \Delta C(a, p)] \to 0$, one has directly

$$\int_{C(a,p^n) \Delta C(a,p)} f^n(\alpha) d\nu(\alpha) \to 0.$$

For details, see Dunford and Schwartz [9] p.294. Thanks to E. Balder for this remark.

Since $\Phi_1^{\mathcal{C}}(a, p^n, f^n) \to \Phi_1^{\mathcal{C}}(a, p, f)$ and since, for every $p \in \mathbb{R}_+^H$, $\nu[C(a,p)] > 0$ (by **R(i)**), it suffices to show that $\nu[C(a,p^n)] \to \nu[C(a,p)]$. Indeed, one has

$$|\nu[C(a,p^n)] - \nu[C(a,p)]| = |\int_A \chi_{C(a,p^n)}(\alpha) d\nu(\alpha) - \int_A \chi_{C(a,p)}(\alpha) d\nu(\alpha)|$$
$$\leq \int_A |\chi_{C(a,p^n)}(\alpha) - \chi_{C(a,p)}(\alpha)| d\nu(\alpha) = \nu[C(a,p^n)\Delta C(a,p)],$$

which converges to zero (by **R(iii)**) when $p^n \to p$. □

[EB]: *If* $\{(p^n, f^n)\} \subset \mathbb{R}_+^H \times L_X$ *is a (norm-)bounded sequence, then for every* $a \in A$ *there exists a subsequence of* $\{\Phi_i^{\mathcal{C}}(a, p^n, f^n)\}$ $(i = 1, 2)$ *which is bounded in* $\mathbb{R}_+^H$.

Let $\{(p^n, f^n)\}$ as above. For every $a \in A$ and for every n, one has

$$0 \leq \Phi_1^{\mathcal{C}}(a, p^n, f^n) = \int_{C(a,p^n)} f^n(\alpha) d\nu(\alpha) \leq \int_A f^n(\alpha) d\nu(\alpha).$$

Since $\{f^n\}$ is norm-bounded and $f^n \geq 0$, we deduce that for some $m \geq 0$

$$\sup_n \|\Phi_1^{\mathcal{C}}(a, p^n, f^n)\| \leq m.$$

We now prove that $\Phi_2^{\mathcal{C}}(a, p^n, f^n)$ is bounded. Indeed, from above, we get

$$\|\Phi_2^{\mathcal{C}}(a, p^n, f^n)\| = \|\frac{1}{\nu[C(a,p^n)]} \Phi_1^{\mathcal{C}}(a, p^n, f^n)\| \leq m \frac{1}{\nu[C(a,p^n)]}.$$

Since the sequence $\{p^n\}$ is bounded there exists a subsequence $\{p^{n_k}\}$ which converges to some element $p \in \mathbb{R}_+^H$. We recall that, in the previous step, we have proved that, for every $a \in A$, $\frac{1}{\nu[C(a,p^{n_k})]} \to \frac{1}{\nu[C(a,p)]}$ when $p^{n_k} \to p$. Consequently, there exists $m'_a > 0$ such that for k large enough $\frac{1}{\nu[C(a,p^{n_k})]} \leq m'_a$, hence

$$\sup_k \|\Phi_2^{\mathcal{C}}(a, p^{n_k}, f^{n_k})\| \leq m'_a.m.$$

3.3 Noguchi's reference coalitions model

We now present Noguchi's model (see [17]) and we deduce his existence result from Theorem 3.1. It can be described by a reference coalition model, with a unique reference coalition $C_N(a,p)$, defined, for each consumer a at price system p, by

$$C_N(a,p) := \{\alpha \in A \mid p \cdot \omega(\alpha) \in I(\omega(a), \delta(a), p)\},$$

where $\delta : A \to \mathbb{R}_+^H$ is a fixed function and $I(\omega(a), \delta(a), p)$ is a subset of $\mathbb{R}$. For Noguchi, "intuitively speaking, $I(\omega(a), \delta(a), p)$ represents (for agent a) an income range in the income-scale, relative to income $p \cdot \omega(a)$ and with magnitude $p \cdot \delta(a)$" and among the examples given, we point out the following one defined by the interval $I(\omega(a), \delta(a), p) = (p \cdot \omega(a) + p \cdot \delta(a), \infty)$.

We now state the existence result.

Corollary 3.1 [Noguchi] *The economy* $(\mathcal{E}, \Phi_2^{\mathcal{C}_N})$ *admits an equilibrium, if it satisfies Assumptions* **A**, **C**, **M**, $\mathbf{ECL_0}$ *together with:*
Assumption N*for every* $(a, w, d, p, t) \in A \times (\mathbb{R}^H_+)^3 \times \mathbb{R}_+$*:*
(i) $I(w, d, p)$ *is an open subset of* $(0, \infty)$*;*
(ii) $\nu[C_N(a, p)] > 0$[9]*;* **(iii)** *the function* $\delta : A \to \mathbb{R}^H_+$ *is measurable;*
(iv) *for every* $\lambda > 0$, $I(w, d, \lambda p) = \lambda I(w, d, p)$*;*
(v) *for every sequence* $\{(p_n, t_n)\} \subset \mathbb{R}^H_+ \times \mathbb{R}$, $(p_n, t_n) \to (p, t)$, *if* $t \in I(w, d, p)$, *then* $t_n \in I(w, d, p_n)$ *for* n *large enough;*
(vi) *for every sequence* $\{(p_n, t_n)\} \subset \mathbb{R}^H_+ \times \mathbb{R}$, $(p_n, t_n) \to (p, t)$, $t_n \in I(w, d, p_n)$ *implies* $t \in \overline{I(w, d, p)}$*;*
(vii) *for every sequence* $\{(w_n, d_n)\} \subset \mathbb{R}^H_+ \times \mathbb{R}^H_+$, $(w_n, d_n) \to (w, d)$, *if* $t \in I(w, d, p)$, *then* $t \in I(w_n, d_n, p)$ *for* n *large enough;*
(viii) *for every sequence* $(w_n, d_n) \to (w, d)$ *in* $\mathbb{R}^H_+ \times \mathbb{R}^H_+$, $t \in I(w_n, d_n, p)$ *implies* $t \in \overline{I(w, d, p)}$*;*
(ix) *the set* $\overline{I(w, d, p)} \setminus I(w, d, p)$ *is countable and*
$c \in \overline{I(\omega(a), \delta(a), p)} \setminus I(\omega(a), \delta(a), p)$ *implies* $\nu[\{a \in A \mid p \cdot \omega(a) = c\}] = 0$.

Proof. We define the reference coalitions $\mathcal{C} := (C(a, p))_{(a,p) \in A \times \mathbb{R}^H}$ by

$$C(a, p) := \{\alpha \in A \mid p \cdot \omega(\alpha) \in \overline{I(\omega(a), \delta(a), p)}\}.$$

From Assumption **N(ix)**, for every $(a, p) \in A \times \mathbb{R}^H_+$, we get

$$C_N(a, p) \subset C(a, p) \text{ and } \nu(C(a, p) \setminus C_N(a, p)) = 0,$$

hence, $\int_{C_N(a,p)} f(\alpha) d\nu(\alpha) = \int_{C(a,p)} f(\alpha) d\nu(\alpha)$ for every $f \in L_X$.

Consequently, every equilibrium of $(\mathcal{E}, \Phi_2^{\mathcal{C}})$ is an equilibrium for $(\mathcal{E}, \Phi_2^{\mathcal{C}_N})$. We now obtain the existence of equilibria of $(\mathcal{E}, \Phi_2^{\mathcal{C}})$ from Theorem 3.1 ($K = 1$) and it suffices to prove that the reference coalitions $\mathcal{C}$, defined above, satisfy Assumption **R** of Theorem 3.1. This is proved in Section 5.2 of the Appendix.

4 Proof of the existence theorem

4.1 Proof of Theorem 2.2 in the integrably bounded case

In this section, we provide an intermediary existence result, also of interest for itself, under the following additional assumption:

Assumption IB [Integrably Bounded] *The correspondence* $a \mapsto X(a)$, *from* A *to* $\mathbb{R}^H_+$, *is integrably bounded, that is, for some integrable function* $\rho : A \to \mathbb{R}_+$, $\sup_{x \in X(a)} \|x\| \le \rho(a)$ *for a.e.* $a \in A$.

[9]In fact, Noguchi [17] only assumed that $\nu[C(a, p)] > 0$ for every $(a, p) \in A \times \mathbb{R}^H_+$ such that $p \cdot \omega(a) > 0$. To be able to get Noguchi's existence result in the more general case, we need to weaken Assumption **R** of Theorem 3.1 and, also, Assumption **E** of Theorem 2.2 as in the working paper [8].

Theorem 4.1 *Under Assumptions* **A**, **C**, **EC**, **ECL** *and* **IB**, *the economy* $(\mathcal{E}, \Phi)$ *admits a free disposal quasi-equilibrium* $(f^*, p^*) \in L_X \times \mathbb{R}^H$ *with* $p^* > 0$, *that is:*

(a) [*Preference Maximization*] *(1) for a.e.* $a \in A$, $f^*(a) \in B(a, p^*)$;
(2) for a.e. $a \in A$ *such that* $p^* \cdot \omega(a) > \inf p^* \cdot X(a)$, $f^*(a)$ *is a maximal element for* $\prec_{a,e^*_a}$ *in the budget set* $B(a, p^*)$ *where* $e^*_a := \Phi(a, p^*, f^*)$;

(b) [*Market Clearing*] $\int_A f^*(a) d\nu(a) \leq \int_A \omega(a) d\nu(a)$.

To prepare the proof of Theorem 4.1, we define the "quasi-demand" correspondence D, from $A \times \mathbb{R}^H_+ \times E$ to $\mathbb{R}^H_+$, by

$$D(a,p,e) := \begin{cases} \{x \in B(a,p) \mid \nexists x' \in B(a,p),\ x \prec_{a,e} x'\} & \\ & \text{if } \inf p \cdot X(a) < w(a,p) \\ B(a,p) & \text{if } \inf p \cdot X(a) = w(a,p). \end{cases}$$

We let $\Delta := \{p \in \mathbb{R}^H_+ \mid \sum_h p_h = 1\}$ and we define the correspondence Γ, from $\Delta \times L_X$ to $\Delta \times L_X$, by $\Gamma(p,f) = \Gamma_1(p,f) \times \Gamma_2(p,f)$, where

$$\Gamma_1(p,f) := \{p \in \Delta \mid (p-q) \cdot \int_A (f(a) - \omega(a)) d\nu(a) \geq 0 \ \forall q \in \Delta\} \subset \Delta$$

$$\Gamma_2(p,f) := \{g \in L_X \mid g(a) \in \mathrm{co} D(a,p,\Phi(a,p,f)) \text{ for a.e. } a \in A\} \subset L_X.$$

The next lemmas summarize the properties of the set L_X of consumption allocations and of the correspondence Γ.

Lemma 4.1 *The set* L_X, *endowed with the weak topology of the (locally convex) space* $L^1(A, \mathbb{R}^H)$, *is nonempty, convex, compact and metrizable.*

Proof of Lemma 4.1. First, the set L_X is nonempty, since it contains the mapping ω; indeed $\omega \in L^1(A, \mathbb{R}^H)$ and, for a.e. $a \in A$, $\omega(a) \in X(a)$ (by **C(vi)**). The set L_X is also convex, since for a.e. $a \in A$, $X(a)$ is a convex set (by **C(i)**).

We show now that L_X is compact for the weak topology of $L^1(A, \mathbb{R}^H)$. From the fact that the correspondence $a \mapsto X(a)$ is integrably bounded (by **IB**), the set L_X is (norm-)bounded and uniformly integrable and consequently L_X is weakly sequentially compact (see, for example, Dunford and Schwartz [9], p.294). In view of Eberlein-Smulian's Theorem, this is equivalent to the fact that the weak closure of L_X is weakly compact. The proof will be complete if we show that L_X is weakly closed. But in the normed space $L^1(A, \mathbb{R}^H)$, the convex set L_X is weakly closed if and only if it is closed in the norm topology of $L^1(A, \mathbb{R}^H)$ (see, for example, Dunford and Schwartz [9], p.422). To show that L_X is closed, we consider a sequence $\{f^n\} \subset L_X$ which converges to some $f \in L^1(A, \mathbb{R}^H)$ for the norm topology of $L^1(A, \mathbb{R}^H)$, then there exists a subsequence $\{f^{n_k}\}$, which converges almost everywhere to f. But, for a.e. $a \in A$, $f^{n_k}(a) \in X(a)$, since $f^{n_k} \in L_X$. Taking the limit when $k \to \infty$, for a.e. $a \in A$, $f(a) \in X(a)$, since $X(a)$ is a closed set (by **C(i)**). This ends the proof that L_X is weakly compact.

Finally, L_X is metrizable (for the weak topology) since, in a separable Banach space, the weak topology on a weakly compact set is metrizable (see, for example, Dunford and Schwartz [9], p.434). □

Lemma 4.2 *The correspondence Γ_1, from $\Delta \times L_X$ to Δ, has a closed graph and nonempty, convex, compact values.*

Proof of Lemma 4.2. It is a classical argument using Berge's Maximum Theorem (see Berge [6] p.123) and proving that the function $(p, f) \mapsto p \cdot (\int_A f(a)d\nu(a) - \int_A \omega(a)d\nu(a))$ is continuous on $\Delta \times L_X$. Since the function $(p, f) \mapsto p \cdot \int_A \omega(a)d\nu(a)$ is clearly continuous, we only have to prove that the function $(p, f) \mapsto p \cdot \int_A f(a)d\nu(a)$ is also continuous. Indeed (recalling that L_X is metrizable), let $(p^n, f^n) \in \Delta \times L_X$ be a sequence such that $(p^n, f^n) \to (p, f)$. One has

$$|p^n \cdot \int_A f^n(a)d\nu(a) - p \cdot \int_A f(a)d\nu(a)| \leq$$

$$|p^n \cdot \int_A f^n(a)d\nu(a) - p \cdot \int_A f^n(a)d\nu(a)| + |p \cdot \int_A f^n(a)d\nu(a) - p \cdot \int_A f(a)d\nu(a)|.$$

For the first term one gets

$$\begin{aligned} |p^n \cdot \int_A f^n(a)d\nu(a) - p \cdot \int_A f^n(a)d\nu(a)| &\leq \|p^n - p\| \int_A \|f^n(a)\| d\nu(a) \\ &\leq \|p^n - p\| \int_A \rho(a)d\nu(a) \to 0, \end{aligned}$$

since $p^n \to p$ and $\{f^n\}$ is integrably bounded by ρ (from **IB** and the fact that $\{f^n\} \subset L_X$). For the second term one has

$$|p \cdot \int_A f^n(a)d\nu(a) - p \cdot \int_A f(a)d\nu(a)| \leq \|p\| \| \int_A [f^n(a) - f(a)]d\nu(a)\| \to 0,$$

since $f^n \to f$ for the weak topology of $L^1(A, \mathbb{R}^H)$. □

Lemma 4.3 *The correspondence Γ_2, from $\Delta \times L_X$ to L_X, has a closed graph and nonempty, convex, compact values.*

Proof of Lemma 4.3. The correspondence Γ_2 has clearly convex values and we show hereafter that it has nonempty values. For every $(p, f) \in \Delta \times L_X$

$$\{g \in L_X \mid g(a) \in D(a, p, \Phi(a, p, f)), \text{ for a.e. } a \in A\} \subset \Gamma_2(p, f).$$

The existence of a measurable selection of the correspondence

$$a \mapsto D(a) := D(a, p, \Phi(a, p, f)) \subset B(0, \rho(a))$$

is a consequence of Aumann's Theorem and it suffices to show that (i) for a.e. $a \in A$, $D(a) \neq \emptyset$ and (ii) the correspondence $D(.)$ is measurable. The first assertion is a consequence of Proposition 5.1 of the Appendix. We now prove the second assertion. Indeed,

$$\begin{aligned} \mathrm{G}_D &= \{(a, z) \in A \times \mathbb{R}^H \mid z \in D(a)\} \\ &= \{(a, z) \in A \times \mathbb{R}^H \mid (a, \Phi(a, p, f), z) \in G\} = h^{-1}(G), \end{aligned}$$

where $G := \{(a, e, z) \in A \times E \times \mathbb{R}^H \mid z \in D(a, p, e)\}$ and $h : A \times \mathbb{R}^H \to A \times E \times \mathbb{R}^H$ is defined by $h(a, z) = (a, \Phi(a, p, f), z)$. But the mapping h is clearly measurable, since the mapping $a \mapsto \Phi(a, p, f)$ is measurable (by **EC**(i)), and G$\in \mathcal{A} \otimes \mathcal{B}(E) \otimes \mathcal{B}(\mathbb{R}^H)$, since the correspondence $(a, e) \mapsto D(a, p, e)$ is measurable [Proposition 5.1 of the Appendix]. Consequently, $\mathrm{G}_D = h^{-1}(G) \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^H)$, which ends the proof of Assertion (ii).

Finally, every measurable selection of the correspondence $a \mapsto D(a)$ is integrable, since from Assumption **IB**, for a.e. $a \in A$, $D(a) \subset B(0, \rho(a))$ for some integrable function ρ. This shows that $\Gamma_2(p, f)$ is nonempty.

We now show that the correspondence Γ_2 has a closed graph. Indeed (recalling that L_X is metrizable), let $\{(p^n, f^n, g^n)\}$ be a sequence converging to some element (p, f, g) in $\Delta \times L_X \times L_X$ such that $g^n \in \Gamma_2(p^n, f^n) \subset L_X$ for all n. Since the sequence $\{g^n\}$ is integrably bounded (by **IB**) and converges weakly to g in $L^1(A, \mathbb{R}^H)$, it is a standard result [see, for example, Yannelis [22]] that

$$\text{for a.e. } a \in A,\ g(a) \in \overline{\text{co}}\, Ls\{g^n(a)\}.$$

But, for a.e. $a \in A$, the correspondence $(p, f) \mapsto coD(a, p, \Phi(a, p, f))$ has a closed graph and convex values, since the correspondence $(p, e) \to D(a, p, e)$ has a closed graph and convex values [Proposition 5.1 of Appendix] and the mapping $(p, f) \mapsto \Phi(a, p, f)$ is continuous on $\Delta \times L_X$ (by **EC**(ii), **IB** and the metrizability of L_X). Hence, recalling that, for a.e. $a \in A$, $g^n(a) \in coD(a, p^n, \Phi(a, p^n, f^n))$ for all n, the closed graph property implies

$$Ls\{g^n(a)\} \subset coD(a, p, \Phi(a, p, f)).$$

Consequently, for a.e. $a \in A$

$$g(a) \in \overline{\text{co}}\, Ls\{g^n(a)\} \subset coD(a, p, \Phi(a, p, f)),$$

which shows that $g \in \Gamma(p, f)$ and ends the proof of the lemma. □

From the three above lemmas, recalling that the Cartesian product of two correspondences with closed graph and non-empty, convex, compact values is a correspondence with closed graph and non-empty, convex, compact values (see Berge [6] p.121), the space $L := \mathbb{R}^H \times L^1(A, \mathbb{R}^H)$, the set $K := \Delta \times L_X$ and the correspondence Γ satisfy all the assumptions of the following fixed-point theorem [see, for example, Fan [10] and Glicksberg [11]].

Theorem 4.2 (Fan-Glicksberg) *Let K be a non-empty, convex, compact subset of a Haussdorf locally convex space L and let Γ be a correspondence, from K to K, with a closed graph and non-empty, convex, compact values. Then there exists $x \in K$ such that $x \in \Gamma(x)$.*

Consequently, there exists an element $(p, f) \in \Delta \times L_X$ satisfying:

$$(p - q) \cdot \int_A (f(a) - \omega(a)) d\nu(a) \geq 0 \text{ for all } \ q \in \Delta, \tag{1}$$

$$f(a) \in \text{co}D(a, p, \Phi(a, p, f)) \text{ for a.e. } a \in A. \tag{2}$$

The following lemma shows that we can remove the convex hull in the above assertion.

Lemma 4.4 *There exists $f^* \in L_X$ satisfying:*

$$(p-q) \cdot \int_A (f^*(a) - \omega(a)) d\nu(a) \geq 0 \text{ for all } q \in \Delta, \tag{3}$$

$$f^*(a) \in D(a, p, \Phi(a, p, f^*)) \text{ for a.e. } a \in A. \tag{4}$$

Proof of Lemma 4.4. From Assertion (2) and the fact that the correspondence $a \mapsto D(a) := D(a, p, \Phi(a, p, f))$, from A to $\mathbb{R}^H$, is measurable [Proposition 5.1 of the Appendix], there exist finitely many measurable selections f_i $(i \in I)$ of the correspondence D such that, for a.e. $a \in A$, $f(a) \in \text{co}\{f_i(a) \mid i \in I\}$. Indeed, consider the correspondence F, from A to $(\mathbb{R}^H \times \mathbb{R})^{\#H+1}$, defined by

$$\begin{aligned} F(a) := \{(f_i, \lambda_i)_{i=1,..,\#H+1} \quad | \quad & (f_i, \lambda_i) \in D(a) \times \mathbb{R}_+, \text{ for all } i \\ & \sum_i \lambda_i = 1 \text{ and } f(a) = \sum_i \lambda_i f_i\}. \end{aligned}$$

Then, clearly F is measurable and nonempty valued, from Caratheodory's theorem and the fact that $f(a) \in \text{co}D(a)$. Consequently, from Aumann's theorem, there exists a measurable selection of the correspondence F, which defines the measurable selections f_i of D.

From Assumption **ECL**, there exists a measurable set $C \subset A_{na}$ such that: (i) for a.e. $a \in A_{na} \setminus C$, the preference relation $\prec_{a,\Phi(a,p,f)}$ is convex and (ii) there exists $f^* \in L_X$ such that, for a.e. $a \in A$, $\Phi(a, p, f) = \Phi(a, p, f^*)$,

$$\begin{aligned} \text{for a.e. } a \in C, f^*(a) \in \{f_i(a) \mid i \in I\} \quad & \subset \quad D(a, p, \Phi(a, p, f)) \\ & = \quad D(a, p, \Phi(a, p, f^*)), \end{aligned} \tag{5}$$

$$\text{for a.e. } a \in A \setminus C, f^*(a) = f(a) \text{ and } \int_A f(a) d\nu(a) = \int_A f^*(a) d\nu(a). \tag{6}$$

Since the preference relation $\prec_{a,\Phi(a,p,f)}$ is convex for a.e. $a \in A \setminus C$ (first, for a.e. $a \in A \setminus A_{na}$ by **C(iii)** and, second, for a.e. $a \in A_{na} \setminus C$ by **ECL**(i)), the set $D(a, p, \Phi(a, p, f))$ is convex. Then, from above

$$\begin{aligned} \text{for a.e. } a \in A \setminus C, \; f^*(a) = f(a) \quad & \in \quad coD(a, p, \Phi(a, p, f)) \\ & = \quad D(a, p, \Phi(a, p, f^*)). \end{aligned} \tag{7}$$

The Assertions (3) and (4) of the lemma follow from Assertions (1),(6) and (5),(7) respectively. □

We come back to the proof of Theorem 4.1 and we show that, for $p^* = p$, (p^*, f^*) is a free disposal quasi-equilibrium of $(\mathcal{E}, \Phi)$. Indeed, from Assertion (4), for a.e. $a \in A$, $f^*(a) \in D(a, p^*, \Phi(a, p^*, f^*))$, hence the equilibrium preference maximization condition is satisfied. This implies, in particular, that for a.e. $a \in A$, $f^*(a) \in B(a, p^*)$, hence $p^* \cdot f^*(a) \leq p^* \cdot \omega(a)$. Integrating over A, one gets $p^* \cdot \int_A (f^*(a) - \omega(a)) d\nu(a) \leq 0$. Using Assertion (3), one deduces that

$$q \cdot \int_A (f^*(a) - \omega(a)) d\nu(a) \leq 0 \text{ for all } q \in \Delta,$$

which implies the equilibrium market clearing condition

$$\int_A f^*(a)d\nu(a) \leq \int_A \omega(a)d\nu(a).$$

4.2 Proof of Theorem 2.2 in the general case

4.2.1 Truncation of the economy

For each integer $k > 1$ and for every $a \in A$, we let

$$X^k(a) := \{x \in X(a) \mid x \leq k[\mathbf{1} \cdot \omega(a)]\mathbf{1}\}$$

and, for every $e \in E$, we consider the restriction of the preference relation $\prec_{a,e}$ on the set $X^k(a)$, which will be denoted identically $\prec_{a,e}$ in the following. We define the truncated economy $\mathcal{E}^k$, by

$$\mathcal{E}^k = \{\mathbb{R}^H, E, (A, \mathcal{A}, \nu), (X^k(a), (\prec_{a,e})_{e\in E}, \omega(a))_{a\in A}\},$$

where the characteristics of $\mathcal{E}^k$ are the same as in the economy $\mathcal{E}$, but the consumption sets $X^k(a)$ and the preferences $(\prec_{a,e})_{e\in E}$ of the agents, which are defined as above.

The externality mapping $\Phi^k : A \times \mathbb{R}^H \times L^k_X \to E$ is defined as the restriction of Φ to $A \times \mathbb{R}^H \times L^k_X$, where

$$L^k_X := \{f \in L^1(A, \mathbb{R}^H) \mid f(a) \in X^k(a), \text{ for a.e. } a \in A\}.$$

It is easy to see that, if $(\mathcal{E}, \Phi)$ satisfies the assumption of Theorem 2.2, then for every k, $(\mathcal{E}^k, \Phi^k)$ satisfies all the assumptions of Theorem 4.1. Consequently, from Theorem 4.1, for every k there exists a free-disposal quasi-equilibrium (p^k, f^k) of $(\mathcal{E}^k, \Phi^k)$ with $p^k > 0$.

4.2.2 For k large enough, $p^k >> 0$

Lemma 4.5 *There exists $\delta > 0$ such that $p^k \geq \delta\mathbf{1}$ for k large enough.*

Proof. Without any loss of generality, we can assume that the sequence $\{p^k\}$ converges to some element p^* in the compact set Δ. To prove the lemma it suffices to show that $p^* >> 0$.

We first show that, for a.e. $a \in A$

$$\exists(f(a), e(a)) \in \mathbb{R}^H_+ \times E,\ (p^*, f(a), e(a)) \in \text{ Ls}\{(p^k, f^k(a), e^k(a))\}. \tag{8}$$

Indeed, since (p^k, f^k) is a quasi-equilibrium of $(\mathcal{E}^k, \Phi^k)$ for every k, one has

$$\text{for a.e. } a' \in A,\ 0 \leq f^k(a') \text{ and } \int_A f^k(a')d\nu(a') \leq \int_A \omega(a')d\nu(a'),$$

hence the sequence $\{\int_A f^k(a')d\nu(a')\}$ is bounded in $\mathbb{R}^H$ and, without any loss of generality, we can assume that it is convergent. Consequently, from Schmeidler's version of Fatou's lemma, there exists $f : A \to \mathbb{R}^H$ and a null set $N \in \mathcal{A}$ such that, for every $a \in A \setminus N$, $f(a) \in Ls\{f^k(a)\}$, that is, there exists a subsequence $\{k_n\}$, which depends on a, such that $f^{k_n}(a) \to f(a)$.

Let $a \in A \setminus N$ be fixed, we will show that Assertion (8) holds. We first prove that $\sup_n \|f^{k_n}\|_1 < \infty$. Defining in $\mathbb{R}^H$, $\|x\|_1 = \sum_h |x_h|$ and, recalling that, for some $m > 0$, $\|x\| \leq m\|x\|_1$ for every x, we get

$$\begin{aligned} \|f^{k_n}\|_1 := \int_A \|f^{k_n}(a')\| d\nu(a') &\leq m \int_A \sum_{h\in H} f_h^{k_n}(a') d\nu(a') \\ &= m \| \int_A f^{k_n}(a') d\nu(a')\|_1, \end{aligned}$$

since $f^{k_n}(a') \geq 0$, for a.e. $a' \in A$. Consequently, $\sup_n \|f^{k_n}\|_1 < \infty$, since the sequence $\{\int_A f^{k_n}(a') d\nu(a')\}$ is convergent.

For this fixed $a \in A \setminus N$, since the sequence $\{(p^{k_n}, f^{k_n})\}$ is (norm-)bounded in $\mathbb{R}^H \times L^1(A, \mathbb{R}^H)$, from Assumption **EB**, there exists a subsequence of $\{k_n\}$, denoted identically, such that $e^{k_n}(a) := \Phi(a, p^{k_n}, f^{k_n})$ converges to some element $e(a) \in E$. Recalling that $f^{k_n}(a) \to f(a)$, we deduce

$$(p^*, f(a), e(a)) \in \text{ Ls}\{(p^k, f^k(a), e^k(a))\},$$

hence Assertion (8) holds for every $a \in A \setminus N$.

We now choose a particular agent $a_0 \in A$ for whom the following properties hold: (i) the preferences of agent a_0 are continuous, (ii) the preferences of agent a_0 are monotonic; (iii) $p^* \cdot \omega(a_0) > 0$ and there exists a subsequence $\{k_n\}$, depending on a_0, such that (iv) $(p^{k_n}, f^{k_n}(a_0), e^{k_n}(a_0)) \to (p^*, f(a_0), e(a_0))$, for some $(f(a_0), e(a_0)) \in \mathbb{R}_+^H \times E$, (v) for every n, $f^{k_n}(a_0) \in D^{k_n}(a_0, p^{k_n}, e^{k_n}(a_0))$ with $e^{k_n}(a_0) = \Phi(a_0, p^{k_n}, f^{k_n})$. Such an agent a_0 clearly exists, since each of the above Assertions $(i) - (v)$ hold for a.e. $a \in A$; they correspond, respectively, to Assumption **C(iv)**, **M(i)**, **M(ii)**, Assertion (8) and the equilibrium preference maximization condition for (p^{k_n}, f^{k_n}) for every n.

We will now use this particular agent a_0 to show that $p^* >> 0$. Suppose it is not true, then there exists h such that $p_h^* = 0$. From the above properties of agent a_0, for all n, $p^{k_n} \cdot f^{k_n}(a_0) \leq p^{k_n} \cdot \omega(a_0)$, $p^{k_n} \to p^*$ and $f^{k_n}(a_0) \to f(a_0)$, and at the limit one gets $p^* \cdot f(a_0) \leq p^* \cdot \omega(a_0)$. Since agent a_0 has monotonic preferences, there exists $z = f(a_0) + t\mathbf{e^h}$, for some $t > 0$ such that $f(a_0) \prec_{a_0, e(a_0)} z$ and clearly $p^* \cdot z = p^* \cdot f(a_0) \leq p^* \cdot \omega(a_0)$. We now show that

$$\exists\ z' \in \mathbb{R}_+^H,\ p^* \cdot z' < p^* \cdot \omega(a_0),\ f(a_0) \prec_{a_0, e(a_0)} z'. \tag{9}$$

Indeed, if $p^* \cdot z < p^* \cdot \omega(a_0)$, we take $z' = z$. If $p^* \cdot z = p^* \cdot \omega(a_0) > 0$, we can choose $i \in H$ such that $p_i^* > 0$ and $z_i > 0$. Since agent a_0 has continuous preferences, there exists $\varepsilon > 0$ such that $z' = z - \varepsilon \mathbf{e^i} \in \mathbb{R}_+^H$ and $f(a_0) \prec_{a_0, e(a_0)} z'$. We have also $p^* \cdot z' = p^* \cdot z - \varepsilon p_i^* < p^* \cdot \omega(a_0)$. This ends the proof of Assertion (9).

We end the proof by contradicting the fact that $f^{k_n}(a_0)$ belongs to $D^k(a_0, p^{k_n}, e^{k_n}(a_0))$. Indeed, from $p^* \cdot \omega(a_0) > 0$ (by (iii)) and Assertion (9), recalling that the sequence

$\{(p^{k_n}, f^{k_n}(a_0), e^{k_n}(a_0))\}$ converges to $(p^*, f(a_0), e(a_0))$ (by (vi)) and using the continuity of preferences of agent a_0, for n large enough, we get $p^{k_n} \cdot \omega(a_0) > 0$, $z' \in \mathbb{R}^H_+$, $p^{k_n} \cdot z' \leq p^{k_n} \cdot \omega(a_0)$ and $x^{k_n}(a_0) \prec_{a_0, e^{k_n}(a_0)} z'$. Moreover, we can also assume that $z' \in X^{k_n}(a_0)$. All together, these conditions contradict the fact that $f^{k_n}(a_0) \in D^k(a_0, p^{k_n}, e^{k_n}(a_0))$ and this ends the proof of the lemma.

4.2.3 For k large enough, (p^k, f^k) is an equilibrium for $(\mathcal{E}, \Phi)$

It is a consequence of the following lemma.

Lemma 4.6 *For every k large enough and for a.e. $a \in A$, one has:*
(i) $B(a, p^k) \subset X^k(a)$;
(ii) $f^k(a)$ is a maximal element in $B(a, p^k)$ for $\prec_{a, e^k(a)}$,
where $e^k(a) = \Phi(a, p^k, f^k)$;
(iii) $p^k \cdot f^k(a) = p^k \cdot \omega(a)$;
(iv) $\int_A f^k(a) d\nu(a) = \int_A \omega(a) d\nu(a)$.

Proof. From Lemma 4.5, there exists K such that, for every $k \geq K$

$$p^k_h > \delta \text{ for each } h \in H \text{ and } \frac{1}{\delta} \leq k.$$

In the following we fix $k \geq K$.
(i) For a.e. $a \in A$, let $x \in B(a, p^k)$, i.e., $x \in \mathbb{R}^H_+$ and $p^k \cdot x \leq p^k \cdot \omega(a)$. From above, recalling that $p^k \in \Delta$, one gets

$$\delta x_h \leq p^k_h x_h \leq p^k \cdot x \leq p^k \cdot \omega(a) \leq \sum_h \omega_h(a) = \mathbf{1} \cdot \omega(a),$$

which implies that

$$0 \leq x \leq \frac{1}{\delta}[\mathbf{1} \cdot \omega(a)]\mathbf{1} \leq k[\mathbf{1} \cdot \omega(a)]\mathbf{1}$$

or equivalently $x \in X^k(a)$.
(ii) For a.e. $a \in A$ such that $p^k \cdot \omega(a) > 0$, $f^k(a)$ is a maximal element in $B(a, p^k)$ for $\prec_{a, e^k(a)}$, since $B(a, p^k) \subset X^k(a)$ (by Part (i)) and the fact that (p^k, f^k) is a free-disposal quasi-equilibrium for $(\mathcal{E}^k, \Phi^k)$. For a.e. $a \in A$ such that $p^k \cdot \omega(a) = 0$, recalling that $p^k >> 0$ (by Lemma 4.5), we get $B(a, p^k) = \{0\}$ and the result follows from the Irreflexivity Assumption **C(ii)**.
(iii) The result is obvious for a.e. $a \in A$ such that $p^k \cdot \omega(a) = 0$. Assume now that $p^k \cdot \omega(a) > 0$. From the Monotonicity Assumption **M(i)**, there exists a sequence $\{f^n(a)\} \subset \mathbb{R}^H_+$ such that $f^n(a) \to f^k(a)$ and $f^k(a) \prec_{a, e^k(a)} f^n(a)$. From Part (ii), $f^k(a)$ is a maximal element of $\prec_{a, e^k(a)}$ in $B(a, p^k)$, consequently $p^k \cdot f^n(a) > p^k \cdot \omega(a)$. Passing to the limit one gets $p^k \cdot f^k(a) \geq p^k \cdot \omega(a)$, which together with $f^k(a) \in B(a, p^k)$ implies that $p^k \cdot f^k(a) = p^k \cdot \omega(a)$.
(iv) Integrating over A the equalities of Part (iii), one gets

$$p^k \cdot \left(\int_A f^k(a) d\nu(a) - \int_A \omega(a) d\nu(a)\right) = 0.$$

Since (p^k, f^k) is a free-disposal quasi-equilibrium for $(\mathcal{E}^k, \Phi^k)$, one has

$$\int_A f^k(a)d\nu(a) \leq \int_A \omega(a)d\nu(a)$$

and, recalling that $p^k >> 0$ (by Lemma 4.5), we get

$$\int_A f^k(a)d\nu(a) = \int_A \omega(a)d\nu(a).$$

5 Appendix

5.1 Properties of the quasi-demand correspondence

Let $(A, \mathcal{A}, \nu)$ be a measure space of consumers, and assume that each consumer a is endowed with a consumption set $X(a) \subset \mathbb{R}^H$, a preference relation $\prec_{a,e}$ on $X(a)$ (for each externality $e \in E$) and a wealth mapping $w : A \times \mathbb{R}^H \to \mathbb{R}$. In the following, we let

$$P := \{p \in \mathbb{R}^H \mid \inf p \cdot X(a) \leq w(a,p) \text{ for a.e. } a \in A\},$$

$$B(a,p) := \{x \in X(a) \mid p \cdot x \leq w(a,p)\},$$

$$D(a,p,e) := \begin{cases} \{x \in B(a,p) \mid \nexists x' \in B(a,p),\, x \prec_{a,e} x'\} & \\ & \text{if } \inf p \cdot X(a) < w(a,p) \\ B(a,p) & \text{if } \inf p \cdot X(a) = w(a,p). \end{cases}$$

The properties of the quasi-demand correspondence D is summarized in the following proposition, which extends standard results (see, for example, Hildenbrand [13]) in the no-externality case (say $E = \{0\}$).

Proposition 5.1 *Let* $\{(A, \mathcal{A}, \nu), E, (X(a), (\prec_{a,e})_{e\in E},)_{a\in A}, w\}$ *satisfy Assumptions* **A***,* **C** *and* **IB** *and assume that the wealth distribution* $w : A \times \mathbb{R}^H \to \mathbb{R}$ *is a Caratheodory function*[10]*. Then:*
(i) for every $p \in P$ *the correspondence* $(a,e) \mapsto D(a,p,e)$*, from* $A \times E$ *to* $\mathbb{R}^H$*, is measurable;*
(ii) for a.e. $a \in A$ *the correspondence* $(p,e) \mapsto D(a,p,e)$*, from* $P \times E$ *to* $\mathbb{R}^H$*, has a closed graph and nonempty, compact values.*

Proof. In the following, for a.e. $a \in A$ and for every $p \in P$, we let

$$P_a := \{p \in P \mid \inf p \cdot X(a) < w(a,p)\},$$

$$A_p := \{a \in A \mid \inf p \cdot X(a) < w(a,p)\}.$$

Proof of (i) Let $p \in P$, we prove that

$$G := \{(a,e,d) \in A \times E \times \mathbb{R}^H \mid d \in D(a,p,e)\} \in \mathcal{A} \otimes B(E) \otimes \mathcal{B}(\mathbb{R}^H)$$

[10]That is, for every $p \in \mathbb{R}^H$, the function $a \mapsto w(a,p)$ is measurable and, for a.e. $a \in A$, the function $p \mapsto w(a,p)$ is continuous. We note that the wealth distribution considered in our model $w(a,p) := p \cdot \omega(a)$ satisfies this property, when ω is assume to be measurable.

and we first notice that $\mathrm{G} = \mathrm{G}_1 \cup \mathrm{G}_2$, where

$$\mathrm{G}_1 := \{(a, e, d) \in (A \setminus A_p) \times E \times \mathbb{R}^H \mid d \in X(a),\ p \cdot d \leq w(a, p)\},$$

$$\mathrm{G}_2 := \{(a, e, d) \in A_p \times E \times \mathbb{R}^H \mid d \in D(a, p, e)\}.$$

We notice that $\mathrm{G}_1 \in \mathcal{A} \otimes B(E) \otimes \mathcal{B}(\mathbb{R}^H)$, since the mapping $(a, d) \to p \cdot d - w(a, p)$ and the correspondence $a \to X(a)$ are measurable and $A_p \in \mathcal{A}$.

To show that $\mathrm{G}_2 \in \mathcal{A} \otimes \mathcal{B}(E) \otimes \mathcal{B}(\mathbb{R}^H)$, we apply the argument used by Hildendrand [13]. Since the correspondence $B(., p)$, from A_p to $\mathbb{R}^H$, has nonempty values and is measurable, there exists a sequence of measurable mappings $\{f_n\}$, from A_p to $\mathbb{R}^H$, such that for a.e. $a \in A_p$, $\{f_n(a)\}$ is dense in $B(a, p)$ (see, for example, [7]). We now define the correspondences ξ_n, from $A_p \times E$ to $\mathbb{R}^H$, by

$$\xi_n(a, e) = \{x \in B(a, p) \mid \text{not}[x \prec_{a,e} f_n(a)]\}$$

and we claim that: $D(a, p, e) = \cap_{n=1}^{\infty} \xi_n(a, e)$, for a.e. $a \in A_p$.

Clearly, for every n, $D(a, p, e) \subset \xi_n(a, e)$. Conversely, let $x \in \cap_{n=1}^{\infty} \xi_n(a, e)$ and suppose that $x \notin D(a, p, e)$. Then, the set $U = \{x' \in B(a, p) \mid x \prec_{a,e} x'\}$ is nonempty and is open relative to $B(a, p)$ (by $\mathbf{C(iv)}$). Since the sequence $\{f_n(a)\}$ is dense in $B(a, p)$, we deduce that for some n_0, $x \prec_{a,e} f_{n_0}(a)$, but this contradicts the fact that $x \in \xi_{n_0}(a, e)$. Thus, we have

$$\begin{aligned} \mathrm{G}_2 &= \{(a, e, d) \in A_p \times E \times \mathbb{R}^H \mid d \in D(a, p, e)\} \\ &= \cap_{n=1}^{\infty} \{(a, e, d) \in A_p \times E \times \mathbb{R}^H \mid d \in \xi_n(a, e)\}. \end{aligned}$$

Hence, the set G_2 is measurable, since $\prec_{a,e}$ is measurable (by $\mathbf{C(v)}$), the mappings f_n and the correspondence $a \mapsto B(a, p)$ are measurable and recalling that $A_p \in \mathcal{A}$.

Proof of (ii) We first show that $D(a, p, e) \neq \emptyset$ for a.e. $a \in A$ and every $(p, e) \in P \times E$. For a.e. $a \in A \setminus A_p$, $D(a, p, e) = B(a, p) \neq \emptyset$ since $\inf p \cdot X(a) \leq w(a, p)$. We now consider $a \in A_p$ and we simply denote $B := B(a, p)$, which is clearly a nonempty, compact set (by **IB**). We suppose, by contraposition, that $D(a, p, e) = \emptyset$, that is, for every $x \in B$, there exists $x' \in B$, $x \prec_{a,e} x'$. Then $B = \cup_{x' \in B} V_{x'}$, where $V_{x'} = \{x \in B \mid x \prec_{a,e} x'\}$ is open in B (by $\mathbf{C(iv)}$). Since B is compact, there exists a finite subset $\{x'_i \mid i \in N\} \subset B$ such that $B = \cup_{i \in N} V_{x'_i}$. We now claim that there exists $i \in N$ such that not $[x'_i \prec_{a,e} x'_j]$ for every $j \in N$. Indeed, if such a maximal element does not exist, for every $i \in N$, there exists $\sigma(i) \in N$ such that $x'_i \prec_{a,e} x'_{\sigma(i)}$. The mappings $\sigma : N \to N$ clearly admits a cycle, that is, for some i and some integer k one has $i = \sigma^k(i)$ (the composition of σ with itself k times). The transitivity (by $\mathbf{C(ii)}$) of $\prec_{a,e}$ implies that $x'_i \prec_{a,e} x'_{\sigma^k(i)} = x'_i$ which contradicts the irreflexivity (by $\mathbf{C(ii)}$) of $\prec_{a,e}$. We end the proof by considering such a maximal element $x'_i \in B$, which belongs to some set $V_{x'_j}$ ($j \in N$), that is, $x'_i \prec_{a,e} x'_j$ for some $j \in J$. But this is in contradiction with the maximality of x'_i. This ends the proof that $D(a, p, e)$ is nonempty.

We now show that, for a.e. $a \in A$, the correspondence $(p, e) \to D(a, p, e)$, from $P \times E$ to $\mathbb{R}^H$, has a closed graph. Let $(p^n, e^n, x^n) \to (p, e, x)$ in $P \times E \times \mathbb{R}^H$ such that, for all n, $x^n \in D(a, p^n, e^n)$. From $p^n \cdot x^n \leq w(a, p^n)$, passing to the limit and recalling that the mapping $w(a, .)$ is continuous, one gets $p \cdot x \leq w(a, p)$. Recalling that $X(a)$ is closed, we get that $x \in B(a, p)$. Thus, if $\inf p \cdot X(a) = w(a, p)$, we have

$x \in D(a,p,e) = B(a,p)$. We assume now that $\inf p \cdot X(a) < w(a,p)$. Since $p^n \to p$, for n large enough, $w(a,p^n) > \inf p^n \cdot X(a)$. Suppose now that $x \notin D(a,p,e)$. This implies that there exists $x' \in B(a,p)$ such that $x \prec_{a,e} x'$. From the fact that $w(a,p) > \inf p \cdot X(a)$ and the Continuity Assumption **C(iv)**, we can find $x'' \in X(a)$ such that $x \prec_{a,e} x''$ and $p \cdot x'' < w(a,p)$. Since $p^n \to p$, for n large enough, $p^n \cdot x'' < w(a,p^n)$. Since $e^n \to e$, from the Continuity Assumption **C(iv)**, for n large enough, $x_n \prec_{a,e^n} x''$. Consequently, we can choose n (large enough) such that $w(a,p^n) > \inf p^n \cdot X(a)$, $x'' \in B(a,p^n)$ and $x_n \prec_{a,e^n} x''$, but this contradicts the fact that $x^n \in D(a,p^n,e^n)$. □

5.2 Properties of Noguchi's reference coalitions

In this section, we end the proof of Corollary 3.1 (of Section 3.3) and it only remains to show that the reference coalitions, defined by

$$C(a,p) = \{\alpha \in A \mid p \cdot \omega(\alpha) \in \overline{I(\omega(a),\delta(a),p)}\}$$

satisfy Assumption **R** of Theorem 3.1.

Proof. **R(i)** is a consequence of **N(ii)** since $C_N(a,p) \subset C(a,p)$ and **R(ii)** is a direct consequence of **N(iv)**. □

Proof of **R(iii)**. Let $(a,p) \in A \times \mathbb{R}^H_+$, we define

$$W(a,p) := \{\omega' \in \mathbb{R}^H_+ \mid p \cdot \omega' \in I(\omega(a),\delta(a),p)\}.$$

Clearly, one has

$$\overline{W(a,p)} \setminus W(a,p) \subset \{\omega' \in \mathbb{R}^H_+ \mid p \cdot \omega' \in \overline{I(\omega(a),\delta(a),p)} \setminus I(\omega(a),\delta(a),p)\}$$

$$\omega^{-1}(\overline{W(a,p)} \setminus W(a,p) \subset \cup_{c \in \overline{I(\omega(a),\delta(a),p)} \setminus I(\omega(a),\delta(a),p)} \{\alpha \in A \mid p \cdot \omega(\alpha) = c\}$$

and using Assumption **N(ix)**, one gets

$$\nu[\omega^{-1}(\overline{W(a,p)} \setminus W(a,p))] = 0.$$

Since the measure $\tau := \nu o \omega^{-1}$ is a finite Borel measure on $\mathbb{R}^H_+$, from Noguchi [17] (see Lemma 2), for every sequence $\{p_n\} \subset \mathbb{R}^H_+$ converging to p, one has

$$\tau(W(a,p_n)\Delta W(a,p)) := \nu[\omega^{-1}(W(a,p_n)\Delta W(a,p))] \to 0.$$

Noticing that $C_N(a,p^n) = \omega^{-1}(W(a,p_n))$ and $C_N(a,p) = \omega^{-1}(W(a,p))$, one gets

$$\begin{aligned} \nu[C_N(a,p^n)\Delta C_N(a,p)] &= \nu[\omega^{-1}(W(a,p_n))\Delta\omega^{-1}(W(a,p))] \\ &= \nu[\omega^{-1}(W(a,p_n))\Delta W(a,p))] \to 0. \end{aligned}$$

Recalling now that $\nu[C(a,p) \subset C_N(a,p)] = 0$ for every $(a,p) \in A \times \mathbb{R}^H_+$, from above, we get $\nu[C(a,p^n)\Delta C(a,p)] \to 0$ when $p_n \to p$ in $\mathbb{R}^H_+$. □

Proof of **R(iv)**. It is a consequence of the following lemma, defining, for a fixed $p \in \mathbb{R}^H_+$, the mappings $f : A \to (\mathbb{R}^H_+)^2$, $g : A \to \mathbb{R}^H_+$ and the correspondence F, from $(\mathbb{R}^H_+)^2$ to $\mathbb{R}$, by $f(a) = (\omega(a),\delta(a))$, $g(\alpha) = p \cdot \omega(\alpha)$ and $F(\omega,\delta) = I(\omega,\delta,p)$ and noticing that

$$C(a,p) = \{\alpha \in A \mid g(\alpha) \in \overline{F(f(a))}\}$$

and that, Condition **N** implies that f, g and F satisfy the assumption of the lemma. (We only notice that, **N(vii)** implies that for every $t \in \mathbb{R}_+^H$, the set $F^{-1}(t) := \{(\omega, \delta) \in (\mathbb{R}_+^H)^2 \mid t \in I(\omega, \delta, p)\}$ is open, hence measurable.)

Lemma 5.1 *Let $f : A \to \mathbb{R}^m$, $g : A \to \mathbb{R}^n$ be two measurable mappings and let F be a correspondence, from $\mathbb{R}^m$ to $\mathbb{R}^n$, such that, for every $(x, t) \in \times\mathbb{R}^m \times \mathbb{R}^n$, $F(x)$ is open and $F^{-1}(t)$ is measurable. Then the set*

$$G := \{(a, \alpha) \in A \times A \mid g(\alpha) \in \overline{F(f(a))}\}$$

is measurable.

Proof. Note that $(a, \alpha) \in G$ if and only if

$$\forall k \in \mathbb{N},\ B(g(\alpha), \frac{1}{k}) \cap F(f(a)) \neq \emptyset$$

and, using the fact that $F(f(a))$ is an open set, if and only if

$$\forall k \in \mathbb{N},\ \exists t_k \in \mathbb{Q}^n,\ \|t_k - g(\alpha)\| < \frac{1}{k} \text{ and } t_k \in F(f(a)).$$

Consequently

$$G = \cap_k \cup_{t \in \mathbb{Q}^n} [A \times \{\alpha \in A \mid \|t - g(\alpha)\| < \frac{1}{k}\} \cap \{a \in A \mid t \in F(f(a))\} \times A],$$

which is measurable since the set $\{\alpha \in A \mid \|t - g(\alpha)\| < \frac{1}{k}\}$ is measurable (since the mapping g is measurable) and the set $\{a \in A \mid t \in F(f(a))\}$ is measurable (since the set $F^{-1}(t)$ is measurable and the mapping f is measurable). □

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