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Abstract:

Euler equation models represent an important class of macroeconomic systems. Our ongoing research (He and Barnett (2003)) on the Leeper and Sims (1994) Euler equations macroeconometric model is revealing the existence of singularity-induced bifurcations, when the model's parameters are within a confidence region about the parameter estimates. Although known to engineers, singularity bifurcation has not previously been seen in the economics literature. Knowledge of the nature of singularity-induced bifurcations is likely to become important in understanding the dynamics of modern macroeconometric models. This paper explains singularity-induced bifurcation, its nature, and its identification and contrasts this class of bifurcations with the more common forms of bifurcation we have previously encountered within the parameter space of the Bergstrom and Wymer (1976) continuous time macroeconometric model of the UK economy. (See, e.g., Barnett and He (1999, 2002)).

1. Introduction

Since the appearance of the Lucas critique, there has been growing interest in Euler equation models with estimated deep parameters. Our currently ongoing analysis of the Leeper and Sims (1994) Euler equations macroeconometric model is revealing the existence of singularity-induced bifurcation, when the model's parameters are near their estimated values (He and Barnett (2003)). Although known in engineering, singularity-induced bifurcations have not previously been encountered in economics. Bifurcation analysis of parameter space stratification is a fundamental and frequently overlooked approach to exploring model dynamic properties and can provide surprising results, as we have previously found with other macroeconometric models (Barnett and He (1999, 2002)). In particular, the existence of bifurcation boundaries within the parameter space can have important implications for robustness of inferences regarding model dynamic properties. Based upon our currently ongoing research with the Leeper and Sims model, we believe that singularity bifurcation may become particularly important in understanding the properties of modern Euler equations macroeconometric models.

The theory of singularity-induced bifurcation is not well known and is subject to ongoing development in the engineering literature. In this paper, we use examples to illustrate the effects of the presence of this type of bifurcation on dynamic systems behavior. The dramatic nature of this type of bifurcation is most easily understood, when related to and contrasted with the nature of the more familiar types previously encountered in economic models. The availability to the economics profession of this information about singularity bifurcation will be needed in understanding our ongoing research on the Leeper and Sims model, as more of our ongoing results with that model become available.

We do not believe that the singularity bifurcation phenomenon is specific to the Leeper and Sims model, but rather can be expected to be found in future research with other Euler equations macroeconometric models. As we show, the implicit functions structure inherent to Euler equations models naturally raises the possibility of singularity bifurcation, while the older reduced-form and analytically-solvable structural-form macroeconometric models do not. Now that we have developed the iterative numerical procedures needed for locating and identifying singularity bifurcation boundaries, we anticipate that this surprising phenomenon will be found to be present in other modern macroeconometric models. Our current and previous work in this area has been concentrated on the analysis of the dynamic properties of various countries' policy-relevant estimated macroeconometric models and hence is of more than just theoretical interest. In each case, we have explored and are exploring the properties of the models' dynamics in the vicinity of the parameter estimates, as estimated by the originators of the models.

2. Stability

Many existing dynamic macroeconomic models can be written in the following general form

$$\mathbf{Dx} = \mathbf{f}(\mathbf{x},\theta), \tag{1}$$

where $\mathbf{D}$ is the vector-valued differentiation operator, $\mathbf{x}$ is the state vector, $\boldsymbol{\theta}$ is the parameter vector, and $\mathbf{f}$ is the vector of functions that governs the dynamics of the system. With t defined to be time, the differentiation operator $\mathbf{D}$ is defined by the vector valued gradient, $\mathbf{Dx} = \frac{\partial \mathbf{x}}{\partial t}$, and analogously $Dx_i = \frac{\partial x_i}{\partial t}$, for i = 1, … , n, where $\mathbf{x} = (x_1,\ldots,x_i,\ldots x_n)'$. Every component of $\mathbf{f}(\mathbf{x},\boldsymbol{\theta})$ is assumed to be smooth (infinitely continuously differentiable) in a local region of interest. For example, the well-known Bergstrom and Wymer (1976) continuous time UK macroeconometric model can be written in the form of the dynamical system (1). See, e.g., Barnett and He (1999). In the language of systems theory, system (1) is the class of first-order autonomous systems.

For system (1), there may exist a point $\mathbf{x}^*$ such that $\mathbf{f}(\mathbf{x}^*,\boldsymbol{\theta}) = \mathbf{0}$. Then $\mathbf{x}^*$ is an equilibrium of the system in the steady state sense. Without loss of generality, we may assume that $\mathbf{x}^* = \mathbf{0}$ (by replacing $\mathbf{x}$ with $\mathbf{x} - \mathbf{x}^*$). The value of the parameter vector $\boldsymbol{\theta}$ can affect the dynamics of the system, (1). Let us assume that $\boldsymbol{\theta}$ can take values within a possible set Θ. It can be important to know how the value of the parameter vector $\boldsymbol{\theta}$ can change the behavior of system (1), especially if a small change in the parameters can alter the nature of the dynamic solution path in fundamental ways (i.e., through a bifurcation in dynamical properties in state space).

Basic properties of any dynamic system are its stability and the nature of its disequilibrium dynamics, whether or not stable. If $\mathbf{x}^*$ is an equilibrium of the system (1), the system will remain at $\mathbf{x}^*$ forever, if the system starts at the equilibrium. Stability analysis tells us what will happen, if the system starts not exactly at $\mathbf{x}^*$, but in a neighborhood of it. Just knowing whether the system will return stably to the equilibrium or will diverge unstably is not enough. We need to know the nature of the dynamic paths, when the system is perturbed away from the equilibrium.

We now introduce theory regarding stability of a system, such as (1), around the equilibrium $\mathbf{x}^*=\mathbf{0}$. For this purpose, let us rewrite (1) as

$$\mathbf{Dx} = \mathbf{A}(\theta)\mathbf{x} + \mathbf{F}(\mathbf{x},\theta), \qquad (2)$$

where $\mathbf{A}(\theta)$ is the Jacobian matrix of $\mathbf{f}(\mathbf{x}, \theta)$ acquired by differentiating $\mathbf{f}$ with respect to $\mathbf{x}$ and evaluating the resulting matrix at the equilibrium, $\mathbf{x}^* = \mathbf{0}$. The matrix $\mathbf{A}(\theta)$ is the coefficient matrix of the linear terms, and

$$\mathbf{F}(\mathbf{x},\theta) = \mathbf{f}(\mathbf{x},\theta) - \mathbf{A}(\theta)\mathbf{x} = o(\mathbf{x};\theta) \qquad (3)$$

is the vector of higher order terms. In nonlinear systems theory, the local stability of (1) can be studied by examining the eigenvalues of the Jacobian matrix $\mathbf{A}(\theta)$ along with certain transversality conditions. See, e.g., Barnett and He (1999,2002).

Because $\mathbf{A}(\theta)$ is a matrix-valued function of the parameter vector, θ, stability of the system (1) could be locally dependent upon θ through $\mathbf{A}$ or more generally through nonlinear dependence of $\mathbf{f}(\mathbf{x},\theta) = \mathbf{A}(\theta)\mathbf{x} + o(\mathbf{x};\theta)$ upon θ. It is important to know for what parameter values, θ, the system, (1), is stable and for what values it is not. But it also is important to know the nature of the instability, when the system is unstable (e.g., periodic, multiperiodic, or chaotic), and the nature of the stability, when the system is stable (e.g., monotonically convergent, damped single-periodic convergent, or damped multiperiodic convergent).

In global analysis, the higher order terms must be considered in determining the dynamics of system (1), when subjected to large perturbations away from equilibrium.

3. Bifurcations in Macroeconomics

An important means of studying dynamic system properties, when the values of a system's parameters are not known with certainty, is through bifurcation analysis. Bifurcation refers to the existence of fundamentally different dynamic solution properties at nearby settings of parameters. This phenomenon can occur when the parameter settings are on different sides of a boundary, called a bifurcation boundary. Robustness of inferences about dynamics becomes critically dependent upon the location of such boundaries, and whether the parameters are close to such a boundary.

Many such boundaries can exist within the parameter space, so that the parameter space becomes stratified. The nature of the dynamics near or on a bifurcation boundary defines the type of bifurcation boundary. Bifurcation boundaries can be located and the type of bifurcation identified by use of Jacobian eigenvalue conditions and certain transversality conditions. Regarding those conditions and our numerical procedure for locating and identifying bifurcation boundaries, see Barnett and He (1999,2002) and He and Barnett (2003).

The types of bifurcation boundaries previously encountered in our work include Hopf, pitchfork, saddle-node, and transcritical bifurcation. Also see Benhabib and Nishimura (1979), Boldrin and Woodford (1990), Dockner and Feichtinger (1991), Nishimura and Takahashi (1992), Bala (1997), and Scarf (1960). Bifurcations are especially important to dynamic macroeconomic systems, since several well-known models, including Bergstrom and Wymer's (1976) UK continuous time model, operate at parameter point-estimates known to be close to bifurcation boundaries. See Barnett and He (1999, 2002). As a means of highlighting the nature of our finding of singularity bifurcation in Euler equation models, we first illustrate and contrast the natures of the more familiar, and less dramatic, Hopf, pitchfork, saddle-node, and transcritical bifurcations, which we have encountered previously with older, structural macroeconometric models.

3.1. Transcritical Bifurcations

We begin by illustrating transcritical bifurcation in a one dimensional state space. For a more mathematical presentation, see Sotomayor (1973). With a one-dimensional system of the form,

$$Dx = G(x,\theta),$$

the transversality conditions for a transcritical bifurcation at $(x, \theta)=(0,0)$ are

$$G(0,0) = G_x(0,0) = 0,\ G_\theta(0,0) = 0,\ G_{xx}(0,0) \neq 0, \text{ and } G_{\theta x}^2 - G_{xx}G_{\theta\theta}(0,0) > 0. \tag{4}$$

The form of one such system is

$$Dx = \theta x - x^2. \tag{5}$$

By setting $Dx = 0$, we immediately see from (5) that the steady state equilibria of the system are at $x^* = 0$ and $x^* = \theta$. System (5) is stable around the equilibrium, $x^* = 0$, for all $\theta < 0$, and unstable for $\theta > 0$. The equilibria along $x^* = \theta$ are stable for $\theta > 0$ and unstable for $\theta < 0$.

Figure 1 illustrates the resulting transcritical bifurcation. The solid lines represent stable equilibrium points, while the dashed lines show unstable equilibria. As θ moves along the horizontal axis from between – ∞ and + ∞ , the system will bifurcate from stable to unstable as θ crosses the origin. But observe that if (x*,θ) remains along the kinked solid line, as θ moves from – ∞ to + ∞, the system will remain stable at all equilibria along that path, despite the fact that the system will pass through and bifurcate at the origin. Although the system will be stable in that case on both sides of the origin, the nature of the stable dynamics can change at the origin. Similarly, if (x*,θ) remains along the kinked dashed line as θ moves from – ∞ to + ∞, the system will remain unstable at all equilibria along that path, despite the fact that the system will pass through and bifurcate at the origin. The nature of the unstable dynamics nevertheless can change as the system bifurcates at the origin.

Transcritical bifurcations have been found in high-dimensional continuous-time macroeconometric systems. In high dimensional cases, transversality conditions have to be verified on a manifold. See Guckenheimer and Holmes (1983) for details.

In general dynamics, there is an infinite number of types of unstable dynamics, including periodic, multiperiodic, chaotic, etc., and there similarly are many forms of stable dynamics, such as monotonically convergent, damped single-periodic oscillatory, etc. Bifurcation or any type does not necessarily imply a shift between stability and instability, but does imply a change in the nature of the disequilibrium dynamics. As a result, if a confidence region around parameter estimates includes a bifurcation point, various kinds of dynamics can be consistent with the parameters being within the confidence region. All may be stable, all may be unstable, or some may be stable and some unstable. In any such case, robustness of inferences about dynamics is damaged.

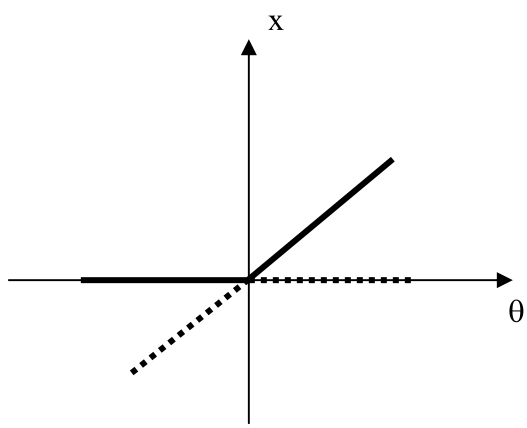

Figure 1. Transcritical Bifurcation Diagram of System (5)

3.2. Pitchfork Bifurcations

We now illustrate pitchfork bifurcation. A well known one-dimensional system with a pitchfork bifurcation is

$$Dx = \theta x - x^3. \tag{6}$$

Letting Dx = 0, we can solve for the system's steady state equilibria. For each $\theta > 0$, this system has three equilibria: $x^* = 0$ (unstable), $x^* = +\sqrt{\theta}$ (stable), and $x^* = -\sqrt{\theta}$ (stable). For every $\theta < 0$, there is only one equilibrium, $x^* = 0$, and it is stable. Figure 2 is the system's bifurcation diagram. Observe the similarity to a pitchfork, turned on its side. Solid lines represent stable equilibrium points, while the dashed line denotes unstable equilibria.

Note that as θ increases to the right from negative values towards the origin, the system will bifurcate at the origin. That bifurcation will cause the system to become unstable, if θ continues along the horizontal axis, or will keep the system stable, if θ moves along either of the two other possible paths. The dynamics may change in some ways, even if the system remains stable on both sides of the bifurcation point. As a result, it is important to know on which side of the origin the system may be operating, even if the system's dynamics are observed to be stable.

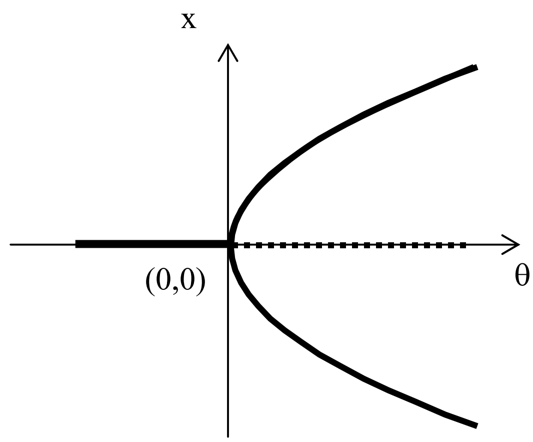

Figure 2. Pitchfork Bifurcation Diagram of System (6).

The Jacobian and transversality conditions for pitchfork bifurcation can be obtained as follows. Consider a one-variable, one-parameter differential equation of the form,

$$Dx = f(x,\theta).$$

Suppose that there exists an equilibrium, x^*, and a parameter value, θ^*, such that (x^*,θ^*) satisfies the following conditions:

$$(a) \frac{\partial f(x,\theta^*)}{\partial x}\Big|_{x=x^*} = 0,$$
$$(b) \frac{\partial^3 f(x,\theta^*)}{\partial x^3}\Big|_{x=x^*} \neq 0,$$
$$(c) \frac{\partial^2 f(x,\theta)}{\partial x \partial \theta}\Big|_{x=x^*,\theta=\theta^*} \neq 0.$$

If the Jacobian condition (a) and the transversality conditions (b) and (c) are satisfied at (x^*,θ^*), then (x^*,θ^*) is a pitchfork bifurcation point. Depending on the signs of the derivatives in (b) and (c), the equilibrium x^* could change from stable to unstable, when the parameter θ crosses θ^*.

Consider again the differential equation

$$Dx = \theta x - x^3.$$

Recall that $x^* = 0$ and $x^* = \pm\sqrt{\theta}$ are equilibria. The Jacobian is $\frac{\partial}{\partial x}(\theta x - x^3) = \theta\text{-}3x^2$, which is equal to zero at the bifurcation point $(x^*,\theta) = (0,0)$, as is required by condition (a). The transversality conditions (b) and (c) also are satisfied at (0,0). Hence the point (0,0) is a pitchfork bifurcation point. Judging by the sign of $\theta\text{-}3x^2$, we can confirm that the equilibrium $x^* = 0$ is stable, when $\theta < 0$ and unstable when $\theta > 0$. The two other equilibria $x^* = \pm\sqrt{\theta}$ are stable for $\theta > 0$, as illustrated in our Figure 2.

Bala (1997) explains how pitchfork bifurcation occurs in the tatonnement process. Chaos also can exist in the tatonnement process, as shown in Bala and Majumdar (1992).

3.3. Saddle-Node Bifurcations

We now turn to saddle-node bifurcation. A simple system with a saddle-node bifurcation is

$$Dx = \theta - x^2. \tag{7}$$

Note that it differs from the basic system for transcritical bifurcation by replacing the first order term with the zero order parameter and from the basic system for pitchfork bifurcation by lowering the orders of both terms. To explore the system's equilibria, set $Dx = 0$. Then $x^2 = \theta$, and therefore $x^* = \pm\sqrt{\theta}$, which requires θ to be nonnegative. Hence, there exist no equilibria for $\theta < 0$. For any given $\theta > 0$, this system has two equilibria at $x^* = \pm\sqrt{\theta}$.

Figure 3 displays the bifurcation diagram. The solid line represents stable equilibrium points, while the dashed one shows unstable ones. Clearly the diagram does not display a pitchfork, since there are no equilibria to the left of the origin. Movement along the path of the equilibria is restricted to movement along the saddle to the right of the origin, with bifurcation along that path occurring at the origin. The origin is called the saddle node. Bifurcation of this system necessarily causes transition between stability and instability.

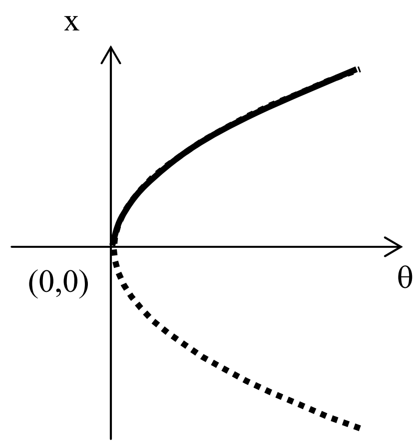

Figure 3. Saddle-Node Bifurcation Diagram for System (7).

For a general one-dimensional system,

$$Dx = f(x,\theta),$$

let x^* be an equilibrium, and let the parameter value θ^* be such that (x^*, θ^*) satisfies the equilibrium condition

$$f(x^*,\theta^*) = 0,$$

and the Jacobian condition

$$\frac{\partial f(x,\theta^*)}{\partial x}\Big|_{x=x^*} = 0.$$

Then the transversality conditions for saddle-node bifurcation at (x^*, θ^*) are

$$(a)\ \frac{\partial f(x,\theta)}{\partial \theta}\Big|_{x=x^*,\theta=\theta^*} \neq 0,$$
$$(b)\ \frac{\partial^2 f(x,\theta)}{\partial x^2}\Big|_{x=x^*,\theta=\theta^*} \neq 0.$$

Transversality conditions for high-dimensional systems can also be formulated [see Sotomayor (1973)].

The following economic system (Gandolfo (1996)),

$$Dr = v[F(r,\alpha) - S(r)],$$

exhibits saddle-node bifurcation, where r is the spot exchange rate defined as domestic currency per foreign currency, $v > 0$ is the adjustment speed, α is a parameter, and $\partial F/\partial\alpha > 0$. The differential equation indicates that the exchange rate adjusts according to excess demand.

3.4. Hopf Bifurcations

Hopf bifurcation is the most studied type of bifurcation in economics. Regarding the eigenvalue and transversality conditions that must be satisfied for Hopf bifurcation, see the Hopf Theorem in Guckenheimer and Holmes (1983). Hopf bifurcation requires the presence of a pair of purely imaginary Jacobian eigenvalues. Hence the dimension of a system needs to be at least two. The transversality conditions, which are rather lengthy, are given in Glendinning (1994).

An example of such a system in the 2-dimensional state-space case with one parameter is

$$Dx = -y + x(\theta - (x^2 + y^2)),$$
$$Dy = x + y(\theta - (x^2 + y^2)). \qquad (8)$$

The equilibria are found by setting $Dx = Dy = 0$. All equilibria are found to satisfy $x^* = y^* = 0$, with the stable equilibria occurring for $\theta < 0$ and the unstable equilibria occurring for $\theta > 0$.

The Hopf bifurcation boundaries can be determined numerically. Consider the case of $\det(\mathbf{A}(\theta^*)) \neq 0$ at the equilibrium (x^*,y^*), when $\mathbf{A}(\theta^*)$ has at least one pair of purely imaginary eigenvalues. If $\mathbf{A}(\theta^*)$ has exactly one such pair, and if some additional transversality conditions hold, the point $(\mathbf{x}^*,\mathbf{y}^*,\theta^*)$ is on a Hopf bifurcation boundary. A numerical procedure to find Hopf bifurcation boundaries was provided in Barnett and He (1999).

Figure 4 shows the phase portrait diagram for Hopf bifurcations in the 2-dimensional state space case with one parameter. The phase portrait not only shows the stable and unstable equilibria, but also the disequilibrium paths followed by (x,y) as they approach or diverge from any of the steady state equilibria. The stable equilibria, designated by a solid dark line, are along the θ axis for negative θ, and the unstable equilibria, designated by the dashed line, are along the θ axis for positive θ. The unstable disequilibrium dynamics to the right of the origin converge to a limit cycle, with the magnitude of the cycle growing as θ increases. The bifurcation point is at the origin. In this case, bifurcation necessarily causes transition between stability and instability. In the literature on chaos, Hopf bifurcation is fundamental, since the first bifurcation along the route to chaos is the loss of stability to a simple single-periodic limit cycle, as produced by Hopf bifurcation. As a result, Hopf bifurcation boundaries tend to be encountered as boundaries between stability and instability, rather than between two forms of stability or between two forms of instability.

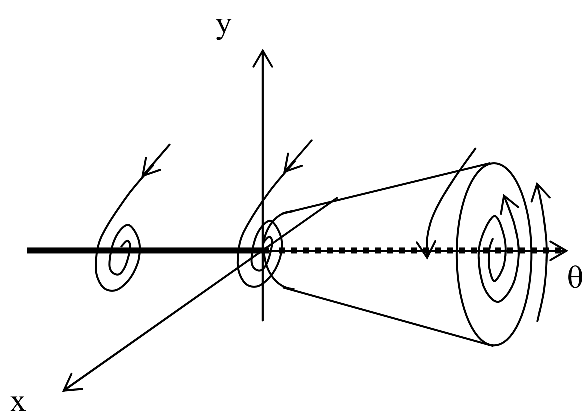

Figure 4. Hopf Bifurcation Phase Diagram for System (8)

4. Singularity-Induced Bifurcations

In Section 3, we reviewed some well-documented bifurcation regions encountered in macroeconomic models. We devote this section to a recently discovered surprising bifurcation region found in the Leeper and Sims (1977) model: a singularity-induced bifurcation.

Some macroeconomic models, such as the widely recognized dynamic Leontief model (Luenberger and Arbel (1977)) and the Leeper and Sims (1994) model, have the form

$$\mathbf{B}\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t), \tag{9}$$

in which $\mathbf{x}(t)$ is the state vector, $\mathbf{f}(t)$ is the vector of driving variables, t is time, and **B** and **A** are constant matrices of appropriate dimensions. The general form of system (9) is not in the class of autonomous systems unless $\mathbf{f}(t) = \mathbf{0}$, but we illustrate only autonomous cases in which $\mathbf{f}(t) = \mathbf{0}$.

The most significant aspect of system (9) is the possibility that the matrix **B** could be singular. If **B** is always invertible, then (9) will be consistent with the discrete-time form of the system (1), as is easily shown by inverting **B** to acquire:

$$\mathbf{x}(t+1) = \mathbf{B}^{-1}\mathbf{A}\mathbf{x}(t) + \mathbf{B}^{-1}\mathbf{f}(t),$$

so that

$$\mathbf{x}(t+1) - \mathbf{x}(t) = \mathbf{B}^{-1}\mathbf{A}\mathbf{x}(t) - \mathbf{x}(t) + \mathbf{B}^{-1}\mathbf{f}(t)$$

$$= [\mathbf{B}^{-1}\mathbf{A} - \mathbf{I}]\mathbf{x}(t) + \mathbf{B}^{-1}\mathbf{f}(t),$$

which clearly is in the form of (1).

Generalizing to permit nonlinearity, the model (9) in continuous time has the following form:

$$\mathbf{B}(\mathbf{x}(t),\theta)\mathbf{D}\mathbf{x} = \mathbf{F}(\mathbf{x}(t),\mathbf{f}(t),\theta). \quad (10)$$

where $\mathbf{f}(t)$ is the vector of driving variables and t is time. The general form of system (10) again is not in the class of autonomous systems unless $\mathbf{f}(t) = \mathbf{0}$, but we consider only autonomous cases in which $\mathbf{f}(t) = \mathbf{0}$. Singularity-induced bifurcation occurs, when the rank of $\mathbf{B}(\mathbf{x},\theta)$ changes, as from an invertible matrix to a singular one. For such changes in rank of $\mathbf{B}(\mathbf{x},\theta)$ to occur, that matrix must depend upon the setting of θ. In such cases, the dimension of the dynamical part of the system changes accordingly.

The dependency of **B** upon θ need not be through a closed form algebraic dependence of the elements of **B** upon θ, but can be through any form of point-to-matrix mapping producing a dependence of **B** upon θ. In fact in our Example 5 below, we provide an example of such a non-algebraic dependence causing singularity bifurcation. If $\mathbf{B}(\mathbf{x},\theta)$ does not depend at all upon θ, then singularity of $\mathbf{B}(\mathbf{x},\theta)$ is not sufficient for (10) to be able to produce singularity bifurcation, since the rank of $\mathbf{B}(\mathbf{x},\theta)$ will not change as θ changes. For example, the Leontief model described by Luenberger and Arbel (1977) is in the class of systems (9) with a singular matrix **B**, but no singularity bifurcation boundary has been found within that model.

In general, the structural properties of the dynamical implicit function system (10) can be substantially more complex than those for the closed form system (1). When **B** = **I**, system (10) becomes system (1). In that case, bifurcations can be classified according to the dynamical forms obtained solely from transforming **A**. When **B** ≠ **I**, the matrix **B** can take values producing a large number of dynamical possibilities for (10).

The systems (9) and (10) are often referred to as differential-algebraic systems. To illustrate the reason for that terminology, consider the two-dimensional state-space case, with **x** = (x_1,x_2). We can perform an appropriate coordinate transformation such that (10) becomes equivalent to the following form, containing one differential equation and one algebraic equation:

$$
\begin{aligned}
B_1(x_1,x_2,\boldsymbol{\theta})Dx_1 &= F_1(x_1,x_2,\boldsymbol{\theta}) \\
0 &= F_2(x_1,x_2,\boldsymbol{\theta}).
\end{aligned}
$$

We use the following examples to demonstrate the complexity of bifurcation behaviors that can be produced from system (10). The first two examples are in that class, but do not produce singularity bifurcation, since **B** does not depend upon the parameters. In the second two examples, $\mathbf{B}(\mathbf{x},\boldsymbol{\theta})$ does depend upon $\boldsymbol{\theta}$, and those two models are found to have singularity bifurcation regions within their parameter spaces.

Example 1. Consider the following system modified from system (5), which we have shown can produce transcritical bifurcation:

$$Dx = \theta x - x^2, \tag{11}$$

$$0 = x - y^2. \qquad (12)$$

Comparing with the general form of (10), observe that

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

which is singular but does not depend upon the value of θ. Observe that the system is a mixture of a differential and an algebraic system.

By setting Dx = 0, we see that the equilibria become $(x^*,y^*) = (0,0)$ and $(x^*,y^*) = (\theta,\pm\sqrt{\theta})$. Figure 5 shows the three-dimensional bifurcation diagram for this system. In this case, (11)-(12) is stable around the equilibrium $(x^*,y^*) = (0,0)$ for $\theta < 0$, as designated by the thick solid straight line, and unstable for $\theta > 0$, as designated by the dashed line. The equilibria for $(x^*,y^*) = (\theta,\pm\sqrt{\theta})$ are undefined when $\theta < 0$ and stable when $\theta > 0$, and are designated by the thick solid parabolic line.

The bifurcation point is at $(x,y,\theta) = (0,0,0)$, where the thick solid line, the dashed solid line, and the parabola all meet. Observe that movement from the stable equilibria at $(x^*,y^*) = (0,0)$ with negative θ to the unstable equilibria at $(x^*,y^*) = (0,0)$ with positive θ will cause bifurcation from stability to instability. But it is also possible to bifurcate at the origin from the stable equilibria at $(x^*,y^*) = (0,0)$ with negative θ to the stable equilibria along the three dimensional parabola $\{(x,y,\theta): x = \theta, y = \pm\sqrt{\theta}, \theta > 0\}$. In that case, bifurcation can change the nature of the dynamics in some ways, although the dynamics will remain stable before and after bifurcation. If a confidence region for estimated θ contains the point (0,0,0), three kinds of equilibria are possible within the confidence region: one unstable and two stable. Different forms of disequilibrium dynamics are likely to exist around each.

Although **B** is singular, the bifurcation point does not produce singularity bifurcation, since **B** does not depend upon θ. Before and after bifurcation, the number of differential equations and the number of algebraic equations remain unchanged. As a result, at any value of θ, the disequilibrium dynamics remain in two dimensional (x,y) state space. Singularity bifurcation cause change in the dimension of the state space.

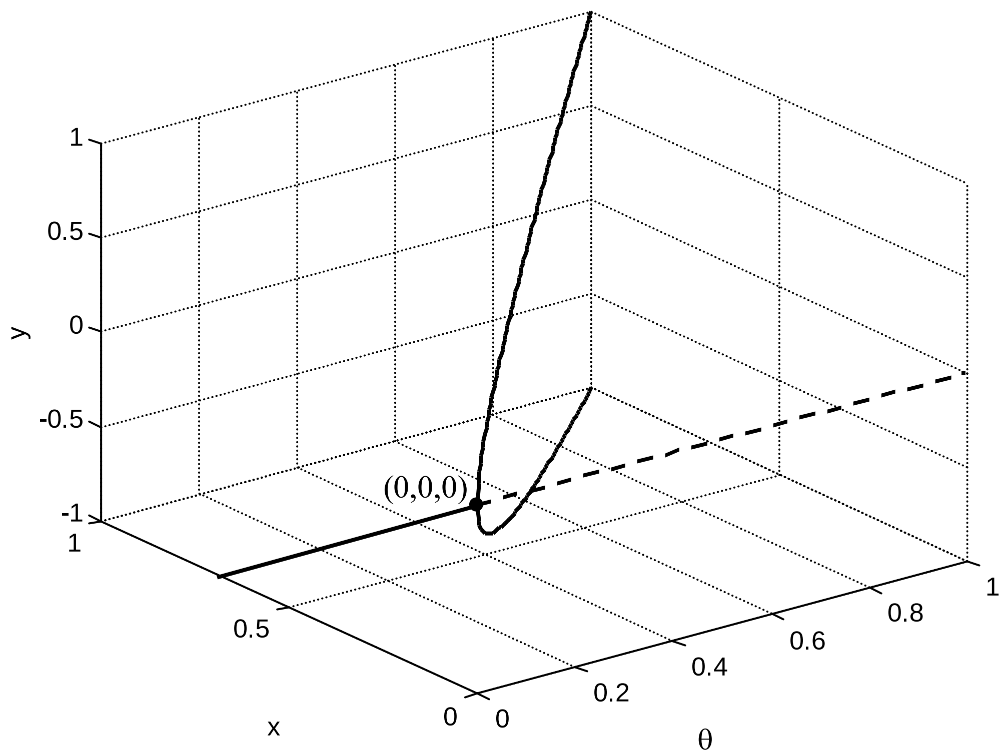

Figure 5. Bifurcation Diagram for (11)-(12).

Example 2. The following system is modified from the system (7) for saddle-node bifurcation.

$$Dx = \theta - x^2, \quad (13)$$

$$0 = x - y^2. \quad (14)$$

Setting Dx = 0, we find that the equilibria are at (x,y) = ($\sqrt{\theta}$,$\pm\sqrt[4]{\theta}$), which is defined only for $\theta \geq 0$. In this case, (13)-(14) is stable around both of the equilibria, (x,y) = ($\sqrt{\theta}$,$+\sqrt[4]{\theta}$) and (x,y) = ($\sqrt{\theta}$,$-\sqrt[4]{\theta}$). The bifurcation point between the two stable regions is (x,y,θ) = (0,0,0). Within the range $0 \leq \theta \leq 1$, the Figure 6 bifurcation diagram displays the equilibria as a thick solid curved line. Observe that there is no discontinuity or change in dimension at the origin in the three dimensional bifurcation diagram.

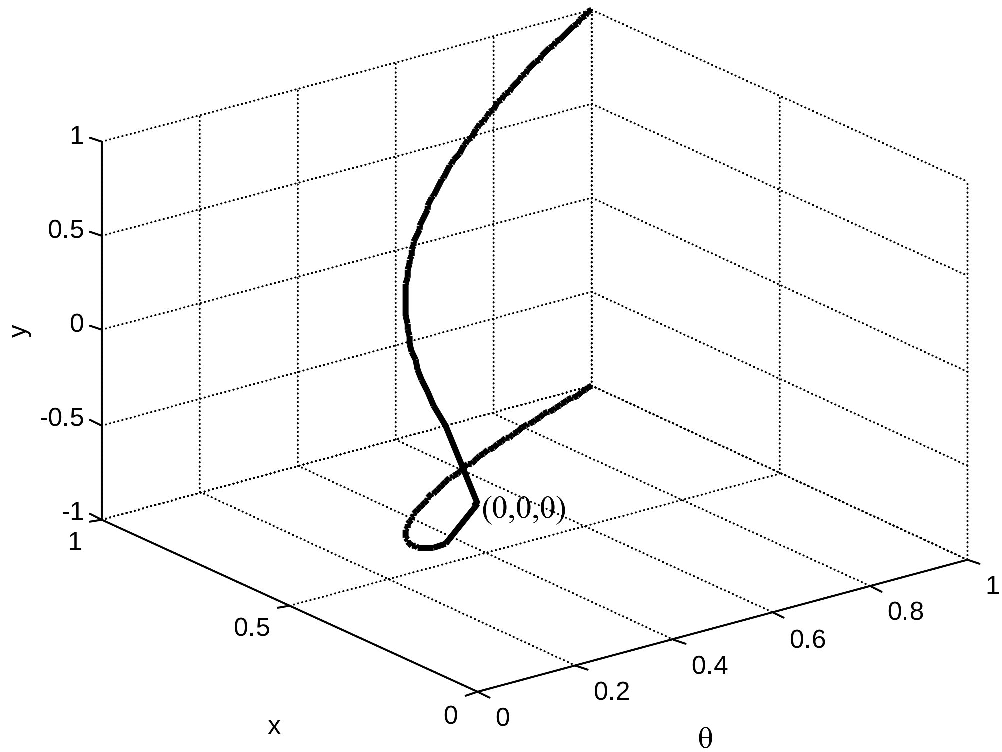

Figure 6. Bifurcation Diagram for the System (13)-(14), when $0 \leq \theta \leq 1$.

The form of matrix **B** again is fixed at

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and again is independent of the parameter, θ. For the same reason as in Example 1, the bifurcation point, which is at the origin, does not produce singularity bifurcation. The dimension of the state space dynamics remains unchanged on either side of the origin.

However, in some systems, such as the Leeper and Sims model, the matrix **B** is also parameterized. A result is the possibility of true singularity bifurcation, with a change in the mix of algebraic and differential equations and the resulting dramatic change in the dimension of the state space dynamics. The following example illustrates bifurcation in such cases.

Example 3. Consider the system

$$Dx = ax - x^2, \tag{15}$$

$$\theta Dy = x - y^2, \tag{16}$$

in which $a > 0$. In this case,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & \theta \end{bmatrix},$$

which does depend upon the parameter θ. Equations (15) and (16) consist of two differential equations with no algebraic equations for nonzero θ. But when θ = 0, the system has one differential equation and one algebraic equation.

By setting Dx = Dy = 0, we can find that for every θ, the equilibria are at (x,y) = (0,0) and (x,y) = (a, $\pm\sqrt{a}$). In this case, (15)-(16) is unstable around the equilibrium $(x^*,y^*) = (0,0)$ for any value of θ. The equilibrium $(x^*,y^*) = (a, +\sqrt{a})$ is unstable for θ < 0 and stable for θ > 0. Note that the location of the equilibrium does not depend upon θ. The third equilibrium $(x^*,y^*) = (a, -\sqrt{a})$ is unstable for θ > 0 and stable for θ < 0.

The effect of adding the second dynamic equation is more visible if we consider the system (15)-(16) in phase to display the disequilibrium dynamics for the state variables (x,y). We do so with a normalization at a = 1. Figure 7 displays those dynamics with positive θ, while Figure 8 displays the dynamics with θ = 0. When θ is negative, Figure 7 remains valid, but with the diagram flipped over about the x axis, so that (1,1) becomes unstable and (1,-1) becomes stable. The equilibrium (0,0) remains unstable for either positive or negative θ.

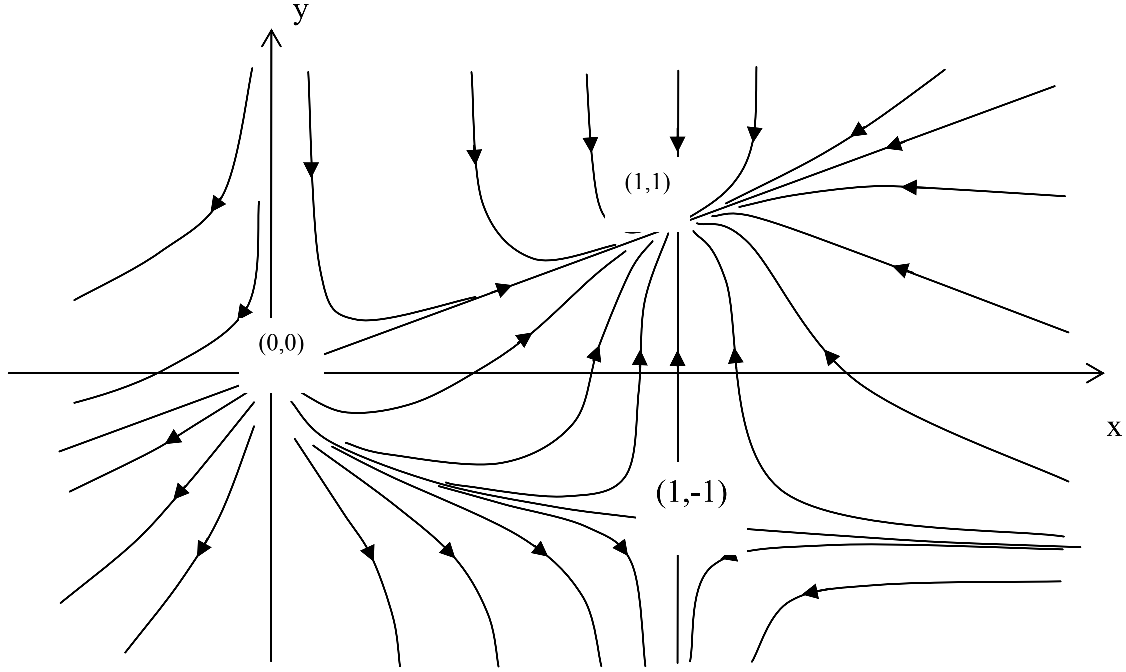

Figure 7. Phase Portrait of (15)-(16) in State Space with a = 1, for θ > 0

Figure 7 clearly shows the two-dimensional state-space dynamics in (x,y) for nonzero values of θ. However, when θ = 0, the system's behavior degenerates into movement along the curve, $x - y^2 = 0$, as shown in Figure 8, since the differential equation (16) changes into the algebraic constraint, $x - y^2 = 0$. That constraint must hold regardless of whether the system is in steady state equilibrium, or out of equilibrium. Figure 8 displays the one unstable equilibrium at (0,0) and the two stable equilibria at (1,1) and (1,-1), with the disequilibrium dynamics constrained to the path, $x - y^2 = 0$. The singularity bifurcation point is not displayed in either Figure 7 or 8, since θ is not an axis of either figure. The singularity bifurcation, produced by the transition from nonzero θ to zero value of θ, results in the dramatic drop in the dimension of the dynamics from Figure 7 to Figure 8. In Figure 7, the dynamics of the system move throughout the two-dimensional state space, while in Figure 8 the dynamics are constrained to move along the one dimensional curve, $x - y^2 = 0$.

It is very important to note the change in dynamical properties produced by singularity bifurcation, even if the bifurcation does not change between stability and instability. For example, if θ changes from positive to zero, when (x,y) is at the equilibrium (1,1), the system will remain stable, but disequilibrium dynamics will drop in dimension to a lower dimensional space. If θ changes from positive to zero, when (x,y) is at the equilibrium (0,0), the dynamics will remain unstable both before and after the bifurcation, but the dimension of the dynamics will drop. If θ changes from positive to zero, when (x,y) is at the equilibrium (1,-1), the dynamics will change from unstable to stable and the dimension of the dynamics also will drop. In all of those cases, the nature of the disequilibrium dynamics changes dramatically, even if there is no transition between stability and instability.

Unless economic theory provides a reason to consider the dynamics from setting parameters directly on a bifurcation point, the change in dynamics from one side of a bifurcation point to the other side is more important than the change in dynamics from parameter settings on one side of a bifurcation point to settings directly on a bifurcation point. Bifurcation regions are measure zero subsets of the parameter space. Hence, the effect on Figure 7 of changing the parameter between strictly negative settings of θ and strictly positive settings of θ is of particular importance. The comparison of the dynamics between two such nonzero settings does not display the dramatic drop into the "black hole" space of Figure 8, but the shift between positive and negative values of θ does cause the stability and instability of the equilibria (1,1) and (1,-1) to be interchanged. Observing the direction of the arrows of the disequilibrium paths around the unstable equilibrium (0,0), we can see that even in the vicinity of that always unstable equilibrium, the nature of the unstable dynamics will change substantially, when the sign of θ changes.

This observation will be important in understanding our ongoing research with the Leeper and Sims model, which we are finding is unstable on both sides of the singularity bifurcation boundary that is within the neighborhood of the parameter estimates (He and Barnett (2003)).

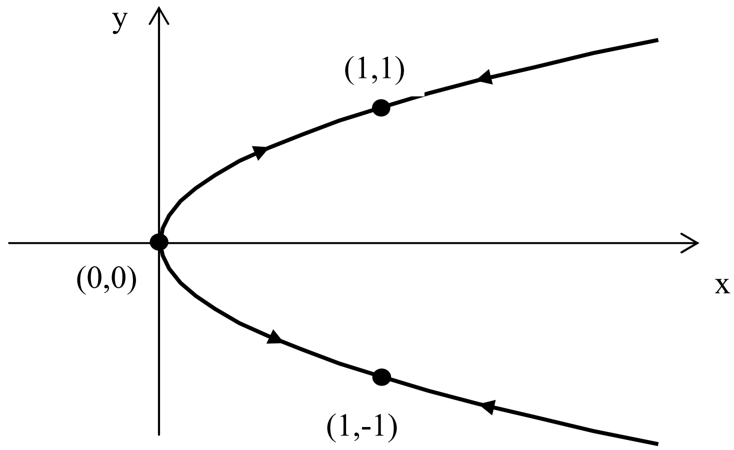

Figure 8. Phase Portrait of (15)-(16) in State Space with a = 1, for θ = 0

Example 4. If the second equation in (15)-(16) is changed to be linear, such that

$$Dx = ax - x^2, \quad (17)$$

$$\theta Dy = x - y, \quad (18)$$

we have a less complicated example of singularity bifurcation. In this case, for every θ the equilibria are at $(x^*,y^*) = (0,0)$ and $(x^*,y^*) = (a,a)$. The system (17)-(18) is unstable around the equilibrium $(x^*,y^*) = (0,0)$ for any value of θ. The equilibrium $(x^*,y^*) = (a,a)$ is unstable for $\theta < 0$ and stable for $\theta \geq 0$. To illustrate, we again normalize by setting $a = 1$. Figures 9 and 10 show the phase portraits in state space for (17)-(18) with $\theta > 0$ or $\theta = 0$, respectively. When $\theta < 0$, the system is everywhere unstable.

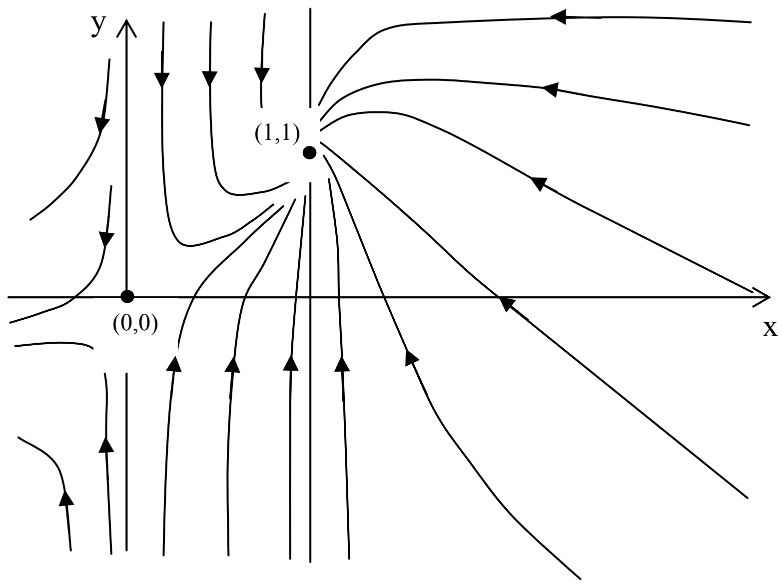

Figure 9. Phase Portrait of (17)-(18) in State Space with a = 1, for θ > 0.

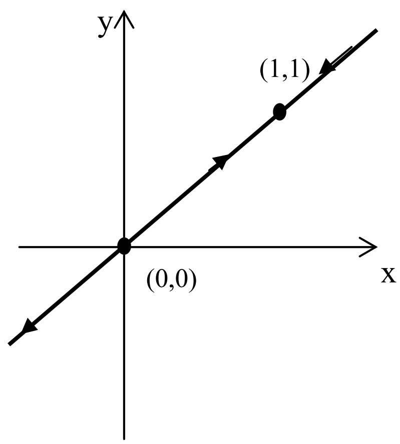

Figure 10. Phase Portrait of (17)-(18) in State Space with a = 1, for θ = 0.

Again, Figures 9 and 10 demonstrate the drastic changes of dynamical properties, when the parameter traverses the bifurcation boundary. When $\theta = 0$, the variable y in (17)-(18) is just a replica of the variable x in (17)-(18), since equation (18) becomes the algebraic constraint, $y = x$. The disequilibrium dynamics in Figure 10, whether stable in the vicinity of (1,1) or unstable in the neighborhood of (0,0), are just one-dimensional along the ray through the origin. However, when $\theta \neq 0$, the system moves into the two-dimensional space in Figure 9.

Observe that (0,0) remains unstable in both Figures 9 and 10, and (1,1) remains stable in both Figures 9 and 10. The singularity bifurcation that causes transition between the two dimensional space in Figure 9 and the one dimensional path in Figure 10 need not cause a change between stability and instability. Stability can remain stable, and instability can remain unstable, but with dramatic change in the nature of the dynamics. Also observe that the nature of the dynamics with θ small and positive is very different from that with θ small and negative. In particular, the equilibrium at $(x^*,y^*) = (1,1)$ is stable in the former case and unstable in the latter case. There is little robustness of dynamical inference to small changes of θ in the vicinity of the bifurcation boundary, even if the startling drop into the measure-zero "black hole" at exactly $\theta = 0$ is never encountered. On the more general subject of robustness of inference in dynamic models, see Barnett and Binner (2004, part 4).

Changes in the dynamical properties of (10) through singularity bifurcation can occur, even when the parameters $\boldsymbol{\theta}$ do not appear directly within the matrix $\mathbf{B} = \mathbf{B}(\mathbf{x},\boldsymbol{\theta})$ itself, but rather affect $\mathbf{B}$ through a mapping from outside $\mathbf{B}$, as illustrated in the following example.

Example 5. Consider the system:

$Dx_1 = x_3,$

$Dx_2 = -x_2,$

$0 = x_1 + x_2 + \theta x_3,$ (19)

which has the following singular **B** matrix:

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{20}$$

where $\mathbf{Dx} = (Dx_1,Dx_2,Dx_3)'$.

Solving $\mathbf{Dx} = \mathbf{0}$, we see that the only equilibrium is at $\mathbf{x}^* = (x_1^*,x_2^*,x_3^*) = (0,0,0)$. For any $\theta \neq 0$, solving the last equation for x_3 and substituting into the first equation results in the two equation system

$Dx_1 = -(x_1 + x_2)/\theta$

$Dx_2 = -x_2,$ (21)

which is stable at its 2-dimensional equilibrium $\mathbf{x}^* = (x_1^*,x_2^*) = (0,0)$ for $\theta > 0$ and unstable at that equilibrium for $\theta < 0$. Observe that the **B** matrix now is the nonsingular matrix $\mathbf{B} = \mathbf{I}$.

But now consider what happens on the singularity bifurcation boundary with $\theta = 0$. Setting $\theta = 0$, we find that system (19) becomes

$x_1 = -x_2,$

$Dx_2 = -x_2,$

$x_3 = x_2,$ (22)

for all t > 0. This system has the following singular **B** matrix:

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{23}$$

Note the different order of the dynamics in (22) from that of (21). In system (22), there are two algebraic constraints and one differential equation, while system (21) has two differential equations and no algebraic constraints. Clearly the **B** matrix is different in the two cases and the rank of the **B** matrix has changed between the two cases. Yet one would not have anticipated this change from inspection of the general form of the system, (19), since its **B** matrix, (20), does not contain the model's parameter within the matrix itself. In short, the **B** matrix can depend upon the parameters, and singularity bifurcation can occur, even if there does not exist a direct closed-form algebraic representation of the dependence of **B** upon the parameters.

5. Conclusion

In this paper, we first summarize those bifurcation phenomena in macroeconomic models that we previously have encountered in our research. We then introduce singularity-induced bifurcation. That class of bifurcation has not previously been encountered in economics. He and

Barnett (2003) recently found singularity bifurcation in their ongoing research on the Leeper and Sims Euler-equations macroeconometric model. We have contrasted the nature of the previously encountered forms of bifurcation with the dramatically different nature of singularity bifurcation. We believe singularity bifurcations will be found to have important implications for robustness of dynamic inferences with other modern Euler-equations macroeconometric models. Euler equation systems are first order equation systems that inherently are in implicit function form and rarely can be solved for closed form representations. We have shown that the implicit function systems (9) and (10) can produce singularity bifurcation, while the closed form differential equations system (1) cannot produce singularity bifurcation. Singularity bifurcation did not appear with older algebraically-solvable macroeconometric models. It is clear why singularity bifurcation needs to be taken seriously with modern Euler equations models.

In the unlikely case that the parameters fall exactly on the measure-zero singularity bifurcation boundary (perhaps as a result of a theoretical constraint), the dynamics of the system drop into a "black hole" lower-dimensional state space. Although that dimensional collapse does not occur on either side of the boundary, the dynamical properties on one side of the boundary can be very different from those on the other side. It is important to recognize that the startling differences in dynamics on the two sides of a singularity bifurcation boundary need not imply a difference in stability on the two sides of the boundary. The dynamics can be unstable on both sides, or stable on both sides, but with very different dynamical properties on the two sides of the boundary. This can occur, even with the parameters being very close to the boundary on each side of the boundary.

In short, even with very high precision of parameter estimates, the nature of dynamics can be dramatically different within different subsets of the parameter estimates' confidence region.

Robustness of dynamical inferences is severely damaged, when a singular bifurcation boundary enters within the confidence region of a model's parameter estimates.

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