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Abstract

This paper examines two related models, which permit analytical investigation, to gain some insights into a relationship between endogenous growth and business cycle. Using two simple models which incorporate Fisher Black's idea that a significant part of business cycle fluctuations is a consequence of an economy's choice, we obtained a negative relationship between uncertainty and growth. Hence a country with large fluctuation will have a lower growth rate, and vice versa. If other factors like elasticity of substitution or average total factor productivity changes, however, a country with large fluctuation will have a higher growth rate, and vice versa. It is also shown that the relationship between volatility of output and growth rate can be either positive or negative depending on parameter values and the distribution of the shock. Thus measured relationship between volatility and growth rate might give a false relationship between uncertainty and growth.

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1 Introduction

The purpose of this paper is to gain some insights into the relationship between growth and business cycle by examining a few simple examples, which permit analytical investigation. Fisher Black (1987) suggested an interesting idea regarding the relationship suggesting that a significant part of business cycle fluctuations is a consequence of people's rational choice, so there is a trade-off between stabilization and growth.

> The business cycle is largely a matter of choice... [S]ociety can avoid large fluctuations in output by agreeing to lower output and lower expected growth in output...[T]he choices that the typical individual makes involve the trade-offs discussed in the introduction. The individual wants:
> **A** high expected growth in output,
> **B** small fluctuations in output, and
> **C** low average unemployment.
> ...Business cycles are the result of people balancing their interests in A, B, and C. (pp. 111-112 Black, 1987)

He used the following line of argument to support his suggestion:

> If people chose to invest in sectors both high and low expected growth, the economy would be more diversified and exhibit lower fluctuation in output, but also lower growth, than it would if people invested only or primarily in sectors with high growth. (p. 104 Black, 1987)

This idea makes sense. If people choose sectors in an economy for their investment or resource allocation based on fluctuation and growth, sectors with large fluctuation and low growth potential will be dominated away, and there will be a trade-off between stabilization and growth. One of the major factors that determine people's preference over fluctuation and growth would be risk-aversion. Thus a country consisting of very risk-averse people will choose a steady and smooth path although it yields a low average growth rate. On the other hand, a country with less risk-averse people will choose a rather rough path, in which higher average growth rate is achieved. Elasticity of substitution between consumption and leisure is another factor, which will be shown in this paper.

To my knowledge, however, Black's idea did not get much attention in the literature. I think part of the reason is that he did not lay out any concrete model for a serious investigation of his idea. In few citations he received, he is also interpreted as suggesting a positive relationship between uncertainty and growth, while most of research results, although not many in total, on the relationship show negative relationship.

Aizenman and Marion (1993), Bernanke (1983), and Pindyck (1991) analyze irreversible choice (investment) under uncertainty and its consequences on growth. The common implication of their models is that increased volatility reduces growth rates. Ramey and Ramey (1991) also get the same implication in a model in which firms commit to their technology in advance, and supports the implication with empirical studies. Ramey and Ramey (1995) further presents empirical evidence on the relationship, and show that a country with a larger output volatility is associated with a lower growth rate, and vice versa. Then they comment that there could be reasons to believe that "growth and volatility could be positively linked," and cite Black (1987) as a leading example.

In this paper I present two models that incorporate Black's (1987) idea in a way as simple as possible and show that the negative relationship obtained by other authors are not an evidence against Black (1987). Furthermore, I show that the measured relationship between volatility and growth rate of output might give a false relationship between uncertainty and growth. Most empirical research in the literature is usually conducted using data on volatility and growth rate.

The models in this paper has a feature of that the impact of a shock to the economy can be controlled by individuals living in it. If the impact of a shock can be controlled by allocating some of resources in the economy which could be otherwise used to enhance growth, then a country consisting of very risk-averse people will choose a steady and smooth path although it yields a low average growth rate. On the other hand, a country consisting of less risk averse people will choose a rather rough path in which higher average growth rate can be obtained.

Representative agent lives in the models, and production takes place according to a simple production function which involves labor and human capital only. Physical capital does not exist. One unit of time can be, and will be, divided into many different activities including leisure, production, and learning. Learning will increase human capital and so it is the source of growth. Then a shock is given to the economy, and it is assumed that the impact of the shock can be reduced if the representative agent allocates some of his resources.

The only resource available in the models is one unit of time endowed in each period. Thus the impact of the shock will depend on the time the representative agent uses. More specifically, the variance of the shock will be an increasing function of time the agent allocates for growth in the first model, and a decreasing function of time the agent allocates separately to reduce the variance. One can imagine an economy with two sectors. One sector has a growth potential and is susceptible to a shock, and another one produces a basic good in the economy without growth possibility and is not susceptible to the shock. If representative agent increases time spent in the basic sector, then overall impact of the shock will be mitigated because more resource is devoted into the non-fluctuating sector.

With models that has such features, I show that the equilibrium relationship between uncertainty and growth can be negative, instead of positive, although people can choose between high growth with more fluctuation and low growth with less one. This means Black (1987) did not suggest a positive relationship when he discussed a trade-off between stabilization and growth. Furthermore, I show that the relationship between volatility and growth rate can be positive although the equilibrium relationship between uncertainty and growth is negative. This means the measured relationship between volatility and growth rate might not coincide with the relationship between uncertainty and growth.

The rest of the paper is as follows. In section 2, we investigate a model in which the shock hits the engine of growth, and in section 3 we compare it with a model in which it hits production function, which is a more conventional setup. Section 4 concludes the paper.

2 Model 1

2.1 Setup

All consumers are identical and the representative agent's utility is given as

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t (\ln c_t + \alpha \ln d_t)$$

where c_t and d_t are consumption and leisure in period t. Of course, β is time preference parameter and α is elasticity of substitution between consumption and leisure. Each agent is endowed with one unit of time.

The production function is

$$y_t = h_t l_t,$$

where h_t and l_t are human capital level and a fraction of time allocated to production, respectively in period t. In this model we do not have productivity shock. Rather we will have a shock in the engine of growth; human capital accumulation technology. The next model will have a more standard setup. i.e., the shock will hit production function.

The human capital accumulation technology is given as

$$h_{t+1} - h_t = (\gamma u_t + \tilde{\epsilon}_{t+1}) h_t,$$

where u_t is a fraction of time devoted to learning, and $\tilde{\epsilon}_{t+1}$ is an i.i.d. shock realized at the beginning of period $t + 1$.

The only source of fluctuation in our economy is $\tilde{\epsilon}_{t+1}$. It is assumed that it has mean zero, and its variance is an increasing function of u_t, i.e.,

$$\mathrm{Var}(\tilde{\epsilon}_{t+1}) = \Omega(u_t), \quad \Omega' > 0. \tag{1}$$

It means that if people choose to grow fast, then they must also choose to have a large fluctuation. It incorporates Black's (1987) idea discussed in the introduction in a way: increase in u_t will increase the human capital available in the next period, so the country will grow faster, but it also increase the variance of the shock. Thus there is a trade-off between stabilization and growth. Another way of controlling the impact of a shock is considered in the next model, in which the variance of a shock is a decreasing function of a separate time devoted to reduce the impact of the shock. However, it will be seen that the basic results are the same.

We further postulate a very simple form of functional relationship between variance of $\tilde{\epsilon}_{t+1}$ and u_t. It is represented by the equation

$$\tilde{\epsilon}_{t+1} = \sigma \epsilon_{t+1} u_t,$$

where ϵ_{t+1} is an i.i.d. shock realized at the beginning of period $t + 1$,which takes a value on a small interval around 0 with mean zero and a finite variance. σ is a parameter to allow mean preserving spread. It is introduced to study the impact of an increase in underlying uncertainty on the choice of u_t and hence on the growth rate. This specification means the function Ω is simply,

$$\mathrm{Var}(\tilde{\epsilon}_{t+1}) = \Omega(u_t) = \sigma^2 \xi u_t^2,$$

where ξ is the variance of ϵ_{t+1}.

We assume that $1 + \gamma + \sigma\epsilon > 0$ for all possible ϵ. It ensures that human capital h_{t+1} is positive no matter what value u_t takes on. Refer to the human capital accumulation technology above.

2.2 Solution

We will seek a steady state solution of the problem. Bellman equation for it is

$$v(h) = \max_{u,l}\{\ln h + \ln l + \alpha \ln(1 - u - l) + \beta \int v(\{\gamma u + \sigma\epsilon u + 1\}h) f(\epsilon) d\epsilon\}, \quad (2)$$

where f is the density function of the shock ϵ.

The first order conditions are

$$\begin{aligned} \frac{1}{l} &= \frac{\alpha}{1 - l - u} \\ \frac{\alpha}{1 - l - u} &= \beta \int v'\left(\{\gamma u + \sigma\epsilon u + 1\}h\right)\left(\gamma + \sigma\epsilon\right) h f(\epsilon) d\epsilon. \end{aligned}$$

These will give

$$\frac{1 + \alpha}{1 - u} = \beta \int v'(\{\gamma u + \sigma u\epsilon + 1\}h)(\gamma + \sigma\epsilon) h f(\epsilon) d\epsilon. \quad (3)$$

By envelope theorem, we have

$$v'(h) = \frac{1}{h} + \beta \int v'(\{\gamma u + \sigma u\epsilon + 1\}h)(\gamma u + \sigma u\epsilon + 1) f(\epsilon) d\epsilon.$$

The solution of this is

$$v'(h) = \frac{1}{1 - \beta}\frac{1}{h}. \quad (4)$$

Substituting it into equation (3), we have

$$\frac{1 + \alpha}{1 - u} = \frac{\beta}{1 - \beta} \int \left(\frac{\gamma + \sigma\epsilon}{\gamma u + \sigma\epsilon u + 1}\right) f(\epsilon) d\epsilon. \quad (5)$$

The left hand side (LHS) of this equation is marginal utility loss due to one unit increase in u_t, and the right hand side (RHS) is infinite sum of marginal utility gain due to an increase in h_{t+1}. This fact can be seen more clearly by the following heuristic reasoning.

Increase in u_t by Δu_t will lose utility by

$$\mathrm{MU}_t \Delta u_t = \frac{\alpha}{1 - l_t - u_t} \Delta u_t = \frac{1+\alpha}{1 - u_t} \Delta u_t.$$

The second equality is obtained through marginal utility balancing in the choice of l_t. The last term is the LHS of the equation (5) except for Δu_t.

On the other hand, increase in u_t by Δu_t will increase tomorrow human capital h_{t+1} by $\Delta h_{t+1} = (\gamma + \sigma\epsilon_{t+1}) h_t \Delta u_t$. Then tomorrow's consumption will increase by $\Delta c_{t+1} = \Delta h_{t+1} l_{t+1} = (\gamma + \sigma\epsilon_{t+1}) h_t l_{t+1} \Delta u_t$, and tomorrow expected utility gain is

$$\begin{aligned}
& \beta \mathrm{E}_t \Big[\mathrm{MU}_{t+1} \Delta c_{t+1} \Big] \\
= \;& \beta \mathrm{E}_t \Big[\frac{1}{c_{t+1}} (\gamma + \sigma\epsilon_{t+1}) h_t l_{t+1} \Delta u_t \Big] \\
= \;& \beta \mathrm{E}_t \Big[\frac{1}{(1 + \gamma u_t + \sigma\epsilon_{t+1} u_t) h_t l_{t+1}} (\gamma + \sigma\epsilon_{t+1}) h_t l_{t+1} \Delta u_t \Big] \\
= \;& \beta \mathrm{E}_t \Big[\frac{1}{(1 + \gamma u_t + \sigma\epsilon_{t+1} u_t)} (\gamma + \sigma\epsilon_{t+1}) \Delta u_t \Big].
\end{aligned}$$

Since increase in tomorrow's human capital level will also increase the utility levels in each of the subsequent periods approximately in equal amount, the overall marginal utility gain is discounted sum of the above quantity, which is

$$\frac{\beta}{1-\beta} \mathrm{E}_t \Big[\frac{1}{(1 + \gamma u_t + \sigma\epsilon_{t+1} u_t)} (\gamma + \sigma\epsilon_{t+1}) \Delta u_t \Big],$$

which is the RHS of equation (5) except for Δu_t. Note that this can be written as

$$\frac{\beta}{1-\beta} \mathrm{E}_t [\mathrm{MU}_{t+1} R_{t+1}], \tag{6}$$

where R_{t+1} is the rate of return to an increase in learning.

Thus we see that the LHS of (5) is today's marginal utility loss in leisure due to increase in today's learning time, and the RHS is the total of future marginal utility gains in consumption because of the increase in learning. They must be equated if a fraction of time is allocated optimally.

Denote the LHS by $\Phi(u,\alpha)$ and the RHS by $\Psi(u,\sigma,\gamma)$. The graphs of $\Phi(u,\alpha)$ and $\Psi(u,\sigma,\gamma)$ are drawn in figure 1. $\Phi(u,\alpha)$ is strictly increasing to infinity, and $\Psi(u,\sigma,\gamma)$ is decreasing as u increases from 0 to 1. Marginal utility gain is diminishing as u increases more and more, while marginal loss is increasing.

By glancing at the graphs, we see that an interior solution, $0 < u < 1$, will exist if and only if $\Phi(0,\alpha)$ is smaller than $\Psi(0,\sigma,\gamma)$. Since $\Phi(0,\alpha)$ $= 1+\alpha$ and $\Psi(0,\sigma,\gamma) = \{\beta/(1-\beta)\}\,\gamma$, an interior solution exists if and only if $1+\alpha < \frac{\beta}{1-\beta}\gamma$.

If the condition is not satisfied, then we will have a corner solution, which is $u = 0$. The reason can be understood easily if we recall that $\Phi(u,\alpha)$ is marginal utility loss due to an increase in today's learning and $\Psi(u,\sigma,\gamma)$ is future marginal utility gain. Overall marginal utility gain is $\Psi(u,\sigma,\gamma)-\Phi(u,\alpha)$. If it is always negative in the relevant interval, $[0,1]$, the best solution is $u = 0$. Thus in that case, people do not want to grow at all because they dislike fluctuations so much.

2.3 Increase in Uncertainty

Assuming that the condition for the interior solution is satisfied, we now consider the effect of an increase in uncertainty, i.e., an increase in σ. To do that, we differentiate $\Psi(u,\sigma,\gamma)$ with respect to σ, and examine its sign.

$$\begin{aligned}\frac{\partial\Psi(u,\sigma,\gamma)}{\partial\sigma} &= \frac{\beta}{1-\beta}\int\frac{\epsilon}{(\gamma u+\sigma\epsilon u+1)^2}f(\epsilon)d\epsilon, \quad \text{and}\\ \frac{\partial^2\Psi(u,\sigma,\gamma)}{\partial\sigma^2} &= -\frac{2\beta}{1-\beta}\int\frac{u\epsilon^2}{(\gamma u+\sigma\epsilon u+1)^3}f(\epsilon)d\epsilon < 0.\end{aligned}$$

The second partial derivative is less than 0 because $(\gamma u+\sigma\epsilon u+1) > 0$. Since $\partial\Psi(u,0,\gamma)/\partial\sigma = 0$, we get that $\partial\Psi(u,\sigma,\gamma)/\partial\sigma < 0$ for $\sigma > 0$, which means that Ψ is decreasing in σ. Thus, as σ increases Ψ shifts down. See figure 2.

It means an increase in the uncertainty in the investment technology of human capital results in a decrease in the investment and consequently growth rate decreases. Thus in a model incorporating Black's (1987) idea, the equilibrium relationship between uncertainty and growth is negative. This is a results that many share in their research. We see that Black was not suggesting a positive equilibrium relationship, and Ramey and Ramey (1995) were wrong in their interpretation of Black.

Let's compare the result here with that in a model which is identical with the current one except the assumption that the impact of the shock cannot be controlled. This will change the human capital accumulation technology to

$$h_{t+1} - h_t = (\gamma u_t + \sigma\epsilon_{t+1})h_t.$$

Then the first order condition of the Bellman equation will give

$$\frac{1+\alpha}{1-u} = \frac{\beta}{1-\beta}\int(\frac{\gamma}{\gamma u + \sigma\epsilon + 1})f(\epsilon)d\epsilon.$$

Again, call LHS $\Phi(u,\alpha)$ and RHS $\Psi(u,\sigma,\gamma)$.[1]

To see the direction of the change in u as σ increases, we differentiate Ψ with respect to σ.

$$\frac{\partial\Psi(u,\sigma,\gamma)}{\partial\sigma} = -\frac{\beta}{1-\beta}\int\frac{\gamma\epsilon}{(\gamma u + \sigma\epsilon + 1)^2}f(\epsilon)d\epsilon.$$

The positivity of this can be seen by differentiating it with respect to σ and noting that $\partial\Psi(u,0,\gamma)/\partial\sigma = 0$. Thus as variance of the shock increases, investment in human capital increases, and growth rate increases.

[1]In this case, the necessary and sufficient condition for an interior solution in the main model $\{\beta/(1-\beta)\}\gamma > 1+\alpha$ becomes a sufficient condition. The reason is as follows.

$$\Psi(0,\sigma,\gamma) = \frac{\beta}{1-\beta}\int\frac{\gamma}{\sigma\epsilon+1}f(\epsilon)d\epsilon = \frac{\beta}{1-\beta}\Big(\gamma - \int\frac{\gamma\sigma\epsilon}{\sigma\epsilon+1}f(\epsilon)d\epsilon\Big).$$

The second equality utilizes that $\int f(\epsilon)d\epsilon = 1$. Now making use of the fact that $0 = \mathrm{E}\epsilon = \int \epsilon f(\epsilon)d\epsilon$, we have

$$\int\frac{\epsilon}{\sigma\epsilon+1}f(\epsilon)d\epsilon = -\int\frac{\sigma\epsilon^2}{\sigma\epsilon+1}f(\epsilon)d\epsilon < 0,$$

because we must have $\sigma\epsilon + 1 > 0$ for all ϵ. Thus

$$\Psi(0,\sigma,\gamma) > \frac{\beta}{1-\beta}\gamma.$$

Since $\Phi(0,\alpha) = 1+\alpha$, we get our sufficient condition.

The reason why increase in variance of the shock increases more investment here is that precautionary saving decision dominates in this case, i.e., the marginal utility is convex, while rate of return does not depend on the variance. $\Psi(u, \sigma, \gamma)$ is marginal utility gain from an increase in learning, and as in (6) it can be written as

$$\frac{\beta}{1-\beta}\mathrm{E}_t[\mathrm{MU}_{t+1}R_{t+1}],$$

where again MU_{t+1} is the marginal utility in period $t+1$ and R_{t+1} is the rate of return on increase in learning. Here R_{t+1} is constant γ. Thus increase in σ increases or decreases depending on $\mathrm{E}_t[\mathrm{MU}_{t+1}]$ increases or not. If MU is convex, i.e., $U''' > 0$, then $\mathrm{E}_t[\mathrm{MU}_{t+1}]$ increases, and if MU is concave, then $\mathrm{E}_t[\mathrm{MU}_{t+1}]$ will decrease.[2] Since logarithmic utility is used here, MU is convex and as a consequence investment in learning increases as uncertainty increases, which is precautionary saving.

On the other hand, in our original model, the rate of return R_{t+1} depends on the variance of the shock. It is $\gamma + \sigma u_{t+1}$ and precautionary saving plays a less significant role. Because the impact of shock can be reduced by reducing growth, the representative agent do that although precautionary saving motive tells him to increase growth. Consequently, steady state investment in human capital decreases.

2.4 Volatility of Output and Growth Rate

Now we calculate the volatility of output and show that the measured relationship between volatility of output and growth rate can be either positive or negative although we obtained a negative relationship between uncertainty and growth.

Usually, the volatility of output is measured by constructing

$$\tilde{\epsilon}_{t+1} = \ln y_{t+1} - \ln y_t - \{\text{mean growth rate } |I_t\},$$

where I_t is the information set in period t. The conditional variance s_t^2 of $\tilde{\epsilon}_{t+1}$ conditioned at the information I_t available at period t is called volatility of output.

[2]But a positive and decreasing marginal utility cannot be globally concave.

We use , however, the conditional variance of

$$\frac{y_t}{(\gamma u+1)y_{t-1}}-1,$$

as the volatility for calculational purposes. The above expression and $\ln y_{t+1}-\ln y_t-\gamma u$ are roughly the same,[3] and there is no definite reason that one is better than the other.

Now

$$\begin{aligned} & \frac{c_{t+1}}{(\gamma u+1)c_t}-1=\frac{h_{t+1}}{(\gamma u+1)h_t}-1 \\ = & \frac{\gamma u+\sigma u\epsilon_{t+1}+1}{\gamma u+1}-1. \end{aligned}$$

Then the conditional variance of the above, volatility of output, is

$$\begin{aligned} & \mathrm{Var}_t\Big(\frac{c_{t+1}}{(\gamma u+1)c_t}-1\Big) \\ = & \mathrm{Var}_t\Big(\frac{\gamma u+\sigma u\epsilon_{t+1}+1}{\gamma u+1}\Big) \\ = & \Big(\frac{u}{\gamma u+1}\Big)^2\sigma^2\mathrm{Var}(\epsilon). \end{aligned}$$

By looking at the expression we can conclude that we do not have any definite general relationship between volatility of output and growth rate, because u decreases and $u/\gamma u+1$ decreases in turn as σ increases. The volatility and growth rate can have either positive or negative relationship depending on various factors; α, β, γ and distribution of the underlying shock $f(\epsilon)$.

2.5 Elasticity of Substitution between Consumption and Leisure

The elasticity of substitution between consumption and leisure is α in our model. From equation (5), we see $\Phi(w)$ shifts up as α increases, so the equilibrium u decreases. This means given the same shock, an economy with

[3] $\{$mean growth rate $|I_t\}$ in our model is γu.

large elasticity of substitution will grow less and experience less fluctuation. Hence the choice suggested by Black (1987) is seen here; the choice between fast growth with large fluctuation and slow growth with small fluctuation. The choice depends on the elasticity of substitution between consumption and leisure.

In general, since the volatility of output is $\left(\frac{u}{\gamma u+1}\right)^2 \sigma^2 \mathrm{Var}(\epsilon)$, a country with a larger volatility will have a higher growth rate, if countries differ in any factor that affects u except σ. Thus the statement that negative relationship exists between growth and uncertainty in equilibrium is not in contradiction to the statement that some countries grow fast with large fluctuations and other countries do the opposite.

2.6 Welfare Comparison

Obviously, as σ increases the welfare will decrease. We can show it using the value function $v(h)$. Its derivative is $\{1/(1-\beta)\}1/h$ as given in equation (4). So $v(h) = \{1/(1-\beta)\}\ln h + C$, and the unknown constant C is determined by plugging the expression for $v(h)$ into the Bellman equation, equation (2), and solving for C. Then

$$C = \frac{1}{1-\beta}\Big\{\ln l + \alpha \ln(1-u-l) + \frac{\beta}{1-\beta}\int \ln(\gamma u + \sigma u \epsilon + 1) f(\epsilon) d\epsilon\Big\}.$$

If we differentiate this C with respect to σ, we will get the welfare change. It is, due to envelope theorem,

$$\frac{dC}{d\sigma} = \frac{\beta}{(1-\beta)^2}\int \frac{u\epsilon}{\gamma u + \sigma u \epsilon + 1} f(\epsilon) d\epsilon.$$

The negativity of this quantity can be shown by using

$$\begin{aligned} 0 &= \mathrm{E}\epsilon = \int \epsilon f(\epsilon) d\epsilon \\ &= \int \frac{(\gamma u + \sigma u \epsilon + 1)\epsilon}{\gamma u + \sigma u \epsilon + 1} f(\epsilon) d\epsilon. \end{aligned}$$

Then

$$\int \frac{u\epsilon}{\gamma u + \sigma u \epsilon + 1} f(\epsilon) d\epsilon$$

$$
\begin{aligned}
&= \frac{u}{\gamma u+1}\int \frac{(\gamma u+1)\epsilon}{\gamma u+\sigma u\epsilon+1}f(\epsilon)d\epsilon \\
&= -\frac{u}{\gamma u+1}\int \frac{\sigma\epsilon^2}{\gamma u+\sigma u\epsilon+1}f(\epsilon)d\epsilon < 0.
\end{aligned}
$$

Thus, as uncertainty increases average growth rate increases but welfare decreases.

3 Model 2

3.1 Setup

Here, another model is introduced, whose setup would seem more standard than the one in the previous model in the sense that the shock hits production technology instead of learning technology, and the resource used to control the impact of the shock is a separate fraction of time different from the time used in learning or production.

Preference is the same. The production function is

$$y_t = (A_t + \bar{A})h_t l_t,$$

where $A_t + \bar{A}$ is total factor productivity (TFP) in period t, while $\bar{A}$ is a constant.

The human capital accumulation technology is given as

$$h_{t+1} - h_t = \gamma u_t h_t.$$

Technology shock affects TFP as

$$A_{t+1} = \tilde{\epsilon}_{t+1},$$

where $\tilde{\epsilon}_{t+1}$ is an i.i.d. shock with mean zero realized at the beginning of period $t+1$. Thus the constant $\bar{A}$ is an average TFP. As in the first model, the impact of the shock can be controlled. It is assumed that

$$\tilde{\epsilon}_{t+1} = \sigma\epsilon_{t+1}(1-q_t),$$

where q_t is a fraction of time devoted to control the impact of a shock. Then the variance of $\tilde{\epsilon}_{t+1}$ is a decreasing function of time q_t. We assume that $\sigma\epsilon+\bar{A} > 0$ for all possible ϵ. This prevents consumption from being negative.

3.2 Solution

Here again we seek a steady state solution of the problem. Bellman equation for it is

$$\begin{aligned} v(h,A) = \max_{u,l,q} \quad & \{\ln(A+\bar{A})\ln h + \ln l + \alpha \ln(1-u-l-q) \\ & + \beta \textstyle\int v\left(h+\gamma uh, \sigma\epsilon(1-q)\right) f(\epsilon) d\epsilon\}, \end{aligned} \tag{7}$$

where f is the density function of the shock ϵ.

The first order conditions are

$$\begin{aligned} \frac{1}{l} &= \frac{\alpha}{w-l-u} && (8) \\ \frac{\alpha}{w-l-u} &= \beta \int v_1\left(h+\gamma uh, \sigma\epsilon w\right) \gamma h f(\epsilon) d\epsilon. && (9) \\ \frac{\alpha}{w-l-u} &= -\beta \int v_2\left(h+\gamma uh, \sigma\epsilon w\right) \sigma\epsilon f(\epsilon) d\epsilon, && (10) \end{aligned}$$

where w is $1-q$. By envelope theorem, we have

$$\begin{aligned} v_1(h,A) &= \frac{1}{h} + \beta \int v_1\left(h+\gamma uh, \sigma\epsilon w\right)(1+\gamma u) f(\epsilon) d\epsilon, \quad \text{and} && (11) \\ v_2(h,A) &= \frac{1}{A+\bar{A}}. && (12) \end{aligned}$$

The solution of equation (11) is

$$v_1(h,A) = \frac{1}{1-\beta}\frac{1}{h}.$$

Substituting it into equation (9), we have

$$\frac{\alpha}{w-l-u} = \frac{\beta}{1-\beta}\int \frac{\gamma h}{h+\gamma uh} f(\epsilon) d\epsilon = \frac{\gamma}{1+\gamma u}. \tag{13}$$

Substituting equation (12) into equation (10) we have

$$\frac{\alpha}{w-l-u} = -\beta \int \frac{\sigma}{\sigma\epsilon w + \bar{A}} \epsilon f(\epsilon) d\epsilon. \tag{14}$$

Thus our first order conditions become equations (8), (13), and (14), which in turn becomes just

$$\frac{\gamma + (1-\beta)\gamma\alpha)}{(1-\beta)(1+\gamma w)} = -\beta \int \frac{\sigma}{\sigma\epsilon w + \bar{A}} \epsilon f(\epsilon) d\epsilon. \tag{15}$$

Denote the LHS of this equation by $\Phi(w, \alpha)$ and righthand side RHS by $\Psi(w, \sigma)$. The former is a decreasing function of w. The latter is increasing from $w = 0$, which can be seen by differentiating $\Psi(w, \sigma)$ with respect to w. Thus we have a graph like figure 3, and we will have an interior solution if and only if $\Phi(1, \alpha) > \Psi(1, \sigma)$, which is

$$\frac{\gamma + (1-\beta)\gamma\alpha)}{(1-\beta)(1+\gamma)} > -\beta \int \frac{\sigma}{\sigma\epsilon + \bar{A}} \epsilon f(\epsilon) d\epsilon. \tag{16}$$

If this condition is not met, then we will have a corner solution, which is $w = 1$. In such case, it does not payoff to try to reduce the impact of a shock at all. In contrast to the first model, a corner solution exists but in the opposite way. In the first model, eliminating uncertainty completely and not growing at all was a possible solution. In the current model, however, letting uncertainty affect in its full strength and not devoting any resource to reduce it is a possible solution. This is because we separate the time used for growth and the time used for reducing uncertainty.

If condition (16) is met, we have a graph like figure 3, and we have an interior solution.

3.3 Increase in Uncertainty

Now, let's increase σ and see its effect. Differentiating $\Psi(w, \sigma)$ with respect to σ once and twice, we have

$$\begin{aligned} \frac{\partial \Psi(w, \sigma)}{\partial \sigma} &= -\frac{\beta}{1-\beta} \int \frac{\epsilon \bar{A}}{(\sigma\epsilon w + \bar{A})^2} f(\epsilon) d\epsilon, \quad \text{and} \\ \frac{\partial^2 \Psi(w, \sigma)}{\partial \sigma^2} &= \frac{2\beta}{1-\beta} \int \frac{w\epsilon^2 \bar{A}}{(\sigma\epsilon w + \bar{A})^3} f(\epsilon) d\epsilon > 0. \end{aligned} \tag{17}$$

Since $\partial\Psi(w, 0)/\partial\sigma = 0$, we conclude that $\partial\Psi(w, \sigma)/\partial\sigma > 0$. Thus when σ increases, we have a upward shift of $\Psi(w, \sigma)$ graph as in figure 4, and the equilibrium value of w decreases, i.e., the equilibrium value of q increases. Equilibrium value of u decreases by equations (8) and (13). They will give

$$w = D + (1 + \gamma D)u,$$

where $D = \{(1-\beta)(1+\alpha)\}/\{\beta\gamma\}$. This means when uncertainty increases, people put more resources into an activity that reduces its impact and in consequence growth rate decreases. As in the first model, we have a negative relationship between uncertainty and growth.

3.4 Increase in average TFP

What would happen if the average TFP $\bar{A}$ increases? Differentiating Ψ with respect to $\bar{A}$, we get

$$\frac{\partial \Psi}{\partial \bar{A}} = \beta \int \frac{\sigma}{(\sigma \epsilon w + \bar{A})^2} \epsilon f(\epsilon) d\epsilon.$$

We see that this is positive by comparing it with equation (17). Hence as the constant part $\bar{A}$ increases, the Ψ curve shifts up, and as a consequence w decreases, q increases, and u decreases. People put more resources to reduce the impact of a shock and forgo growth. This is because they get more consumption each period, so they are able to afford to put more resources into reducing the impact of a shock.

We will see later, however, that welfare will increase when average TFP increases. Thus a policy that tries to enhance average productivity might sometimes reduce growth rate, although it increases welfare.

3.5 Elasticity of Substitution between Consumption and Leisure

As the elasticity of substitution between consumption and leisure α increases, $\Phi(w, \alpha)$ shifts upward, so the equilibrium w increases, q decreases, and u increases. Our result here is opposite to that in the first model. There, as elasticity of substitution increases, u decreases and growth rate decreases. Nonetheless, we get the same implication in both models; a country with a larger fluctuation will show a higher growth rate, if cross country difference is only due to elasticity of substitution. If a cross country data set is generated from either model, it will show a larger fluctuation is associated higher growth rate and vice versa.

3.6 Volatility of Output and Growth Rate

The volatility of output is

$$\begin{aligned}
& \mathrm{Var}_t\Big(\frac{y_{t+1}}{(\gamma u+1)y_t}-1\Big) \\
= & \mathrm{Var}_t\Big(\frac{(A_{t+1}+\bar{A})h_{t+1}l_t}{(A_t+\bar{A})(\gamma u+1)h_t l_t}-1\Big) \\
= & \mathrm{Var}_t\Big(\frac{(A_{t+1}+\bar{A})h_{t+1}l_t}{(A_t+\bar{A})h_{t+1}l_t}-1\Big) \\
= & \mathrm{Var}_t\Big(\frac{A_{t+1}}{A_t+\bar{A}}\Big)=\mathrm{Var}_t\Big(\frac{\sigma w\epsilon_{t+1}}{A_t+\bar{A}}\Big) \\
= & \Big(\frac{1}{A_t+\bar{A}}\Big)^2\sigma^2 w^2\mathrm{Var}(\epsilon).
\end{aligned}$$

Here again, we do not have any definite relationship between volatility of output and growth rate, because w decreases as σ increases.

3.7 Welfare Comparison

As σ increases, the welfare decreases. We can show this using the value function $v(h, A)$. Its derivative is given in equations (11) and (12). So $v(h, A)$ $= \{1/(1-\beta)\} \ln h + ln(A+\bar{A}) + C$, and the unknown constant C is determined by plugging the expression for $v(h, A)$ into the Bellman equation, equation (7), and solving for C. Then

$$\begin{aligned}
C = \quad & \frac{1}{1-\beta}\Big\{\ln l+\alpha\ln(1-u-l-q) \\
& +\frac{\beta}{1-\beta}\ln(1+\gamma u)+\beta\int\ln(\sigma\epsilon(1-q)+\bar{A})f(\epsilon)d\epsilon\Big\}.
\end{aligned}$$

If we differentiate this C with respect to σ, we will get the welfare change, which is, due to envelope theorem,

$$\frac{\partial C}{\partial\sigma}=\frac{\beta}{(1-\beta)}\int\frac{\epsilon w}{\sigma\epsilon w+\bar{A}}f(\epsilon)d\epsilon.$$

The negativity of this quantity is shown in establishing the shape of $\Psi(w, \sigma)$ graph in connection with equation (15). Thus, as uncertainty increases average growth rate increases but welfare decreases.

If the average TFP $\bar{A}$ increases, then the welfare will increase. This can be seen by differentiating C with respect to $\bar{A}$, which is

$$\beta \int \frac{1}{\sigma \epsilon w + \bar{A}} f(\epsilon) d\epsilon.$$

It is positive since $\sigma \epsilon w + \bar{A} > 0$ for all ϵ and w with $0 < w < 1$. The reader is reminded that the growth rate decreases when average TFP increases.

4 Conclusion

We examined a relationship between endogenous growth and business cycle by adding a new feature in a simple standard framework: the possibility of controlling the impact of a shock. In the first model, we analyzed the case in which disturbance in an economy occurs in the engine of growth, i.e., in the accumulation of human capital. On the other hand, in the second model and we analyzed the case in which disturbance occurs in the production technology.

In both models, we obtained a negative relationship between uncertainty and growth. Hence a country with large fluctuation will have a lower growth rate, and vice versa. But if other factors like elasticity of substitution or average total factor productivity changes, then uncertainty and growth rate move together. For example, as elasticity of substitution changes, the variance of the shock and growth rate changes in the same direction. Thus a country with a larger fluctuation will have a higher growth rate, and vice versa. Thus depending on how data is generated, a large fluctuation could be associated with low growth rate or high growth rate.

We also saw that the relationship between volatility of output and growth rate can be either positive or negative depending on parameter values and the distribution of the shock. Thus one should be careful when using measured relationship between volatility and growth rate to infer the true relationship between uncertainty and growth.

Overall, I tried to rescue Black's idea (1987) about the relationship between business cycle and growth. In order to complete the mission, I intend to extend the models in this paper to one which include many elements assumed away here, like physical capital, government, or monetary aspects. Then it will be interesting to compare the results in such an extended model with those in the RBC or new Keynesian models. Of course, the results will be obtained by using calibration and numerical method.

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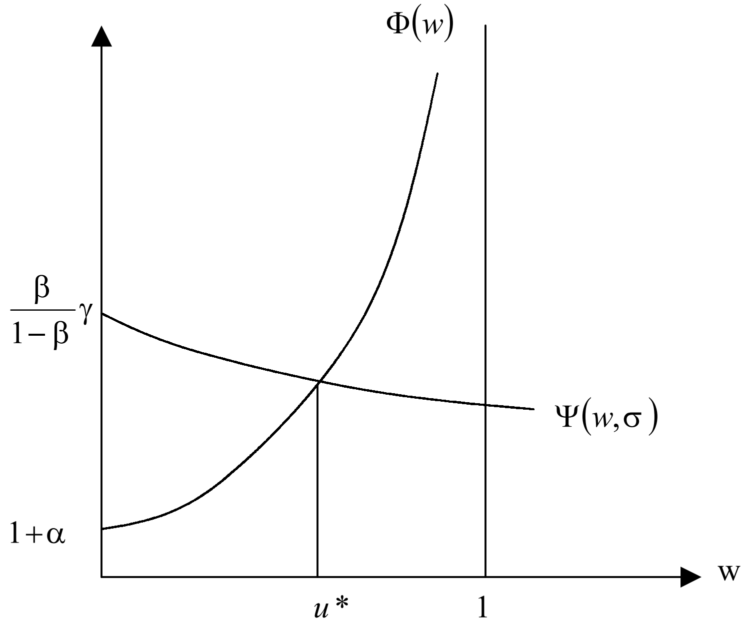

Figure 1

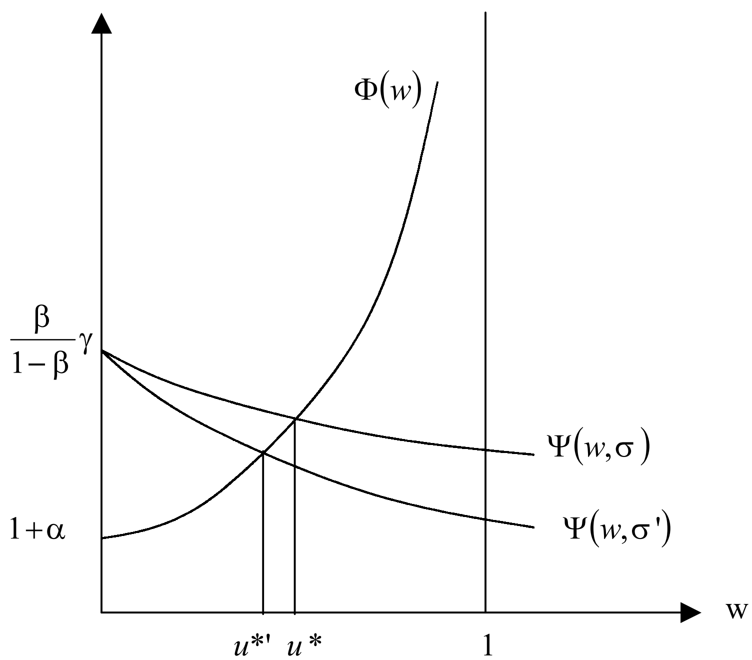

Figure 2

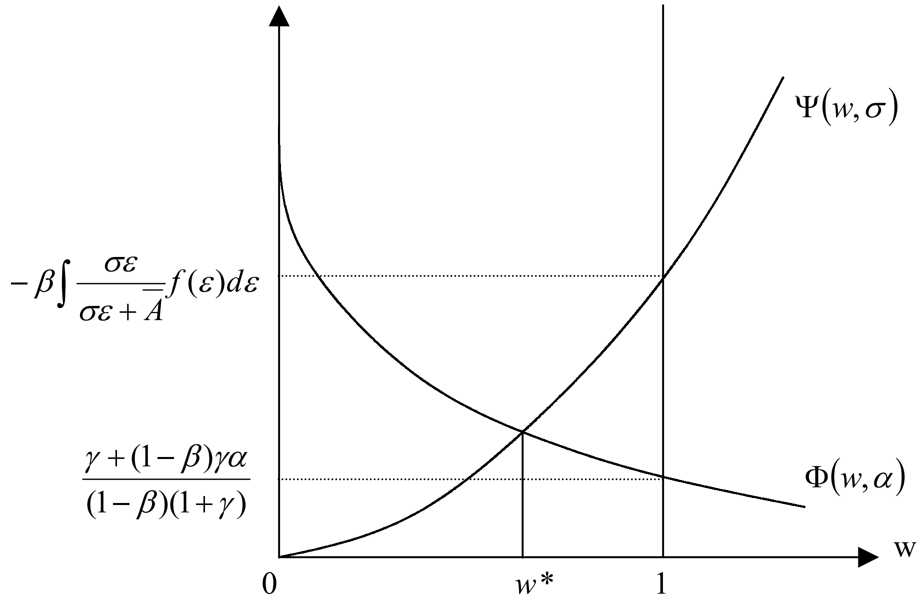

Figure 3

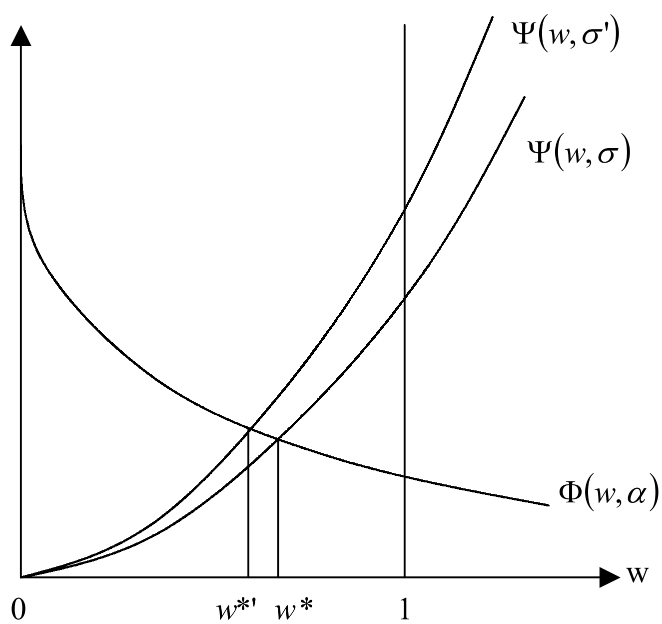

Figure 4