Bifurcation Analysis of Endogenous Growth Models

Abstract:
This paper investigates the dynamics of Uzawa-Lucas endogenous growth model “On the Mechanics of Economic Development”, Journal of Monetary Economics, 1988, 22, pp 3-42, and a variant of Jones semi-endogenous growth model “Sources of US Economic growth in a World of Ideas” The American Economic Review, March 2002, Vol 92 No. 1, pp 220-239. A detailed bifurcation analysis is done within the feasible parameter space of the models. I showed the existence of codimension-1 bifurcations (Hopf, Branch Point, Limit Point of Cycles, and Period Doubling). In addition some codimension-2 (Bogdanov-Takens and Generalized Hopf) bifurcations are detected in the modified Jones model. While the aforementioned are all local bifurcations, the Uzawa-Lucas model also shows the presence of global bifurcation.

1. Introduction
The Uzawa-Lucas (Uzawa, 1965 and Lucas, 1988) model is one of the most important endogenous growth models. It is a two sector model with two production functions devoted respectively to produce human capital and physical capital. Individuals are homogenous and have the same level of work qualification and expertise (H). They divide their time between working to produce final good and dedicate the remaining to training and studying.

The social planner solution for the Uzawa-Lucas model is saddle path stable but indeterminacy arises for the representative agent’s equilibrium as shown by Benhabib and Perli (1994). The presence of externalities in human capital introduces this difference between the market solution and social planner solution. There is a wedge between return on capital as perceived by the representative agent and a return on capital from a social point of view due to the presence of externality.

I solve for the steady states and provide a detailed bifurcation analysis of the model. In steady state I show that consumption, output and physical capital grow at the same rate while human capital grows at a lower rate. The task of this paper is to examine whether the dynamics of the model change within the feasible parameter space of the model. A system undergoes a bifurcation if a small, smooth change in a parameter value(s) produces a sudden 'qualitative' or topological change in the nature of singular points and trajectories of the system. The presence of bifurcation damages the inference robustness of the dynamics when inferences are based on point estimates of the model. Hence, knowing the stability boundaries in the feasible region of the parameters point estimates may lead to better inferences of policies.

I locate the transcritical bifurcation boundary and the Hopf bifurcation boundary, corresponding to different combination of parameters which are displayed in two-dimension and three-
dimension diagrams using Mathematica. Numerical continuation package Matcont is used to further analyze the stability properties of limit cycles generated by Hopf bifurcations and presence of other kinds of bifurcations. I showed the existence of Hopf, Branch Point, Limit Point of Cycles and Period Doubling bifurcations within the feasible parameter range of the model. While the above mentioned are all local bifurcations, presence of global bifurcation is also confirmed. There is even possibility of having chaotic dynamics through the series of Period Doubling bifurcations found.

In Uzawa-Lucas model, it is the human capital formation itself that, by non-decreasing marginal returns, creates endogenous growth. On the other hand, Romer (1990) revolutionized the idea of growth being driven by technological change that results from research and development of profit maximizing agents. The model suggest that the long-run growth rate of per capita income should be rising with the increases in R&D intensity or time spent accumulating skills, thereby establishing a relation between the level of human capital and economic growth. These changes should generate temporarily high growth rates and long-run level effects.

Knowledge is a nonrival good as knowledge can be used by many people simultaneously without degradation. This indicates the presence of increasing returns to scale in production associated with any new idea which in turn depends on population (of educated workers/number of researchers). This is the so called “strong” scale effect of the first generation idea based growth models (Romer(1990) and Grossman and Helpman(1991)) where the growth rate of the economy is an increasing function of scale (population). Contrary to these results, the US data shows that the economy is fluctuating around its balanced growth path even though educational attainment and research intensity is steadily rising for last 50 years. Jones (2002) model tries to reconcile these facts with the stability of U.S. growth rates using a model that exhibits “weak” scale effects. Jones found that long-run growth arises from the worldwide discovery of ideas, which depends on rate of population growth of the countries contributing to world research. Nevertheless, constant growth can temporarily proceed at a faster rate, provided research intensity and educational attainment rise steadily over time. Such models are often called semi-endogenous growth models.

I incorporate a human capital accumulation in Jones model which explicitly takes into account the possibility that the investment in skill acquisition by agents might be positively, negatively or not influenced at all by technological progress (the invention of new varieties of intermediate goods). Hence the direction of technological progress is ultimately driven by human capital investment. Compared to Bucci (2008), I introduce the possibility of decreasing returns to scale associated with human capital itself and time spent accumulating it in the human capital production equation. Along the balanced growth path of this modified Jones model, I have shown that the effect of population growth on long run growth of the economy depends on the combination of preference and technological parameters. The long run growth can be even positive with no population growth. Hence reinforcing Bucci’s (2008) result that economic growth is no longer semi-endogenous and is ultimately driven by private incentives to invest in human capital.

I showed the existence of codimension-1 bifurcations (Andronov-Hopf, Branch Point, Limit Point of Cycles, and Period Doubling). In addition some codimension-2 (Bogdanov-Takens and
Generalized Hopf bifurcations are detected in the modified Jones model. Some of these results found in this paper are never seen before in the literature of endogenous and semiendogenous growth models.

Stability analysis is critical in understanding the dynamics of the model. Benhabib and Perli (1994) analyzed the stability property of the long-run equilibrium in Lucas (1988) model. Arnold (2000a,b) has analyzed the stability of equilibrium in Romer (1990) model. Arnold (2006) has done the same for the Jones (1995) model. Mondal (2007) examined the dynamics of Grossman-Helpman (1991b) model of endogenous product cycles. The results derived in these papers provide important insights to researchers considering different policies. But a detailed bifurcation analysis has not been done so far for most of these popular endogenous and semi-endogenous growth models. The current paper aims to fill this gap.

As pointed out by Banerjee et al (2011) “Just as it is important to know for what parameter values a system is stable or unstable, it is equally important to know the nature of stability (e.g. monotonic convergence, damped single periodic convergence, or damped multi-periodic convergence) or instability (periodic, multi-periodic, or chaotic)”. Barnett and his coauthors have made significant contribution in this area. Barnett and He (1999, 2001, 2002) examined the dynamics of Bergstrom-Wymer continuous-time dynamic macroeconometric model of the UK economy. Both transcritical bifurcation boundary and the Hopf bifurcation boundary for the model were found. Barnett and He (2008) have estimated the singularity bifurcation boundaries within the parameter space for Leeper and Sims (1994) model. Barnett and Duzhak (2010) found Hopf and Period Doubling bifurcations using local bifurcation analysis in a New Keynesian model. More recently, Banerjee et al (2011) examined the possibility of cyclical behavior that is, Hopf bifurcation in the Marshallian Macroeconomic Model.

Section 2, describes the Uzawa-Lucas model and the derivations of the dynamic equations for the centralized and the decentralized economy. In Section 3, I discuss the possibility of the existence of various bifurcations in the model. Section 4, describes the modified Jones model with section 5 and 6 presenting balanced growth path and bifurcation analysis for the model. Finally, Section 7 concludes the paper.

2. **The Uzawa-Lucas Model**

The production function in the physical sector is defined as follows

\[ Y = AK^\alpha (\varepsilon hL)^{1-\alpha}h_a^\zeta, \quad 0 < \alpha < 1 \]

Where \( Y \) is output, \( A \) is constant technology level, \( K \) is physical capital, \( \alpha \) is the share of capital, \( \varepsilon \) is the fraction of labor time devoted to produce output and \( (1 - \varepsilon) \) is the fraction of labor time devoted to produce human capital and \( 0 < \varepsilon < 1 \). \( \varepsilon hL \) is the quantity of labor measured in efficiency units employed to produce output, \( h_a^\zeta \) measures the externality (average human capital
of the work force) and $\zeta$ is the positive externality parameter in the production of human capital. In per capita terms, $y = Ak^\alpha (\varepsilon h)^{1-\alpha} h_\alpha^\zeta$.

The physical capital accumulation equation is $\dot{K} = AK^\alpha (\varepsilon hL)^{1-\alpha} h_\alpha^\zeta - C - \delta K$. In per capita terms, $\dot{k} = Ak^\alpha (\varepsilon h)^{1-\alpha} h_\alpha^\zeta - c - (n + \delta)k$ and the human capital accumulation equation is $\dot{h} = \eta h(1 - \varepsilon)$, where $\eta$ is defined as schooling productivity.

The problem is

$$\max_{c_t, k_t} \int_t^{\infty} e^{-(\rho-n)t} [c(\tau)^{1-\sigma} - 1] / (1 - \sigma) \, d\tau$$

Subject to

$$\dot{k} = Ak^\alpha (\varepsilon h)^{1-\alpha} h_\alpha^\zeta - c - (n + \delta)k \quad \text{and} \quad \dot{h} = \eta(1 - \varepsilon)$$

where $\rho (\rho > n > 0)$ is the subjective discount rate, and $\sigma (\geq 0)$ is the inverse of intertemporal elasticity of substitution in consumption.

2.1 Social Planner Problem

The social planner takes externality associated with human capital in to account in solving the maximization problem (1) subject to physical capital accumulation equation and human capital accumulation equation. From the first order conditions (see Appendix 1.2) we derive the equations describing the economy of the Uzawa-Lucas model from a social planner’s perspective

$$\frac{\dot{k}}{k} = Ak^{\alpha-1} \varepsilon^{1-\alpha} h^{1-\alpha+\zeta} - \frac{c}{k} - (n + \delta)$$

$$\frac{\dot{h}}{h} = \eta(1 - \varepsilon)$$

$$\frac{\dot{c}}{c} = \frac{\alpha Ak^{\alpha-1} \varepsilon^{1-\alpha} h^{1-\alpha+\zeta} - (\rho + \delta)}{\sigma}$$

$$\frac{\dot{\varepsilon}}{\varepsilon} = \eta \frac{(1 - \alpha + \zeta)}{(1 - \alpha)} \varepsilon + \eta \frac{(1 - \alpha + \zeta)}{\alpha} - \frac{c}{k} + \frac{(1 - \alpha)}{\alpha}(n + \delta)$$

$$\frac{\dot{L}}{L} = n$$

Let $m = \frac{y}{k} = Ak^{\alpha-1} \varepsilon^{1-\alpha} h^{1-\alpha+\zeta}$ & $g = \frac{c}{k}$. Taking logarithms of $m$ and $g$ and differentiating with respect to time, the following 2 equations define the dynamics of Uzawa Lucas model.
Steady state \((m^*, g^*)\) is given by \(\dot{m} = \dot{g} = 0\).

**Theorem 1.** A unique steady state exists if

\[
\Lambda = \frac{(1-\alpha + \zeta)}{\alpha} (\sigma - 1) \eta (1 - \varepsilon) + \rho > 0
\]

**Proof:** \(\Lambda\) is the necessary and sufficient for the transversality condition for the consumer’s utility maximization problem to hold (see Appendix 1)

\[
m^* = \frac{n + \delta}{\alpha} + \frac{(1 - \alpha + \zeta)}{\alpha}
\]

\[
g^* = \frac{\rho - n}{\sigma} + \frac{(1 - \alpha)}{\alpha} (n + \delta) + \frac{\eta (1 - \alpha + \zeta)}{\alpha (1 - \alpha)} (\sigma - \alpha)
\]

**Theorem 2:** The social planner solution is saddle path stable.

Proof: I linearize around the steady state \(s^* = (m^*, g^*)\) to analyze the local stability properties of the system (I) and (II),

\[
[\begin{bmatrix} \dot{m} \\ \dot{g} \end{bmatrix} = \begin{bmatrix} \frac{\partial \dot{m}}{\partial m} & \frac{\partial \dot{m}}{\partial g} \\ \frac{\partial \dot{g}}{\partial m} & \frac{\partial \dot{g}}{\partial g} \end{bmatrix} \begin{bmatrix} m_t - m^* \\ g_t - g^* \end{bmatrix} \text{where,} \quad J_s = \begin{bmatrix} -(1 - \alpha)m^* & 0 \\ \frac{\alpha}{\sigma - 1} g^* & g^* \end{bmatrix}
\]

Using Vieta’s theorem, we have a relationship between the roots \((q_1, q_2)\) of the Jacobian \(J_s\) and the coefficient of the characteristic equation associated with it, \(q^2 - a_1 q + a_2 = 0\)

\[
\text{Trace}(J) = q_1 + q_2 = a_1 = -(1 - \alpha)m^* + g^*
\]

\[
\text{Determinant}(J) = q_1 q_2 = a_2 = -(1 - \alpha)m^* g^*
\]

As \(m^*, g^* > 0\), \(\text{Det}(J) < 0\), which guarantees saddle path stability. Specifically, \(q_1 = g^*, q_2 = -(1 - \alpha)m^*\)

**2.2 Representative Agent Problem**

From the first order conditions (see Appendix 1.3) and setting \(h = h_a\), I derive the equations describing the economy of the Uzawa-Lucas model from a decentralized economy’s perspective.
Theorem 3. A unique steady state exists if

\[ \Lambda = \frac{(1 - \alpha + \zeta)}{\alpha} (\sigma - 1) \eta(1 - \varepsilon) + \rho > 0 \]

\[ 0 < \varepsilon < 1 \]

Proof: \( \Lambda \) is the necessary and sufficient for the transversality condition for the consumer’s utility maximization problem to hold (appendix 1.1)

Let \( \dot{m} = \frac{v}{k} \) & \( g = \frac{c}{k} \). Taking logarithms of \( m \) and \( g \) and differentiating with respect to time, the following 3 equations define the dynamics of Uzawa Lucas model

(I) \[ \frac{\dot{m}}{m} = -(1 - \alpha)m + \frac{(1 - \alpha)}{\alpha} (n + \delta) + \eta \frac{(1 - \alpha + \zeta)}{\alpha} - \eta \frac{\zeta}{\alpha} \varepsilon \]

(II) \[ \frac{\dot{g}}{g} = \left( \frac{\alpha}{\sigma} - 1 \right) m - \frac{\rho \sigma}{\sigma} - \delta \left( \frac{1}{\sigma} - 1 \right) + g + n \]

(III) \[ \frac{\dot{\varepsilon}}{\varepsilon} = \eta \frac{(a - \zeta)}{\alpha} \varepsilon + \eta \frac{(1 - \alpha + \zeta)}{\alpha} - g + \frac{(1 - \alpha)}{\alpha} (n + \delta) \]

Steady state \((m^*, g^*, \varepsilon^*)\) is given by \( \dot{m} = \dot{g} = \dot{\varepsilon} = 0 \) and derived to be

\[ \varepsilon^* = 1 - \frac{(1 - \alpha)(\rho - n - \eta)}{\eta[\zeta - \sigma(1 - \alpha + \zeta)]} \]

\[ m^* = \eta \frac{[1 - \alpha + \zeta(1 - \varepsilon^*)]}{\alpha(1 - \alpha)} + \frac{n}{\alpha} \]

\[ g^* = \eta \frac{[1 - \alpha + \zeta(1 - \varepsilon^*) + \alpha \varepsilon^*]}{\alpha} + \frac{n(1 - \alpha)}{\alpha} \]
Note that \( 0 < \frac{(1-\alpha)(\rho-n-\eta)}{\eta[\xi-\sigma(1-\alpha+\zeta)]} < 1 \) is necessary for \( 0 < \varepsilon^* < 1 \) which in turn guarantees that \( m^*, g^* > 0 \). I linearize the system (I), (II) and (III) around the steady state \( s^* = (m^*, g^*, \varepsilon^*) \) to analyze its local stability properties,

\[
\begin{bmatrix}
\dot{m} \\
\dot{\eta} \\
\dot{g} \\
\dot{\varepsilon}
\end{bmatrix} =
J_m
\begin{bmatrix}
\frac{\partial m}{\partial m_{s^*}} \\
\frac{\partial m}{\partial g_{s^*}} \\
\frac{\partial m}{\partial \varepsilon_{s^*}} \\
\frac{\partial m}{\partial \xi_{s^*}}
\end{bmatrix}
\begin{bmatrix}
\dot{m} - m^* \\
\dot{\eta} \\
\dot{g} - g^* \\
\dot{\varepsilon} - \varepsilon^*
\end{bmatrix}
\]

where,

\[
J_m =
\begin{bmatrix}
-(1-\alpha)m^* & 0 & -\frac{\xi}{\alpha}m^* \\
\left(\frac{a}{\sigma} - 1\right)g^* & g^* & 0 \\
0 & -\varepsilon^* & \frac{(\alpha-\xi)}{\alpha} \varepsilon^*
\end{bmatrix}
\]

The characteristic equation associated with \( J_m \), \( q^3 + c_2q^2 + c_1q + c_0 = 0 \), where

\[
c_0 = \eta \frac{[\sigma(1-\alpha + \xi) - \xi]}{\sigma} m^* g^* \varepsilon^*
\]

\[
c_1 = \eta^2 \frac{(\alpha - \xi)}{\alpha} \varepsilon^2 - (1-\alpha)m^* g^*
\]

\[
c_2 = -\eta \frac{(2\alpha - \xi)}{\alpha} \varepsilon^2
\]

3. **Bifurcation Analysis of Uzawa-Lucas Model**

In this section, we examine the existence of codimension 1 and 2 transcritical and Hopf bifurcations in the system (I), (II) & (III). The codimension, as defined by Kuznetsov (2004), is the number of independent conditions determining the bifurcation boundary. This procedure of varying a single parameter helps us to identify codimension-1 bifurcation and varying 2 parameters helps us to identify codimension-2 bifurcation.

An equilibrium point \( s^* \) of the system is called **hyperbolic** if the coefficient matrix \( J_m \) has no eigenvalues with zero real parts. For small perturbations of parameters, there are no structural changes in the stability of a hyperbolic equilibrium, provided that the perturbations are sufficiently small. Therefore, bifurcations occur at nonhyperbolic equilibria only.

A **transcritical** bifurcation occurs when a system has a nonhyperbolic equilibrium with a geometrically simple zero eigenvalue at the bifurcation point, and additional transversality conditions are satisfied given by the Sotomayor’s Theorem [Barnett and He (1999)]. So the first condition we are going to use to find the bifurcation boundary is \( c_0 = \det(J_m) = 0 \)

**Theorem 4:** \( J_m \) has a zero eigenvalue if

\[
\eta \left[ \frac{\sigma(1-\alpha + \xi) - \xi}{\sigma} \right] m^* g^* \varepsilon^* = 0 \tag{a}
\]
Hopf bifurcations occur at points at which the system has a nonhyperbolic equilibrium with a pair of purely imaginary eigenvalues, but without zero eigenvalues. Also additional transversality conditions must be satisfied. We shall use the following theorem, based upon the version of the Hopf Bifurcation Theorem in Guckenheimer and Holmes (1983): $J_m$ has precisely one pair of pure imaginary eigenvalues if $c_0 - c_1 c_2 = 0$ and $c_1 > 0$. If $c_0 - c_1 c_2 \neq 0$ and $c_1 > 0$, $J$ has no pure imaginary eigenvalues.

**Theorem 5:** $J_m$ has precisely one pair of pure imaginary eigenvalues if

\[
\begin{align*}
\alpha m^* g^* ((\alpha - 1) a \sigma + \zeta (\sigma - \alpha)) + \eta^2 \sigma \epsilon^{*2} (2 \alpha - \zeta) (\alpha - \zeta) &= 0 \quad \text{and} \\
\frac{\eta^2}{\alpha} \epsilon^{*2} (\alpha - \zeta) - (1 - \alpha) m^* g^* &> 0.
\end{align*}
\]

(b)

**3.1 Case Studies**

In order to be able to view the boundaries, we only consider two or three parameters. The procedure is applicable to any number of parameters.

Let $\theta^* = \{\eta = 0.05, \zeta = 0.1, \alpha = 0.65, \rho = 0.0505, \sigma = 0.15, n = 0, \delta = 0\}$

$\omega^* = \{\eta = 0.05, \zeta = 0.1, \alpha = 0.75, \rho = 0.0505, \sigma = 0.15, n = 0, \delta = 0\}$

Case I: Free Parameter $\alpha, \eta$

Assume that other parameters operate at $\theta^*$. The result is illustrated in Figure 1, the red line gives a range of $\alpha$ and $\eta$ satisfying Hopf bifurcation condition while the blue line depicts the value of $\alpha$ and $\eta$ satisfying transcritical bifurcation boundary.

Similarly, the following cases gives the range of parameter values satisfying condition (a) and condition (b) in blue and red respectively while the rest of the parameters are set at $\theta^*$.

Case II: Free Parameters $\zeta, \alpha$ (figure 2)

Case III: Free Parameters $\sigma, \alpha$ (figure 3)

Case IV: Free Parameters $\zeta, \rho$ (figure 4): Notice that for case IV we do not have a Hopf bifurcation boundary.

We now add another parameter as free parameter and continue with the analysis. The following cases give the range of parameter values satisfying condition (a) and condition (b) in blue and red regions respectively while the rest of the parameters are assumed to be at $\theta^*$.

Case V: Free Parameters $\alpha, \zeta, \rho$ (figure 5)
Case VI: Free Parameters $\eta, \zeta, \sigma$ (figure 6)

Case VII: Free Parameters $\alpha, \eta, \rho$ (figure 7): Notice that for case IV we do not have a Hopf bifurcation boundary.

Case VIII: Free Parameters $\alpha, \sigma, \rho$ (figure 8)

Case IX: Free Parameters $\alpha, \eta, \sigma$ (figure 9)

The following is one of the ways we can explain cyclical behavior obtained in the model. Suppose there is a change in policy encouraging increase in savings rate. Consumption decreases initially, when intertemporal substitution for consumption is high ($\sigma$ is low) as people start saving more. This will encourage a movement of labor from output production to human capital production. Human capital starts increasing. This implies faster accumulation of physical capital provided sufficient amount of externality to human capital in the production function for physical capital is present. If people care about the future more (subjective discount rate $\rho$ is low), consumption starts rising gradually with speedier capital accumulation (leading to greater consumption goods production in the future). This will eventually lead to a decline in savings rate. Hence two opposing effects are playing when the savings rate is different from the equilibrium rate. Interaction between different parameters can cause cyclical convergence to equilibrium (figure 10) or may cause instability, and for some parameter values we may have convergence to cycles (figure 11).

I further investigate the stability properties of cycles generated by different combination of parameters using the numerical continuation package Matcont. While some of the limit cycles generated by Andronov-Hopf bifurcation are stable (supercritical bifurcation), there could be some unstable limit cycles (subcritical bifurcation) created as well. Table 1 reports the values of share of capital ($\alpha$), externality in production of human capital ($\zeta$) and inverse of intertemporal elasticity of substitution in consumption ($\sigma$) at which Hopf bifurcations are generated when they are treated as free parameters.

A positive value of the first Lyapunov coefficient indicates creation of subcritical Hopf bifurcation. Thus for each of the cases reported in Table 1, an unstable limit cycle (periodic orbit) bifurcates from the equilibrium. All of these cases also give rise to Branch Points (Pitchfork/Transcritical bifurcations). Continuation of limit cycle from the Hopf point for the case when $\alpha$ is the free parameter gives rise to Limit Point (Fold/Saddle Node) bifurcation of Cycles. From the family of limit cycles bifurcating from the Hopf point, Limit Point Cycle (LPC) is a fold bifurcation of the cycle where two limit cycles with different periods are present near LPC point at $\alpha = 0.738$.

Continuing computation further from the Hopf point, gives rise to series of Period Doubling (flip) bifurcations. Period doubling bifurcations is defined as a situation when a new limit cycle emerges from an existing limit cycle, and the period of the new limit cycle is twice that of the old
one. The first period doubling bifurcation at $\alpha = 0.7132369$ has a positive normal form coefficients indicating unstable double period cycles are involved. Whereas the rest of the period doubling bifurcations generated have negative normal form coefficients giving rise to stable double-period cycles.

The period of the cycle rapidly increases for a very small perturbation in parameter $\alpha$ as is evident in figure 12(C). The limit cycle approaches a homoclinic orbit which is a global bifurcation. A homoclinic orbit is a trajectory of a flow of the dynamical system which joins a saddle equilibrium point to itself. In other words, a homoclinic orbit lies in the intersection of the stable manifold and the unstable manifold of an equilibrium. There is even possibility of having chaotic dynamics through the series of Period Doubling bifurcations.

I carry out the continuation of limit cycle from the first Hopf point for the cases when $\zeta$ and $\sigma$ are treated as the free parameters. Both the cases give rise to Limit Point Cycles with a nonzero normal form coefficient indicating the limit cycle manifold has a fold at the LPC point. Similar results are found if we carry out the continuation of limit cycles from the second Hopf point for each of these cases and hence are not reported.

<table>
<thead>
<tr>
<th>Parameters Varied</th>
<th>Equilibrium Bifurcation</th>
<th>Bifurcation of Limit Cycle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$ (Figure 12)</td>
<td>Figure 12 (A)</td>
<td>Limit point cycle (LPC)</td>
</tr>
<tr>
<td>Other parameters set at $\beta^*$</td>
<td>First Lyapunov coefficient = 0.00242, $\alpha = 0.738207$</td>
<td>period = 231.206, $\alpha = 0.7382042$</td>
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<tr>
<td></td>
<td></td>
<td>Normal form coefficient = 0.007</td>
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<tr>
<td></td>
<td></td>
<td>Period Doubling (PD)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>period = 584.064, $\alpha = 0.7132369$</td>
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<tr>
<td></td>
<td></td>
<td>Normal form coefficient = 0.910</td>
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<td>Period Doubling (PD)</td>
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<td></td>
<td></td>
<td>period = 664.005, $\alpha = 0.7132002$</td>
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<tr>
<td></td>
<td></td>
<td>Normal form coefficient = -0.576</td>
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<td></td>
<td>Period Doubling (PD)</td>
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<td></td>
<td>period = 693.988, $\alpha = 0.7131958$</td>
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<td>Normal form coefficient = -0.469</td>
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<td>Period Doubling (PD)</td>
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<td></td>
<td>period = 713.978, $\alpha = 0.7131940$</td>
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<td>Normal form coefficient = -0.368</td>
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<td>Period Doubling (PD)</td>
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<td>period = 725.667, $\alpha = 0.7131932$</td>
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<td>Normal form coefficient = -0.314</td>
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<td>Period Doubling (PD)</td>
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<td></td>
<td></td>
<td>period = 784.104, $\alpha = 0.7131912$</td>
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<tr>
<td></td>
<td></td>
<td>Normal form coefficient = -0.119</td>
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<p>| Branch Point (BP) | |
|-------------------| |</p>
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Figure 13 (A)</th>
<th>Figure 13 (B)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Hopf (H)</strong></td>
<td>First Lyapunov coefficient = 0.00250, $\zeta = 0.107315$</td>
<td>Limit point cycle (LPC) period = 215.751, $\zeta = 0.1073147$ Normal form coefficient = 0.009</td>
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<tr>
<td><strong>Branch Point (BP)</strong></td>
<td>$\zeta = 0.047059$</td>
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<table>
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<tr>
<th>Parameter</th>
<th>Figure 14 (A)</th>
<th>Figure 14 (B)</th>
</tr>
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<tbody>
<tr>
<td><strong>Hopf (H)</strong></td>
<td>First Lyapunov coefficient = 0.00246 $\zeta = 0.052623$</td>
<td>Limit point cycle (LPC) period = 213.83, $\sigma = 0.1394026$ Normal form coefficient = 0.009</td>
</tr>
<tr>
<td><strong>Branch Point (BP)</strong></td>
<td>$\zeta = 0.0264$</td>
<td></td>
</tr>
</tbody>
</table>

Figure 1: Free Parameters $\alpha, \eta$

Figure 2: Free Parameters $\zeta, \alpha$
Figure 3: Free Parameters $\sigma, \alpha$

Figure 4: Free Parameters $\zeta, \rho$

Figure 5: $\alpha, \zeta, \rho$ are free parameters
Figure 6: Free Parameters $\eta, \zeta, \sigma$

Figure 7: $\alpha, \eta, \rho$ are free parameters
Figure 8: $\alpha, \sigma, \rho$ are free parameters

Figure 9: Free Parameters $\alpha, \eta, \sigma$
Figure 10: Parameters in the Stable Region
Figure 11: Parameters on the Hopf Bifurcation Boundary
Figure 12

(A)

(B)

(C)
Figure 13
Figure 14
4. The Modified Jones Model

The economy is populated by $N_t$ identical, infinitely lived agents. The number of agents are initially positive and subsequently grows at a rate $\frac{N_t}{N_t} = n (> 0)$. Each individual is endowed with one unit of time and divides this unit among producing goods, producing ideas and producing human capital.

\begin{equation}
L_\alpha t + L_\gamma t = L_t = \epsilon t N_t
\end{equation}

Where, at time $t$, $L_t$ is employment, $L_\gamma t$ is the total amount of raw labor employed in producing output, $L_\alpha t$ is the total number of researchers and $(1 - \epsilon_t)$ represents the amount of time the individual spends accumulating human capital.

Physical capital is accumulated by foregoing consumption.

\begin{equation}
\dot{K} = s_{kt} Y_t - dK_t, \quad K_0 > 0
\end{equation}

where $s_{kt}$ is the fraction of output that is invested ($(1 - s_{kt})$ is the fraction consumed). $d$ is the exogenous, constant rate of depreciation. $Y_t$ is the aggregate production of homogenous final good and $K_t$ is capital stock. Hence, we can also write

\begin{equation}
\dot{K} = Y_t - C_t - dK_t
\end{equation}

Output is produced using total quantity of human capital, $H_\gamma t$ and a set of intermediaries $j$, which are obtained one-from-one from capital. $H_\gamma t$ is given by

\begin{equation}
H_\gamma t = h_t L_\gamma t
\end{equation}

where $h_t$ is human capital per person and $L_\gamma t$ is total amount of raw labor employed in producing output. An individual’s human capital ($h_t$) is produced by foregoing time in the labor force. Since individual spend $(1 - \epsilon_t)$ amount of time accumulating human capital,

\begin{equation}
\dot{h}_t = \eta h_t \beta_1 (1 - \epsilon_t) \beta_2 - \theta g_A h_t, \quad 0 < \beta_1, \beta_2, \epsilon_t < 1, \eta > 0, (1 + \theta) > 0
\end{equation}

Where $\eta$ is the productivity of human capital in the production of new human capital $\theta$ reflects the effect of technological progress on human capital investment and $g_A$ is the growth rate of technology.

Equation (5) builds on the human capital accumulation equation from the Uzawa-Lucas (Uzawa, 1965 and Lucas, 1988) model. Firstly, it is modified to show that higher the level of human capital or time spent accumulating human capital, the more difficult it is to generate additional human capital (Gong, Greiner and Semmler, 2004). This is reflected in the equation by $0 < \beta_1, \beta_2 < 1$. Values of $\beta_1$ or $\beta_2 = 1$ imply that an increase in the time spent for education or higher level of human capital itself, raises the growth rate of human capital accumulation monotonically which in turn, raises the balanced growth rate. This can be interpreted as “strong” scale effect. US data clearly reject that as shown by Jones(2002). The US economy is fluctuating
around its balanced growth path even though educational attainment and research intensity is steadily rising for last 50 years. Secondly, we incorporate the fact that faster technological progress \((g_t)\) may influence the rate of human capital accumulation. This depends on the technological parameter \(\theta\). We restrict \(\theta > -1\) to prevent explosive or negative long run growth rates as in Bucci (2008). Hence faster technological progress may increase, decrease or have no effect on human capital investment.

The production function is given by

\[
Y_t = H_{yt} \int_0^A x(j)^\alpha \, dj
\]

\(x(j)\) is the input of intermediate \(j\) and \(A\) is the number of available intermediates, \(\alpha \in (0,1)\) and \(\frac{1}{1-\alpha}\) is the elasticity of substitution for any pair of intermediates. Research and development (R&D) enables firms to produce new intermediates. The R&D technology is

\[
\dot{A} = \gamma H_{At}^\lambda A_t^{1-\phi}
\]

New ideas produced at any point in time depends on the number of researchers \((H_{At})\) and existing stock of ideas \((A_t)\). \(\phi\) captures the externalities emanating from R&D, \(\phi > 0, I\) allow past discoveries to either increase or decrease current research productivity. \(0 < \lambda \leq 1\) captures the possibility of duplication in research. \(H_{At}\) is effective research effort given by

\[
H_{At} = h_t L_{At}
\]

### 4.1 Final Goods Sector

Faced with a price list \(\{p(i): i \in R_+\}\) for all the producer durables, the representative final output firm choose a profit–maximizing quantity \(x(i)\) for each durable

\[
\max_x \int_0^\infty [H_y^{1-\alpha} x(i)^\alpha - p(i)x(i)] \, di - wH_y
\]

Where ‘\(w\)’ is the rental rate per unit of human capital. Solving the maximization problem gives

\[
p(i) = \alpha H_y^{1-\alpha} x(i)^{\alpha-1}
\]

\[
w = (1 - \alpha) \frac{Y}{H_y}
\]

### 4.2 Intermediate Goods Sector

The demand curve in equation (9) is what the producer of each specialized durable takes as given in choosing the profit maximizing output to set. Faced with a given value of \(H_y\) and \(r\), a firm that has already incurred the fixed cost of investment in a design will choose a level of output \(x\) to maximize its revenue minus variable cost at every date.
π = max p(x)x - rx

where ‘r’ is the interest rate on loans denominated in goods. Solving the Monopoly profit maximization problem gives

\[ p(i) = \bar{p} = \frac{r}{\alpha} \]

The flow of monopoly profit is

\[ \pi(i) = \bar{\pi} = \bar{p}\bar{x} - r\bar{x} = (1 - \alpha)\bar{p}\bar{x} \]

### 4.3 R&D Sector

The decision to produce new specialized input depends on a comparison of the discounted stream of net revenue and the cost of the initial investment in a design. Because the market for designs is competitive, the price for designs \( P_A \) will bid up until it is equal to present value of the net revenue that a monopoly can extract. Hence,

\[ \int_t^\infty e^{-\int_t^r r(s)ds} \pi(\tau) d\tau = P_A(t) \]

Let spot prices at any point in time be measured in units of current output. Because goods can be converted one-for-one, the spot price for capital is one and its rate of return is r. Because of the assumption that anyone engaged in research can freely take advantage of the entire existing stock of designs in doing research to produce new designs, its follow from R&D technology equation (7),

\[ wH_A = P_A \gamma H_A^\lambda A^{1-\phi} \]

If \( v(t) \) denote the value of the innovation

\[ v(t) = \int_t^\infty e^{-\int_t^r r(s)ds} \pi(\tau) d\tau \]

Therefore, equation (14) can be equivalently written as

\[ wH_A = v\gamma H_A^\lambda A^{1-\phi} \]

Also because of symmetry with respect to different intermediate, \( K = Ax \), equation (6) is written as

\[ Y = (AH_Y)^{1-\alpha}(K)^\alpha \]

Hence, from equation (10) and (17),

\[ w = (1 - \alpha)A \left( \frac{K}{AH_Y} \right)^\alpha \]
From zero profits in the final goods sector $\pi = H^{1-\alpha}Ax^\alpha - pAx - wH = 0$ and equation (10)

(19) \hspace{1cm} Y - wH = Ap\epsilon = \alpha Y

Notice that wages equalize across sectors due to free entry and exit.

**4.4 Consumers**

Each agent supply labor and receive some amount of consumption, $c(t)$. Individual maximize the intertemporal utility function choosing consumption and the fraction of time to devote in human capital production (or the fraction of time to devote in market work). Hence, the agent’s problem is

$$\max_{c_t,\epsilon_t} \int_t^{\infty} e^{-(\rho-n)t} \left[c(t)^{1-\sigma} - 1\right] / (1 - \sigma) \ dt$$

Subject to

$$\dot{K} = r_t[K_t + v_tA_t] + w_tH_t - c_tN_t - v_t\dot{A}_t - \epsilon_tA_t$$

$$\dot{h}_t = \eta h_t^{\beta_z}(1 - \epsilon_t)^{\beta_z} - \theta g_A h_t$$

and,

$$\epsilon_t \in [0,1]$$

where $\rho (\rho > n > 0)$ is the subjective discount rate, and $\sigma (\geq 0)$ is the inverse of intertemporal elasticity of substitution in consumption.

**5. Balanced Growth Path (BGP) Analysis**

**Definition 1.** (Balanced Growth Path (BGP)) I define a BGP as a state where variables $A$, $K$, $H$ and $Y$ grow at a constant (possibly positive) rates, (ii) technological progress ($A$) and the available stock of human capital ($H$) grow at the same rate, $g_A = g_H$ and, (iii) $r$, $\frac{A}{K}$ and $\frac{H}{K}$ are constants (iv) the amount of time the individual spends on accumulating human capital is constant $\dot{\epsilon} = 0$.

**Proposition 1:** BGP equilibrium exists when the R&D technology has constant returns to scale in $H_A$ and A together.

Proof: From equation (7) and (8) and using the fact that $\frac{L_A}{L_A} = n$, the growth rate of $A$, $g_A = \gamma \frac{H_A^2}{A^\phi}$, is constant along a BGP when, $g_A = \frac{\lambda}{\phi} [g_h + n]$ is constant, that is when $\lambda = \phi$.

Using Definition 1 and Proposition 1, we get the following result along the BGP (see Appendix 2.2),
\[ g_c = g_K = g_H = g_A = \frac{\rho - n(\sigma - \xi)}{(1 + \theta)\xi + 1 - \sigma - \theta(1 - \beta_1)} \quad \text{and} \quad g_Y = (2 - \alpha)g_A, \]

where \( \xi = \frac{\beta_2 L_Y}{(1 - \varepsilon)N} = \text{constant} \)

Hence along a BGP, consumption \((C)\), physical capital \((K)\), human capital \((H)\) and technology \((A)\) grow at a same constant rate.

**Proposition 2:** The long run growth rate depends on the preference and technological parameters. A positive long run growth rate exists under certain combination of these parameters. While the effect of population growth on growth rate of output is ambiguous, positive long run output growth rate is still achievable with zero population growth.

Proof: A simple inspection of equation \((1_{BGP})\) proves the results of proposition 1.

### 6. Bifurcation (Local) Analysis

Let \( m = \frac{Y}{K} \) and \( g = \frac{cN}{K} \). Using equations (11), (19) and \( K = Ax \) implies

\[
(20) \quad r = \alpha^2 m
\]

And the physical capital equation can be written as,

\[
(21) \quad \frac{\dot{K}}{K} = m - g - d
\]

The consumers intertemporal optimization conditions are (for proof see appendix 2.1)

\[
(22) \quad \frac{\dot{c}}{c} = \frac{r - \rho}{\sigma} = \frac{\alpha^2 m - \rho}{\sigma}
\]

\[
(22') \quad -r + \frac{h}{h} \left( \frac{\beta_2 L_Y}{(1 - \varepsilon)N} + 1 \right) + \theta g_A \left( \frac{\beta_2 L_Y}{(1 - \varepsilon)N} - (1 - \beta_1) \right) = (\beta_2 - 1) \frac{(-\varepsilon)}{(1 - \varepsilon)} - \frac{w}{w} - n
\]

Substituting equations (20), (21), (22) and using \( g = \frac{cN}{K} \), we can derive \( \frac{\dot{g}}{g} = \frac{\dot{c}}{c} - \frac{\dot{N}}{N} - \frac{\dot{K}}{K} \).

\[
(23) \quad \frac{\dot{g}}{g} = \left( \frac{\alpha^2}{\sigma} - 1 \right) m - \frac{\rho}{\sigma} + n + g + d
\]

Now multiplying both sides of equation (1) by \( h_t \), and using the definitions of equations (4) and (8).
Using equations (10), (16) and (24) in equation (7), \( \frac{\dot{A}}{A} = \frac{\gamma h_A}{A \phi} \), and setting \( \lambda = 1 \) for the rest of the analysis,

\[
\frac{\dot{A}}{A} = \frac{\gamma h \epsilon N}{u} - \frac{(1-\alpha)Y}{vA}
\]

The following can be shown from equation (15) and using \( \pi = \frac{\alpha(1-\alpha)Y}{A} \) (which can be derived from equations (12) and (19)),

\[
\frac{\dot{v}}{v} = r - \frac{\pi}{v} = \alpha^2 m - \alpha v
\]

Let \( f = \frac{\epsilon_t}{(1-\epsilon_t)} \). Using equation (10) and (16), it can be shown,

\[
\frac{L_{\gamma t}}{(1-\epsilon_t)N_t} = \frac{1}{(1-\epsilon_t)h_t N_t} \frac{(1-\alpha)Y_t}{w_t} = \frac{vf}{u}
\]

Let \( z = \frac{\eta(1-\epsilon_t)\beta_2}{h_t(1-\beta_1)} \), equation (5) can be written as,

\[
\frac{h}{h} = z - \theta g_A
\]

We can derive \( \frac{\dot{w}}{w} = \frac{\dot{v}}{v} + (1-\phi) \frac{\dot{A}}{A} \) from equation (16) and substitute equation (25) and (26) in it, to get

\[
\frac{\dot{w}}{w} = \alpha^2 m - \alpha v + (1-\phi)(u - v)
\]

Equation (22’) is simplified in the following way by using (28), (29) and (30)

\[
\frac{\dot{e}}{e} = \frac{1}{f(\beta_2 - 1)} \left[ -z - \theta g_A(\beta_1 - 2) + \alpha v - \beta_2 \frac{zf}{u} - (1-\phi)(u - v) - n \right]
\]

\[
m = \frac{Y}{K} = \left( \frac{AH_Y}{K} \right)^{1-\alpha}
\]

from equation (17)

Using equation (16), (18) and (31), \( \nu A^{1-\phi} = w = (1-\alpha)A \left( \frac{K}{AH_Y} \right)^\alpha \Rightarrow m^{\frac{\alpha}{1-\alpha}} = \frac{(1-\alpha)A}{\nu A} \)
Substituting equations (25) and (26) in, \[ \frac{\dot{m}}{m} = \frac{(1 - \alpha)}{\alpha} \left[ -\alpha^2 m + \alpha v + \phi (u - v) \right], \] derived from the above relation

(32) \[ \frac{\dot{m}}{m} = \frac{(1 - \alpha)}{\alpha} \left[ -\alpha^2 m + \alpha v + \phi (u - v) \right] \]

From equations (21) and (32) and using \[ \frac{\dot{v}}{v} = \frac{\dot{m}}{m} + \frac{\dot{k}}{k}, \]

(33) \[ \frac{\dot{v}}{v} = \frac{(1 - \alpha)}{\alpha} \left[ -\alpha^2 m + \alpha v + \phi (u - v) \right] + (m - g - d) \]

Plugging in results from (25), (26) and (33) in \[ \frac{\dot{v}}{v} = \frac{\dot{Y}}{Y} - \frac{\dot{\phi}}{\phi} - \frac{\dot{A}}{A}, \]

(34) \[ \frac{\dot{v}}{v} = \left[ (1 - \alpha) m + v - g + \left( \frac{(1 - \alpha) \phi}{\alpha} - 1 \right) (u - v) - d \right] \]

Using equation (30) in \[ \frac{\dot{z}}{z} = -\beta_2 f \frac{\dot{e}}{e} - (1 - \beta_1) \frac{\dot{h}}{h} \] and \[ \frac{\dot{f}}{f} = \frac{\dot{e}}{e} (1 + f), \] we derive,

(35) \[ \frac{\dot{z}}{z} = \frac{1}{f(\beta_2 - 1)} \left[ -z - \theta g_A (\beta_1 - 2) + \alpha v - \beta_2 \frac{zf}{u} - (1 - \phi) (u - v) - n \right] - (1 - \beta_1) (z - \theta g_A) \]

(36) \[ \frac{\dot{f}}{f} = \frac{(1 + f)}{f(\beta_2 - 1)} \left[ -z - \theta g_A (\beta_1 - 2) + \alpha v - \beta_2 \frac{zf}{u} - (1 - \phi) (u - v) - n \right] \]

Using equations (25), (28), (30) in \[ \frac{\dot{u}}{u} = \frac{h}{h} + \frac{N}{N} - \phi \frac{A}{A} + \frac{e}{e}, \]

(37) \[ \frac{\dot{u}}{u} = z - \theta g_A + n - \phi (u - v) + \left[ -z - \theta g_A (\beta_1 - 2) + \alpha v - \beta_2 \frac{zf}{u} - (1 - \phi) (u - v) - n \right] \]

Equations (23), (32), (34), (35), (36), and (37) represent the dynamic equations for the model.

6.1 Steady State

Definition 2. (Steady State) I define a steady state as a state where variables \( g, m, v, z, f \) and \( u \) grow at a constant (possibly zero) rates. A steady state is a BGP with zero growth rate.

Therefore, the steady state \( s^* = (g^*, m^*, v^*, z^*, f^*, u^*) \) is such that, \( \dot{g} = \dot{m} = \dot{v} = \dot{z} = \dot{f} = \dot{u} = 0 \). It is derived by solving the following equations (I)-(VI).

(I) \[ \left( \frac{\alpha^2}{\sigma} - 1 \right) m - \frac{\rho}{\sigma} + n + g + d = 0 \]

(II) \[ -\alpha^2 m + \alpha v + \phi (u - v) = 0 \]
Theorem 6. A unique steady state exists if

\[ \Lambda = (1 + \phi)(\sigma - 1) g_A + \rho - n > 0 \]

**Proof:** \( \Lambda \) is the necessary and sufficient for the transversality condition for the consumer’s utility maximization problem to hold (appendix 2.1)

### 6.2 Bifurcation

I examine the existence of codimension 1 and 2 bifurcations in the dynamical system defined by equations (23), (32), (34), (35), (36), and (37). As discussed earlier *Andronov-Hopf* bifurcation is the birth of a limit cycle from an equilibrium, when the equilibrium changes stability via a pair of purely imaginary eigenvalues. Table 2 reports the values of subjective discount rate (\( \rho \)), share of human capital and share of time devoted for the human capital production (\( \beta_1 \& \beta_2 \), respectively), effect of technological progress on human capital accumulation (\( \theta \)) and the depreciation rate of capital, (\( d \)) at which Hopf bifurcation occurs when they are treated as free parameters.

Some of the limit cycles generated by Andronov-Hopf bifurcation are unstable (subcritical bifurcation) identified by a positive value of the first Lyapunov coefficient. Thus for each of the cases reported in Table 2, an unstable limit cycle (periodic orbit) bifurcates from the equilibrium. When \( \rho, \beta_1, \theta \& d \) are treated as free parameters, a slight perturbation of them give rise to *Branch Points (Pitchfork/Transcritical bifurcations)*. Notice that some of the Hopf points detected are neutral saddles and are not bifurcations.

The cyclical behavior could occur for various reasons. For instance, suppose profit for monopolist increases. As the market for designs is competitive, the price for designs \( P_A \), bids up until it is equal to present value of the net revenue that a monopoly can extract. From equation (14), wages
in the R&D sector rises. As a result of higher wages in the research sector, labor move out of output production to research sector. When sufficient amount of externalities to R&D (\(1 - \phi > 0\) in equation (7)) is present, the growth rate of technology \(g_A\) starts rising. If there is a negative effect of technical progress on human capital investment \((\theta > 0)\), human capital accumulation start declining. The price of final good durables is a positive function of the average quality of labor given by equation (4) and (9). This implies that prices start falling in the final goods sector due to decline in average quality of labor which in turn, implies that monopoly profits start falling.

I further investigate the stability properties of cycles generated by different combination of such parameters. Continuation of limit cycle from the Hopf point for the case when \(\rho\) is the free parameter gives rise to two Period Doubling (flip) bifurcations. Period doubling bifurcation is defined as a situation when a new limit cycle emerges from an existing limit cycle, and the period of the new limit cycle is twice that of the old one. The initial period doubling bifurcations occur at \(\rho = 0.0257\) and \(\rho = 0.0258\) with a negative normal form coefficients indicating stable double-period cycles are involved.

Continuing computation further from the Hopf point, gives rise to Limit Point (Fold/ Saddle Node) bifurcation of Cycles. From the family of limit cycles bifurcating from the Hopf point, Limit Point Cycle (LPC) is a fold bifurcation of the cycle where two limit cycles with different periods are present near LPC point at \(\rho = 0.0258\). We get another Period Doubling (flip) bifurcations upon further computation.

I carry out the continuation of limit cycle from the second Hopf point for the case when \(\theta\) is treated as the free parameter. I investigate the existence of codimension-2 bifurcations by allowing two free parameters \(\theta\) and \(\rho\) for the first case and \(\theta \& \beta_1\) for the second. Two points were detected corresponding to codim 2 bifurcations: Bogdanov-Takens and Generalized Hopf (Bautin) for each of the cases. At each Bogdanov-Takens point the system has an equilibrium with a double zero eigenvalue and the normal form coefficients \((a; b)\) are reported in Table 2 which are all nonzero. The Generalized Hopf points are nondegenerate since the second Lyapunov coefficient \(l_2\) are nonzero. The Generalized Hopf (Bautin) bifurcation is a bifurcation of an equilibrium at which the critical equilibrium has a pair of purely imaginary eigenvalues and the first Lyapunov coefficient for the Andronov-Hopf bifurcation vanishes. The bifurcation point separates branches of sub- and supercritical Andronov-Hopf bifurcations in the parameter plain. For nearby parameter values, the system has two limit cycles which collide and disappear via a saddle-node bifurcation of periodic orbits.
<table>
<thead>
<tr>
<th>Parameters Varied</th>
<th>Equilibrium Bifurcation</th>
<th>Continuation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_1 )</td>
<td>Branch Point (BP)</td>
<td></td>
</tr>
<tr>
<td>(Figure i)</td>
<td>( \beta_1 = 1 )</td>
<td></td>
</tr>
<tr>
<td>{ \alpha = 0.4, \rho = 0.055, \beta_2 = 0.04, n = 0.01, d = 0, \theta = 0.4, \phi = 1, \sigma = 8 }</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \beta_1 )</td>
<td>Hopf (H)</td>
<td></td>
</tr>
<tr>
<td>(Figure ii)</td>
<td>First Lyapunov coefficient = 0.0000230, ( \beta_1 = 0.19 )</td>
<td></td>
</tr>
<tr>
<td>{ \alpha = 0.4, \rho = 0.025772, \beta_2 = 0.04, n = 0.01, d = 0, \theta = 0.4, \phi = 0.8, \sigma = 0.08 }</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \beta_2 )</td>
<td>Hopf (H)</td>
<td></td>
</tr>
<tr>
<td>{ \alpha = 0.4, \beta_1 = 0.19, \rho = 0.025772, n = 0.01, d = 0, \theta = 0.4, \phi = 0.8, \sigma = 0.08 }</td>
<td>First Lyapunov coefficient = 0.00002302, ( \beta_1 = 0.026698 )</td>
<td></td>
</tr>
<tr>
<td>( d )</td>
<td>Branch Point (BP)</td>
<td></td>
</tr>
<tr>
<td>{ \alpha = 0.4, \beta_1 = 0.19, \rho = 0.055, \beta_2 = 0.04, n = 0.01, \theta = 0.4, \phi = 1, \sigma = 8 }</td>
<td>d = 0.826546</td>
<td></td>
</tr>
<tr>
<td>( \rho )</td>
<td>Hopf (H), Neutral Saddle, ( \rho = 0.026726 )</td>
<td>Figure iii (A)</td>
</tr>
<tr>
<td>(Figure iii)</td>
<td>First Lyapunov coefficient = 0.0000149</td>
<td>Period Doubling</td>
</tr>
<tr>
<td>{ \alpha = 0.4, \beta_1 = 0.19, \rho = 0.055, \beta_2 = 0.04, n = 0.01, \theta = 0.4, \phi = 1, \sigma = 0.08 }</td>
<td>( \rho = 0.025772 )</td>
<td>(period = 1,569.64; ( \rho = 0.0257 ))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Normal form coefficient = -0.056657e-013</td>
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<tr>
<td></td>
<td></td>
<td>(period = 1,741.46; ( \rho = 0.0258 ))</td>
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<tr>
<td></td>
<td></td>
<td>Normal form coefficient = -7.235942e-015</td>
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<tr>
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<td></td>
<td>Limit point cycle (period = 2,119.53; ( \rho = 0.0258 ))</td>
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<tr>
<td></td>
<td></td>
<td>Normal form coefficient = 7.894415e-004</td>
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<td>Period Doubling (period = 2,132.13; ( \rho = 0.0258 ))</td>
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<td></td>
<td>Normal form coefficient = -1.763883e-013</td>
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<td></td>
<td>Branch Point (BP)</td>
<td>( \rho = 0.026726 )</td>
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<td>Hopf (H), Neutral Saddle, ( \rho = 0.026698 )</td>
<td>( \rho = 0.026726 )</td>
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<tr>
<td></td>
<td>Figure iv (A)</td>
<td>Figure iv (B): <strong>Codimension-2 bifurcation</strong></td>
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<td>----------------------</td>
<td>-------------------------------------------------------------------------------</td>
<td>---------------------------------------------</td>
</tr>
<tr>
<td><strong>Hopf (H)</strong></td>
<td>First Lyapunov coefficient: ( \alpha = 0.4, \beta_1 = 0.19, \rho = 0.029710729, \beta_2 = 0.04, n = 0.01, d = 0, \phi = 0.69716983, \sigma = 0.08 )</td>
<td><strong>Generalized Hopf (GH)</strong> ( \theta = 0.000044, \rho = 0.580853 )</td>
</tr>
<tr>
<td></td>
<td>First Lyapunov coefficient: ( \theta = 0.00001973 ) ( \theta = 0.355216 )</td>
<td>( l_2 = (0.000001254) )</td>
</tr>
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<td><strong>Hopf (H)</strong></td>
<td>( \theta = 0.400000 )</td>
<td><strong>Bogdanov-Takens (BT)</strong> ( \theta = 0, \rho = 0.644247 )</td>
</tr>
<tr>
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<td>( (a,b) = (0.000001642, -0.003441) )</td>
<td>( (a,b) = 0.000008949 )</td>
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<tr>
<td><strong>Generalized Hopf (GH)</strong></td>
<td>( \theta = 0.000055, \beta_1 = 0.584660 )</td>
<td><strong>Bogdanov-Takens (BT)</strong> ( \theta = 0, \beta_1 = 0.903003 )</td>
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<tr>
<td></td>
<td>( l_2 = 0.0000008949 )</td>
<td>( (a,b) = (0.000006407790, 0.0329134) )</td>
</tr>
<tr>
<td><strong>Hopf (H)</strong></td>
<td>( \theta = 0.612624 )</td>
<td><strong>Branch Point (BP)</strong> ( \theta = 0.613596 )</td>
</tr>
<tr>
<td><strong>Neutral Saddle</strong></td>
<td>( \theta = 0.612624 )</td>
<td></td>
</tr>
<tr>
<td><strong>Branch Point (BP)</strong></td>
<td>( \theta = 0.613596 )</td>
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</table>
Figure (iv)
7. Conclusion

This paper provides a detailed stability and bifurcation analysis of the Uzawa-Lucas model and the modified Jones model. In the Uzawa-Lucas model, transcritical bifurcation boundary and Hopf bifurcation boundary are located for the decentralized version. The examination of the stability properties of the limit cycles created from various Hopf bifurcations in the model depicts the occurrence of Limit Point of Cycles and Period Doubling bifurcations within the feasible parameter range of the model. The series of Period Doubling bifurcations confirms the presence of global bifurcation. This also highlights the possibility of having chaotic dynamics in the model. On the contrary, the social planner solution for the Uzawa-Lucas model is always saddle path stable with no possibility of occurrence of bifurcation in the feasible parameter range of the model. Thus the externality of human capital parameter plays an important role in determining the dynamics of Uzawa-Lucas model.

In the modified Jones model, I have shown that the long run growth rate does depend on the rate of population growth along the balanced growth path but the direction of the relationship depends on other parameters of the model. More importantly, the long run growth rate can even be positive with no population growth. Several Andronov-Hopf bifurcations and Branch Points are located. Further, the stability properties of the limit cycles created from these Hopf bifurcations are examined. I showed the existence several codimension-1 bifurcations (Limit Point of Cycles and Period Doubling bifurcations) and codimension-2 bifurcations (Bogdanov-Takens and Generalized Hopf) are located. The choice of certain parameters in locating various bifurcations emphasizes the role played by human capital in such a model where growth is driven by technological progress, which in turn, is ultimately driven by human capital investment. The parameters in the human capital accumulation among others equation play a key role in determining the dynamics of the model.

Appendix 1.1:

In steady state, m, g and ε are constant, which implies

\[
\frac{\dot{K}}{K} = \frac{\dot{C}}{C} = \frac{\dot{Y}}{Y} = \frac{\dot{k}}{k} + n = \frac{(1 - \alpha + \xi)}{(1 - \alpha)} \eta(1 - \varepsilon) + n
\]

Transversality condition requires \(\lim_{t \to \infty} \lambda_t K_t = 0\) and \(\lim_{t \to \infty} \mu_t h_t = 0\) which in turn implies

\[
\rho > (1 - \sigma) \frac{(1 - \alpha + \xi)}{(1 - \alpha)} \eta(1 - \varepsilon)
\]

Appendix 1.2:

Social Planner Problem
\[ H = \frac{c(\tau)^{1-\sigma} - 1}{(1-\sigma)} + \lambda [Ak^\alpha e^{1-\alpha}h^{1-\alpha+\zeta} - c - (n + \delta)K] + \mu \eta h(1 - \epsilon) \]

The first order conditions are

1. \( c: \quad c^{-\sigma} e^{-(\rho-n)} = \lambda \)
2. \( \varepsilon: \quad \lambda (1 - \alpha)Ak^\alpha e^{-\alpha}h^{1-\alpha+\zeta} = \mu \eta h \)
3. \( k: \quad \lambda [\alpha Ak^\alpha e^{1-\alpha}h^{1-\alpha+\zeta} - (n + \delta)] = -\dot{\lambda} \)
4. \( h: \quad \lambda (1 - \alpha + \zeta)Ak^\alpha e^{1-\alpha}h^{-\alpha+\zeta} + \mu \eta (1 - \epsilon) = -\dot{\mu} \)

Appendix 1.3:

Decentralized or Market Solution

\[ H = \frac{c(\tau)^{1-\sigma} - 1}{(1-\sigma)} + \lambda [Ak^\alpha e^{1-\alpha}h^{1-\alpha+\zeta} - c - (n + \delta)K] + \mu \eta h(1 - \epsilon) \]

The first order conditions are

1. \( c: \quad c^{-\sigma} e^{-(\rho-n)} = \lambda \)
2. \( \varepsilon: \quad \lambda (1 - \alpha)Ak^\alpha e^{-\alpha}h^{1-\alpha+\zeta} = \mu \eta h \)
3. \( k: \quad \lambda [\alpha Ak^\alpha e^{1-\alpha}h^{1-\alpha+\zeta} - (n + \delta)] = -\dot{\lambda} \)
4. \( h: \quad \lambda (1 - \alpha + \zeta)Ak^\alpha e^{1-\alpha}h^{-\alpha+\zeta} + \mu \eta (1 - \epsilon) = -\dot{\mu} \)

Appendix 2.1:

I use the zero profit condition \( w_tH_{At} = u_t \dot{A}_t \) and equation (26), \( A_tu_t = A_t r_t u_t - A_t \pi_t \) in the wealth accumulation equation of the households \( \dot{K} = r_t [K_t + u_t A_t] + w_t H_t - c_t N_t - \dot{A}_t u_t - A_t u_t \), to get

\[ \dot{K} = r_t K_t + w_t h_t (1 - l_{ht}) N_t - c_t N_t - w_t h_t L_{At} + A_t \pi_t \]

The relevant Hamiltonian for the consumer’s problem is

\[ H = e^{-(\rho-n)t}[c(\tau)^{1-\sigma} - 1] / (1-\sigma) + \lambda [r_t K_t + w_t h_t (1 - l_{ht}) N_t - c_t N_t - w_t h_t L_{At} + A_t \pi_t] + \mu [\eta h_{t}^{\beta_1}(1 - \epsilon_t)^{\beta_2 - \theta} g_{A} h_{t}] \]

The first order conditions are

1. \( c^{-\sigma} e^{-(\rho-n)t} = \lambda N \) \quad \Rightarrow \quad \frac{c}{c} = \frac{r - \rho}{\sigma} \)
2. \( \varepsilon: \quad -\lambda w h N - \mu \eta h^{\beta_1} \beta_2 (1 - \epsilon)^{\beta_2 - 1} = 0 \) \quad \Rightarrow \quad \frac{\lambda}{\mu} = \frac{\eta h^{\beta_1 - 1} (1 - \epsilon) \beta_2 - 1 \beta_2}{w N} \)
3. \( K: \quad \lambda r = -\dot{\lambda} \) \quad \Rightarrow \quad \frac{\dot{\lambda}}{\lambda} = -r
(iv) \[ h: \quad \lambda w N - \lambda w L_A + \mu \eta \beta_1 h^{\beta_1 - 1} (1 - \varepsilon)^{\beta_2} - \mu \theta g_A = -\hat{\mu} \]

Dividing (iv) by \( \mu \) and substituting (ii) in it, \[ \eta h^{\beta_1 - 1} (1 - \varepsilon)^{\beta_2} \left[ \frac{\beta_2 L_Y}{(1 - \varepsilon) N} + \beta_1 \right] - \theta g_A = -\frac{\hat{\mu}}{\mu} \quad (v) \]

Now, from (ii), \[ \frac{\dot{\lambda}}{\lambda} - \frac{\dot{\mu}}{\mu} = (\beta_2 - 1) \left( \frac{(-\varepsilon)}{(1 - \varepsilon)} \right) + (\beta_1 - 1) \frac{\dot{h}}{h} - \frac{\dot{w}}{w} - n \]

and substituting (iii) and (v) in it,

\[ -r + \frac{h}{h} \left( \frac{\beta_2 L_Y}{(1 - \varepsilon) N} + 1 \right) + \theta g_A \left( \frac{\beta_2 L_Y}{(1 - \varepsilon) N} - (1 - \beta_1) \right) = (\beta_2 - 1) \left( \frac{(-\varepsilon)}{(1 - \varepsilon)} \right) - \frac{\dot{w}}{w} - n \]

Transversality Conditions:

\[ \lim_{t \to \infty} \lambda_t \left[ K_t + v_t A_t \right] = 0 \quad \text{and} \quad \lim_{t \to \infty} \mu_t h_t = 0 \]

In a steady state, \( \dot{g} = \dot{m} = \dot{v} = \dot{u} = \dot{f} = \dot{z} = 0 \), and using the fact that in the steady state, \( \frac{H_y}{N} \)

is constant, \( \dot{g} = 0 \) and \( \dot{m} = 0 \) implies \( \frac{\dot{r}}{r} + \frac{\dot{N}}{N} = \frac{\dot{p}}{p} \).

We use \( \ddot{z} = -\beta_2 \frac{\dot{r}}{r} \frac{1}{\beta_1} - (1 - \beta_1) \frac{\dot{h}}{h} = 0 \) and \( \dot{f} = \frac{\dot{e}}{\epsilon} (1 + f) = 0 \) to derive the following

\[ \frac{\dot{e}}{\epsilon} = \frac{-1 + \beta_1}{\beta_2} \frac{\dot{h}}{h} \]

Hence \( \dot{u} = \frac{\dot{h}}{h} + \frac{\dot{N}}{N} - \phi \frac{A}{A} + \frac{\dot{e}}{\epsilon} = 0 \) implies \( \frac{\dot{h}}{h} = \frac{\beta_2}{(1 - \beta_1 + \beta_2)} \frac{\epsilon g_A - n}{A} \)

\[ \frac{\dot{H}_Y}{H_y} = \frac{\dot{h}}{h} + \frac{\dot{N}}{N} + \frac{\dot{e}}{\epsilon} = \phi g_A \]

From equation (32), \( m = \frac{y}{k} = \left( \frac{\alpha H}{K} \right)^{1-\alpha} \)

\[ \frac{\dot{m}}{m} = (1 - \alpha) \left[ \frac{\dot{A}}{A} + \frac{\dot{H}_Y}{H_Y} - \frac{\dot{k}}{K} \right] = 0 \] implies \( \frac{\dot{k}}{K} = \frac{\dot{A}}{A} + \frac{\dot{H}_Y}{H_Y} = (1 + \phi) g_A \)

Hence, \( \frac{\dot{K}}{K} = \frac{\dot{C}}{C} = \frac{\dot{Y}}{Y} = (1 + \phi) g_A \)

Hence, the transversality condition implies that, \( (1 + \phi)(\sigma - 1) g_A + \rho - n > 0 \)

**Appendix 2.2:**

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is constant \( \Rightarrow \bar{x} \) is constant since \( K = Ax \). The flow of monopoly profit is \( \bar{\pi} = (1 - \alpha)\bar{p}\bar{x} \). From equation (13), we know the PDV of this stream of profit must equal to price \( P_A \) of the design and using equation (9),

\[
v = P_A = \frac{1}{r} \pi = \frac{1}{r} (1 - \alpha)\bar{p}\bar{x} = \frac{(1-\alpha)}{r} \alpha H_Y^{1 - \alpha} x^{\alpha - 1} x = \frac{\alpha(1-\alpha)}{r} H_Y^{1 - \alpha} x^\alpha
\]

The condition determining the allocation of \( H_Y \) and \( H_A \) says that wages paid to human capital in each sector must be the same, from equation (16) and (18),

\[
w = vyH_A^{\lambda - 1}A^{1 - \phi} = (1 - \alpha) \frac{H_Y^{1 - \alpha} A^{\alpha}}{H_Y}
\]

\[
\Rightarrow H_Y = \frac{r}{\alpha} \frac{A^\phi}{\gamma H_A^\lambda} H_A = \frac{r}{\alpha g_A} H_A
\]

From equation (11) and (12), we get, \( \bar{\pi} = \frac{(1-\alpha)}{\alpha} \bar{r}\bar{x} \) and from, equation (15),

\[
v(t) = \frac{\pi}{r} \frac{\dot{\bar{v}}}{\bar{v}} = 0
\]

\[
\frac{\dot{w}}{w} = \frac{\dot{v}}{v} + (\lambda - 1) \frac{\dot{H}_A}{H_A} + (1 - \phi) \frac{\dot{A}}{A} = 0 \text{ under CRS}
\]

Dividing the household's wealth accumulation equation \( \dot{K} = rK + wH_Y - cN + A\pi \) by \( K \) and incorporating, \( -\frac{\lambda}{\lambda} = r \), we get

\[
\frac{\lambda}{\lambda} = -g_K + w \frac{H_Y}{K} - cN + A \pi
\]

For the first order conditions from the consumer’s optimization problem (see Appendix 1 for derivation),

\[
\eta h^{\beta_1 - 1}(1 - \epsilon)\beta_2 \frac{\beta_2}{(1 - \epsilon) N + \beta_1} - \theta g_A = -\frac{\dot{\mu}}{\mu}
\]

\[
\frac{\dot{\lambda}}{\lambda} - \frac{\dot{\mu}}{\mu} = (\beta_2 - 1) (-g_\epsilon) \frac{\epsilon}{(1 - \epsilon)} + (\beta_1 - 1) g_h - g_w - \eta
\]

the human capital equation, \( g_h = \eta \frac{(1-\epsilon)\beta_2}{h(1-\beta_1)} - \theta g_A \), implies,

\[
-\frac{\dot{\mu}}{\mu} = (g_h + \theta g_A) \frac{\beta_2}{(1 - \epsilon) N + \beta_1} - \theta g_A
\]

and, as \( H = hN\epsilon \) and \( g_\epsilon = 0 \) along a BGP, \( g_h = g_H - n \) implies

\[
\frac{\dot{\lambda}}{\lambda} - \frac{\dot{\mu}}{\mu} = (\beta_1 - 1) g_h - n
\]
Substituting (a) and (b) in (c),

\[ -g_K + w \frac{H_Y}{K} - c - \frac{\pi_A}{K} + (g_h + \theta g_A) \left[ \frac{\beta_2}{(1-\varepsilon)N} L^Y - \beta_1 \right] - \theta g_A = (\beta_1 - 1) g_h - \eta \]

As \( H_Y = \frac{r}{\alpha g_A} H_A \) and in a BGP, \( r \) and \( g_A \) are constants, \( \frac{H_A}{H_Y} \) constant, that is \( \frac{L_A}{L_Y} \) constant

As \( \varepsilon \) is constant in the labor endowment equation \( \frac{L_Y}{N} \) is also a constant.

I assume \( \frac{H_Y}{K}, \frac{c}{K}, \frac{A}{K} \) is constant along a BGP implying \( g_c = g_k = g_A = g_h + n, g_c = \frac{r-\rho}{\sigma} = g_c - n \). Substituting \( r = \sigma(g_A - n) + \rho \) and using \( g_{\varepsilon} = g_w = 0, g_A = g_h + n \) in the equation below (see consumer’s optimization problem in Appendix I)

\[ -r + \frac{\dot{h}}{h} \left( \frac{\beta_2 L_Y}{(1-\varepsilon)N} + 1 \right) + \theta g_A \left( \frac{\beta_2 L_Y}{(1-\varepsilon)N} - (1 - \beta_1) \right) = (\beta_2 - 1) \frac{(-\dot{e})}{(1-\varepsilon)} - \frac{\dot{w}}{w} - n \]

\[ -\sigma(g_A - n) - \rho + (g_A - n) \left( \frac{\beta_2 L_Y}{(1-\varepsilon)N} + 1 \right) + \theta g_A \left( \frac{\beta_2 L_Y}{(1-\varepsilon)N} - (1 - \beta_1) \right) = -n \]

\[ g_c = g_k = g_h = g_A = \frac{\rho - n(\sigma - \xi)}{(1 + \theta)\xi + 1 - \sigma - \theta(1 - \beta_1)} \text{ where } \xi = \frac{\beta_2 L_Y}{(1-\varepsilon)N} \text{ is constant} \]

\[ \frac{Y}{K} = \left( \frac{AH_Y}{K} \right)^{1-\alpha} \text{ where } \frac{H_Y}{K} \text{ is constant along a BGP.} \]

\[ g_Y = (2 - \alpha) g_A \]

References:


(7) Barnett, W. and He, Y., 2002 “Stabilization Policy as bifurcation selection: Would stabilization policy work if the economy really were unstable?”, Macroeconomic Dynamics, Volume 6, pp 713-747


