

**MONOTONE COMPARATIVE STATICS
FOR GAMES WITH BOTH STRATEGIC COMPLEMENTS
AND STRATEGIC SUBSTITUTES**

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ABSTRACT. I characterize conditions under which equilibria are increasing with respect to an increase in parameter, within the setting of a general class of parametrized games which allows for the presence of both strategic complements and strategic substitutes simultaneously. The results are demonstrated for 2-player and more-than-2-player games, allow for general strategy spaces, and allow for best responses to be either functions or correspondences. These results represent a broadening of the well-known theories of games with strategic complements and games with strategic substitutes, in which all players must have uniform strategic properties.

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Strategic complements and strategic substitutes are prevalent in many game theoretic situations from their introduction in Bulow, Geanakoplos and Klemperer (1985), and formalize very natural interactions between players in game theoretic applications. A player has strategic complements if his optimal strategy *increases* as his opponents play higher strategies, as in a network game, where as more players adopt a technology, it becomes increasingly beneficial for a given player to adopt. A player has strategic substitutes if his optimal strategy *decreases* as his opponents play higher strategies, as in the classic case of Cournot competition. As such, there has been a wide breadth of literature to address properties of games with strategic complements (or GSC), and games with strategic substitutes (or GSS), including the existence and structure of equilibrium sets and the presence of monotone comparative statics.¹

Despite the wealth of research related to GSC and GSS, however, very little has been said on the application of similar results under less restrictive circumstances. In GSC, all players have strategic complements, while in GSS, all players have strategic substitutes. Yet, what if strategic complements and strategic substitutes occur simultaneously? What can be said about a game of this nature which is not covered under the settings above? In fact, there are many applications in the literature which consider games of this nature. The simplest observation of this phenomenon is the classic Matching Pennies game. Bulow et al. (1985) allows for this possibility in their original presentation of strategic substitutes and complements. Dixit (1987) and Fudenberg and Tirole (1984) are examples of pre-commitment games where players' strategic behavior is permitted to be opposed in this way. Singh and Vives (1984) address this scenario in a 2-firm model which mixes Bertrand and Cournot approaches and allows one to strategically choose quantity while the other strategically selects price. Tombak (2006) discusses pre-commitment in industries with

¹For example, Milgrom and Roberts (1990) and Vives (1990) address the existence and ordered properties of the set of Nash equilibria in GSC; Roy and Sabarwal (2008) demonstrate the inability to extend these properties to GSS. There are results in each of these settings with respect to monotone comparative statics as well. Milgrom and Shannon (1994) demonstrate that in parametrized GSC, the set of equilibria is nondecreasing in the parameter; Villas-Boas (1997) and Roy and Sabarwal (2010b) present an analogous result for parametrized GSS. This is by no means an exhaustive list of what has been said regarding GSC and GSS. See Vives (2005) for a helpful survey article.

learning effects under what he terms “strategic asymmetry” - simultaneous strategic substitutes and strategic complements.

The intuition behind the interaction between one player with strategic complements and another with strategic substitutes can be seen in the interaction between a criminal, who chooses a level of criminal activity, and the police, who choose a level of law enforcement. As the criminal engages in more crime, the police want to *increase* the optimal level of law enforcement (strategic complements). However, as the police up their law enforcement efforts, the criminal is likely to respond by *decreasing* his optimal level of crime (strategic substitutes). This interaction, which mixes the coordination efforts of one player (the police) with the oppositional efforts of another player (the criminal), is the prevalent feature of the games considered here.

Monaco and Sabarwal (2011) address this broad class of games, and characterize the non-ordered structure of the set of pure strategy Nash equilibria. The purpose of this work is to extend these results to characterize monotone comparative statics in settings which are similarly not covered in the GSC or GSS literature. In this context, I give conditions under which, given a Nash equilibrium of the game, and given some increase in a complementary parameter,² the equilibria at the new, higher parameter are either *not lower* or, respectively, *higher*, than the original equilibrium.

Determining the presence of monotone comparative statics has policy implications in many applications. For example, in a Cournot-type game, it may be of interest to determine if an increase in government subsidies to an industry leads to a higher level of production for *each firm in the industry*: this is equivalent to asking whether or not there is monotone comparative statics. Similarly, in a model of technological development, this approach can capture if parameter changes lead to, say, a higher level of research and development for all firms in a given industry; or, in a model of environmental standards, the approach can gauge if a change in parameter would prompt more stringent emissions restrictions across all countries in the model.

²Here, I mean, quite simply, a parameter whose increase brings about an increase in the players’ best responses. A formal definition is provided below.

The thrust of this paper lies in the middle ground it addresses. Unlike the results regarding monotone comparative statics which exist for GSC or GSS, for which the requirements on players are uniform (all players must exhibit either strategic complements or strategic substitutes, respectively), the results here examine a much more general class of games, which includes games for which some player (or subset of players) has strategic substitutes, while the remaining players have strategic complements. Therefore, this work provides a generalization of the theory of monotone comparative statics to a broad class of games, many of which cannot be covered by the existing literature on GSC and GSS.

An important theme throughout this analysis is the role of the players with strategic substitutes in determining monotone comparative statics. Essentially, when a complementary parameter increases, each strategic substitutes player is impacted in two ways. There is (1) a **direct effect**: as the parameter increases, the player is incentivized to *increase his best response*. Additionally, there is (2) a **strategic effect**: since the parameter increase will cause opposing players to increase their best responses, the strategic substitutes player will be incentivized to *decrease his best response*. The relationship between these two opposing forces is central to the results, and helps to provide some valuable intuition into how equilibria may (or may not) be increasing in this class of games. Moreover, it relates the work here to existing results for GSS.³

Section 1 provides the framework for the lattice-theoretic approach in this paper, and includes several examples which fall under this general class of games. Section 2 gives results concerning nowhere decreasing equilibria, and Sections 3 and 4 give the primary results on increasing equilibria in 2-player and more-than-2-player games, respectively.

³For example, Theorem 2 here reflects the same necessary condition for the existence of increasing equilibria in Roy and Sabarwal (2010b); however, for increasingly general cases here, this condition is insufficient. See sections 3 and 4 for the appropriate counterexamples, the ensuing necessary conditions, and results.

1. PRELIMINARIES

To begin, I present the mathematical details related to the approach here. These are standard definitions in the literature on GSC and GSS.⁴

1.1. Sets (Strategy spaces). A partially ordered set X is a *lattice* if, for all $a \in X$, for all $b \in X$, $a \vee b \in X$ and $a \wedge b \in X$, where “ \vee ” represents the *join* (or supremum) of two elements, and “ \wedge ” represents the *meet* (or infimum) of two elements. A lattice is *complete* if for any non-empty subset $A \subseteq X$, both the join and meet over all elements of A is included in X .

For a lattice X , and any two subsets $A \subseteq X$, $B \subseteq X$, compare the two sets in the following ways: A is *weakly lower than B in the standard induced set order*,⁵ denoted $A \sqsubseteq B$, if: $\forall a \in A, \forall b \in B, a \vee b \in B$ and $a \wedge b \in A$;⁶ A is *completely lower than B* if: $\forall a \in A, \forall b \in B, a \preceq b$; A is *strictly lower than B* if: $\forall a \in A, \forall b \in B, a \prec b$.

1.2. Functions (Payoff functions). For $f : X \times T \rightarrow \mathbb{R}$, with X lattice, T partially ordered set, f satisfies the *single crossing property (SCP) in $(x; t)$* if, for $x'' \succ x', t'' \succ t'$, $f(x'', t') > f(x', t') \implies f(x'', t'') > f(x', t'')$ and $f(x'', t') \geq f(x', t') \implies f(x'', t'') \geq f(x', t'')$. This condition is given as the *strict single crossing property* if the weak inequality implies the strict one. Similarly, for $f : X \times T \rightarrow \mathbb{R}$, with X lattice, T partially ordered set, f satisfies the *decreasing single crossing property (DSCP) in $(x; t)$* if, for $x'' \succ x', t'' \succ t'$, $f(x', t') > f(x'', t') \implies f(x', t'') > f(x'', t'')$ and $f(x', t') \geq f(x'', t') \implies f(x', t'') \geq f(x'', t'')$. This condition is given as the *strict decreasing single crossing property* if the weak inequality implies the strict one. For $f : X \rightarrow \mathbb{R}$, X lattice, f is *quasisupermodular* in x if for all $x \in X, y \in X$, $f(x) \geq f(x \wedge y) \implies f(x \vee y) \geq f(y)$ and $f(x) > f(x \wedge y) \implies f(x \vee y) > f(y)$. This

⁴See Topkis (1998), Birkhoff (1967), Milgrom and Roberts (1990), Milgrom and Shannon (1994), Shannon (1995), Roy and Sabarwal (2008), or Roy and Sabarwal (2010b) for an additional and more rigorous treatment of the preliminaries.

⁵Sometimes this order is referred to as the “strong set order”.

⁶In particular, A weakly lower than B in the induced set order implies the following useful property, which defines A *weakly lower than B* : $\forall a \in A, \exists b \in B$ such that $a \preceq b$ and $\forall b \in B, \exists a \in A$ such that $a \preceq b$. So, “weakly lower in the induced set order” implies “weakly lower”; however, the converse is not true.

condition is known as *strict quasisupermodularity* if the weak inequality implies the strict one for all unordered x and y in X .

1.3. Correspondences (Best responses). A correspondence $g^i : X_{-i} \rightrightarrows X_i$ is *nonincreasing (in the induced set order)* in x_{-i} if, for $x'_{-i} \preceq x''_{-i}$, $g^i(x''_{-i})$ is weakly lower than $g^i(x'_{-i})$ in the induced set order. A correspondence $g^i : X_{-i} \rightrightarrows X_i$ is *nondecreasing (in the induced set order)* in x_{-i} if, for $x'_{-i} \preceq x''_{-i}$, $g^i(x'_{-i})$ is weakly lower than $g^i(x''_{-i})$ in the induced set order.⁷ A correspondence $g^i : X_{-i} \rightrightarrows X_i$ is *strictly decreasing* if, for $x'_{-i} \prec x''_{-i}$, $g^i(x''_{-i})$ is strictly lower than $g^i(x'_{-i})$. A correspondence $g^i : X_{-i} \rightrightarrows X_i$ is *strictly increasing* if, for $x'_{-i} \prec x''_{-i}$, $g^i(x'_{-i})$ is strictly lower than $g^i(x''_{-i})$. In the setting of a lattice game below, these correspondences will be interpreted as best responses.⁸

1.4. Strategic complements and strategic substitutes. The notions of strategic complements and strategic substitutes are formalized in the following way: player i has *strategic substitutes* if f^i is quasisupermodular in x_i and has the DSCP in $(x_i; x_{-i})$, which is true if and only if g^i is nonincreasing in x_{-i} . Similarly, player i has *strategic complements* if f^i is quasisupermodular in x_i and has the SCP in $(x_i; x_{-i})$, which is true if and only if g^i is nondecreasing in x_{-i} .

To define a stronger strategic relationship, player i has *strict strategic substitutes* if for $x'_{-i} \preceq x''_{-i}$, $g^i(x''_{-i})$ is *strictly lower* than $g^i(x'_{-i})$. Player i has *strict strategic complements* if for $x'_{-i} \prec x''_{-i}$, $g^i(x'_{-i})$ is *strictly lower* than $g^i(x''_{-i})$.⁹ Notice that the case where a player's best response is a constant function can be viewed as either

⁷When not explicitly stated, it is assumed that a nonincreasing or nondecreasing correspondence is so with respect to the induced set order.

⁸When games have singleton-valued best responses, the definitions above simplify to the standard definitions of a nonincreasing or nondecreasing function, and a strictly increasing or strictly decreasing function.

⁹In the most general lattice framework, the concept of a strictly increasing or strictly decreasing cannot be equated with an application of its related version of the strict SCP. Shannon (1995) demonstrates that if payoff function f^i is strictly quasisupermodular in x_i , and satisfies the strict DSCP in $(x_i; x_{-i})$, then for $x'_{-i} \preceq x''_{-i}$, $g^i(x''_{-i})$ is *completely lower* than $g^i(x'_{-i})$; here, the strictness of the inequality cannot be guaranteed under general conditions. This is discussed in Edlin and Shannon (1998), who give differentiable conditions on finite-dimensional Euclidean spaces for which this strictness is upheld.

strategic complements or strategic substitutes. Therefore, in a lattice game with strategic complements, strategic substitutes may be introduced trivially by having some players with constant best response functions. Using the strict versions of the properties given here allows the analysis to sidestep this undesirable situation.

1.5. Parametrized lattice games. Let I be a non-empty set of players, and for each player i , define a strategy space that is a partially ordered set, denoted (X^i, \preceq^i) , and a real-valued payoff function, denoted $f^i(x_i, x_{-i})$. As usual, the domain of each f^i is the product of the strategy spaces, (X, \preceq) , endowed with the product order. The strategic game $\Gamma = (I, (X^i, \preceq^i, f^i)_{i \in I}, T)$ is a *parametrized lattice game* if:

- i.) $\forall i, X_i$ is a non-empty complete lattice;
- ii.) T is a partially ordered set;
- iii.) $\forall i, f^i : X \times T \rightarrow \mathbb{R}$ is order continuous in x_i .¹⁰

Notice that in a parametrized lattice game, no restriction is placed on whether players have strategic complements or strategic substitutes. Consequently, this definition allows for general games with strategic complements, general games with strategic substitutes, and mixtures of the two. The best response correspondence for player i is given by $g^i : X_{-i} \times T \rightrightarrows X_i$, and the joint best response correspondence is given by $g : X \times T \rightrightarrows X$.¹¹ As usual, the set of pure strategy Nash equilibria *at a given parameter level* t is given by the set of fixed points of g , denoted by $FP(t)$ or $\mathcal{E}(t)$.

I will characterize how payoffs to individual players respond to changes in the parameter t with the single crossing property. Player i 's action is *weakly complementary* to parameter t if f^i has the single crossing property in $(x_i; t)$, which is true if and only

¹⁰In the standard order interval topology, which takes as its subbasis for the closed sets all closed rays of the form $(-\infty, b]$ and $[a, +\infty)$. A well-known result via Birkhoff (1967) is the following: a lattice X is compact in its interval topology if and only if it is complete. See Birkhoff (1967) or Erne (1980a) for more details.

¹¹Throughout the paper, both BR^i and g^i will be used to denote the best response function or correspondence for player i . While one is a more direct acronym, the other is often more convenient and streamlined.

if g^i is nondecreasing in t ,¹² player i 's action is *strictly complementary* to parameter t if f^i is strictly quasimodular in x_i and has the strict single crossing property in $(x_i; t)$, which is true if and only if $g^i(x_{-i}; t)$ is *weakly completely increasing*¹³ in t . Notice the second definition here utilizes the Shannon (1995) result relating strict quasimodularity and strict SCP to the best response correspondence. Definitions for best responses which are nonincreasing with respect to t can be analogously defined using the decreasing single crossing property.

1.6. Introductory examples. Here, I present three examples of common games which exhibit both strategic substitutes and strategic complements at the same time. These examples are chosen for their simplicity, to demonstrate the prevalence of these types of games. As mentioned in the introduction, there are many instances where these types of strategic situations can be found in the literature, from investment and entry deterrence (Fudenberg and Tirole (1984)) to innovation in vertically integrated markets (Bhattacharya (1990)), from team project games (Roy and Sabarwal (2010b)) to contests (Dixit (1987)).

Example: Odds and evens

The simplest example of a game for which players can have strategic substitutes and strategic complements at once is the classic Matching Pennies game, or what I term Odds and Evens (as shown in Figure 1).

Strategy spaces for each player are $S_1 = S_2 = \{a, b\}$, where $s_i = a$ corresponds to playing the number 1 and $s_i = b$ corresponds to playing the number 2. This setup establishes an order on the strategy spaces, \succeq , such that $b \succeq a$. Therefore, playing b can be seen as *increasing* one's choice, while playing a decreases the choice (again, with respect to the given order \succeq).

¹²I will use two expressions, interchangeably, to denote the best response of player i at a given t : $g^i(x_{-i}, t)$ and $g_t^i(x_{-i})$. They are equivalent.

¹³A correspondence $g : T \rightrightarrows X$ is *weakly completely increasing*, if for every $t' \prec t''$, $g(t')$ is completely lower than $g(t'')$.

		Player 2	
		a	b
Player 1	a	-1, 1	1, -1
	b	1, -1	-1, 1

FIGURE 1. A game of Odds and Evens.

Player 1 obtains a positive payoff when the sum of strategies played is *odd*, while Player 2 obtains a positive payoff when the sum is *even*. Notice the differing objectives in this simple example: Player 1 always responds optimally by acting in the opposite direction of Player 2. If s_2 increases from a to b , s_1 would optimally decrease his action and play a . On the other hand, Player 2 wants to coordinate his action with the action of player 1. So, an increase of s_1 from a to b will motivate Player 2 to also increase his action to b . In this way, Player 1 exhibits strategic substitutes, while Player 2 exhibits strategic complements - and in fact, these properties are strict for both players.¹⁴

Example: Simultaneous Cournot and Bertrand competition

A second example comes from Singh and Vives (1984). They compare the standard Bertrand (price as strategic variable) and Cournot (quantity as strategic variable) competition cases, and discuss what occurs in a 2-firm game when one firm chooses price while his rival chooses quantity. They remark on the duality between Cournot and Bertrand competition; namely, Cournot (Bertrand) competition with substitutes

¹⁴I acknowledge the limitations of this example, in that the Odds and Evens game has no equilibrium, while the context of the ensuing results typically assume the existence of an equilibrium. Nevertheless, the example helps to establish the presence of this class of games in even simple, 2x2 examples. See Monaco and Sabarwal (2011) for an extension of the Matching Pennies example for which there are pure strategy Nash equilibria.

is the dual of Bertrand (Cournot) competition with complements, in the sense that from one scenario, the equilibrium strategies and reaction functions of the dual can be recovered.

Using this fact, the authors analyze the case where one firm, say firm 1, strategically chooses prices, while his opponent firm 2 chooses quantities. The firms then compete in the market for a differentiated good. Given the duality between the Bertrand and Cournot cases, and given linear inverse demand curves $p_1 = \alpha_1 - \beta_1 q_1 - \gamma q_2$ and $p_2 = \alpha_2 - \gamma q_1 - \beta_2 q_2$, the firms can maximize profit functions in the following way, where q_1 and p_2 are both formulated as functions of p_1 and q_2 :

$$\Pi_1 = p_1 q_1 = p_1 q_1(p_1, q_2)$$

$$\Pi_2 = p_2 q_2 = p_2(p_1, q_2) q_2$$

By substituting out q_1 and p_2 , familiar reaction functions are observed:

$$p_1^* = \frac{\alpha_1 - \gamma q_2}{2}$$

$$q_2^* = \frac{\alpha_2 \beta_1 - \alpha_1 \gamma + \gamma p_1}{2(\beta_1 \beta_2 - \gamma^2)}$$

The reaction function for firm 1 resembles the standard Cournot reaction function, while the firm 2 reaction function resembles that from the Bertrand case, given an appropriate change of variable.¹⁵ It is clear, therefore, that for $\gamma > 0$, the reaction functions slope in opposite directions, as shown below.

Example: Cournot markets with spillovers

Suppose two Cournot firms, an incumbent (firm 1) and an entrant (firm 2), are competing in quantities, x_1 and x_2 . Suppose, in addition, that there is a one-way spillover from the incumbent to the entrant, in the form of a defection of some

¹⁵As per the authors, parameter changes allow for the expression $q_2^* = \frac{a_2 + c p_1}{2}$, where, briefly, $a_2 = \frac{\alpha_2 \beta_1 - \alpha_1 \gamma}{\beta_1 \beta_2 - \gamma^2}$, and $c = \frac{\gamma}{\beta_1 \beta_2 - \gamma^2}$.

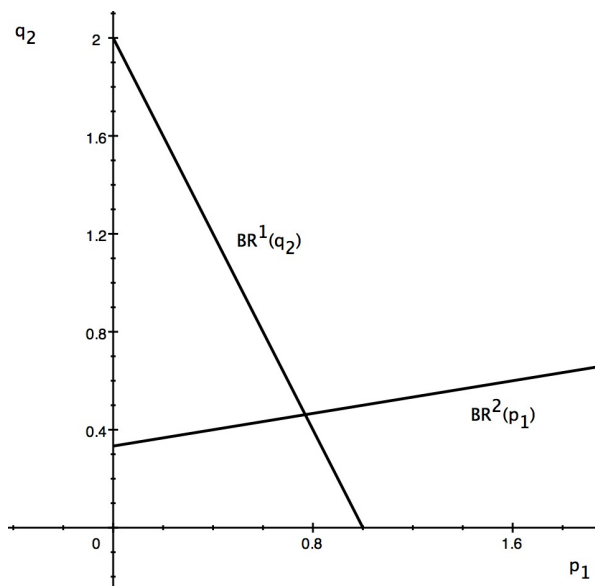


FIGURE 2. Duopoly with price and quantity competition.

employees who may bring with them some technological information from the incumbent; the impact of the spillover is to lower the cost to the entrant at high levels of output.

Let the price in the market be determined by $p = a - b(x_1 + x_2)$. Firm 1 has a constant marginal cost, but firm 2's costs include the spillover in terms of a function $s(x_1)$. We can therefore give the profit functions of each firm in the following way:

$$\begin{aligned}\pi_1 &= (a - b(x_1 + x_2))x_1 - c_1x_1 \\ \pi_2 &= (a - b(x_1 + x_2))x_2 - c_2x_2s(x_1)\end{aligned}$$

Using the usual techniques, the best responses can be found:

$$\begin{aligned}BR^1(x_2) &= \max\left\{\frac{a - bx_2 - c_1}{2b}, 0\right\} \\ BR^2(x_1) &= \max\left\{\frac{a - bx_1 - c_2s(x_1)}{2b}, 0\right\}\end{aligned}$$

Here, I take a particular form for the spillover function to reflect this transfer of technological knowledge: $s(x_1) = -\frac{1}{8}x_1^{1.4}$. Given parameter values $a = 15$, $b = 1$, $c_1 = 1$, $c_2 = 10$, there is an equilibrium at $(2.678, 8.643)$, and Figure 3 demonstrates

the opposing strategic interaction between firms. The incumbent firm, who enjoys no spillover effect, exhibits *strategic substitutes*, while the entrant firm has *strategic complements*.

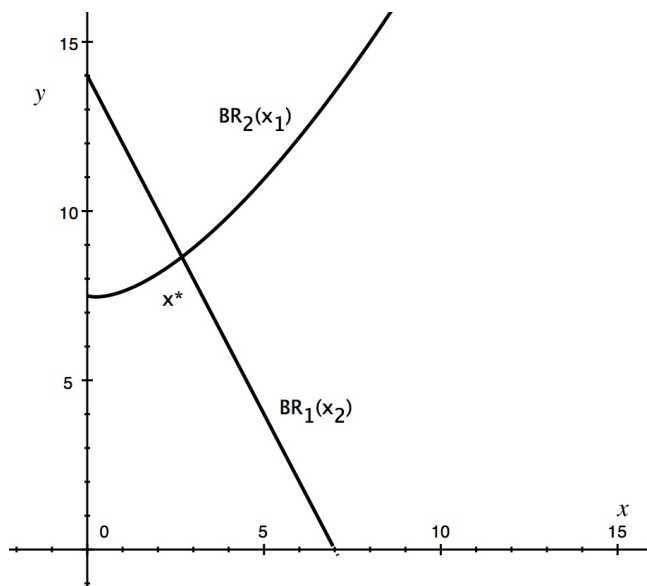


FIGURE 3. Equilibrium in Cournot market with spillovers.

In fact, Tombak (2006) provides a detailed application of one such scenario where these learning effects seem particularly prevalent: the aircraft industry. Tombak discusses the relationship between an existing firm (Boeing) and an entrant firm (Airbus), using a precommitment-type approach similar to that of Fudenberg and Tirole (1984). In doing so, he depicts the market between these two firms as characterized by *strategic asymmetry* which arises from strong learning effects particular to this industry; as such, at the time of Airbus' entry, the best response function for Boeing was downward sloping, while the best response function for Airbus was upward sloping, just as in the above example. Tombak (2006) provides several additional examples of industries with learning curve or experience curve effects, which may lead to similarly asymmetric scenarios, including nuclear power, chemical processing, and shipbuilding.

1.7. Related Result. There is a related result which considers a similarly general class of lattice games. Monaco and Sabarwal (2011) demonstrate a non-robustness with respect to the order structure of equilibria sets in lattice games. In contrast to the standard results for GSC (see Milgrom and Roberts (1990) or Vives (1990)), they show any ordered properties of the set of equilibria are not maintained with even minimal introduction of strategic substitutes; in particular, no two distinct equilibria of a given lattice game are comparable if either:

- there exists at least one player with strict SS and at least one player with strict SC;
- or, there exists at least one player with strict SS for whom the best response is singleton-valued.

So, even in a framework which allows for both strategic substitutes and strategic complements, it is possible to obtain a result with respect to the ordered properties of the set of equilibria. This is useful to note here, as the proceeding work solidifies the collective theory on such a general class of games.

2. NOWHERE DECREASING EQUILIBRIA

The first development gives minimal conditions on the primitives of a parametrized lattice game which guarantee the set of equilibria is **nowhere decreasing** in the parameter t . These results provide an intuitive interpretation: given an increase in the parameter, no equilibrium of the new game (at the higher parameter) can be lower than an equilibrium of the game at the original (lower) parameter.

Theorem 1: Define a parametrized lattice game $\Gamma = (I, (X^i, \preceq^i, f^i)_{i \in I}, T)$, such that for each player i , f^i has SCP in $(x_i; t)$, for fixed x_{-i} . Suppose one of the following conditions is satisfied:

- i.) At least one player has strict strategic substitutes, and let this player's best response be singleton-valued;
- ii.) at least one player has strict strategic substitutes, and let this player's payoff be strictly complementary in t ;

iii.) at least two players have strict strategic substitutes.

Then, for all $t^* \in T$, $\hat{t} \in T$, $t^* \preceq \hat{t}$, for any $x^* \in FP(t^*)$ and any $\hat{x}^* \in FP(\hat{t})$, $x^* \neq \hat{x}^*$, we have $\hat{x}^* \not\prec x^*$.

Proof. The proof is provided in the Appendix.

This theorem has a wide array of applications. Two examples are presented below: a variation of the common Cournot oligopoly case with non-monotonic spillovers, and a 3-player group project game. It is noteworthy that in these examples, equilibria are non-decreasing, but they are not necessarily increasing, which highlights the usefulness of this result: while weaker than an “increasing equilibria” result like those in the ensuing sections, may be the best one can do in many applications.

Example: Cournot market with non-monotone spillovers

In a variant of the above model in section 1, suppose two firms are both existing firms (and therefore may not have an entrant/incumbent relationship) competing in quantities. Here, however, we consider a spillover function for firm 2 $s(x_1)$ which is non-monotone; the spillover increases firm 2’s costs at every level of output, but this negative impact is decreasing at lower levels of x_1 , and increasing as firm 1’s output increases. Notice that condition i.) of Theorem 1 is satisfied, since firm 1 has a strictly decreasing and singleton-valued best response.¹⁶

Beginning with the same profit functions, it is clear best responses are the same:

$$BR^1(x_2) = \max\left\{\frac{a - bx_2 - c_1}{2b}, 0\right\}$$

$$BR^2(x_1) = \max\left\{\frac{a - bx_1 - c_2s(x_1)}{2b}, 0\right\}$$

only here, define $s(x_1) = \frac{2}{3}x_1^3 - x_1^2 - \frac{x_1}{2} + 3$. Given parameter values $a = 15$, $b = \frac{1}{2}$, $c_1 = 11$, $c_2 = 3$, there are three equilibria of the Cournot game: $x^* = (2, 4)$,

¹⁶In fact, condition ii.) is satisfied as well, albeit trivially, since for a player with singleton-valued best responses, strict complementarity in $t \Leftrightarrow$ weak complementarity in t .

$y^* = (\frac{1}{2}, 7)$, and $z^* = (4, 0)$. Now, if the complementary parameter a increases from 15 to 17, both best responses increase, and there are 3 new equilibria: $\hat{x}^* = (1.686, 8.627)$, $\hat{y}^* = (1, 10)$, and $\hat{z}^* = (6, 0)$. Firstly, it is clear that no equilibrium has decreased, so the above property is shown. However, while y^* and z^* have increased, x^* has not. Therefore, while the nondecreasing property holds here, a declaration of “increasing” equilibria cannot be made. The figure below demonstrates the result, with the original best responses in black, and increased best responses in blue.

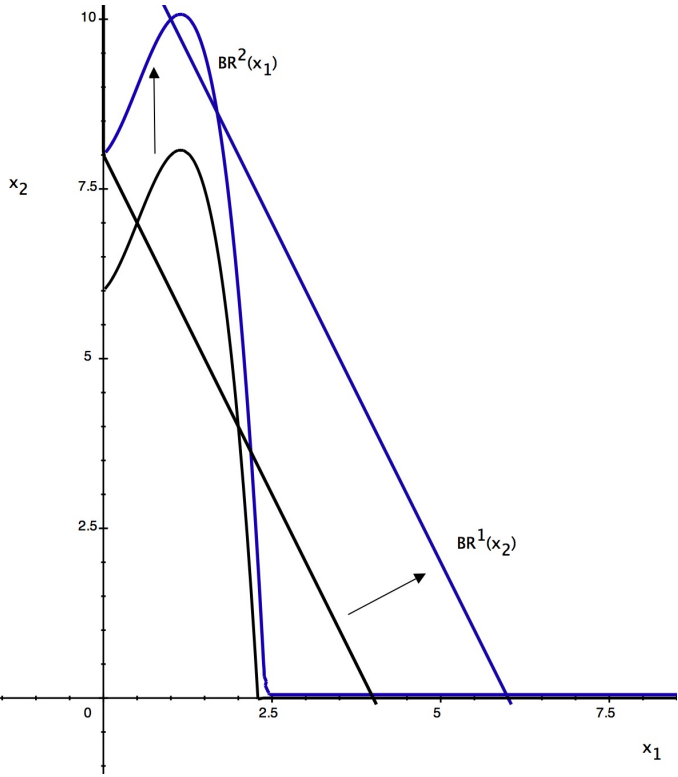


FIGURE 4. Equilibria after increase in a .

Example: Group project game

Examine a group project game between three players:¹⁷ two of the players, say players 1 and 2, are able to complete the project by themselves, while player 3 cannot.

¹⁷This game comes from Sabarwal and Roy (2008).

Any players who successfully complete the project receive some reward, $r(t)$, which is shared among the successful players. For example, should all three players succeed, they each receive a share of the reward, $\frac{r(t)}{3}$.

Strategies for each player are levels of effort $x_i \in [0, 1]$, where x_i also indicates the probability of success. Players have a quadratic cost of effort given by $\frac{c_i x_i^2}{2}$. The expected profit for each player, π_i , is given below, with π_1 symmetric to π_2 :

$$\pi_1 = r(t)x_1(1-x_2)(1-x_3) + \frac{r(t)}{2}x_1x_2(1-x_3) + \frac{r(t)}{2}x_1(1-x_2)x_3 + \frac{r(t)}{3}x_1x_2x_3 - \frac{c_1x_1^2}{2}$$

$$\pi_3 = \frac{r(t)}{2}x_1(1-x_2)x_3 + \frac{r(t)}{2}(1-x_1)x_2x_3 + \frac{r(t)}{3}x_1x_2x_3 - \frac{c_3x_3^2}{2}$$

Best responses are functions here, and each player's best response is expressed as:

$$x_1^* = BR^1(x_{-1}) = \min\left\{\frac{r(t)}{c_1}\left(1 - \frac{1}{2}(x_2 + x_3) + \frac{1}{3}x_2x_3\right), 1\right\}$$

$$x_2^* = BR^2(x_{-2}) = \min\left\{\frac{r(t)}{c_2}\left(1 - \frac{1}{2}(x_1 + x_3) + \frac{1}{3}x_1x_3\right), 1\right\}$$

$$x_3^* = BR^3(x_{-3}) = \min\left\{\frac{r(t)}{c_3}\left(\frac{1}{2}(x_1 + x_2) - \frac{2}{3}x_1x_2\right), 1\right\}$$

With explicit best response functions, it is straightforward to observe that both players 1 and 2 have strict strategic substitutes, which means condition iii.) of Theorem 1 is satisfied. Also notice that the reward $r(t)$ is parametrized by some t ; for $r(t)$ increasing in t , all three players' payoffs have the single crossing property in t . According to Theorem 1, as t increases, one should not observe any decreasing equilibria. To examine further, suppose the reward is constant with respect to t , such that $r(t) = r$ (and thus, the parameter is now r); in addition, suppose for positive constants \bar{c} and c , $c_i = \bar{c} - \frac{2}{3}cx_i$, for $i = 1, 2$. Cost of effort for player 3 remains the same. Using parameter values $r = 0.9$, $\bar{c} = 0.984$, $c = 0.6$, $c_3 = 0.48$, observe the best responses in the figure below,¹⁸ where ABC gives the best response of players 1 and 2, and DE gives the best response of player 3. There is one equilibrium of the game: $(0.7623, 0.7623, 0.7028)$. Now, increasing the reward such that $r = 1$, observe three equilibria, none of which are lower (using the standard product order)

¹⁸Since this game is symmetric between players 1 and 2, x_1 and x_2 will be plotted on the same axis in the figures.

than the equilibrium at $r = 0.9$: $(0.906, 0.906, 0.7474)$, $(0.9957, 0.9957, 0.6954)$, and $(1, 1, 0.6944)$. This is consistent with Theorem 1: although only one of the 3 equilibria is greater than the equilibrium at $r = 0.9$, none are lower.

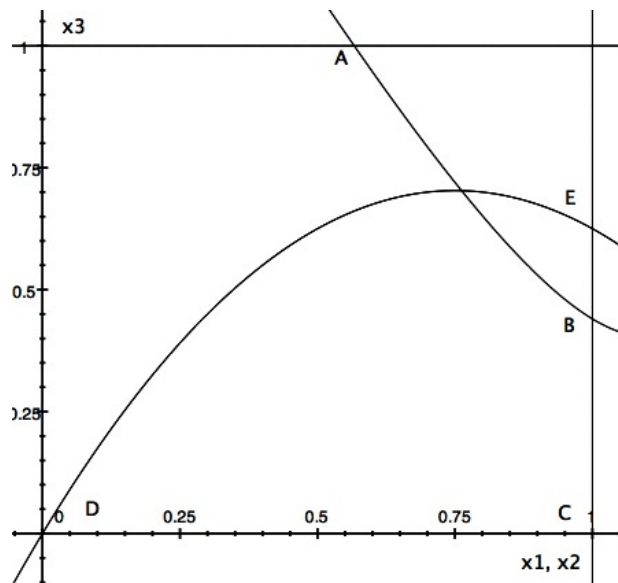


FIGURE 5. One equilibrium at $r = 0.9$.

3. INCREASING EQUILIBRIA: 2-PLAYER GAMES

At this point, it is useful to examine where individual equilibria of a parametrized lattice game are not just *nondecreasing* in the parameter, but are in fact *increasing* in the parameter. This is a logical extension of the previous result, and addresses an important gap in the literature. While monotone comparative statics results exist in GSC and GSS, the general nature of the framework of this paper permits a much broader class of games not covered in the above cases. Additionally, there are many applications for which it is of interest to determine when an increase in parameter causes equilibria of the game to increase. This section initiates the approach by analyzing two-player games.

3.1. Linearly ordered strategy spaces. Begin with a 2-player, parametrized lattice game, where players' strategy spaces (X_1 and X_2 , respectively,) are linearly

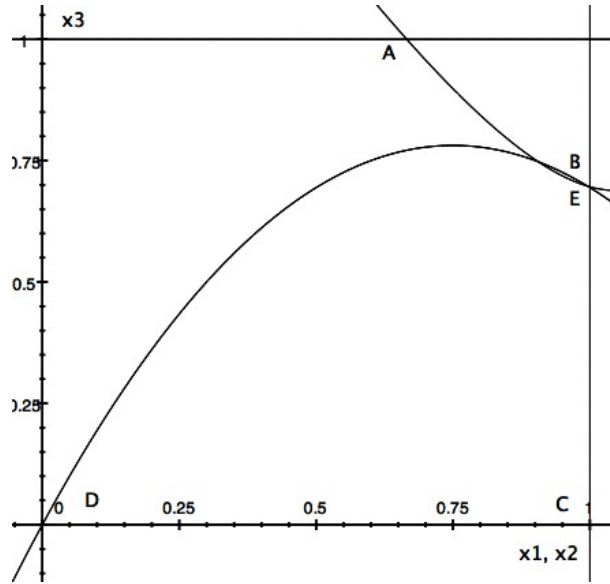


FIGURE 6. Three equilibria at $r = 1.0$, none of which are lower.

ordered. Here, I give one player (say, player 1) strict strategic substitutes, and give the other player, player 2, strategic complements. Notice that the best response function for player 1, $g^1(\cdot, t)$, is (strictly) decreasing in x_2 , and the best response function for player 2, $g^2(\cdot, t)$, is increasing in x_1 .¹⁹ Also, I consider (as above) that best responses are increasing in the parameter t . Here, suppose an equilibrium exists at $x^* = (x_1^*, x_2^*) = (g^1(x_2^*, t^*), g^2(x_1^*, t^*)) = (g_{t^*}^1(x_2^*), g_{t^*}^2(x_1^*))$. This setup is useful to gain an intuition on these types of games, and can be useful for the reader to keep in mind in the sections which follow. Figure 7 below provides a visual representation.²⁰

In this setting, the question of concern can be formulated as follows: given this setup, and given an equilibrium x^* at some parameter t^* , examine an increase in the parameter to, say, $\hat{t} \succ t^*$. Best responses will both increase at \hat{t} , but since the relative sizes on these increases may differ, the new equilibrium may not have increased. So, if there is an equilibrium (say, \hat{x}^*) at this higher parameter value, when will this

¹⁹Here, best responses are singleton-valued. The 2-player game analysis will consider best responses which are correspondences later in this section.

²⁰While the figure gives *linear* best responses for simplicity, the same intuition can be gained from a graph of non-linear best responses.

new equilibrium be higher than the original equilibrium? In other words, when is $\hat{x}^* \succeq x^*$?

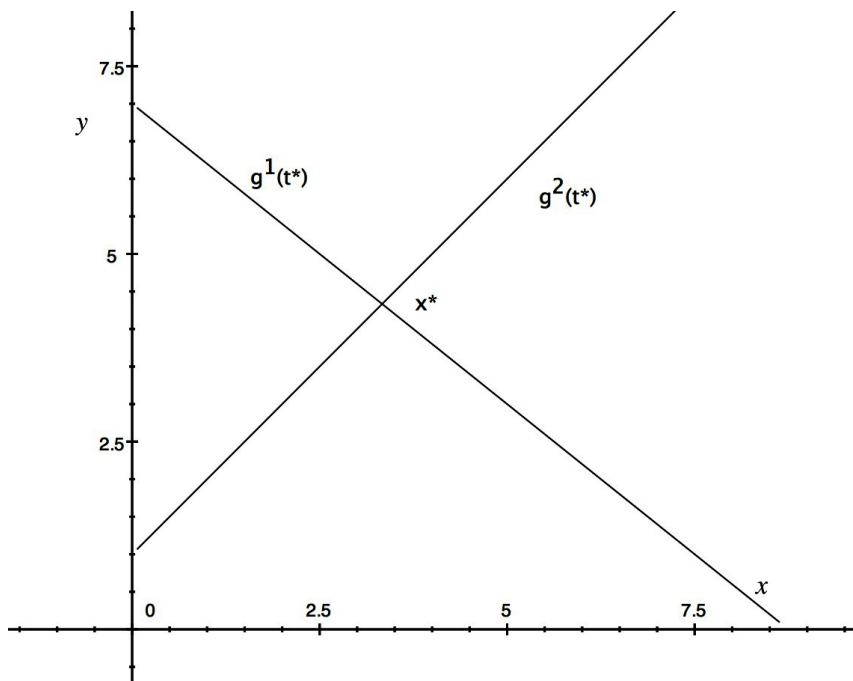


FIGURE 7. Intuitive, 2-player game.

Firstly, as the non-decreasing result above shows, when the parameter increases, $\hat{x}^* \neq x^*$. This leaves three potential regions of the joint strategy space in which the new equilibrium can lie, denoted by A, B, and C in Figure 8 below. If the new equilibrium falls in area A, then $\hat{x}_1^* \prec x_1^*$ and $\hat{x}_2^* \succeq x_2^*$; if the new equilibrium falls in area B, then $\hat{x}_1^* \succeq x_1^*$ and $\hat{x}_2^* \prec x_2^*$; if the new equilibrium falls in area C, then $\hat{x}_1^* \succeq x_1^*$ and $\hat{x}_2^* \succeq x_2^*$ (so, $\hat{x}^* \succeq x^*$). In the area C case, it can be said the equilibrium increases.

Since both best responses are increasing in the parameter t , when $t \uparrow$ (from t^* to \hat{t}), both g^1 and g^2 will shift upward. As a result of these shifts, it seems as though the equilibrium strategy for the strategic complements player cannot strictly decrease.

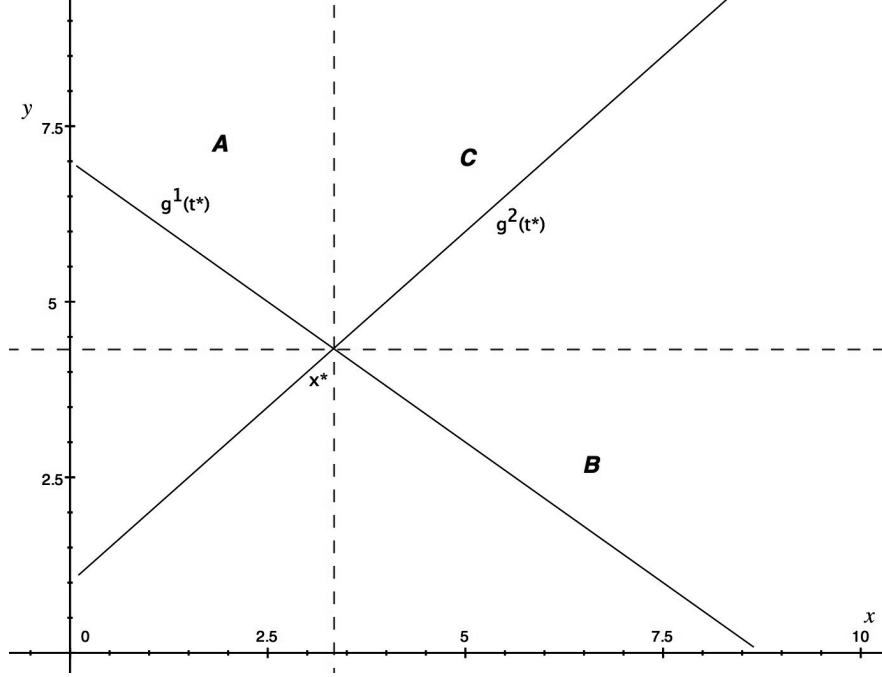


FIGURE 8. Intuitive, 2-player game. Areas A, B, and C are indicated.

Indeed, this is the case:

Lemma 1: Suppose in a 2-player parametrized lattice game, strategy spaces are linearly ordered, player 1 has strict strategic substitutes while player 2 has strategic complements (WLOG), best response functions are increasing in the parameter, and there exists an equilibrium denoted $x^* = (x_1^*, x_2^*) = g_{t^*}(x^*)$. Given x^* , suppose $\exists \hat{x}^* = g_{\hat{t}}(\hat{x}^*)$ for some $\hat{t} \succ t^*$. Then, $\hat{x}_2^* \succeq x_2^*$.

Proof. Since $\hat{x}_1^* = g_{\hat{t}}^1(\hat{x}_2^*)$ exists by hypothesis, and since strategy spaces are linearly ordered, consider 2 cases:

- (i.) $\hat{x}_1^* \succeq x_1^* \implies \hat{x}_2^* = g_{\hat{t}}^2(\hat{x}_1^*) \succeq g_{\hat{t}}^2(x_1^*) \succeq g_{t^*}^2(x_1^*) = x_2^*$.
- (ii.) $\hat{x}_1^* \prec x_1^*$. Here, since by Theorem 1, the equilibrium at $\hat{t} \succ t^*$ cannot be lower than the equilibrium at t^* , it must be such that $\hat{x}_2^* \not\prec x_2^* \implies \hat{x}_2^* \succeq x_2^*$. ■

When the parameter increases, and both best responses shift, there are two effects on the optimal strategy for player 2: a *direct effect*, which implies that for a higher t , the optimal x_2 will be higher for any level of x_1 ; and a *strategic effect*, which implies that when player 1 increases his action (in response to the increase in t), due to strategic complements, player 2 will also want to increase x_1 . In player 2's case, both effects work in the same (increasing) direction, which leads to the unambiguous result that x_2^* will not go down. So, area B can be ruled out as a potential location for \hat{x}^* .

Following a similar argument, it is clear why the case for player 1 may not be as straightforward. The direct effect here will cause x_1 to increase for any level of x_2 in response to the increase in t ; however, the strategic (or indirect) effect works in the *opposite* direction. The parameter increase will cause player 2's best response to shift upward, but this increase in x_2 will push the optimal response of player 1 *downward*, due to strategic substitutes.

One way of viewing the opposing forces here is to notice that for $\hat{x}_1^* \succeq x_1^*$ to be true, the (increasing) direct effect of $t \uparrow$ should outweigh or dominate, in some sense, the (decreasing) strategic effect of $t \uparrow$. If so, then in tandem with the result of Lemma 1, $\hat{x}^* \in C$. If not, then $\hat{x}^* \in A$. Theorem 2 below utilizes this fact, and demonstrates that a characterization on these effects, in terms of player 1's second iterate best response, is equivalent to an increased equilibrium.

Theorem 2: Suppose in a 2-player parametrized lattice game, strategy spaces are linearly ordered, player 1 has strict strategic substitutes while player 2 has strategic complements (WLOG), best response functions are increasing in the parameter, and there exists an equilibrium denoted $x^* = g_{t^*}(x^*)$. Given x^* , suppose $\exists \hat{x}^* = g_{\hat{t}}(\hat{x}^*)$ for some $\hat{t} \succ t^*$. Then the following equivalence is true:

$$g_{\hat{t}}^1(g_{\hat{t}}^2(x_1^*)) \succeq x_1^* \iff \hat{x}^* \succeq x^*.$$

Proof. (\Rightarrow) By Lemma 1, $\hat{x}_2^* \succeq x_2^*$. By hypothesis, $g_{\hat{t}}^1(g_{\hat{t}}^2(x_1^*)) \succeq x_1^* \implies g_{\hat{t}}^1(g_{\hat{t}}^2(x_1^*)) \not\prec x_1^* \implies \hat{x}_1^* \not\prec x_1^*$. This last implication follows since $\hat{x}_1^* \prec x_1^* \implies \hat{x}_2^* \preceq g_{\hat{t}}^2(x_1^*) \implies \hat{x}_1^* \succeq$

$g_t^1(g_t^2(x_1^*))$, where $x_1^* \succ \hat{x}_1^* \implies x_1^* \succ g_t^1(g_t^2(x_1^*))$; that is, $\hat{x}_1^* \prec x_1^* \implies g_t^1(g_t^2(x_1^*)) \prec x_1^*$. Now, $\hat{x}_1^* \not\prec x_1^* \iff \hat{x}_1^* \succeq x_1^*$, by linearly ordered X_1 . Therefore, $\hat{x}^* \succeq x^*$.

(\Leftarrow) $\hat{x}^* \succeq x^* \implies \hat{x}_1^* \succeq x_1^* \implies g_t^2(\hat{x}_1^*) \succeq g_t^2(x_1^*) \implies g_t^1(g_t^2(\hat{x}_1^*)) \preceq g_t^1(g_t^2(x_1^*)) \iff \hat{x}_1^* \preceq g_t^1(g_t^2(x_1^*))$. Since by hypothesis, $x_1^* \preceq \hat{x}_1^*$, $x_1^* \preceq g_t^1(g_t^2(x_1^*))$. ■

The intuition here is fundamental: in parametrized lattice games where both strategic substitutes and strategic complements are present, in response to an increase in a complementary parameter t , only the strategic substitutes player experiences opposing (and therefore, ambiguous) effects of the increase. The *direct effect* causes him to want to increase his optimal strategy; this is captured by the $g_t^1(\cdot)$ since $\hat{t} \succ t^*$. Yet, the *strategic effect* causes him to want to decrease his optimal strategy, where this effect is observed since $g_t^2(x_1^*) \succeq x_2^*$ causes player 1's best response at $g_t^2(x_1^*)$ to be lower than his best response at x_2^* . Theorem 2 demonstrates that in this 2-player setup, the incidence of an increasing equilibrium is equivalent to checking one condition on the second-iterate of the strategic substitutes player's best response at the new parameter value. The second iterate captures these two opposing effects, and can be used to determine which effect dominates.

It is worthy to note the relationship between this result and Theorem 1 in Roy and Sabarwal (2010b). In games with strategic substitutes, the second iterate of the joint best response is known to have many useful properties.²¹ Roy and Sabarwal (2010b) obtain an increasing equilibrium result by characterizing this second iterate in a similar fashion to Theorem 2. Since in the GSS framework, all players have strategic substitutes, this implies that all strategic substitutes players must reconcile their direct and strategic effects in the same way shown here. There is only one such player in the case of Theorem 2 here, so the only required condition is the second iterate characterization on this player. Therefore, the result here highlights a relationship with the literature on games with strategic substitutes, and stresses the central role of any player with strategic substitutes to achieve monotone comparative

²¹See Zimper (2007) and Roy and Sabarwal (2010a) for the role the second iterate plays in rationalizability, dominance solvability, and convergence of best response and adaptive dynamics in GSS.

statics within this general framework. In fact, this helps justify why, in the ensuing results, the only necessary characterizations will be on the best responses of strategic substitutes players.²² Below is an application of the theorem.

Example: Simultaneous Cournot and Bertrand competition

To see an example applying Theorem 2, consider the model of Singh and Vives, presented in section 1.6. In particular, let $\alpha_1 = \alpha_2$, which gives best responses as:

$$g_\alpha^1(q_2) = \frac{\alpha - \gamma q_2}{2}$$

$$g_\alpha^2(p_1) = \frac{\alpha(\beta_1 - \gamma) + \gamma p_1}{2(\beta_1 \beta_2 - \gamma^2)}$$

Given the Singh and Vives assumptions on parameters, namely, that $\beta_1 - \gamma > 0$ and $(\beta_1 \beta_2 - \gamma^2) > 0$, we can say that both best responses are increasing in the common demand parameter α . Then for initial parameter values $\alpha = 2$, $\beta_1 = \beta_2 = 2$, $\gamma = 1$, $g_\alpha^1(q_2) = \frac{2 - q_2}{2}$, $g_\alpha^2(p_1) = \frac{2 + p_1}{6}$, the goods are substitutes, and there is an equilibrium at $(p_1^*, q_2^*) = (\frac{2}{3}, \frac{2}{5})$. Given an increase in the complementary parameter from $\alpha = 2$ to $\hat{\alpha} = 3$, checking for an increasing in equilibrium via Theorem 2 is equivalent to checking the following condition: $g_{\hat{\alpha}}^1(g_{\hat{\alpha}}^2(p_1^*)) \geq p_1^*$. Since $g_{\hat{\alpha}}^2(p_1^*) = \frac{11}{18}$, and $g_{\hat{\alpha}}^1(\frac{11}{18}) = \frac{43}{36} \geq \frac{2}{3}$, the condition is satisfied, and as predicted, the equilibrium increases as shown below: $(\hat{p}_1^*, \hat{q}_2^*) \approx (1.15, 0.69)$.

A noteworthy extension of Theorem 2 can be made in the case where both players are permitted to have set-valued best responses, with two modifications to the assumptions.

²²I acknowledge it would be preferred to have these characterizations on the *primitives* of the game (say, payoff functions,) rather than on best responses. However, the determination of an increased equilibrium depends on the initial equilibrium as well as the initial parameter level and the degree of its increase. The characterization on best responses allows for a straightforward calculation in applications where an equilibrium has been identified. In addition, as the 3-player crime and punishment application demonstrates in section 4, there may be some initial parameter values for which equilibria do *not increase*.

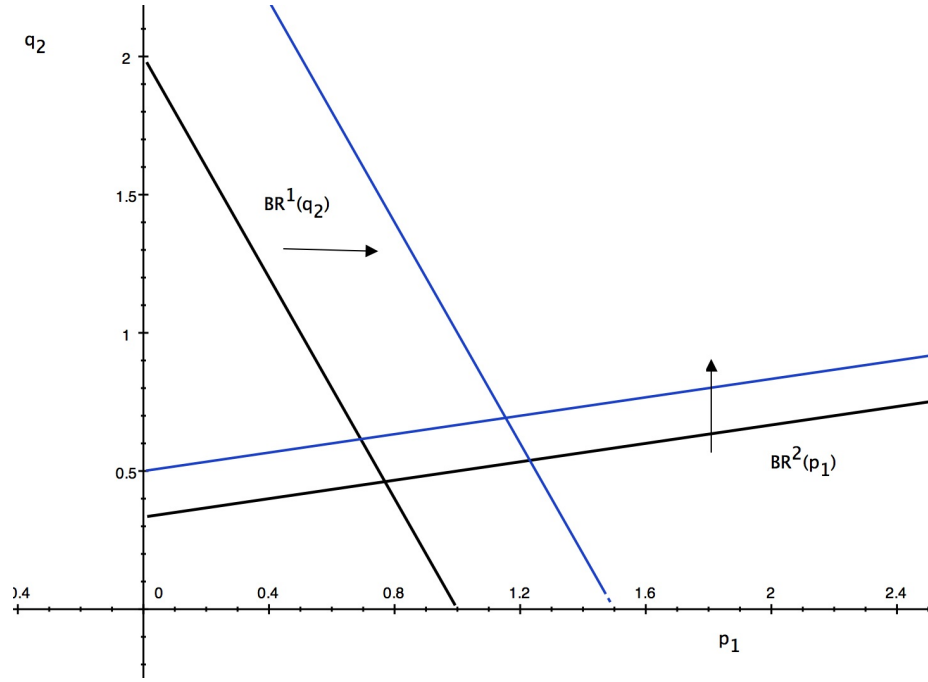


FIGURE 9. Equilibrium after increase in α .

Theorem 2*: Suppose in a 2-player parametrized lattice game, strategy spaces are linearly ordered, player 1 has strict strategic substitutes while player 2 has *strict* strategic complements (WLOG), best responses are correspondences, and x_1, x_2 complementary in the parameter, with x_1 *strictly* complementary. Suppose there exists an equilibrium $x^* \in g_{t^*}(x^*)$, and suppose $\exists \hat{x}^* \in g_{\hat{t}}(\hat{x}^*)$ for some $\hat{t} \succ t^*$. Then the following equivalence is true:

$$\begin{aligned} &\text{for some } x_1^* \in g_{t^*}^1(x_2^*), \exists \tilde{x}_1 \in g_{\hat{t}}^1(\inf g_{\hat{t}}^2(x_1^*)) \text{ such that } \tilde{x}_1 \succeq x_1^* \\ &\iff \\ &\hat{x}^* \succeq x^*, \forall \hat{x}^*, \forall x_2^*, \text{ where } x^* = (x_1^*, x_2^*). \end{aligned}$$

Proof. The proof is provided in the Appendix.

To gain some intuition about Theorem 2*, I want to demonstrate that the condition in the left-hand side of the equivalence is a fairly weak requirement on a second iterate of the best response of player 1 at \hat{t} . The primary difficulty introduced by

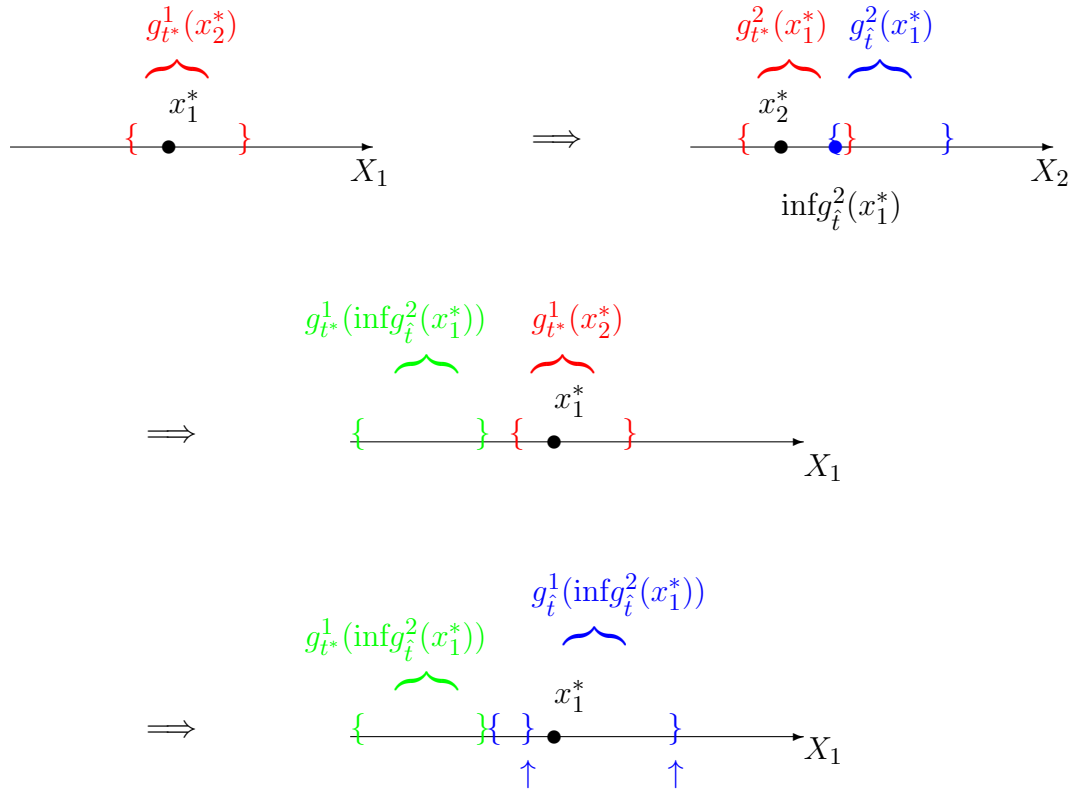
the presence of correspondences is potential overlap in players' best response sets (it is still possible that they do not overlap, but in this case, analysis is straightforward and similar to the function case). While forming the condition above in Theorem 2*, I address this by looking at the infima and suprema of the sets of interest. By taking the infimum of the set $g_t^2(x_1^*)$, when player 1 takes his best response to this at \hat{t} , strategic substitutes will cause this best response to be higher than any other best response to an element of $g_t^2(x_1^*)$. That is, the condition presents the *weakest strategic effect* possible. Similarly, by taking the supremum of $g_t^1(\cdot)$ in the proof of the theorem, I am allowing for the *strongest direct effect* possible. Recall that the strategic effect pushes the second iterate lower for player 1 (via strategic substitutes), while the direct effect pushes the second iterate higher. Hence, the left-hand side requirement is fairly weak in the sense that the equivalence holds while permitting the strategic (decreasing) effect to be minimal (by taking the inf), and allowing the direct (increasing) effect to be great.

To demonstrate in a picture below, take some x^* , and let $x_1^* \in g_{t^*}^1(x_2^*)$. This is indicated in red in the picture below, where the linearly ordered strategy space X_1 is represented by a linear spectrum. Notice that x_1^* may lie in the interior of the best response set. Now, when player 2 takes his best response to x_1^* at \hat{t} , denoted in blue in the second frame, this set will be higher than $g_{t^*}^2(x_1^*)$. While it is possible that $\text{inf}g_{\hat{t}}^2(x_1^*) \prec x_2^*$,²³I focus on the case of interest where $\text{inf}g_{\hat{t}}^2(x_1^*) \succeq x_2^*$.

Whether $\text{inf}g_{\hat{t}}^2(x_1^*)$ lies outside of $g_{t^*}^2(x_1^*)$ bears no weight here: due to strict strategic substitutes, when player 1 best responds (at t^*) to $\text{inf}g_{\hat{t}}^2(x_1^*)$, this set, marked in green below in the third frame, will now be strictly lower than $g_{t^*}^1(x_2^*)$. This is exactly the *strategic effect*, which, even while driven low, is kept as high as possible by player 1 considering the infimum of $g_t^2(x_1^*)$. Now, to consider the *direct effect*, recall that x_1 is strictly complementary in t , so $g_{\hat{t}}^1(\text{inf}g_{\hat{t}}^2(x_1^*))$, shown in blue in the final frame, is completely higher than $g_{t^*}^1(\text{inf}g_{\hat{t}}^2(x_1^*))$. It should now be clear how these two effects, strategic and direct, interact: if the direct effect is not strong enough, then the supremum of $g_{\hat{t}}^1(\text{inf}g_{\hat{t}}^2(x_1^*))$ will remain below x_1^* , as shown by the leftmost

²³This case is not of great interest here because, if this infimum lies sufficiently low (below x_2^* to be precise), the strategic effect impacting player 1 will actually be *positive*.

arrow, indicating the strategic effect dominates. In this instance, the equilibrium strategy will not increase for player 1. If, however, the direct effect dominates, and $\sup g_t^1(\inf g_t^2(x_1^*))$ is driven above x_1^* , then the condition of Theorem 2* will be met, and player 1 will always take a higher equilibrium strategy \hat{x}_1^* at \hat{t} .



PICTURE 1: Illustrated intuition of Theorem 2*.

3.2. Lattice strategy spaces. To extend this section on 2-player games, I consider a generalization which allows players to have strategy spaces which may not be linearly ordered. However, a simple second iterate characterization, like that of Theorem 2, will not be sufficient to guarantee increasing equilibria in this context. To see why, consider the following counterexample:

Counterexample:

Consider the game on the left of the figure below:

	a_2 (low)	b_2 (high)
a_1 (no crime)	1, <u>5</u>	<u>1</u> , 0
b_1 (GTA)	<u>10</u> , <u>2</u>	0, 0
c_1 (robbery)	3, <u>2</u>	0, 0
d_1 (GTA + robbery)	5, 0	0, <u>10</u>

	a_2 (low)	b_2 (high)
a_1 (no crime)	1, <u>5</u>	1, 0
b_1 (GTA)	10, <u>2</u>	0, 0
c_1 (robbery)	5, 2	<u>5</u> , <u>3</u>
d_1 (GTA + robbery)	<u>15</u> , 0	0, <u>15</u>

FIGURE 10. Original game on left; higher parameter game on right.

Player 1 (on the left side of the matrix) is a criminal, deciding whether or not to engage in criminal activity. a_1 corresponds to no criminal activity, b_1 corresponds to committing grand theft auto, c_1 corresponds to robbing a bank, and d_1 corresponds to committing both acts (GTA and bank robbery), with $a_1 \prec b_1 \prec d_1$, $a_1 \prec c_1 \prec d_1$, b_1 and c_1 unordered (and thus, X_1 is a lattice but not linearly ordered). Player 2 (on the top of the matrix) represents law enforcement, deciding between a level of low (a_2) or high (b_2) law enforcement, $a_2 \prec b_2$. Notice that in this game, unless the criminal commits both acts (plays d_1), the low level of law enforcement is optimal for player 2. For player 1, a_1 (no crime) is the optimal response to a high level of law enforcement, while grand theft auto is optimal when only a low level of law enforcement occurs. It is clear that player 1 has strategic substitutes, while player 2 has strategic complements. The sole Nash equilibrium is $x^* = (b_1, a_2)$.

Now, suppose the bank receives a new, large, cash deposit - this is equivalent to saying that some parameter of the game t increases, where t represents a measure of the potential value of the bank's deposits. The new game is given on the right side of the above figure. Notice that the payoffs to the criminal for selecting c_1 and d_1 have increased. Likewise, the law enforcement payoff to using a high level of law

enforcement against c_1 and d_1 has increased. In other words, the t parameter is complementary in the sense defined previously: best responses are nondecreasing for both players.

Suppose one tries to implement the condition of Theorem 2 on the second iterate best response of the strategic substitutes player (player 1). This would give: $g_t^1(g_t^2(x_1^*)) = g_t^1(g_t^2(b_1)) = g_t^1(a_2) = d_1 \succ b_1 = x_1^*$. The condition of Theorem 2 is satisfied, so it would necessarily imply that the new equilibrium at the higher t , denoted by \hat{x}^* , should be higher than x^* . However, $\hat{x}^* = (c_1, b_2)$, which is *not higher*. The presence of merely two unordered strategies for player 1, b_1 and c_1 , creates a hiccup, and the second iterate condition of Theorem 2 fails to accurately predict that the new equilibrium does not increase.

In light of this, what follows will require a different approach and characterization in the result. I will apply a fixed point theorem here which will achieve the desired result and eliminate the need to assume the existence of an equilibrium at the higher parameter. In particular, examine a parametrized lattice game $\Gamma = (I, (X^i, \preceq^i, f^i)_{i \in I}, T)$ as in section 1.5, where $I = 2$. Allow the following assumptions here as well, and call them $\mathcal{A} = \{(I.), (II.), (III.)\}$:

- (I.) For every $i = 1, 2$, $g^i(x_{-i}, \cdot)$ is nondecreasing in t and continuous in x_{-i} ;
- (II.) Let player 1 have strategic substitutes, player 2 have strategic complements, without loss of generality;
- (III.) Let each player's strategy space, X_i , be a non-empty, compact,²⁴ convex²⁵ sublattice of a Banach lattice.²⁶

Clearly, with these assumptions, a class of 2-player games is defined in which players can be permitted to have more general strategy spaces, and in which players have opposing strategic characterizations. Consider an equilibrium of Γ at some t^* , denoted x^* . Let $\hat{t} \succ t^*$, and define for any player i , $\hat{y}_i = g_{\hat{t}}^i(x_{-i}^*)$. Notice that $\forall i$, $\hat{y}_i \succeq x_i^*$, since the parameter is complementary. In addition, define for any player i ,

²⁴Compactness here is with respect to the interval topology, as described above.

²⁵Convexity of a lattice is defined in the following way: lattice X is *convex* if $\forall a, b$ in X , $a \preceq b$, $[a, b] \subset X$; a subset $S \subseteq X$ is a *convex sublattice* of X if $a, b \in S \Rightarrow [a \wedge b, a \vee b] \subseteq S$.

²⁶A *Banach lattice* is a vector lattice endowed with a lattice norm, which is also a real Banach space. See Topkis (1998) or Birkhoff (1967) for a more detailed description.

$\hat{x}_i = g_{\hat{t}}^i(\hat{y}_{-i})$. The following theorem then gives the condition under which given an increase in parameter to some $\hat{t} \succeq t^*$, the equilibrium x^* increases:

Theorem 3: Let there be a parametrized lattice game Γ as defined above, with initial equilibrium x^* at t^* , and suppose there exists $\hat{t} \succeq t^*$. Suppose assumptions \mathcal{A} are satisfied. Suppose that for the strategic substitutes player, player 1, $x_1^* \preceq g_{\hat{t}}^1(\hat{x}_2)$. Then there exists an equilibrium at \hat{t} , denoted \hat{x}^* , such that $\hat{x}^* \succeq x^*$.

Proof. For player 1, define $A_1 = [x_1^*, \hat{y}_1]$. For player 2, define $A_2 = [x_2^*, \hat{x}_2]$. Note that via *III.*), these are closed and convex intervals. Let $A = A_1 \times A_2$. To apply Schauder's fixed point theorem, it must be shown that the joint best response, $g : X \rightarrow X$, restricted to A , is a self-map.

For player 1, notice that $g_{\hat{t}}^1(x_2^*) = \hat{y}_1$, and by assumption, $g_{\hat{t}}^1(\hat{x}_2) \succeq x_1^*$. As a result, $g_{\hat{t}}^1(A_2) \subseteq A_1$. For player 2, observe that $g_{\hat{t}}^2(x_1^*) \succeq x_2^*$. Also, $g_{\hat{t}}^2(\hat{y}_1) = \hat{x}_2$. Therefore, $g_{\hat{t}}^2(A_1) \subseteq A_2$. Combining these two facts yields $g_{\hat{t}} : A \rightarrow A$ as a self-map; by Schauder's Theorem, there exists a fixed point, $\hat{x}^* = g_{\hat{t}}(\hat{x}^*)$, such that $\hat{x}^* \succeq x^*$. ■

This result shows that even if strategy spaces are not linearly ordered in a 2-player setting, a straightforward characterization on the *third iterate best response* for the strategic substitutes player can be checked to determine if the equilibrium has increased. First, allow player 1 to best respond to x_2^* at \hat{t} (yielding \hat{y}_1), then allow player 2 to best respond to \hat{y}_1 at \hat{t} (yielding \hat{x}_2). Since $\hat{x}_2 \succeq x_2^*$, $g_{\hat{t}^*}^1(\hat{x}_2) \preceq x_1^*$, pushed lower by the strategic effect. However, $g_{\hat{t}}^1(\hat{x}_2) \succeq g_{\hat{t}^*}^1(\hat{x}_2)$, which captures the direct effect of the parameter increase on player 1. If the condition of Theorem 3 is satisfied, then in essence, the strategic effect is outweighed by the direct effect. Notice that once again, the only required characterization in this setting is on the strategic substitutes player, which simplifies its application.

3.3. Lattice strategy space & best response correspondences. Theorem 3 above can be generalized to the case where best responses are permitted to be set-valued. Allow the following assumptions here, and call them $\mathcal{A}^* = \{(I.)^*, (II.)^*, (III.)^*\}$:

- (I.)^{*} For every $i = 1, 2$, $g^i(x_{-i}, \cdot)$ is a convex-valued correspondence which is nondecreasing in t and has closed graph;
- (II.)^{*} Let player 1 have strategic substitutes, and player 2 have strategic complements, without loss of generality;
- (III.)^{*} Let each player's strategy space, X_i , be a non-empty, compact, convex sublattice of a Banach lattice.

I utilize the same definitions as above, with one addition. For any given x^* , let $\bar{x}_i^* = \sup g_{t^*}^i(x_{-i}^*)$, and let $\bar{x}^* = (\bar{x}_1^*, \bar{x}_2^*)$. Notice that $\bar{x}_i^* \succeq x_i^*$. The ensuing result is analogous to Theorem 3, while permitting best responses to be set-valued.

Theorem 3*: Let there be a parametrized lattice game Γ as defined above, with initial equilibrium x^* at t^* , and suppose there exists $\hat{t} \succeq t^*$. Suppose assumptions \mathcal{A}^* are satisfied. Additionally, suppose x_2 is strictly complementary in t . Suppose that for the strategic substitutes player, player 1, $\bar{x}_1^* \preceq \inf g_{\hat{t}}^1(\sup g_{\hat{t}}^2(\sup g_{\hat{t}}^1(\bar{x}_2^*)))$. Then there exists an equilibrium at \hat{t} , denoted \hat{x}^* , such that $\hat{x}^* \succeq x^*$.

Proof. For player 1, define $A_1^* = [\bar{x}_1^*, \sup g_{\hat{t}}^1(\bar{x}_2^*)]$. For player 2, define $A_2^* = [\bar{x}_2^*, \sup g_{\hat{t}}^2(\sup g_{\hat{t}}^1(\bar{x}_2^*))]$. To get existence of a fixed point $\hat{x}^* \succeq \bar{x} \succeq x^*$, and hence, to apply Glicksberg-Fan fixed point theorem, it remains to be checked if the joint best response correspondence restricted to $A^* = A_1^* \times A_2^*$ is a self-map.

For player 1, $\sup g_{\hat{t}}^1(\bar{x}_2^*) = \sup g_{\hat{t}}^1(\bar{x}_2^*)$. In addition, $\bar{x}_1^* \preceq \inf g_{\hat{t}}^1(\sup g_{\hat{t}}^2(\sup g_{\hat{t}}^1(\bar{x}_2^*)))$, by assumption. Therefore, $g_{\hat{t}}^1(A_2^*) \subseteq A_1^*$. For player 2, $\sup g_{\hat{t}}^2(\sup g_{\hat{t}}^1(\bar{x}_2^*)) = \sup g_{\hat{t}}^2(\sup g_{\hat{t}}^1(\bar{x}_2^*))$. Now consider: $\bar{x}_1^* \succeq x_1^* \implies g_{\hat{t}^*}^2(\bar{x}_1^*) \supseteq g_{\hat{t}^*}^2(x_1^*)$. By x_2 strictly complementary with t , $g_{\hat{t}}^2(\bar{x}_1^*)$ is strictly higher than $g_{\hat{t}^*}^2(x_1^*)$. This gives $g_{\hat{t}}^2(\bar{x}_1^*)$ strictly higher than $g_{\hat{t}^*}^2(x_1^*)$, which implies $\inf g_{\hat{t}}^2(\bar{x}_1^*) \succeq \sup g_{\hat{t}^*}^2(x_1^*) = \bar{x}_2^*$. Therefore, $g_{\hat{t}}^2(A_1^*) \subseteq A_2^*$. By Glicksberg-Fan fixed point theorem, there exists some fixed point, $\hat{x}^* \in g_{\hat{t}}(\hat{x}^*)$ such

that $\hat{x}^* \succcurlyeq \bar{x}^* \succcurlyeq x^*$. ■

Here, it is still possible to extract some intuition behind the requirement of Theorem 3*, despite its appearance. Recall that Theorems 2 and 2* required a condition on a second iterate of player 1's best response, which was used to capture the tug of war between the direct and strategic effects; also recall that this condition was fairly weak. The required condition for Theorem 3*, $\bar{x}_1^* \preceq \inf g_t^1(\sup g_t^2(\sup g_t^1(\bar{x}_2^*)))$, is stronger than that required for Theorems 2 and 2*, in the following sense.

First, examine the inside term, $g_t^1(\bar{x}_2^*)$. The best response taken at \hat{t} illustrates the first direct effect on player 1, while the strategic effect comes from $\bar{x}_2^* \succeq x_2^*$. Notice that by taking this best response at \bar{x}_2^* , the strategic effect is strong (recall that in Theorem 2*, the strategic effect was somewhat weak since an infimum was taken). The $g_t^1(\bar{x}_2^*)$ term is pushed lower by the strategic effect, but pulled higher by the direct effect. Yet, Theorem 3* requires a condition on the term $\inf g_t^1(\sup g_t^2(\sup g_t^1(\bar{x}_2^*)))$. Upon further examination, it appears as though the full condition applies a *second iteration of the strategic effect on player 1*. By taking the best response at the supremum in $g_t^2(\sup g_t^1(\bar{x}_2^*))$, player 2's best response set is pushed high due to strategic complements. The additional strategic effect comes in when player 1 takes his best response to the supremum of this set, which will push $g_t^1(\sup g_t^2(\sup g_t^1(\bar{x}_2^*)))$ even lower. So in a sense, incorporating the suprema here strengthen the second iteration of this strategic (decreasing) effect. To then compare the infimum of this set, which has been pushed rather low from two iterations of strategic effects, with x_1^* , suggests that the condition of Theorem 3* is stronger than the analogous condition in Theorem 2*.

One additional aspect of the intuition in 2-player games is notable. Recall that in the 2-player, linearly ordered strategy space case without correspondences, the only characterization needed to ensure increasing equilibria is on the second iterate best response of the player with strategic substitutes. This remains constant throughout this section. The only required condition for the theorems above is a characterization for this player, and the reason should be clear. Only a player with strategic substitutes faces an ambiguous overall effect from the combination of the direct and

strategic effects of a parameter increase. Therefore, only a characterization on this player is necessary, to ensure the direct effect is not outweighed by the strategic effect. In the following section, when the setting is expanded to $I > 2$ players, this fact will remain: the only necessary characterizations will be on the subset of players who have strategic substitutes.

4. INCREASING EQUILIBRIA: GAMES WITH MORE THAN 2 PLAYERS

This section expands on the previous one by allowing for games with an arbitrary I number of players. In the 2-player, linearly ordered strategy space case, characterizing increasing equilibria was equivalent to a condition on the second iterate best response of the strategic substitutes player. However, section 3.2 demonstrates such a condition will not work when strategy spaces are not linearly ordered. A simple counterexample here will show that likewise, this condition will also fail in a game with more than 2 players, even if strategy spaces are linearly ordered.

Counterexample:

Consider a counterexample similar to that in section 3.2, only introduce a third player, who acts as a second law enforcement player. In addition, now consider $X_1 = \{a_1, b_1, c_1, d_1\}$ such that $a_1 \prec b_1 \prec c_1 \prec d_1$: the range of criminal activities available to the criminal player are now increasing from some a_1 (no crime) to d_1 (highest level of crime). As before, each law enforcement player can choose either a low or high level of law enforcement. Let the game be represented by Figure 11. Player 1 has strategic substitutes, players 2 and 3 have strategic complements, and $x^* = (c_1 a_2 a_3)$.

Now, similar to earlier, increase a parameter t of this game, which may be interpreted as the worth of the criminal activity (recall that earlier, this was given as the value of bank deposits at the bank which may be robbed). The increase in t leads to higher best responses for player 1, since he will want to engage in a higher level of criminal activity. In addition, the higher stakes lead the law enforcement players

	a ₂ (low)	b ₂ (high)
a ₁	1, 5, 5	1, 0, 2
b ₁	5, 2, 2	5, 0, 2
c ₁	10, 2, 2	0, 0, 2
d ₁	7, 0, 0	0, 10, 0

	a ₂ (low)	b ₂ (high)
a ₁	1, 5, 0	1, 0, 0
b ₁	5, 2, 0	0, 0, 0
c ₁	3, 2, 0	0, 0, 0
d ₁	2, 0, 10	0, 10, 10

a₃ (low)
b₃ (high)

FIGURE 11. Counterexample: original parameter.

to weakly increase their optimal strategies as well. This new game, at some higher parameter $\hat{t} \succ t$, is given by Figure 12.

	a ₂ (low)	b ₂ (high)
a ₁	1, 5, 5	1, 0, 2
b ₁	5, 2, 2	5, 0, 2
c ₁	10, 2, 2	10, 0, 2
d ₁	15, 0, 0	7, 10, 0

	a ₂ (low)	b ₂ (high)
a ₁	1, 5, 0	1, 6, 0
b ₁	5, 2, 0	3, 4, 4
c ₁	10, 2, 4	0, 4, 4
d ₁	7, 0, 10	0, 10, 10

a₃ (low)
b₃ (high)

FIGURE 12. Counterexample: higher parameter.

What if the second iterate condition of Theorem 2 were to be effective here? This would require checking $g_t^1(g_t^2(x_{-2}^*), g_t^3(x_{-3}^*)) = g_t^1(g_t^2(c_1 a_3), g_t^3(c_1 a_2)) = g_t^1(a_2 b_3) = c_1 \succeq c_1 = x_1^*$. That is, the condition would be satisfied, which would predict a higher equilibrium at \hat{t} . Yet, the new equilibrium is $\hat{x}^* = (b_1 b_2 b_3)$, which is not higher. Therefore, once again, a condition on the second iterate best response of the strategic substitutes player will not be sufficient to guarantee an accurate prediction of an equilibrium increase.²⁷

Due to this fact, the results in this section will require a slightly stronger characterization of the players in the game. First, the results will impose higher orders of rationality on the players, represented by best response characterizations past the second iterate. In addition, while both strategic complements and strategic substitutes can both be present, it will be necessary to ensure that all I players in the game have at least one property or the other.

4.1. Lattice strategy spaces. Examine a parametrized lattice game $\Gamma = (I, (X^i, \preceq^i, f^i)_{i \in I}, T)$ as in section 1.5 however, allow any finite I such that $I \geq 2$. In addition, the following assumptions will be made throughout this section, and call them $\mathcal{B} = \{(i.), (ii.), (iii.)\}$:

- (i.) For every player i , $g^i(x_{-i}, \cdot)$ is nondecreasing in t and continuous in x_{-i} ;
- (ii.) Let some subset of players, say indices $l = 1, \dots, k$ have strategic substitutes, while the remaining players, $j = k + 1, \dots, I$ have strategic complements;²⁸
- (iii.) Let each player's strategy space, X_i , be a non-empty, compact, convex sublattice of a Banach lattice.

With a few restrictions, these types of games include the generalized lattice games which allow for many players and permit both players with strategic substitutes and

²⁷An additional counterexample is provided in the Appendix, which demonstrates that if a 3-player game has two strategic *substitutes* players (with one strict), and one strategic complements player, even in the context of linearly ordered strategy spaces and best response functions, a second iterate condition is similarly insufficient to characterize increasing equilibria. Therefore, this counterexample is not specifically dependent on having giving the additional player strategic complements.

²⁸This notation is not meant to confuse: for the purposes of this section, index l will be used for a player with strategic substitutes, while index j will be used for a player with strategic complements. Index i will be used when the player can be any player in I .

players with strategic complements. Assumption (ii.) ensures that all players have either strategic substitutes or strategic complements, but allows for an arbitrary number of each. As in the previous section, consider an equilibrium of Γ at some t^* , denoted x^* . Let $\hat{t} \succ t^*$, and define for any player $i \in I$, $\hat{y}_i = g_i^i(x_{-i}^*)$. Notice that $\forall i$, $\hat{y}_i \succeq x_i^*$, since the parameter is complementary. Denote the greatest element of each X_i by \bar{x}_i , which exists as each strategy space is a complete lattice. Also, define, for any player $j = k + 1, \dots, I$, $\tilde{x}_j = g_j^j((\hat{y}_i)_{i=1}^k, (\bar{x}_{j'})_{j' \neq j})$.

The following theorem establishes when an increase of parameter to \hat{t} leads to an increased equilibrium in this multi-player setting:

Theorem 4: Let there be a parametrized lattice game Γ as defined above, with initial equilibrium x^* at t^* , and suppose there exists $\hat{t} \succeq t^*$. Suppose assumptions \mathcal{B} are satisfied. Suppose that for each strategic substitutes player $l = 1, \dots, k$, $x_l^* \preceq g_l^l((\hat{y}_{l'})_{l' \neq l}, (\tilde{x}_j)_{j=k+1}^I)$. Then there exists an equilibrium at \hat{t} , denoted \hat{x}^* , such that $\hat{x}^* \succeq x^*$.

Proof. For each player $l = 1, \dots, k$, define $B_l = [x_l^*, \hat{y}_l]$; for each player $j = k + 1, \dots, I$, define $B_j = [x_j^*, \tilde{x}_j]$. Define the products $B = \prod_{i=1}^I B_i$ and $B_{-i} = \prod_{i' \neq i} B_{i'}$ in the typical way.

For each player $l = 1, \dots, k$, $g_l^l(x_{-l}^*) = \hat{y}_l$. In addition, $g_l^l((\hat{y}_{l'})_{l' \neq l}, (\tilde{x}_j)_{j=k+1}^I) \succeq x_l^*$ by assumption. Therefore, for each $l = 1, \dots, k$, $g_l^l(B_{-l}) \subseteq B_l$. For players $j = k + 1, \dots, I$, first notice that, $g_j^j(x_{-j}^*) = \hat{y}_j \succeq x_j^*$. In addition, $g_j^j((\hat{y}_i)_{i=1}^k, (\tilde{x}_{j'})_{j' \neq j}) \preceq g_j^j((\hat{y}_i)_{i=1}^k, (\bar{x}_{j'})_{j' \neq j}) = \tilde{x}_j$, where the inequality comes from $\tilde{x}_j \preceq \bar{x}_j$, $\forall j$, and the nondecreasing best response of player j . Therefore, for each $j = k + 1, \dots, I$, $g_j^j(B_{-j}) \subseteq B_j$. Hence, $g_i : B \rightarrow B$ is indeed a self-map; by Schauder's Theorem, there exists a fixed point, $\hat{x}^* = g_i(\hat{x}^*)$, such that $\hat{x}^* \succeq x^*$. ■

As mentioned above, notice how the only required conditions are on the players $l = 1, \dots, k$, who are the players with strategic substitutes. This arises once again because of the opposing direct and strategic effects for these players. As long as the direct effect is great enough to dominate the strategic effect, the existence of an

increased equilibrium can be achieved.

Example: 3-firm competition

Take a discrete, 3-player game, and consider firms competing in quantities. Let firm 1 be a large firm (or an incumbent firm, for example), which has an advanced technology allowing it to produce one of three levels of output: Low, Medium, and High (denoted L_1 , M_1 , and H_1). Let firms 2 and 3 be smaller (or potentially entrant) firms, who are only capable of producing either the Low or Medium level of output. These smaller firms experience a kind of technological spillover if enough output is produced by their respective rival firms, and as such exhibit strategic complements. In addition, suppose the firms each respond, potentially asymmetrically, to a demand parameter, t , which is complementary to each firm's output choice. This gives $X_1 = \{L_1, M_1, H_1\}$, $X_2 = \{L_2, M_2\}$, and $X_3 = \{L_3, M_3\}$.²⁹ Define the game with the matrices below:

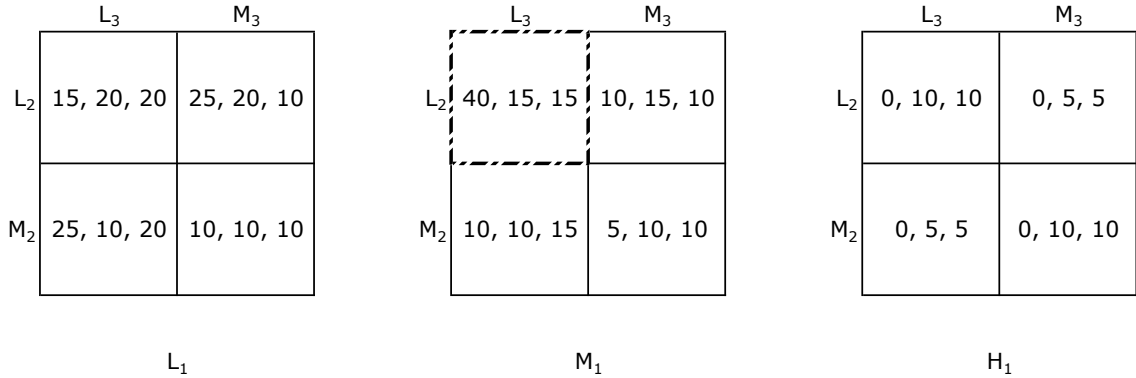


FIGURE 13. 3-player game at original parameter t .

Notice, first, that either smaller firm is only willing to produce the medium level of output if both competitors produce their maximum levels. Also, notice the strategic substitutes of the large firm: it is only willing to produce a level other than L_1 if both competitors produce low levels. It is clear from the matrices that the equilibrium is

²⁹While Theorem 4 can be applied to more general strategy spaces, I use an example with linearly ordered strategy spaces here for simplicity.

$x^* = (M_1, L_2, L_3)$. Now let the parameter t increase to some \hat{t} , $\hat{t} \succ t$, such that best responses for each player are non-decreasing and the game becomes:

	L ₃	M ₃
L ₂	20, 25, 25	25, 25, 15
M ₂	25, 15, 15	20, 15, 25
	L ₁	

	L ₃	M ₃
L ₂	45, 20, 15	30, 20, 20
M ₂	30, 15, 15	10, 15, 20
	L ₁	

	L ₃	M ₃
L ₂	5, 10, 10	5, 10, 15
M ₂	5, 15, 10	5, 15, 15
	L ₁	

FIGURE 14. 3-player game at increased parameter \hat{t} .

It is useful to note, firstly, that the parameter increase affects the two small firms differently. Firm 2 is now willing to produce M_2 if and only if firm 1 produces H_1 . However, firm 3 is more willing to increase its output: it will produce M_3 as long as one of its competitors is producing more than the low level of output. Since the assumptions of Theorem 4 are met, it only remains to be checked if the condition above is satisfied for all strategic substitutes players (which here, means just firm 1), and indeed it is. To see this: $\hat{y}_1 = M_1$, $\tilde{x}_2 = L_2$, $\tilde{x}_3 = M_3$, and therefore, $x_1^* = M_1 \preceq M_1 = g_1^1(\tilde{x}_2, \tilde{x}_3) = g_1^1(L_2, M_3)$. Since the condition is satisfied, and Theorem 4 can be applied, there should be a higher equilibrium, some \hat{x}^* , at this new parameter, which there is: $\hat{x}^* = (M_1, L_2, M_3)$. It is interesting to note that only firm 3, which is most dramatically impacted by the parameter increase, selects a higher level of output in the new equilibrium.

Example: A general 3-player game

Consider a 3-player parametrized lattice game, in which players' strategy spaces are linearly ordered, and best responses are singleton valued. Suppose $X_1 = \{a_1, b_1, c_1\}$, $X_2 = \{a_2, b_2\}$, and $X_3 = \{a_3, b_3\}$, where $a_i \prec b_i (\prec c_i, \text{ for } i = 1), \forall i$. Now, suppose

player 1 has strategic substitutes, while players 2 and 3 have strategic complements. Define best responses at t in the following way:

$$\begin{aligned} \text{P1.) } & g_t^1(a_2a_3) = b_1; g_t^1(a_2b_3) = g_t^1(b_2a_3) = g_t^1(b_2b_3) = a_1; \\ \text{P2.) } & g_t^2(a_1a_3) = g_t^2(a_1b_3) = g_t^2(b_1a_3) = g_t^2(b_1b_3) = g_t^2(c_1a_3) = a_2; g_t^2(c_1b_3) = b_2; \\ \text{P3.) } & g_t^3(a_1a_2) = g_t^3(a_1b_2) = g_t^3(b_1a_2) = g_t^3(b_1b_2) = g_t^3(c_1a_2) = a_3; g_t^3(c_1b_2) = b_3; \end{aligned}$$

The equilibrium at t is $x^* = (b_1a_2a_3)$. Now, increase the (complementary) parameter to some $\hat{t} \succ t$, and give new best responses as follows:

$$\begin{aligned} \text{P1.) } & g_{\hat{t}}^1(a_2a_3) = g_{\hat{t}}^1(a_2b_3) = g_{\hat{t}}^1(b_2a_3) = b_1; g_{\hat{t}}^1(b_2b_3) = a_1; \\ \text{P2.) } & g_{\hat{t}}^2(a_1a_3) = g_{\hat{t}}^2(a_1b_3) = g_{\hat{t}}^2(b_1a_3) = g_{\hat{t}}^2(b_1b_3) = a_2; g_{\hat{t}}^2(c_1a_3) = g_{\hat{t}}^2(c_1b_3) = b_2; \\ \text{P3.) } & g_{\hat{t}}^3(a_1a_2) = a_3; g_{\hat{t}}^3(a_1b_2) = g_{\hat{t}}^3(b_1a_2) = g_{\hat{t}}^3(b_1b_2) = g_{\hat{t}}^3(c_1a_2) = g_{\hat{t}}^3(c_1b_2) = b_3; \end{aligned}$$

The condition of Theorem 4 holds here: $\hat{y}_1 = b_1$, $\tilde{x}_2 = a_2$, $\tilde{x}_3 = b_3$, and therefore, $x_1^* = b_1 \preceq b_1 = g_{\hat{t}}^1(\tilde{x}_2, \tilde{x}_3) = g_{\hat{t}}^1(a_2b_3)$. Hence, the increase in t should bring about a higher equilibrium, which it does: $\hat{x}^* = (b_1a_2b_3)$. The preceding example is a special case of this example.

Since the necessary conditions for Theorem 4 above may not be straightforward to check in some applications, an alternate version is presented here as Corollary 1. While this version is unambiguously stronger than Theorem 4 (and therefore, there may be some applications for which the Corollary cannot be used yet Theorem 4 can), it provides an easier method of checking, by using \bar{x}_j , for each player $j = k + 1, \dots, I$, instead of \tilde{x}_j .

Corollary 1: Let there be a parametrized lattice game Γ as defined above, with initial equilibrium x^* at t^* , and suppose there exists $\hat{t} \succeq t^*$. Suppose assumptions \mathcal{B} are satisfied. Suppose that for each strategic substitutes player $l = 1, \dots, k$, $x_l^* \preceq g_{\hat{t}}^l((\hat{y}_{l'})_{l' \neq l}, (\bar{x}_j)_{j=k+1}^I)$. Then there exists an equilibrium at \hat{t} , denoted \hat{x}^* , such that $\hat{x}^* \succeq x^*$.

Proof. The proof is provided in the Appendix.

Notice that in the above example, the condition to apply Corollary 1 does not hold: $x_1^* = b_1 \not\leq a_1 = g_{\bar{t}}^1(\bar{x}_2, \bar{x}_3)$. However, as mentioned, the weaker condition of Theorem 4 is met.

Example: Crime and punishment

Consider a 3-player game, modeled in the spirit of Becker (1968). Becker famously characterizes a model in which criminal offenses in society are a function of many interacting forces, including the potential damage to society of the offense, the cost of apprehending and convicting the offenders, the probability of capture, and the punishment with regard to the offense.³⁰ Here, I introduce a game-theoretic approach which captures many of these interactions, and which fits in the context of the approach in the paper.

Suppose player 1 is the government player (G), who selects a level of law enforcement $x \geq 0$. Law enforcement incurs a cost of c^4 per unit. Players 2 and 3 are two criminals (C_1 and C_2 , respectively), who select a level of criminal activity, $y_1 \geq 0$ for C_1 and $y_2 \geq 0$ for C_2 . The payoff functions for each player are given as:

$$u_G = -C(x) - D(x, y_1, y_2, t) = -xc^4 - \frac{t(y_1^2 + y_2^2)}{x}$$

$$u_{C_1} = \tilde{p}(y_1, x, t)G(y_1) = \frac{y_1^{\frac{1}{2}}}{1 + t + xy_1}$$

$$u_{C_2} = \tilde{p}(y_2, x, t)G(y_2) = \frac{y_2^{\frac{1}{2}}}{1 + t + xy_2}$$

In the terminology of Becker, a government player would aim to minimize a loss function which incorporates the “total social loss in real income from offenses, convictions, and punishments.” Here, I interpret the payoff u_G in two terms: the first, C , reflects the cost of law enforcement, which increases in x ; the second, D , reflects

³⁰For the sake of simplification, I will suppress some aspects of the original model, such as distinctions which incorporate legal or criminal justice-related aspects of crime and punishment. This would mean, for example, that instead of discussing *probability of conviction*, as in Becker, I use the *probability of capture*, which does not account for the role of legal institutions or the criminal justice system in going from capture to conviction.

the social damage incurred due to criminal activity, which is increasing in each y_i but decreasing in x . This permits an interpretation of x as an endogenously-chosen level of law enforcement which impacts the probability of capture: it is more costly to hire additional police officers or improve crime fighting technology, but doing so lowers the potential social damage of crime by lowering the potential rate of success of the criminal activity. In fact, Becker considers the probability of (conviction) capture be used explicitly as a decision variable in his model.

Becker suggests the number of offenses committed by an individual may be represented as a function of the probability of conviction, punishment, and other factors, such as available potential income from both legal and illegal activities. Here, payoffs to the criminals can be seen as the expected utility from engaging in a certain level of criminal activity, which is a function of the probability of evading capture.³¹ To simplify, there is no punishment component here as in Becker; hence, if caught, the criminal receives payoff of zero. The two components of u_{C_1} and u_{C_2} , therefore, are as follows. $\tilde{p}(y_i, x, t)$ represents the probability of evading capture: it is decreasing in y_i (a larger-scale crime is more difficult to get away with) and x (a greater number of police officers would be more difficult to evade). G represents the gain to the criminals from engaging in criminal activity, and is increasing and concave in y_i , as in Becker.

The game also has an additional parameter $t \geq 0$, which provides a measure of the stakes of the criminal activity. For the government, a higher t represents an increase in the amount of potential damages of the criminal activity being undertaken; for the criminal, higher stakes increase the chances of getting caught, since a “higher stakes” crime is likely to be more difficult to execute. For example, if the criminal activity is some type of terrorist attack on a public place, t could be representative of the number of citizens or bystanders who could be affected or injured by the act. An attack on a crowded place would increase t and therefore, heighten the negative impact of the attack on society. A higher t would also escalate the difficulty in executing a terrorist attack (since there may be many witnesses to the criminal

³¹Thus, here, the criminal players have already decided to engage in criminal activity, presumably because it provides a higher expected payoff than the payoff to engaging in some legal alternative.

activity), and thus decrease the probability of evading capture. It is in this light that as the parameter t increases, there is greater potential harm to society from the criminal activity, but the crime becomes more difficult to get away with (and the probability of evasion drops).

Given the setup, best response functions can be found to be:

$$\begin{aligned} g^G(y_1, y_2) &= \frac{1}{c^2} \sqrt{t(y_1^2 + y_2^2)} \\ g^{C_1}(x, y_2) &= \frac{1+t}{x} \\ g^{C_2}(x, y_1) &= \frac{1+t}{x} \end{aligned}$$

which indicates that the government player (G) has strategic complements with respect to (y_1, y_2) , while each criminal player has strategic substitutes with respect to x (and a best response which is constant with respect to the other criminal's choice variable). Also note that each best response is increasing in t : that is, as the stakes get higher, both the government and the criminals are incentivized to increase their respective action. In particular, setting $t = \frac{1}{2}$ (and $c = 2$), there is an equilibrium at $(x^*, y_1^*, y_2^*) = (\frac{1}{2}\sqrt{\frac{3}{2}}, 2\sqrt{\frac{3}{2}}, 2\sqrt{\frac{3}{2}}) \approx (0.612, 2.449, 2.449)$. Now, suppose the parameter t increases to $\hat{t} = 1$. Since each best response is a function and nondecreasing in this parameter, and supposing the strategies are taken from a closed interval of non-negative reals, Theorem 4 can be applied to determine if there is an increased equilibrium strategy profile \hat{x}^* corresponding to the new parameter \hat{t} . Technically, there are two conditions to check here, but taking advantage of symmetry across C_1 and C_2 , only checking 1 will suffice: $y_1^* \leq g_{\hat{t}}^{C_1}(\hat{y}_2, \tilde{x}_1)$. It is easy to see that $\hat{y}_2 = g_{\hat{t}}^{C_2}(x^*, y_1^*) = \frac{4\sqrt{2}}{\sqrt{3}} = \hat{y}_1$; $\tilde{x}_1 = g_{\hat{t}}^G(\hat{y}_1, \hat{y}_2) = \frac{2}{\sqrt{3}}$; and finally, $g_{\hat{t}}^{C_1}(\hat{y}_2, \tilde{x}_1) = g_{\hat{t}}^{C_1}(\frac{2}{\sqrt{3}}, \frac{4\sqrt{2}}{\sqrt{3}}) = \sqrt{3} < 2\sqrt{\frac{3}{2}}$. Therefore, the condition is not satisfied; pursuant to this fact, the equilibrium at \hat{t} is $(\hat{x}^*, \hat{y}_1^*, \hat{y}_2^*) = (\frac{\sqrt{2\sqrt{2}}}{2}, \frac{4}{\sqrt{2\sqrt{2}}}, \frac{4}{\sqrt{2\sqrt{2}}}) \approx (0.840, 2.378, 2.378)$, which is not higher than the original equilibrium.

However, if we start at the equilibrium at \hat{t} , $(\hat{x}^*, \hat{y}_1^*, \hat{y}_2^*)$ given above, and increase t again to $\hat{\hat{t}} = 2$, then the equilibrium will increase. In fact, a simple calculation

demonstrates that $\frac{\partial x^*}{\partial t} > 0, \forall t > 0$, so the government will always increase its equilibrium level of law enforcement when t increases. Similarly, it is easy to see that for $i = 1, 2$, $\frac{\partial y_i^*}{\partial t} \geq 0 \iff t \geq 1$: if the stakes are sufficiently high, then both criminals will be willing to increase their equilibrium level of criminal activity as stakes get marginally greater. This is consistent with the intuition given throughout the paper, and demonstrates that given an increase in the parameter t , the equilibrium strategies of the strategic substitutes players will go up if and only if the parameter is high enough (the *direct effect* once again) to outweigh the *strategic effect* which pushes y_1^* and y_2^* lower as x^* increases. Graphically, this can be seen in the figure below, with variable x on the horizontal axis, and variables $y = y_1 = y_2$ on the vertical axis. The figure on the left demonstrates the increase from \hat{t} to $\hat{\hat{t}}$, for which the equilibrium increases, while that on the right shows the increase from t to \hat{t} , for which the equilibrium does not increase.

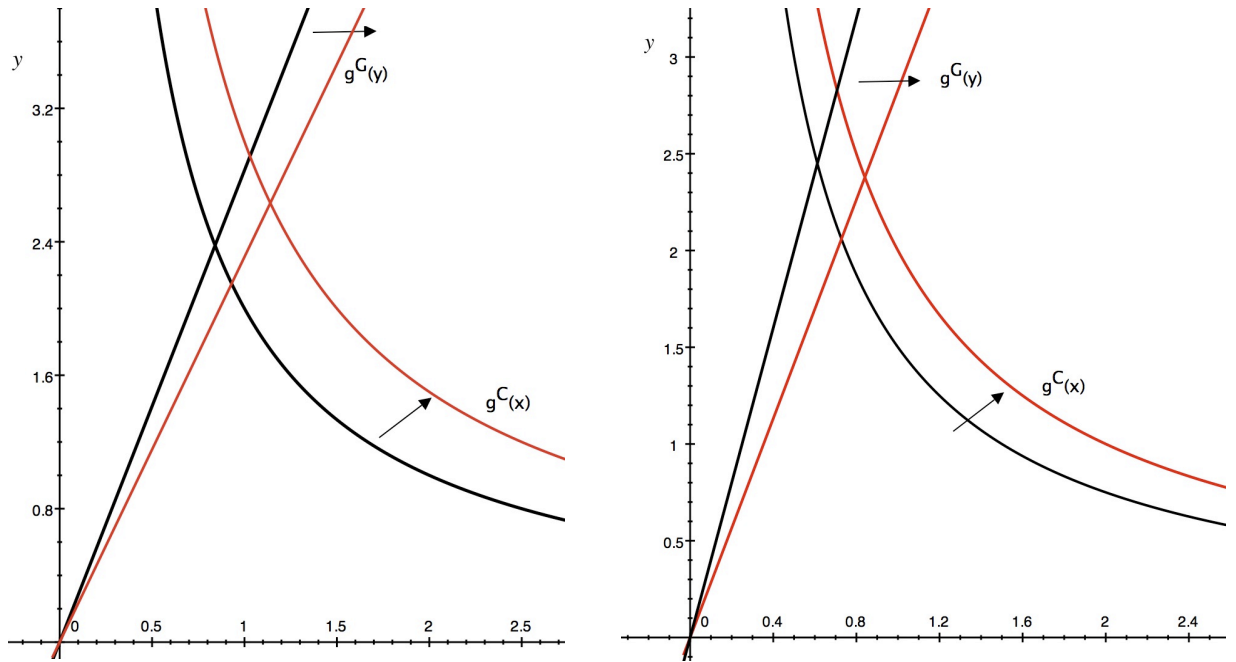


FIGURE 15. Two parameter increases in the crime and punishment game. Left: $\hat{t} \rightarrow \hat{\hat{t}}$, Right: $t \rightarrow \hat{t}$.

4.2. Lattice strategy space & best response correspondences. I extend the results of the previous section to allow for best response correspondences here. In alliance with the previous results, increasing equilibria can be achieved, provided a condition is met for all of the players with strategic substitutes. To apply Glicksberg-Fan, a certain subset of our previous assumptions will be necessary: $\mathcal{B}^* = \{(i.)^*, (ii.), (iii.)\}$:

- (i.)^{*} For every player $i = 1, 2, \dots, I$, $g^i(x_{-i}, \cdot)$ is a convex-valued correspondence which is nondecreasing in t and has closed graph;
- (ii.) Let some subset of players, say indices $l = 1, \dots, k$ have strategic substitutes, while the remaining players, $j = k + 1, \dots, I$ have strategic complements;
- (iii.) Let each player's strategy space, X_i , be a non-empty, compact, convex sublattice of a Banach lattice.

Recall for any given x^* , let $\bar{x}_i^* = \sup g_{t^*}^i(x_{-i}^*)$, and let \bar{x}_i be the greatest element in X_i . In addition, define $\tilde{x}_j = \sup g_t^j((\sup g_t^l(\bar{x}_{-l}^*))_{l=1}^k, (\bar{x}_{j'})_{j' \neq j})$.

Theorem 4*: Let there be a parametrized lattice game Γ as defined above, with initial equilibrium x^* at t^* , and suppose there exists $\hat{t} \succeq t^*$. Suppose assumptions \mathcal{B}^* are satisfied. Suppose x_2 is strictly complementary in t . Also, suppose that for each strategic substitutes player $l = 1, \dots, k$, $\bar{x}_l^* \preceq \inf g_t^l((\sup g_t^{l'}(\bar{x}_{-l'}^*))_{l' \neq l}, (\tilde{x}_j)_{j=k+1}^I)$. Then there exists an equilibrium at \hat{t} , denoted \hat{x}^* , such that $\hat{x}^* \succeq x^*$.

Proof. For each player $l = 1, \dots, k$, define $C_l = [\bar{x}_l^*, \sup g_t^{l'}(\bar{x}_{-l'}^*)]$; for each player $j = k + 1, \dots, I$, define $C_j = [\bar{x}_j^*, \tilde{x}_j]$. First, to show that $\forall l, g_t^l(C_{-l}) \subseteq C_l$, simply notice that $\sup g_t^{l'}(\bar{x}_{-l'}^*) \preceq \sup g_t^{l'}(\bar{x}_{-l'}^*)$, and that by assumption, $\bar{x}_l^* \preceq \inf g_t^l((\sup g_t^{l'}(\bar{x}_{-l'}^*))_{l' \neq l}, (\tilde{x}_j)_{j=k+1}^I)$. To see that $\forall j, g_t^j(C_{-j}) \subseteq C_j$, first notice that $\bar{x}_{-j}^* \succeq x_{-j}^* \implies g_t^j(\bar{x}_{-j}^*) \supseteq g_t^j(x_{-j}^*)$, which is then completely higher than $g_{t^*}^j(x_{-j}^*)$ by strict complementarity in t . This implies $g_t^j(\bar{x}_{-j}^*)$ is completely higher than $g_{t^*}^j(x_{-j}^*)$, which gives $\inf g_t^j(\bar{x}_{-j}^*) \succeq \sup g_{t^*}^j(x_{-j}^*) = \bar{x}_j^*$. In addition $(\tilde{x}_{j'})_{j' \neq j} \preceq (\bar{x}_{j'})_{j' \neq j} \implies g_t^j((\sup g_t^{l'}(\bar{x}_{-l'}^*))_{l=1}^k, (\tilde{x}_{j'})_{j' \neq j}) \sqsubseteq g_t^j((\sup g_t^{l'}(\bar{x}_{-l'}^*))_{l=1}^k, (\bar{x}_{j'})_{j' \neq j})$, which naturally implies $\sup g_t^j((\sup g_t^{l'}(\bar{x}_{-l'}^*))_{l=1}^k, (\tilde{x}_{j'})_{j' \neq j}) \preceq \sup g_t^j((\sup g_t^{l'}(\bar{x}_{-l'}^*))_{l=1}^k, (\bar{x}_{j'})_{j' \neq j}) = \tilde{x}_j$.

Therefore, since $g_{\hat{t}}$ is a self-map from C to C , it has a fixed point, $\hat{x}^* \succeq \bar{x}^* \succeq x^*$. ■

The intuition for this result is quite reminiscent to that of Theorem 3*. The condition is strong, in the sense that suprema are chosen to increase the impact of the strategic effect of the parameter increase on each strategic substitutes player. Similar to Theorem 4, this theorem can be sharpened using the greatest strategy for the strategic complements players instead of the \tilde{x}_j . Although it requires a stronger characterization on the strategic substitutes players, it is potentially more easily applicable, since no additional calculations are required to find the \bar{x}_j .

Corollary 1*: Let there be a parametrized lattice game Γ as defined above, with initial equilibrium x^* at t^* , and suppose there exists $\hat{t} \succeq t^*$. Suppose assumptions \mathcal{B}^* are satisfied. Suppose x_2 is strictly complementary in t . Also, suppose that for each strategic substitutes player $l = 1, \dots, k$, $\bar{x}_l^* \preceq \inf g_{\hat{t}}^l((\sup g_{\hat{t}}^{l'}(\bar{x}_{-l}^*))_{l' \neq l}, (\bar{x}_j)_{j=k+1}^I)$. Then there exists an equilibrium at \hat{t} , denoted \hat{x}^* , such that $\hat{x}^* \succeq x^*$.

Proof. See Proof of Theorem 4*.

5. SUMMARY

This paper extends work addressing a class of generalized lattice games which are not covered in the previously existing literature on games with strategic complements and games with strategic substitutes. Given the theoretical context, I provide robust results with respect to monotone comparative statics in this class of games; in particular, the results highlight conditions under which equilibria are either nondecreasing or increasing with respect to a complementary parameter. The characterizations of these results are straightforward to apply, and several examples are provided to highlight this fact. Importantly, the results here highlight the central role of the strategic substitutes player(s) in particular, and the relationship between this class of games and GSS. In the simplest case considered here, section 3.1, increasing equilibria arise

under similar conditions as those imposed by Roy and Sabarwal (2010b); however, beyond this case, additional characterizations are needed.

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6. APPENDIX

6.1. Proof of Theorem 1. The proof will be shown separately under the assumptions of conditions (i.), (ii.), and (iii.), respectively. The approach to the proof will be to assume the contrapositive: suppose $\exists \hat{x}^* \in FP(\hat{t})$, $\exists x^* \in FP(t^*)$, such that $\hat{x}^* \prec x^*$. Then, a contradiction is reached through the application of the two fundamental properties of the admissible family of games:

- a.) By applying the fact that g is nondecreasing in t , show that for every $x^* \in FP(t^*)$, \exists a strategy profile in $g(\hat{x}^*; t^*)$ which is lower than x^* ;
- b.) By applying the fact that one player exhibits strict SS, show that for every $x^* \in FP(t^*)$, there does not exist a strategy profile in $g(\hat{x}^*, t^*)$ which is lower than x^* .

If this contradiction can be reached, then the Theorem is shown.

Proof under assumption (i.)

Assume $\exists \hat{x}^* \in FP(\hat{t})$, $\exists x^* \in FP(t^*)$, such that $\hat{x}^* \prec x^*$.

(a.) First, apply the property that g is nondecreasing in t . Fix $x = \hat{x}^*$. Since $t^* \preceq \hat{t}$, $g(\hat{x}^*; t^*)$ is weakly smaller than $g(\hat{x}^*; \hat{t})$. By definition, this gives: $\forall x \in g(\hat{x}^*; \hat{t})$, $\exists \bar{x} \in g(\hat{x}^*; t^*)$ with $\bar{x} \preceq x$. In particular, since $\hat{x}^* \in g(\hat{x}^*; \hat{t})$, $\exists \bar{x} \in g(\hat{x}^*; t^*)$ with $\bar{x} \preceq \hat{x}^*$. By assumption, given x^* , $\hat{x}^* \prec x^* \Rightarrow \bar{x} \prec x^*$. Or, given $x^* \in g(x^*; t^*)$, $\exists \bar{x} \in g(\hat{x}^*; t^*)$ such that $\bar{x} \prec x^*$. (Note that the inequality need not be strict here for all components). The implication is that given any equilibrium $x^* \in g(x^*; t^*)$, there is some element $\bar{x} \in g(\hat{x}^*; t^*)$ which is lower. Call this result (a.).

(b.) Now, apply the property that at least 1 player has strict SS. Without loss of generality, let player 1 have strict strategic substitutes. By hypothesis, $\hat{x}^* \prec x^* \iff (\forall i, \hat{x}_i^* \preceq x_i^* \text{ and } \exists \bar{i}, \text{ such that } \hat{x}_{\bar{i}}^* \prec x_{\bar{i}}^*)$.

Case 1: What if $\exists \bar{i} \neq 1$ such that $\hat{x}_{\bar{i}}^* \prec x_{\bar{i}}^*$? Then, $\hat{x}_{-1}^* \prec x_{-1}^*$. Fix $t = t^*$. By strict strategic substitutes of Player 1, $g^1(x_{-1}^*; t^*)$ is strictly lower than $g^1(\hat{x}_{-1}^*; t^*) \iff$

for $\hat{x}_1 = g^1(x_{-1}^*; t^*)$, for $\bar{x}_1 = g^1(\hat{x}_{-1}^*; t^*)$, $\hat{x}_1 \prec \bar{x}_1$. Therefore, $\forall \hat{x} \in g(x^*; t^*)$, $\forall \bar{x} \in g(\hat{x}^*; t^*)$, $\bar{x} \not\prec \hat{x}$, since the first component of \hat{x} will always be smaller than the first component of \bar{x} . In particular, for $x^* \in g(x^*; t^*)$, $\bar{x} \not\prec x^*$, **for every** $\bar{x} \in g(\hat{x}^*; t^*)$. This directly contradicts result (a.).

Case 2: What if $\forall j \neq 1$, $\hat{x}_j^* = x_j^*$? Then we must have $\hat{x}_1^* \prec x_1^*$. But, $\hat{x}_{-1}^* = x_{-1}^*$. By g^1 as a singleton-valued best response, it must be such that $\hat{x}_1^* = x_1^*$; this contradicts the assumption that $\hat{x}_1^* \prec x_1^*$. ■

Proof under assumption (ii.)

Assume $\exists \hat{x}^* \in FP(\hat{t})$, $\exists x^* \in FP(t^*)$, such that $\hat{x}^* \prec x^*$. The proof of part (a.) is identical to the proof of part (a.) under assumption (i.).

(b.) Now, apply the property that at least 1 players has strict SS. Without loss of generality, let player 1 have strict strategic substitutes. By hypothesis, $\hat{x}^* \prec x^* \iff (\forall i, \hat{x}_i^* \preceq x_i^* \text{ and } \exists \bar{i}, \text{ such that } \hat{x}_{\bar{i}}^* \prec x_{\bar{i}}^*)$.

Case 1: What if $\exists \bar{i} \neq 1$ such that $\hat{x}_{\bar{i}}^* \prec x_{\bar{i}}^*$? Then, $\hat{x}_{-1}^* \prec x_{-1}^*$. Fix $t = t^*$. By strict strategic substitutes of Player 1, $g^1(x_{-1}^*; t^*)$ is strictly lower than $g^1(\hat{x}_{-1}^*; t^*) \iff \forall \hat{x}_1 \in g^1(x_{-1}^*; t^*)$, $\forall \bar{x}_1 \in g^1(\hat{x}_{-1}^*; t^*)$, $\hat{x}_1 \prec \bar{x}_1$. Therefore, $\forall \hat{x} \in g(x^*; t^*)$, $\forall \bar{x} \in g(\hat{x}^*; t^*)$, $\bar{x} \not\prec \hat{x}$, since the first component of \hat{x} will always be smaller than the first component of \bar{x} . In particular, for $x^* \in g(x^*; t^*)$, $\bar{x} \not\prec x^*$, **for every** $\bar{x} \in g(\hat{x}^*; t^*)$. This directly contradicts result (a.).

Case 2: What if $\forall j \neq 1$, $\hat{x}_j^* = x_j^*$? Then we must have $\hat{x}_1^* \prec x_1^*$. For player 1, then, $\hat{x}_{-1}^* = x_{-1}^* = \bar{x}_{-1}$ (so we fix x_{-1} , in essence). Now, $t^* \preceq \hat{t} \Rightarrow g^1(\bar{x}_{-1}; t^*)$ is completely lower than $g^1(\bar{x}_{-1}; \hat{t})$. So, $\forall x_1 \in g^1(\bar{x}_{-1}; t^*)$, $\forall x'_1 \in g^1(\bar{x}_{-1}; \hat{t})$, $x_1 \preceq x'_1$. In particular, $x_1^* \preceq \hat{x}_1^*$, which we can apply since \bar{x}_{-1} has been fixed equal to its value at both t^* and \hat{t} ; however, this contradicts the above statement that we must have $\hat{x}_1^* \prec x_1^*$. ■

Proof under assumption (iii.)

Assume $\exists \hat{x}^* \in FP(\hat{t})$, $\exists x^* \in FP(t^*)$, such that $\hat{x}^* \prec x^*$. This proof then proceeds identically to the Proof under assumption (ii.), with the only difference applied to Case 2 as follows:

Case 2: What if $\forall j \neq 1$, $\hat{x}_j^* = x_j^*$? Then we must have $\hat{x}_1^* \prec x_1^*$. But, we know that there is a second player, say player 2 (without loss of generality), who also has strategic substitutes. In particular, $\hat{x}_{-2}^* \prec x_{-2}^*$. Fix $t = t^*$. Then, $g^2(x_{-2}^*; t^*)$ is strictly lower than $g^2(\hat{x}_{-2}^*; t^*) \iff \forall \hat{x}_2 \in g^2(x_{-2}^*; t^*)$, $\forall \bar{x}_2 \in g^2(\hat{x}_{-2}^*; t^*)$, $\hat{x}_2 \prec \bar{x}_2$. Therefore, as in case 1, $\forall \hat{x} \in g(x^*; t^*)$, $\forall \bar{x} \in g(\hat{x}^*; t^*)$, $\bar{x} \not\prec \hat{x}$, since now the **second** component of \hat{x} will always be smaller than the second component of \bar{x} . In particular, for $x^* \in g(x^*; t^*)$, $\bar{x} \not\prec x^*$, **for every** $\bar{x} \in g(\hat{x}^*; t^*)$. This directly contradicts result (a). ■

6.2. Proof of Theorem 2*.

Proof. (\implies) It can be shown under these assumptions,³² in a similar fashion to Lemma 1, that $\hat{x}_2^* \succeq x_2^*$. And by assumption, for some x_1^* , $\exists \tilde{x}_1 \in g_{\hat{t}}^1(\text{inf}g_{\hat{t}}^2(x_1^*))$ with $\tilde{x}_1 \succeq x_1^* \implies \text{sup}(g_{\hat{t}}^1(\text{inf}g_{\hat{t}}^2(x_1^*))) \not\prec x_1^* \implies \hat{x}_1^* \not\prec x_1^*$, for any \hat{x}_1^* . To see the last implication here, notice the following: $\hat{x}_1^* \prec x_1^* \implies \text{sup}g_{\hat{t}}^1(\text{inf}g_{\hat{t}}^2(x_1^*)) \prec x_1^*$. Now, let $\bar{t} = \hat{t}$. This gives $\hat{x}_2^* \prec \text{inf}g^2(x_1^*, \hat{t}) \implies g_{\hat{t}}^1(\hat{x}_2^*)$ strictly higher than $g_{\hat{t}}^1(\text{inf}g^2(x_1^*, \hat{t}))$, $\forall \bar{t}$. Again, let $\bar{t} = \hat{t}$. This gives $g_{\hat{t}}^1(\hat{x}_2^*)$ strictly higher than $g_{\hat{t}}^1(\text{inf}g^2(x_1^*, \hat{t})) \implies x_1^* \succ \hat{x}_1^* \succ \text{sup}g_{\hat{t}}^1(\text{inf}g_{\hat{t}}^2(x_1^*)) \implies x_1^* \succ \text{sup}g_{\hat{t}}^1(\text{inf}g_{\hat{t}}^2(x_1^*))$. That is, $\hat{x}_1^* \prec x_1^* \implies x_1^* \succ \text{sup}g_{\hat{t}}^1(\text{inf}g_{\hat{t}}^2(x_1^*))$. By linearly ordered X_1 , this gives $\hat{x}_1^* \succeq x_1^*$, again for any \hat{x}_1^* .

(\Leftarrow) $\hat{x}^* \succeq x^* \implies \hat{x}_1^* \succeq x_1^* \implies g_{\hat{t}}^2(\hat{x}_1^*)$ is strictly higher than $g_{\hat{t}}^2(x_1^*) \implies \hat{x}_2^* \succ \text{sup}g_{\hat{t}}^2(x_1^*) \succeq \text{inf}g_{\hat{t}}^2(x_1^*)$. This implies $g_{\hat{t}}^1(\hat{x}_2^*)$ is strictly lower than $g_{\hat{t}}^1(\text{inf}g_{\hat{t}}^2(x_1^*))$, which gives $x_1^* \preceq \hat{x}_1^* \prec \tilde{x}_1$, $\forall \tilde{x} \in g_{\hat{t}}^1(\text{inf}g_{\hat{t}}^2(x_1^*))$, and completes the proof. ■

³²Given some x^* , \hat{x}^* , since strategy spaces are linearly ordered, only 2 cases need to be considered:

- (i.) $\hat{x}_1^* \prec x_1^*$. Here, since player 1 has strict strategic substitutes and x_1 is strictly complementary in t , by Theorem 1, $\hat{x}_2^* \not\prec x_2^* \implies \hat{x}_2^* \succeq x_2^*$.
- (ii.) $\hat{x}_1^* \succeq x_1^* \implies g_{\hat{t}}^2(\hat{x}_1^*)$ is strictly larger (in the set sense) than $g_{\hat{t}}^2(x_1^*)$ by strict strategic complements. By t complementary for player 2, $g_{\hat{t}}^2(x_1^*) \supseteq g_{\hat{t}^*}^2(x_1^*) \implies g_{\hat{t}}^2(\hat{x}_1^*)$ is strictly higher than $g_{\hat{t}^*}^2(x_1^*) \implies \hat{x}_2 \succ x_2^*$, $\forall \hat{x}_2 \in g_{\hat{t}}^2(\hat{x}_1^*)$, $\forall x_2^* \in g_{\hat{t}^*}^2(x_1^*)$. ■

6.3. Additional counterexample. The counterexample at the onset of section 4 demonstrates why the condition of Theorem 2 will not apply to a simple extension with 3 players instead of 2. It does so in the context of a game with two strategic substitutes players and one strategic complements player. This counterexample shows that even with linearly ordered strategy spaces and singleton-valued best responses, a characterization on the second iterate best response for strategic substitutes players is similarly not sufficient to guarantee an increase in equilibrium, in the case where there is more than 1 strategic substitutes player.

Take a 3-player game, similar to the counterexample mentioned above. Here, allow players 1 and 2 to both be criminals, while player 3 is the lone law enforcement player. Allow player 1 to be big-time criminal, who chooses a level of criminal activity from $X_1 = \{a_1, b_1, c_1, d_1\}$, with $a_1 \prec b_1 \prec c_1 \prec d_1$. Player 2 can only engage in smaller-scale criminal activities, and thus chooses merely between a_2 (low) and b_2 (high), $a_2 \prec b_2$. Player 3 can, as above, choose between low and high levels of law enforcement. In this context, players 1 and 2 have strategic substitutes, with player 1 having strict strategic substitutes, while player 3 has strategic complements.

The original game is in the first figure below, with an equilibrium at $x^* = (c_1 a_2 a_3)$. Now, given an increase in a complementary parameter t , it is possible to arrive at a new game, given in the second figure below. To see if the second iterate condition of Theorem 2 were to work here, it is straightforward to calculate: $\hat{y}_1 = g_t^1(a_2 a_3) = d_1$, $\hat{y}_2 = g_t^2(c_1 a_3) = b_2$, and $\hat{y}_3 = g_t^3(c_1 a_2) = a_3$. Therefore, the condition of Theorem 2 would need to hold for both players 1 and 2: $g_t^1(\hat{y}_2, \hat{y}_3) = c_1 \succeq c_1 = x_1^*$; $g_t^2(\hat{y}_1, \hat{y}_3) = a_2 \succeq a_2 = x_2^*$. Indeed, the condition would be satisfied, suggesting an increase in equilibrium should take place. However, as the figure demonstrates, the new equilibrium at \hat{t} , $\hat{x}^* = (b_1 b_2 b_3)$, has not increased.

6.4. Proof of Corollary 1.

Proof. For each player $l = 1, \dots, k$, define $B_l = [x_l^*, \hat{y}_l]$; for each player $j = k + 1, \dots, I$, define $B_j = [x_j^*, \bar{x}_j]$. Let their products be defined naturally: $B = \prod_{i=1}^I B_i$ and $B_{-i} = \prod_{i' \neq i} B_{i'}$. To apply Schauder's fixed point theorem, it remains to be shown that the joint best response function $g : X \rightarrow X$ is indeed a self-map from B to B .

	a ₂	b ₂
a ₁	1, 5, 5	1, 5, 2
b ₁	4, 1, 2	4, 0, 2
c ₁	6, 1, 2	3, 0, 2
d ₁	3, 1, 0	2, 0, 0

a₃ (low)

	a ₂	b ₂
a ₁	1, 1, 0	1, 5, 0
b ₁	4, 1, 0	0, 0, 0
c ₁	3, 1, 0	0, 0, 0
d ₁	2, 1, 10	0, 0, 10

b₃ (high)

FIGURE 16. Counterexample: original parameter t .

	a ₂	b ₂
a ₁	1, 1, 5	1, 5, 2
b ₁	4, 1, 2	4, 4, 2
c ₁	6, 1, 2	7, 4, 2
d ₁	10, 1, 0	6, 0, 0

a₃ (low)

	a ₂	b ₂
a ₁	1, 1, 0	1, 5, 4
b ₁	4, 1, 0	2, 4, 4
c ₁	7, 1, 0	0, 4, 4
d ₁	6, 1, 10	0, 0, 10

b₃ (high)

FIGURE 17. Counterexample: higher parameter \hat{t} .

For each player $l = 1, \dots, k$, $g_t^l(x_{-l}^*) = \hat{y}_l \succeq x_l^*$. In addition, $g_t^l((\hat{y}_{l'})_{l' \neq l}, (\bar{x}_j)_{j=k+1}^I) \succeq x_l^*$ by assumption. Therefore, for each $l = 1, \dots, k$, $g_t^l(B_{-l}) \subset B_l$. Now, consider each player $j = k + 1, \dots, I$. First, $g_t^j(x_{-j}^*) = \hat{y}_j \succeq x_j^*$. In addition, $g_t^j((\hat{y}_l)_{l=1}^k, (\bar{x}_{j'})_{j' \neq j}) \preceq \bar{x}_j$, since by definition, it must be lower than the highest strategy player j can play.

Therefore, for each $j = k + 1, \dots, I$, $g_{\hat{t}}^j(B_{-j}) \subset B_j$. Hence, $g_{\hat{t}} : B \rightarrow B$ is indeed a self-map; by Schauder's Theorem, there exists a fixed point, $\hat{x}^* = g_{\hat{t}}(\hat{x}^*)$, such that $\hat{x}^* \succ x^*$. ■