ABSTRACT. Generally we can distinguish between two types of comparative statics problems. The first type of problem considers the change of the optimal solution to a maximization problem as the objective function changes, the other type the change due to a change in the constraint set. Lattice-based comparative statics theorems have been developed for both types of problems in the literature. The strengths of these lattice-theoretic comparative statics results are that they don't depend on the usual smoothness, interiority and concavity assumptions as required by the classical approach based on the Implicit Function Theorem, as well as convexity of the constraint set. Moreover, these comparative statics results also apply in the case of non-unique solutions.

Quah (2007) expanded existing results by Milgrom and Shannon (1994) by making them applicable to some non-lattice constraint sets. In this paper, I extend existing comparative statics theorems to parameterized objective functions and non-lattice constraint sets. This generalization makes it possible to analyze a variety of economic optimization problems that fall into this class of problems that cannot be addressed using existing lattice-based techniques. I provide examples from consumer theory, producer theory and environmental economics that show the result’s broad scope of applications.

JEL: C61, D11, D21, Q58

Keywords: Lattices, comparative statics, supermodularity, Single Crossing Property, consumer problem, LeChatelier principle
1. Introduction

Comparative statics of constrained optimization problems is a question at the heart of economic analysis. Oftentimes we are not only interested in the optimizers of the problem themselves, but also how they will be affected by changes in exogenous parameters. Generally, we distinguish between two types of comparative statics problems. The first type of problem considers the change of the optimal solution to a maximization problem as the objective function changes, the other type the change due to a change in the constraint set.

Comparative statics theorems based on lattice programming methods have been developed for both types of problems. The strengths of these results are that they don’t depend on the usual smoothness, interiority and concavity assumptions as required by the classical approach based on the Implicit Function Theorem, as well as convexity of the constraint set. Moreover, these comparative statics results also apply in the case of non-unique solutions.

Based on Topkis (1978), Vives (1990) and Milgrom and Roberts (1990) use a lattice-based approach to establish monotone increasing best responses in games with strategic complements under the cardinal assumptions of supermodularity and increasing differences on the objective function when strategy spaces are lattices. Milgrom and Roberts (1990) also focus on monotone comparative statics results under these cardinal assumptions in supermodular games. Milgrom and Shannon (1994) then extend these results to the ordinal case. They provide necessary and sufficient conditions on the primitives for nondecreasing solutions in the parameters of the problem, generalizing the previously used cardinal counterparts. In many cases however, constraint sets in economic optimization problems, such as the budget set in
the consumer problem, are not lattices and therefore Milgrom and Shannon’s result cannot be applied. Quah (2007) addresses these types of problems and considers comparative statics with respect to changes in non-lattice constraint sets. His result provides necessary and sufficient conditions for nondecreasing solutions using a weaker set order, that in particular, can be used to establish normality of demand under assumptions on the primitives.

The contribution of this paper is that it extends Quah’s result to include both non-lattice constraint sets and parameterized objective functions. The result shows its power through many applications.

Quah (2007) generalizes the monotonicity theorem of Milgrom and Shannon (1994) to some types of non-lattice constraint sets, i.e. the consumer’s budget set. However, his result does not address comparative statics with respect to parameters in the objective function. In economics, we naturally encounter a variety of questions that involve parameter changes in the objective function, such as changes in consumer preferences or technology shocks that affect production costs, while the constraint sets are not lattices. My result extends Quah’s work to these types of optimization problems by including parameterized objective functions.

Importantly, this generalization allows us to analyze a variety of economic optimization problems that fall into this class of problems that cannot be addressed using existing lattice-based techniques. A natural example from consumer theory is the consumer’s utility maximization problem with Stone-Geary preferences. The question, how income and essential consumption basket changes affect consumer demand, for example, cannot be addressed by existing results. My comparative statics theorem provides necessary and sufficient conditions for nondecreasing demand. Another
application from consumer theory is the question whether the stakes in a dictator game with inequality aversion preferences matter and my result provides conditions on the primitives when this is the case. In producer theory, my result can be applied to multiple-plant production problems and price discrimination with capacity constraints to answer the question when production quantities are nondecreasing in demand or cost parameters and the constraint. Many other applications can be found in the area of environmental economics such as production regulation through emissions standards and cost-efficient emissions regulation. Here the result provides necessary and sufficient conditions for nondecreasing factor demand or emissions reduction as demand, cost parameters and emissions standards change. Moreover, I can use this comparative statics result to generalize known lattice-based versions of the LeChatelier principle, that expresses the idea that long run factor demand is more responsive to price changes than in the short run.

The remainder of the paper is organized as follows. Section 2 introduces the theoretical framework, section 3 gives the generalized monotone comparative statics results and section 4 provides a variety of applications.
2. Theoretical Background

To obtain monotone comparative statics results with respect to changes in the constraint set we need to be able to order these sets. Milgrom and Shannon (1994) uses the strong set order by Veinott (1989), while Quah (2007) introduces a weaker concept to extend the comparative statics results from Milgrom and Shannon (1994) to problems with non-lattice constraint sets. This case is easily encountered even in simple optimization problems like the consumer’s utility maximization problem.

**Definition 1.** Set Orders

1. **Strong Set Order**

   Let $X$ be a lattice and consider two subsets $S$ and $S'$. Then $S'$ dominates $S$ in the strong set order ($S' \succeq_S S$) if and only if for all $x \in S$ and for all $y \in S'$, $x \land y \in S$ and $x \lor y \in S'$.

2. **$(\Delta, \nabla)$-induced Strong Set Order**

   Let $X$ be a partially ordered set and consider two non-empty subsets of $X$, $S$ and $S'$. The operations $\Delta$ and $\nabla$ induce the following set order on $S$ and $S'$. $S'$ dominates $S$ by the $(\Delta, \nabla)$-induced strong set order ($S' \succeq_{\Delta, \nabla} S$) if and only if for all $x \in S$ and for all $y \in S'$, $x\Delta y \in S$ and $x\nabla y \in S'$.

3. **$C_i(C)$-flexible Set Order**

   Let $X \subseteq \mathbb{R}^l$ be a convex set and define the operations $\nabla^\lambda_i$ and $\Delta^\lambda_i$ on $X$ as follows for $\lambda$ in $[0,1]$:

   $$x \nabla^\lambda_i y = \begin{cases} 
y & \text{if } x_i \leq y_i \\
\lambda x + (1 - \lambda)(x \lor y) & \text{if } x_i > y_i
\end{cases}$$
\[ x \Delta_i^\lambda y = \begin{cases} 
  x & \text{if } x_i \leq y_i \\
  \lambda y + (1 - \lambda)(x \land y) & \text{if } x_i > y_i 
\end{cases} \]

(a) Let \( S' \) and \( S \) be subsets of the convex sublattice \( X \). Then \( S' \) dominates \( S \) in the \( C_i \)-flexible set order \((S' \succeq_i S)\) if for any \( x \) in \( S \) and \( y \) in \( S' \), there exists \((\nabla_i^\lambda, \Delta_i^\lambda)\) in \( C_i \) such that \( x \nabla_i^\lambda y \) is in \( S' \) and \( x \Delta_i^\lambda y \) is in \( S \).

(b) \( S' \) dominates \( S \) in the \( C \)-flexible set order \((S' \succeq S)\) if \( S' \succeq_i S \) for all \( i \).

To graphically add some intuition to the operations \( \nabla_i^\lambda \) and \( \Delta_i^\lambda \), the points \( x \), \( y \), \( x \Delta_i^\lambda y \) and \( x \nabla_i^\lambda y \) form a backward-bending parallelogram instead of the rectangle generated by \( x \), \( y \) and their join and meet (see Figure 2.1).

\textbf{Figure 2.1.}

Quah (2007) describes the requirements of the \( C_i \)-flexible set order as that for any pair of unordered points \( x \in S \) and \( y \in S' \), I can find \( x \Delta_i^\lambda y \) and \( x \nabla_i^\lambda y \) such that
$x\Delta^\lambda_i y$ is in $S$ and $x\nabla^\lambda_i y$ is in $S'$, with the four points forming a backward-bending parallelogram.

Besides an order on the constraint sets, we also need assumptions on the objective function for monotone comparative statics. The monotonicity theorem in Milgrom and Shannon (1994) requires quasisupermodularity, which can be interpreted as weak complementarity between the choice variables. Quah (2007) introduces a similar property needed when extending Milgrom and Shannon’s result to a non-lattice context.

**Definition 2.** $(\Delta, \nabla)$-(Quasi)Supermodularity

Let $X, T$ be partially ordered sets and let $\Delta$ and $\nabla$ be two operations on $X$. Consider a function $f : X \to \mathbb{R}$. Then I can define the following properties for $f$.

1. $f$ is $(\Delta, \nabla)$-supermodular if for all $x, y \in X$, $f(x \nabla y) - f(y) \geq f(x) - f(x \Delta y)$.
2. $f$ is $(\Delta, \nabla)$-quasisupermodular if for all $x, y \in X$, $f(x) \geq (> f(x \Delta y) \Rightarrow f(x \nabla y) \geq (> f(y)$.

Analogously to the previous case, Quah defines properties of a function $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^l$ is a convex set, with respect to the operations $\nabla^\lambda_i, \Delta^\lambda_i$. 

Definition 3. $C_i$-(Quasi)Supermodularity

1. A function is $(\nabla_\lambda^i, \Delta_\lambda^i)$-supermodular for some $\lambda \in [0, 1]$ if $f(x\nabla_\lambda^iy) - f(y) \geq f(x) - f(x\Delta_\lambda^i y)$ for all $x, y$ in $X$.

2. A function $f : X \to R$ is $C_i$-supermodular if it is $(\nabla_\lambda^i, \Delta_\lambda^i)$-supermodular for all $(\nabla_\lambda^i, \Delta_\lambda^i)$ in $C_i = \{ (\nabla_\lambda^i, \Delta_\lambda^i) : \lambda \in [0, 1] \}$. 
   If $f$ is $C_i$-supermodular for all $i$, then it is $C$-supermodular.

3. A function $f : X \to R$ is $C_i$-quasisupermodular if it is $(\nabla_\lambda^i, \Delta_\lambda^i)$-quasisupermodular for all $(\nabla_\lambda^i, \Delta_\lambda^i)$ in $C_i = \{ (\nabla_\lambda^i, \Delta_\lambda^i) : \lambda \in [0, 1] \}$, that is $f(x) \geq (>)f(x\Delta_\lambda^i y) \Rightarrow f(x\nabla_\lambda^i y) \geq (>)f(y)$.
   If $f$ is $C_i$-quasisupermodular for all $i$, then it is $C$-quasisupermodular.

Intuitively, Definition 3 (1) requires the function’s value to increase more along the right side than the left side of the parallelogram formed by these four points. Definition 3 (3) says that for all possible parallelograms that can be formed based on $x$ and $y$, if the function is nondecreasing along the left side of the parallelogram, then it will also be nondecreasing along the right side.

Quah (2007) also shows that $C_i$-quasisupermodularity follows from properties that can easily be verified. The first one is supermodularity, the other a form of concavity.

Definition 4. $i$-Concavity

A function $f$ is $i$-concave if it is concave in direction $v$ for any $v > 0$ with $v_i = 0$. 

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While Quah (2007) addresses comparative statics with regard to changes in the constraint set using this framework, Milgrom and Shannon (1994) also include comparative statics with respect to parameter changes in the objective function. Their result requires an additional assumption on the objective function, which is that for a parametrized objective function, \( f : X \times T \to \mathbb{R} \), where \( X \) is a lattice and \( T \) is a partially ordered set, \( f \) needs to satisfy the Single Crossing Property in \((x,t)\). To work in the more general, non-lattice context, I introduce new versions of the Single Crossing Property to use throughout the remainder of the paper.

**Definition 5.** Single Crossing Property

(1) Standard Single Crossing Property

\( f \) satisfies the Single Crossing Property (SCP) if for every \( x \leq y \) and for every \( t \leq t' \), \( f(y,t) \geq f(x,t) \Rightarrow f(y,t') \geq f(x,t') \).

(2) \((\Delta, \nabla)\)-Single Crossing Property

\( f \) satisfies the \((\Delta, \nabla)\)-Single Crossing Property (\((\Delta, \nabla)\)-SCP) if for every \( x, y \) with \( x \leq_{(\Delta, \nabla)} y \) and for every \( t \leq t' \), \( f(y,t) \geq f(x,t) \Rightarrow f(y,t') \geq f(x,t') \).

(3) \(i\)- Single Crossing Property

\( f \) satisfies the \(i\)-Single Crossing Property (\(i\)-SCP) if for every \( x, y \) with \( x_i \leq y_i \) and for every \( t \leq t' \), \( f(y,t) \geq f(x,t) \Rightarrow f(y,t') \geq f(x,t') \).

For twice continuously differentiable functions \( f \), the \(i\)-Single Crossing Property (\(i\)-SCP) is implied by \( \frac{\partial^2 f}{\partial x_i \partial t} \geq 0 \).

\[ \frac{\partial^2 f}{\partial x_i \partial t} \geq 0 \]

The standard Single Crossing Property is implied by increasing differences, which for twice continuously differentiable functions \( f \) is equivalent to \( \frac{\partial^2 f}{\partial x_i \partial t} \geq 0 \) for all \( i \). As the \(i\)-Single Crossing Property only focuses on the \(i\)-th component, it is implied by \( \frac{\partial^2 f}{\partial x_i \partial t} \geq 0 \).
3. Comparative Statics

Consider the following general constrained optimization problem:

$$\max f(x, t) \text{ subject to } x \in S$$

For convenience, denote $M(t, S) := \arg\max_{x \in S} f(x, t)$.

The main theorem of this paper generalizes the main monotone comparative statics result in Quah (2007) for optimization problems with constraint sets that can be ordered by the $C_i$-flexible set order to problems with parametrized objective functions. I provide necessary and sufficient conditions for nondecreasing optimal solutions for parameter changes in the objective function and the above mentioned type of constraint sets; thus the main theorem of the paper provides conditions on the objective function that characterize nondecreasing optimal solutions. The proofs follow Milgrom and Shannon (1994) with some simple modifications.

**Theorem 1.** $f(x, t)$ is $C_i$-quasisupermodular and has the $i$-Single Crossing Property in $(x, t)$ if and only if whenever $S' \succeq_i S$ and $t' \geq t$, $\arg\max_{x \in S'} f(x, t') \geq_i \arg\max_{x \in S} f(x, t)$ (argmax $f(x, t)$ is nondecreasing in $(t, S)$).

**Proof.** ($\Rightarrow$) Suppose $S' \succeq_i S$ for $t' \geq t$ and let $x \in M(t, S)$, $y \in M(t', S')$. Since $x \in M(t, S)$ and $S \preceq_i S'$, $f(x, t) \geq f(x \Delta^*_i y, t)$. By $C_i$-quasisupermodularity of $f$, this implies $f(x \nabla^*_i y, t) \geq f(y, t)$ with $x_i > y_i$. As $f$ also has the $i$-Single Crossing Property in $(x, t)$, $f(x \nabla^*_i y, t) \geq f(y, t) \Rightarrow f(x \nabla^*_i y, t') \geq f(y, t')$ for $t' \geq t$. Since $y \in M(t', S')$ it follows that $x \nabla^*_i y \in M(t', S')$. 

Now suppose \( x \Delta^i y \notin M(t, S) \) and hence \( f(x, t) > f(x \Delta^i y, t) \). \( C_i \)-quasisupermodularity of \( f \) implies \( f(x \nabla^i y, t) > f(y, t) \) and by the \( i \)-Single Crossing Property it follows that \( f(x \nabla^i y, t') > f(y, t') \) for any \( t' \geq t \). This contradicts the assumption that \( y \in M(t', S') \). Therefore, \( x \Delta^i y \in M(t, S) \) and \( M(t, S) \leq_i M(t', S') \).

\((\Leftarrow)\) Fix \( t \). Let \( x \) and \( y \) be two elements in \( X \) and suppose that \( f \) is not \( C_i \)-quasisupermodular for some \( \lambda^* \in [0, 1] \). The only case I need to look at is when \( x_i > y_i \) and \( x \) and \( y \) are unordered. Also, \( x \Delta^i y \neq x \) and \( x \nabla^i y \neq y \).

Let \( S = \{ x, x \Delta^i y \} \) and \( S' = \{ y, x \nabla^i y \} \). Then \( S' \geq_i S \). \( C_i \)-quasisupermodularity of \( f \) can be violated in the following two ways. First, suppose \( f(x, t) \geq f(x \Delta^i y, t) \), but \( f(x \nabla^i y, t) < f(y, t) \). In this case \( x \) is a maximizer of \( f \) in \( S \) and \( y \) maximizes \( f \) uniquely in \( S' \), which violates the \( i \)-increasing property (since \( x_i > y_i \)). Alternatively, suppose \( f(x, t) > f(x \Delta^i y, t) \), but \( f(x \nabla^i y, t) = f(y, t) \). Now \( y \) maximizes \( f \) in \( S' \) while \( x \) is the unique maximizer in \( S \). This again contradicts the \( i \)-increasing property. So \( f \) is \( C_i \)-quasisupermodular.

Now let \( S \equiv \{ x, \bar{x} \} \) with \( x_i \leq \bar{x}_i \). Then \( f(\bar{x}, t) - f(x, t) \geq 0 \) implies \( \bar{x} \in M(t, S) \). Since \( M(t, S) \leq_i M(\bar{t}, S) \) for \( \bar{t} \geq t \) it follows that \( f(\bar{x}, \bar{t}) - f(x, \bar{t}) \geq 0 \) for all \( \bar{t} \geq t \). Thus \( f \) has the \( i \)-Single Crossing Property. \( \square \)

Notice that the solution to the optimization may not be unique, as I have made no assumptions to guarantee uniqueness. In the case of a solution set, \( \argmax_{x \in S'} f(x, t') \) dominates \( \argmax_{x \in S} f(x, t) \) in the \( C_i \)-flexible set order. By Proposition 3 in Quah (2007), this implies that \( \argmax_{x \in S'} f(x, t') \) is \( i \)-higher than \( \argmax_{x \in S} f(x, t) \).

\(^2\)Quah (2007) defines that a set \( S' \) is \( i \)-higher than a set \( S \), if whenever both sets are nonempty, for any \( x \in S \) there exists \( x' \in S' \) such that \( x'_i \geq x_i \) and for any \( x' \in S' \) there exists \( x \in S \) such that \( x'_i \geq x_i \).
Similarly, consider the general case, where $\Delta$ and $\nabla$ are two operations on $X$. This result gives necessary and sufficient conditions for nondecreasing optimal solutions using the $(\Delta, \nabla)$-Single Crossing Property for operations $\Delta$ and $\nabla$ on $X$.

**Theorem 2.** $f(x, t)$ is $(\Delta, \nabla)$-quasisupermodular and has the $(\Delta, \nabla)$-Single Crossing Property in $(x, t)$ if and only if whenever $S'$ dominates $S$ in the $(\Delta, \nabla)$-induced set order ($S' \geq_{(\Delta, \nabla)} S$) for $t' \geq t$, $\argmax_{x \in S'} f(x, t') \geq_{(\Delta, \nabla)} \argmax_{x \in S} f(x, t)$ ($\argmax_{x \in S} f(x, t)$ is nondecreasing in $(t, S)$).

**Proof.** See Appendix. $\square$

To demonstrate the applicability of his result, Quah (2007) also shows that his new concept of $C_i$-supermodularity arises from the combination of supermodularity and a form of concavity, two assumptions that are reasonable under many circumstances.

**Proposition 1.** (Quah)

The function $f : X \rightarrow R$ is $C_i$-supermodular if it is supermodular and $i$-concave.

This proposition allows us to easily verify $C_i$-supermodularity, which is important in applications. While Quah’s result is able to address problems with non-lattice constraint sets, the number of applications remains limited as it does not include commonly occurring parameterized objective functions. My monotone comparative statics theorem is able to address these types of problems and opens the door for a variety of applications.
4. Applications

Applications for my monotone comparative statics result can be found in many different areas of economics. In consumer theory, parameterized utility functions such as Stone-Geary preferences of inequality aversion preferences are very natural applications. Multi-plant production allocation and price discrimination with capacity constraints are examples from producer theory. Many applications can also be found in environmental economics, i.e. production decisions and technological change under emissions standards, the effect of changes in the ethanol quota and the cost-efficient emissions allocation. Finally I also provide a lattice-based LeChatelier Principle that generalizes existing versions.

4.1. Parametrized Consumer Utility Maximization Problem. The standard utility maximization problem as discussed in Quah (2007) can be extended by parametrization of the utility function by some parameter $\theta$. The parameter vector of this problem is two-dimensional, $t = (\theta, w)$.

As shown in Quah (2007), the budget set for $w' \geq w$ does not dominate the initial one by the strong set order, because the join of two arbitrary elements may lie outside of the larger set. However, $B(p, w')$ does dominate the smaller budget set $B(p, w)$ in the $C$-flexible set order as illustrated in Figure 4.1.

For my comparative statics theorem to apply, the utility function needs to satisfy $C\gamma$-quasisupermodularity and the $i$-Single Crossing Property. If these conditions are satisfied, by Theorem 1, we have nondecreasing solutions to the consumer’s utility maximization problem.
The class of Stone-Geary utility functions is an example of such parametrized utility functions and in the following possible interpretations of the parameter $\theta$ will be discussed.

**Example 1. Stone-Geary Utility**

Consider utility functions of the form $u(x) = \sum_{i=1}^{n} \alpha_i \log(x_i - b_i)$ with $\alpha_i > 0$, $x_i - b_i > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. In this class of utility functions, the parameters $b_1, ..., b_n$ can be interpreted as a necessary consumption basket. An interesting question in this context is how the consumer’s demand for a good changes if his income increases, but so does his necessary consumption basket. Thus, $t = (b, w) \leq t' = (b', w')$. Theorem \[\Box\] readily provides comparative statics results for this case. I can easily check that $u$ is $C_i$-quasisupermodular\[^3\] and satisfies the $i$-Single Crossing Property\[^4\]. Thus, since

\[^3\]u is $C_i$-quasisupermodular if it is supermodular and $i$-concave. In this case, I see that $u$ is supermodular since $\frac{\partial^2 u}{\partial x_i \partial x_j} = 0$, $\forall i, j$, $i \neq j$. Moreover, $u$ is $i$-concave as $\frac{\partial^2 u}{\partial x_i^2} = -\frac{\alpha_i}{(x_i - b_i)^2} \leq 0$, $\forall j \neq i$.

\[^4\] $\frac{\partial^2 u}{\partial x_i \partial b_i} = \frac{\partial^2 u}{\partial x_i \partial b_i} = \frac{\alpha_i}{(x_i - b_i)^2} \geq 0$. 

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**Figure 4.1. Budget set**
\[ B(p, w') \geq_i B(p, w) \], my result guarantees nondecreasing demand for good \( i \) as income and necessary consumption basket go up. Notice that this comparative statics result also applies for changes in the parameter vector \( b \) only. Then my theorem yields nondecreasing demand in \( b \) as income remains unchanged. This type of comparative statics could not have been analyzed using earlier lattice-based results.

An application of Stone-Geary preferences can be found in Harbaugh (1998), that analyzes the prestige motive behind charitable donations. The paper considers two possible types of benefits to the donor, intrinsic benefit and prestige benefit. His utility hence depends on the amount of the public good \( x \) that he consumes, prestige \( p \) and amount donated \( d \). The donor’s utility maximization problem can then be written as

\[
\text{max } u(x, p, d) = \log x + b \log(p + k_1) + c \log(d + k_2)
\]

subject to the budget constraint \( x + qd \leq w \), where \( q \) is the after-tax price of giving and \( k_1 \) and \( k_2 \) are non-negative constants that capture how much the individual values prestige and his intrinsic benefit. In the framework of the above discussion of Stone-Geary utility, \(-k_1\) and \(-k_2\) are the necessary amounts of prestige and intrinsic benefit for the donor. The amount of prestige resulting from a donation depends on how charities report donations and how society converts these reports into prestige. Harbaugh (1998) considers different reporting schemes by the charities, one of which is exact reporting. In this case, prestige is equal to the amount donated \((p = d)\). The optimization problem can then be simplified to two variables of choice, the amount of the public good \( x \) and the donation \( d \). The main comparative statics question that
Harbaugh’s paper wants to address is the impact of the prestige motive on donations, but it also provides results for the effect of income.

Notice that the question how a person’s increased preference for prestige affects the amount donated cannot be addressed using existing monotone comparative statics results. Milgrom and Shannon’s monotonicity theorem applies to parameter changes in the objective functions when the constraint set is a lattice, which this budget set clearly is not. Quah’s result is able to address the isolated effect of an income change, but does not apply to parameter changes in the utility function and hence cannot answer the question regarding the prestige motive for donations.

My comparative statics theorem however is able to address parameter changes in the objective function when constraint sets are not lattices. As shown above, this class of utility functions satisfies $C_i$-quasisupermodularity and has the $i$-SCP in $-k_i$. Since $B(q, w') \geq_i B(q, w)$, the optimal donation is nondecreasing in $w$ and $-k_i$ by Theorem 1. Hence, if a person puts more emphasis on prestige (i.e. $-k_i$ increases) as his wealth increases, my comparative statics result yields nondecreasing donations, which is what the empirical analysis in Harbaugh (1998) concludes as well. I see from the data, that the model with the higher parameter estimate for prestige preference predicts more donations and more donations as income increasing. Overall, these effects are confirmed by the data.

**Example 2. Inequality Aversion**

Another type of preferences that my result nicely applies to is inequality aversion as described in Fehr and Schmidt (1999). Empirical evidence shows that people don’t
always exclusively pursue their self-interest, but also consider fairness in certain situations. Fehr and Schmidt (1999) propose preferences where the individuals’ utility depends on both their payoff and fairness; both advantageous and disadvantageous inequality negatively impact on their utility.

In their model, players experience inequity whenever they are better or worse off in material terms. Moreover, they dislike inequity that is to their disadvantage more than advantageous inequity.

For simplicity, consider the utility function in the 2 player case. Player \( i \)'s utility is given by

\[
u_i(x) = x_i - \alpha_i \max\{x_j - x_i; 0\} - \beta_i \max\{x_i - x_j; 0\},\]

where \( 0 \leq \beta_i < 1 \) and \( \beta_i \leq \alpha_i \).

The second and third term in this utility function measure the utility loss from disadvantageous and advantageous inequity respectively. \( \beta_i \leq \alpha_i \) ensures that player \( i \) suffers more if he is worse off than player \( j \) than if he is better off.

First consider the simple two player dictator game, where one player, the dictator, decides how to split up a given amount \( w \) into shares \( s_1 \) and \( s_2 \). To determine the (for him) optimal allocation, he maximizes his utility \( u_1(s_1, s_2) \) given the available amount \( s_1 + s_2 \leq w \). With the above specified linear inequity aversion and \( w = 1 \), the model predicts the extreme solutions of total fairness \( (s_1^* = s_2^* = \frac{w}{2}) \) if \( \beta_1 > \frac{1}{2} \) and no sharing at all \( (s_1^* = w) \) if \( \beta_1 < \frac{1}{2} \). Since this is not consistent with empirical findings, consider a modification of the above utility function that is concave in the amount of advantageous inequality, in which case optimal solutions between \( \frac{w}{2} \) and \( w \) are possible. For example, for \( s_1 \geq \frac{w}{2} \) the dictator solves

\[\frac{16}{16}\]

\[\small\text{According to Fehr and Schmidt (1999), offers of more than half were practically never observed. Therefore, I focus on the more relevant case where } s_1 \geq \frac{w}{2}.\]

\[\frac{5}{5}\]
\[
max u_1(s_1, s_2) = s_1 - \beta_1 (s_1 - s_2)^2
\]

subject to \( s_1 + s_2 \leq w \).

One question that has oftentimes been addressed in the literature is whether the stakes in such a game matter and people become more selfish as the stakes increase. While the earlier literature\(^6\) found no significant effect of the stakes on the dictator’s offer shares, some recent results in Blake and Rand (2010) and Novakova and Flegr (2013) suggest that if the stakes are sufficiently high, people will act more selfishly. In terms of the parameters of the dictator’s utility maximization problem, this corresponds to an increase in \( w \) and decrease in \( \beta_1 \) or an increase in the parameter vector \( t = (-\beta_1, w) \).

As the constraint set is not a lattice this problem cannot be addressed by the monotone comparative statics result in Milgrom and Shannon (1994); Quah’s result that allows for non-lattice constraint sets is not applicable to problems involving changes in a parameter in the objective function.

I can verify that the utility function is \( C_i \)-quasisupermodular\(^7\) and has the \( i \)-SCP in \( x_1 \). Moreover, the constraint set \( S(w') \) dominates \( S(w) \) in the \( C_i \)-flexible set order for \( w' \geq w \). Hence, Theorem \(^8\) applies and yields nondecreasing optimal amounts for the dictator to keep for himself.

Most of the experimental studies report shares rather than amounts. To account for that, normalize \( w \) to 1 and only consider an increase of \(-\beta_1\) as the stakes increase. In this case, the monotone comparative statics result yields nondecreasing optimal

---

\(^6\)see for example Forsythe et al. (1994), List and Cherry (2008)  
\(^7\)\( u \) is supermodular, as \( \frac{\partial^2 u}{\partial x_1 \partial x_2} = \frac{\partial^2 u}{\partial x_2 \partial x_1} = 2\beta_1 \geq 0 \). Moreover, \( u \) is \( i \)-concave, since \( \frac{\partial^2 u}{\partial x_i^2} = -2\beta_1 \leq 0 \). Hence \( u \) is \( C_i \)-quasisupermodular by Proposition 2 in Quah (2007).  
\(^8\)\( \frac{\partial^2 u}{\partial x_1 \partial (-\beta_1)} = 2 > 0 \)
shares for the dictator. Thus my result shows that the inequity aversion preferences are consistent with recent findings on high-stakes dictator games. As the \( \hat{s} \)-SCP does not hold for \( s_2 \) (\( \frac{\partial^2 u}{\partial s_i \partial (-t_1)} = -1 \)), the utility function does not have the standard SCP and therefore this result could not have obtained using Milgrom and Shannon’s result. However, the \( \hat{s} \)-SCP is satisfied for \( s_1 \) and my result yields nondecreasing optimal shares for the dictator as the stakes in the dictator game rise as some of the empirical studies find.

Thus inequity aversion preferences are another application where my monotone comparative statics result is able to theoretically replicate empirical findings. Another example of similar preferences can be found in Bolton and Ockenfels (2000). In their specification, an individual’s utility depends on their own payoff and their relative share. For the 2 player case, the utility function can be written as

\[
  u_i(c \sigma_i, \sigma_i) = a_i c \sigma_i - \frac{b_i}{2} (\sigma_i - \frac{1}{2})^2,
\]

where \( c \) denotes the total payout to both players and \( \sigma_i \) is player \( i \)’s relative share. The first term measures how much the player cares about his own payoff, the second term measures the loss of utility from unequal shares. I can easily verify that \( u_i \) is \( \hat{s} \)-quasisupermodular\(^9\) and satisfies the \( \hat{s} \)-SCP in \( a_i \) and \(-b_i\).\(^{10}\) Hence the comparative statics results from above can also be obtained using these preferences.

\(^9\) \( u_i \) is supermodular, as \( \frac{\partial^2 u_i}{\partial \sigma_i \partial \sigma_j} = 0 \) and \( \frac{\partial^2 u_i}{\partial \sigma_i \partial c} = a_i > 0 \).
\( u_i \) is \( \hat{s} \)-concave, since \( \frac{\partial^2 u_i}{\partial \sigma_i \partial \sigma_i} = -b_i < 0 \) and \( \frac{\partial^2 u_i}{\partial \sigma_i \partial c} = 0 \).
\(^{10}\) \( \frac{\partial^2 u_i}{\partial \sigma_i \partial a_i} = c > 0 \) and \( \frac{\partial^2 u_i}{\partial \sigma_i \partial (-b_i)} = \sigma_i - \frac{1}{2} \geq 0 \), as \( \sigma_i \in [\frac{1}{2}, 1] \) by Statement 1 and 3 in Bolton and Ockenfels (2000).
4.2. **Multiple-Plant Production.** Another example that my comparative statics result can be applied to is production allocation in a multiple-plant firm. Basic two firm models can be found in Patinkin (1947) and Sattler and Scott (1982). In the latter, the firm is allocating production of a given production target $\bar{q}$ between an old and a new plant, where the new plant has lower costs than the old one. The firm faces the following cost minimization problem:

$$\min \ C(q_{new}, q_{old}) = C_{new}(q_{new}) + C_{old}(q_{old})$$

subject to $q_{new} + q_{old} \geq \bar{q}$.

Both Patinkin (1947) and Sattler and Scott (1982) mostly focus on the “broken” marginal cost curve as the firm switches from only operating one plant to both plants as total output increases. my comparative statics theorem on the other hand adds more general insights under what assumptions on the cost function and exogenous changes the firm will increase production in both plants in the case that total output is such that both of them are in use.

**Example 3.** Since we are interested in comparative statics results, I include parameters $\omega_{new}$ and $\omega_{old}$ in the cost functions that measure the level of technology (with $\frac{\partial MC_i}{\partial \omega_i} \leq 0$) and rewrite the problem as

$$\max \ \tilde{C}(q_{new}, q_{old}, \omega_{new}, \omega_{old}) = -[C_{new}(q_{new}, \omega_{new}) + C_{old}(q_{old}, \omega_{old})]$$

subject to $q_{new} + q_{old} \geq \bar{q}$.

Now consider a situation where production technology in the old plant gets updated and the firm increases its total production target.

Hence, $t' = (\omega'_{new}, \omega'_{old}, \bar{q}') \geq t$, where $t'$ is the new and $t$ the initial parameter vector.
Since the constraint sets are not lattices, the comparative statics result by Milgrom and Shannon (1994) does not apply here. The result in Quah (2007), while allowing for non-lattice constraint sets, does not address parameter changes in the objective function. My result on the other hand is able to handle these types of problems.

We easily see that the new constraint set $S(\vec{q}')$ dominates the original one $S(\vec{q})$ in the $C_i$-flexible set order. Assuming that the cost function at each plant is convex in quantity, the objective function is $C_i$-quasisupermodular and satisfies the $i$-SCP for all $i$.\[1\] As a result, my comparative statics theorem yields nondecreasing optimal production quantities for either plant in this case.

4.3. **Price Discrimination with Capacity Constraints.** Another simple application of my comparative statics result can be found in the area of price discrimination with capacity constraints. In the airline and lodging industries, this problem is known as yield management. Belobaba (1987) summarizes yield management research in the airline industry; for examples from the lodging industry, see Hanks, Cross and Noland (1992). Reece and Sobel (2000) discusses the example of an airline that practices price discrimination while facing a capacity constraint\[12\] and addresses the question how airlines should adjust the allocation of seats between the customer groups as demand in one of the market segments increases or if costs of operation

---

\[1\] $C$ is $C_i$-quasisupermodular if it is supermodular and $i$-concave.

As each cost function is independent of the quantity produced at the other plant, $\frac{\partial^2 \tilde{C}}{\partial q_i \partial q_j} = 0$ for $i \neq j$, thus $\tilde{C}$ is supermodular. Moreover, assuming convex cost functions, $\frac{\partial^2 \tilde{C}}{\partial x_i^2} \leq 0$ for all $i$. Thus $\tilde{C}$ is $C_i$-quasisupermodular for all $i$.

Additionally, since marginal cost is nonincreasing in $\omega_i$ at both plants, $\frac{\partial^2 \tilde{C}}{\partial x_i \partial \omega_j} = -\frac{\partial MC_i}{\partial \omega_j} = 0$ for $i \neq j$ and $\frac{\partial^2 \tilde{C}}{\partial x_i \partial \omega_i} = -\frac{\partial MC_i}{\partial \omega_i} \geq 0$ for all $i$. Therefore the $i$-SCP holds.

\[12\] This multiple fare class problem is also briefly discussed in Belobaba (1989).
increase. They separately consider the cases of fixed non-binding and binding capacity as well as the possibility of capacity adjustment. They find that in the case of non-binding capacity constraints, changes in the marginal cost of operation directly affect prices and optimal quantities. In the case of binding capacity constraints, as long as marginal cost is below the point of intersection of the marginal revenue curves, changes in marginal cost do not affect the optimal capacity allocation; demand changes for one group however influence the optimal quantity and price of the other group. If the firm also optimally chooses capacity in the long run, prices will only rise or fall in the long run if marginal cost changes as capacity is adjusted. With my comparative statics result, I can easily address a variety of combinations of the above mentioned scenarios.

Example 4. Consider an airline that price-discriminates between two groups of customers, i.e. business and leisure travelers. Since the number of seats on a plane is limited, the airline faces the following constrained profit maximization problem:

\[
\max \pi = p_L(q_L, \phi_L) \cdot q_L + p_B(q_B, \phi_B) \cdot q_B - C(q_L, q_B, \omega)
\]

subject to \(q_L + q_B \leq \bar{q}\)

Since we are interested in comparative statics, the demand functions and the cost function have been parametrized. \(\phi_L\) and \(\phi_B\) capture exogenous demand shocks such as holiday travel or vacation time with \(\frac{\partial p_i}{\partial \phi_i} \geq 0\) for \(i = L, B\). The cost function parameter \(\omega\) accounts for changes in transportation cost inputs such as fuel prices and assume marginal cost of transportation is nonincreasing in \(\omega\). Moreover, assume linear demand functions with \(\frac{\partial p_i}{\partial q_i} \geq 0, i = L, B\) and constant marginal costs of transportation.
A straightforward comparative statics question is how the firms' optimal allocation between business and leisure travelers changes during peak travel season compared to normal traffic. In anticipation of higher demand, the airline increases its capacity by assigning larger planes to popular routes. Additionally, I can add a decrease in input costs, such as lower kerosene prices to the scenario.

Then, \( t' = (\phi'_L, \phi'_B, \omega', \bar{q}') \geq t \).

Once again, since the constraint sets are not lattices, the comparative statics result by Milgrom and Shannon (1994) does not apply here. The result in Quah (2007), while allowing for non-lattice constraint sets, does not address parameter changes in the objective function, while Theorem 1 on the other hand is able to handle these types of problems.

Clearly, the constraint set at higher capacity \( S(\bar{q}') \) dominates \( S(\bar{q}) \) in the \( C_i \)-flexible set order. Under the above assumptions on the objective function, \( \pi \) is \( C_i \)-quasisupermodular and satisfies the \( \varepsilon \)-SCP. Thus, by Theorem 1 the airline’s optimal number of seats allotted to both leisure and business customers is nondecreasing.

When comparing these results to those in Reece and Sobel (2000), we see that when solely focusing on changes in marginal cost my comparative statics theorem yields the same conclusions in the cases of non-binding capacity constraints and when considering capacity adjustments.

Demand changes in the case of fixed and binding capacity are one aspect that cannot be addressed by my result, since at capacity, the marginal cost for an extra seat for either one of the market segments is the marginal revenue of the other. So if as in the above example demand for leisure travel increases during holidays,
and thus $MC_B$ increase. For my result to apply, we need $t' \geq t$, which is not consistent with demand in one market increasing and marginal cost of transportation in the other market increasing.

Besides this special case of binding and fixed capacity, my framework can address comparative statics questions of a more general nature in this model, as it allows for other factors like demand shocks occurring in conjunction with transportation cost and capacity changes. Moreover, unlike the generally used comparative statics approach that requires uniqueness of solution, this result also applies in the case of multiple optimizers, as it could occur for piecewise profit functions that have a linear part.

4.4. **Production Regulation by Efficiency Standards.** Another area of application with a variety of examples is production regulation, particularly in environmental economics. Production can either be regulated by direct restrictions on production such as quotas or indirectly through efficiency standards, value restrictions, etc.

Production quotas directly impose a limit on the quantity a firm may produce to restrict supply and maintain a certain price level. The value of this limit is set by some regulatory agency, hence it depends on the strictness of the regulator. In the model, the rigidity of regulation is captured by the parameter $\theta$. Additionally, the firm’s profit function $\pi$ will also be parametrized by $\phi$ and $\omega$ to capture shifts in demand and changes to the firm’s costs.
A very general version of the firm’s constrained optimization problem can then be written as follows:

$$\max \pi = V(x_R, x_{UR}, \phi) - C(x_R, x_{UR}, \omega)$$

subject to $x_R \leq \bar{q}(\theta)$,

where $x_R$ denotes the regulated commodities the firm produces and $x_{UR}$ is the vector of all unregulated goods the firm produces.

Naturally, a question of interest is how policy changes, economic shocks or technological progress affect the firm’s optimal output, that is what impact do changes of the parameter vector $t = (\phi, \omega, \theta)$ have on the optimal solutions.

Denote the constraint set depending on the parameter vector $t$ by $S(t) = \{x \in \mathbb{R}^n | x_R \leq \bar{q}(\theta)\}$. It can easily be seen that for $t' \geq t$ with $\theta' \geq \theta$, $S(t')$ dominates $S(t)$ in the strong set order. This type of problem can be addressed using the result by Milgrom and Shannon (1994).

Instead of imposing a direct limit on the quantity that a firm may produce of a certain good, an alternative policy to monitor output is to establish efficiency standards. For an efficiency standard, a weighted sum of all commodities with efficiency based weights needs to lie below an upper bound determined by the regulator.

The constraint set depending on the parameter vector in this case can be written as $S(t) = \{x \in \mathbb{R}^n | \alpha \cdot x \leq \bar{\theta}\}$, where $\alpha$ represents either the vector of weights or prices. Notice that these constraint sets are not lattices like in the previous case of a production quota, hence the set at a higher parameter $t'$ does not dominate the initial one at $t$ in the strong set order and Milgrom and Shannon’s result does not apply here. However, $S(t')$ dominates $S(t)$ in the $C$-flexible set order. These types of problems have first been addressed in Quah (2007), but his result does not consider parameter
changes in the objective function. Therefore, the above described comparative statics
problem that involves parameter changes in the objective function and a non-lattice
constraint set cannot be addressed by existing results.

My comparative statics theorem however does apply in this case, given that the
firm’s objective function satisfies $C_i$-quasisupermodularity and the $i$-Single Crossing
Property. For example, this is the case for simple linear demand and cost func-
tions where demand for goods $i$ and $j$ is unrelated and marginal cost of good $i$ is
nonincreasing in the parameter $\omega$.\footnote{See Appendix for a more detailed discussion of $C_i$-quasisupermodularity and the $i$-SCP for profit functions.}

**Example 5.** Consider a car producer that produces two types of cars, one with
high fuel efficiency and one with low fuel efficiency. The produced quantities of each
type are denoted $x_H$ and $x_L$. Let $\phi_H$, $\phi_L$ and $\theta$ be parameters that capture changes
in demand for high and low efficiency cars and the strictness of regulation for the
production of the fuel-inefficient model. Demand for either car models increases in
$\phi_i$, so $\frac{\partial p_i}{\partial \phi_i} > 0$. Moreover, let low values of $\theta$ imply a “green” mindset, which results
in stricter regulation and lower production limits.

In the case of an efficiency standard, the optimization problem of the firm can be
written as

$$\max \pi = p_H(\phi_H)x_H + p_L(\phi_L)x_L - C(x_H, x_L)$$
subject to $\alpha_H x_H + \alpha_L x_L \leq \eta(\theta)$,

where $\alpha_H$ and $\alpha_L$ are weights based on energy consumption and $\eta(\theta)$ is an upper
bound.
A question of interest now is how an increase in all parameters from \( t = (\phi_H, \phi_L, \theta) \) to \( t' > t \), that represents a change to a generally less environmentally conscious attitude in society, affects the firm’s optimal output for both types of cars. It seems straightforward that less emphasis on the environment leads to laxer regulation standards for inefficient cars. Also, a less “green” attitude by society increases the demand for cars. Demand for low efficiency cars is higher because people are not willing to buy the more expensive, but also more efficient cars. On the other hand, people that would not buy cars at all in the greener mindset and rely solely on public transportation, bicycles and walking may now buy high efficiency cars.

First of all, notice that the new constraint set for \( \theta' > \theta \) dominates the previous one in the \( C_i \)-flexible set order as illustrated in Figure 4.2, while these constraint sets cannot be ranked in the strong set order.

![Figure 4.2. Efficiency Standard Constraint Sets](image)

As previously pointed out, the profit function \( \pi \) is \( C_i \)-quasisupermodular if the demand and cost function are linear and if either demand of good \( i \) and good \( j \) are
unrelated or if $\frac{\partial p_i}{\partial x_j} \geq 0$ and $\frac{\partial p_j}{\partial x_i} \geq 0$. Clearly, this is the case if both markets are perfectly competitive and therefore prices are determined by the market and costs are linear.

Moreover, the profit function also needs to satisfy the $i$-Single Crossing Property. If $\pi$ is twice continuously differentiable, it is easy to check if it exhibits the cardinal concept of increasing differences, which implies the $i$-Single Crossing Property. The conditions under which the profit function has increasing differences are

$$\frac{\partial^2 \pi}{\partial x_L \partial \phi_L} = \frac{\partial p_L}{\partial \phi_L} \geq 0 \quad \text{and} \quad \frac{\partial^2 \pi}{\partial x_H \partial \phi_H} = \frac{\partial p_H}{\partial \phi_H} \geq 0.$$ 

These conditions are consistent with the idea of the model that demand is increasing in the parameters $\phi_L$ and $\phi_H$.

Hence the conditions for Theorem 1 are satisfied and we have nondecreasing optimal solutions. Thus production of both high and low efficiency cars is nondecreasing if society cares less about reducing emissions and the environment.

Other possible interpretations of changes in the parameters $\phi_L$ and $\phi_H$ in this example are economic conditions such as changes in household wealth or more or less favorable interest rates that lead to increases or decreases in the demand for cars. Moreover, we can parametrize the cost function by $\omega_H$ and $\omega_L$ as well to account for changes in production technology or government taxes and subsidies for either type.
4.5. Emissions Standards and Production Decisions.

Example 6. Helfand (1991) examines the effect of various different forms of emissions standards on a firm’s optimal output decision. Some of their findings can be replicated and extended using my result. For example, Helfand (1991) finds that the direction of input adjustments after the introduction of a standard depends on the sign of the cross-partial of the production function, which is the change in the marginal product of one input as another one changes. As will be shown in the following, a nonnegative cross-partial of the production function is a sufficient condition for $C_i$-quasisupermodularity, which needs to be satisfied for my result to apply.

Consider a firm with a production function $f(x_1, x_2)$ and output is nondecreasing with nonincreasing marginal returns for each input ($\frac{\partial f}{\partial x_i} \geq 0$, $\frac{\partial^2 f}{\partial x_i^2} \leq 0$ for all $i$). During the production process the firm also produces pollution $A(x_1, x_2)$. First consider the case where both inputs contribute to pollution, so $\frac{\partial A}{\partial x_1} > 0$. Furthermore, suppose the firm has a linear cost function and that the market for this good is perfectly competitive. Without any regulatory constraints the firm maximizes its profit

$$\pi = p \cdot f(x_1, x_2) - \omega_1 x_1 - \omega_2 x_2.$$ 

One type of standard discussed in Helfand (1991) is the case of a set amount of a specific input. Here, the regulator puts a limit on how much of a polluting input can be used in the production process or requires a minimum amount of a pollution-abating input. Suppose input 1 increases pollution, while input 2 reduces emissions.

First, suppose the regulator restricts the use of the polluting input. So the standard takes the form $x_1 \leq \bar{A}_1$. For $\bar{A}_1' \geq \bar{A}_1$, the constraint set $S(\bar{A}_1')$ dominates $S(\bar{A}_1)$ in the strong set order. Alternatively, a minimum amount of the pollution-reducing
input could be required, so $x_2 \geq \bar{A}_2$. Again, for $\bar{A}'_2 \geq \bar{A}_2$, $S(\bar{A}'_2)$ dominates $S(\bar{A}_2)$ in the strong set order.

While these types of standards can be addressed using the result of Milgrom and Shannon (1994), there are other ones that do not fall into this framework. For example, Helfand (1991) also considers the case where the regulator introduces an emissions standard $\bar{A}$ that limits the total amount of allowed pollution by this firm in a given period of time. Moreover, assume that pollution is proportional to the amount of input used in the production process. We can then write this constraint as $a_1x_1 + a_2x_2 \leq \bar{A}$. As both inputs cause pollution, $a_1, a_2 > 0$.

With regard to comparative statics, the natural question to ask is how changes in the emissions standard affect the firm’s input decision. With the comparative statics result in Theorem 1 I can go a little bit further than that and examine how changes in the parameter vector $t = (p, -\omega_1, -\omega_2, \bar{A})$ affect the optimal input decision of the firm. Suppose $t$ increases to $t' = (p', -\omega'_1, -\omega'_2, \bar{A}') \geq t$, that is the price for the finished product increases, input prices decrease and regulation is loosened. As mentioned above, problems with non-lattice constraint sets like this don’t fall into the framework of Milgrom and Shannon (1994). Quah (2007) can also not be applied here, as his result does not account for parameter changes in the objective function.

To see that my comparative statics result applies to this case, we need to verify the assumptions of Theorem 1. Clearly, the new constraint set $S(\bar{A}')$ dominates $S(\bar{A})$ in the $C_i$-flexible set order. For Theorem 1 to apply, the objective function needs to satisfy $C_i$-quasisupermodularity and the $i$-SCP. For the above profit function, this
will be the case given that \( \frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0, \ i \neq j \).

Then the firm’s optimal input quantities are nondecreasing in \( t \).

Hence, when limiting attention to changes in the emissions standard, I find the same results as Helfand (1991). If \( \frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0 \) and the Hessian is negative semi-definite, which in my approach is a sufficient condition for \( C_r \)-quasisupermodularity, more polluting inputs and less of emissions-abating inputs are used when standards are looser (or non-existent) than under stricter regulation. Moreover, my comparative statics theorem can address a more general version of these constrained optimization problems. Price and technology parameters can be included in the objective function and my result addresses changes in optimal inputs for changes in the parameter vector consisting of price, technology and emissions standard.

In addition to that, Helfand (1991) restricts attention to unique optimal solutions of the firm’s constrained optimization problem by assuming that the profit function \( \pi \) is \( C_r \)-quasisupermodular if it is supermodular and \( i \)-concave. The former is the case if

\[
\frac{\partial^2 \pi}{\partial x_i \partial x_j} = p \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0.
\]

By assumption, \( \frac{\partial^2 f}{\partial x_i \partial x_j} \leq 0 \); thus \( \frac{\partial^2 \pi}{\partial x_i \partial x_j} = p \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} \leq 0 \) for all \( i \) and \( \pi \) is \( i \)-concave for all \( i \) in the 2 goods case. In the general case, the Hessian

\[
H_{-i} = \begin{bmatrix}
  f_{1,1} & \cdots & f_{1,i-1} & f_{1,i+1} & \cdots & f_{1,n} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  f_{i-1,1} & \cdots & f_{i-1,i-1} & f_{i-1,i+1} & \cdots & f_{i-1,n} \\
  f_{i+1,1} & \cdots & f_{i+1,i-1} & f_{i+1,i+1} & \cdots & f_{i+1,n} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  f_{n,1} & \cdots & f_{n-1,n} & f_{n+1,n} & \cdots & f_{n,n}
\end{bmatrix}
\]

needs to be negative semi-definite. Moreover, \( \frac{\partial^2 \pi}{\partial x_i \partial p} = \frac{\partial f}{\partial x_i} \geq 0 \), \( \frac{\partial^2 \pi}{\partial x_i \partial (-\omega_j)} = 0 \) for \( i \neq j \) and \( \frac{\partial^2 \pi}{\partial x_i \partial (-\omega_i)} = 1 \geq 0 \) for all \( i \). Hence the \( i \)-SCP holds.
is strictly concave. My result goes beyond that, as it does not require strict concavity and provides comparative statics results also in the case of multiple solutions.

4.6. Emissions Standards and Technological Innovation. Another example in the context of production regulation is technological innovation under emissions standards. Bruneau (2004) compares incentives for innovation in emissions-reducing technology for emissions and performance standards. In the course of the discussion, the paper also addresses changes in output and emissions abatement due to technical innovation.

Example 7. Consider the case of an emissions standard, that restricts emissions by a firm to $\bar{e}$. Suppose units are scaled such that emissions $e$ rise/abatement $a$ falls one-to-one with output $q$, so $e = q - a$. Then a firm, when choosing optimal output and abatement levels faces the constraint $q - a \leq \bar{e}$.

Abatement however is costly, but these costs can be lowered by technological innovation. This is represented by the abatement cost $kC(a)$, where $k \leq 1$ is a technology parameter. Moreover, assume that the market price $P$ depends on industry output $Q$.

The firm then solves the following constraint profit maximization problem:

$$\max_{q,a} \pi = P \cdot q - C(q) - kC(a)$$

subject to $q - a \leq \bar{e}$.

Bruneau (2004) uses quadratic specifications for both cost functions ($C(q) = \frac{1}{2}cq^2$ and $kC(a) = \frac{1}{2}ka^2$) and first considers the simplified case where production costs
are equal to zero. He finds that in both cases, technological progress from \( k = 1 \) to \( k < 1 \) leads to increases in output and abatement.

My comparative statics result can be applied in this context to replicate the comparative statics results in Bruneau (2004) and provides conditions on the profit function that yields nondecreasing optimal solutions. Technical innovation means a decrease in \( k \) (or increase in \(-k\)). As in the previous example, the result in Milgrom and Shannon (1994) cannot be applied to this problem, as the constraint set is not a lattice. Moreover, it cannot be addressed by the main theorem in Quah (2007), as his result does not account for parameter changes in the objective function. For my comparative statics theorem to apply, the profit function needs to satisfy \( C_i \)-quasisupermodularity and the \( i \)-Single Crossing Property.\(^{15}\) We can easily check that \( \pi \) is \( C_i \)-quasisupermodular, as it is supermodular (\( \frac{\partial^2 \pi}{\partial q \partial a} = \frac{\partial^2 \pi}{\partial a \partial q} = 0 \)) and \( i \)-concave (\( \frac{\partial^2 \pi}{\partial q^2} \leq 0 \) for linear demand functions, \( \frac{\partial^2 \pi}{\partial a^2} \leq 0 \)). It also has the \( i \)-SCP, since \( \frac{\partial^2 \pi}{\partial q \partial (-k)} = 0 \) and \( \frac{\partial^2 \pi}{\partial a \partial (-k)} = a \geq 0 \). Thus the profit function satisfies the assumptions for Theorem \( \text{[1]} \) and therefore yields nondecreasing optimal solutions for output and abatement.

With my result, I can easily extend the results in Bruneau (2004) by considering comparative statics with respect to parameter changes such as demand shocks or production costs with simultaneous changes in the constraint set due to regulatory adjustments to the emissions standard \( \bar{e} \).

Rewrite the profit function as \( \pi = P(\alpha) \cdot q - \frac{1}{2}cq^2 - \frac{1}{2}ka^2 \) and let \( t = (\alpha, -c, -k, \bar{e}) \).

An increase in the parameter vector to \( t' \) can be interpreted as a positive demand shock and technological progress that lowers production and abatement costs, while

\(^{15}\)As the constraint set remains unchanged, I do not have to check if the sets can be ranked by the \( C_i \)-flexible set order.
the regulator loosens the emissions standard. Theorem [1] yields nondecreasing optimal solutions for output and abatement, since we can check that the $i$-SCP for the additional parameters is satisfied and the constraint set at $\bar{e}'$ dominates the one at $\bar{e}$ in the $C_i$-flexible set order.

4.7. Ethanol Quota. In the Energy Independence and Security Act of 2007, the Renewable Fuel Standard (RFS) requires the blenders to use increasing amounts of renewable fuel, which in practice mainly means ethanol made from corn. The policy aims to reduce dependence on oil for gasoline production and replace petroleum gasoline by renewable fuels such as ethanol. For conventional cars, fuel can be blended with up to 10% to 15% of ethanol, flex-fuel vehicles can also run on 85% ethanol.

**Example 8.** In this example adapted from Zhang, Qiu and Wetzstein (2010), consider the gasoline blending sector. Blenders provide two types of fuel, E85 (85% ethanol and 15% petroleum gasoline) suitable for flex-fuel vehicles and $E_\gamma$, which contains $\gamma$% ethanol and $(1-\gamma)$% petroleum gasoline, for regular cars. Currently the maximum blend is regulated at 15%. To consider possible adjustments of this “blend wall”, let $0.1 \leq \gamma \leq 0.2$. Let $p_{85}(E_{85}, \phi)$ and $p_{\gamma}(E_{\gamma}, \phi)$ denote inverse market demand for E85 and $E_\gamma$, with $\phi$ being a parameter that captures changes in demand. The blender uses $e_{85}$ and $g_{85}$ in ethanol and petroleum gasoline in the blending process, which is proportional. Hence, $\frac{e_{85}}{e_{85} + g_{85}} = 0.85$ or after rearranging terms, $g_{85} = \frac{3}{17} e_{85}$. Similarly for $E_\gamma$, $g_\gamma = \frac{1-\gamma}{\gamma} e_\gamma$. The production technologies $y_{85}(e_{85})$ and
of blended fuel can be expressed only in terms of amount of ethanol used because of the proportional production process and output is assumed to be increasing with diminishing marginal returns.

Moreover, under the RFS, each blender has to meet his mandated ethanol quota of \( e_0 \), that is his total quantity of ethanol \( e = e_{85} + e_\gamma \) must exceed \( e_0 \). The blenders profit maximization problem can then be written as follows.

\[
\text{max } \pi = p_{85}(E_{85}, \phi) \cdot y_{85}(e_{85}) + p_{\gamma}(E_\gamma, \phi) \cdot y_\gamma(e_\gamma) - c_G\left(\frac{3}{17}e_{85} + \frac{1-\gamma}{\gamma}e_\gamma\right) - c_e \cdot (e_{85} + e_\gamma)
\]

subject to \( e_{85} + e_\gamma \geq e_0 \).

Zhang, Qiu and Wetzstein (2010) finds that under certain conditions on demand elasticities, an increase in the “blend wall” \( \gamma \) likely delays the shift towards flex-fuel vehicles and leads to an increase in \( E_\gamma \) supply and a decrease in \( E_{85} \) provided.

For the same reasons as in the previous examples, while existing comparative statics theorems cannot be applied in this case, my result allows us to consider the impact of raising the “blend wall” \( \gamma \) and increasing the ethanol quota. The constraint set \( S(e'_0) \) under the new policy dominates the original one, \( S(e_0) \) in the \( C_i \)-flexible set order. As discussed in the appendix, the profit function is \( C_i \)-quasisupermodular for linear demand and cost functions and unrelated demand. The \( i \)-Single Crossing Property is satisfied for \( p_{85} \) and \( p_\gamma \) increasing in \( \phi \) and I can check that \( \frac{\partial^2 \pi}{\partial e_\gamma \partial \gamma} \geq 0 \).

Thus, Theorem \([\square]\) applies and yields nondecreasing optimal solutions for both \( e_{85} \) and \( e_\gamma \); hence the policy change favoring the use of renewable fuel leads to an increase in both E85 and regular fuel. my result is the same as in Zhang, Qiu and Wetzstein

\[16\] Unlike Zhang, Qiu and Wetzstein (2010), this example considers the case of a constrained optimization problem with an ethanol quota without the option to buy and sell permits, which eliminates the constraint. The unconstrained problem can be addressed by existing comparative statics result, this type of constrained optimization problem however cannot.
(2010) for regular fuel and also yields the conclusion that these policies promoting renewable fuels don’t necessarily accelerate the transition to flex-fuel vehicles. The advantages of my approach lie in the ability to easily consider a more general problem with additional parameters and changes in the constraint set as well as providing assumptions on the objective function instead of demand elasticities.

4.8. **Cost-Efficient Emissions Regulation.** Another application in environmental economics is the problem of cost-efficient emissions regulation. The goal of the regulator is to limit the amount of total emissions of a given pollutant. As the costs to reduce emissions vary among producers, the regulator’s goal is to achieve the desired emissions limit at the lowest total abatement cost. Theoretical models for efficient emissions allocation can first be found in Baumol and Oates (1971) or Montgomery (1972) and continue to serve as a baseline model as for example in Muller and Mendelsohn (2009). Cost-effectiveness has been the main criterion for the design of permit markets and therefore the cost-efficient allocation continues to be of interest as the baseline comparison even as regulation is moving more and more towards market-based approaches and away from the traditional command-and-control regulation.

The design of regulatory standards depends on the class of pollutants. First, consider the class of uniformly mixed assimilative pollutants, such as greenhouse gases. These types of pollutants do not accumulate in the atmosphere over time and their concentration only depends on the total amount of emissions regardless the source and its location. In the following example, consider a version of this model adapted from Montgomery (1972) and Tietenberg (2006).
Example 9. Suppose there are $J$ producers that omit a uniformly mixed assimilative pollutant, for example carbon dioxide. The regulator sets the target emissions limit at $\bar{A}$. The relationship between firms’ emissions and total pollution can be described as

$$A = a + b \sum_{j=1}^{J} (\bar{e}_j - r_j),$$

where $A$ is pollution per year, $\bar{e}_j$ denotes the uncontrolled emission rate of firm $j$ and $r_j$ is firm $j$’s emissions reduction. The parameters $a$ and $b$ represent “background pollution” from other sources and the degree of proportionality between emissions and total pollution. Moreover, assume that each firm’s cost function $C_j(r_j, \omega)$ is continuous and that the marginal cost of emissions reduction is increasing. The parameter $\omega$ captures technological change. The regulator’s problem can then be written as

$$\max \quad \tilde{C} = - \sum_{j=1}^{J} C_j(r_j, \omega)$$
subject to $a + b \sum_{j=1}^{J} (\bar{e}_j - r_j) \leq \bar{A}, r_j \geq 0$

An interesting comparative statics question in this context is the effect of more environmentally conscious mindset in society. On the firm’s side, this manifests itself in new emissions reducing technology, on the government’s side greener policy means stricter emissions standards. Now consider such an increase in the parameter vector $t = (\omega, -\bar{A})$ caused by new emissions reducing technology and a reduction of the pollution target by the regulator.

Because of the parameter change in the objective function in a problem with a non-lattice constraint set, the existing monotone comparative statics results in Milgrom and Shannon (1994) and Quah (2007) do not apply to this particular problem.
result however is able to address this type of constrained optimization problem, as I can verify that all conditions in Theorem 1 are satisfied.

The constraint set \( S(-\bar{A}') \) dominates \( S(-\bar{A}) \) by the \( C_i \)-flexible set order and the objective function \( \tilde{C} \) is supermodular (\( \frac{\partial^2 \tilde{C}}{\partial r_i \partial r_j} = 0 \), for all \( i, j \)) and \( i \)-concave (in the two firm case, \( \frac{\partial^2 \tilde{C}}{\partial r_i^2} = -\frac{\partial MC_i}{\partial r_i} < 0 \) since marginal abatement cost is increasing; in the general case, we need to verify that the Hessian \( H_{-i} \) is negative semi-definite). Moreover, \( \frac{\partial^2 \tilde{C}}{\partial r_j \partial \omega} = -\frac{\partial MC_j}{\partial \omega} \geq 0 \) as technological progress decreases the firm’s marginal cost of emissions reduction and hence \( \tilde{C} \) exhibits the \( i \)-Single Crossing Property. Therefore, all conditions of Theorem 1 are satisfied and the cost-effective, optimal amounts of emissions reduction are nondecreasing for all firms.

My result can be used in this setting to illustrate the effect of regulation regime changes and technological progress in emissions abatement on the cost-efficient emissions allocation. Under natural assumptions on the primitive, my theorem provides comparative statics results for a common baseline model in emissions regulation. As this is the allocation that other regulatory systems are designed to attain as well, it also suggests how these changes affect the outcome in today’s market-based approaches.

Unlike the standard comparative statics approach that requires uniqueness of solution, this result can also be applied in cases with multiple optimizers. This could occur in the case of a piecewise cost function that is partly linear (see Figure 4.3).

**Example 10.** The LeChatelier principle in economics expresses the idea that long run demand is more responsive to price changes than demand in the short run. For example, consider the impact of an input price reduction on a firm’s demand for that input. The LeChatelier principle now says, that the demand increase in the short run, when some factors of production are fixed, is smaller than the increase in the long run, when all factors can be adjusted freely. Milgrom and Roberts (1996) introduces a lattice-theory based global LeChatelier principle, which is extended to additional classes of constraints faced by the firm in the short run in Quah (2007).

Quah (2007) points out the two different types of comparative statics problems associated with the short run and long run factor demand adjustment following a price change. In the short run, the firm’s objective function changes while the constraint set, based on the optimal point before the price change, stays the same. Between the short run and the long run, the objective function remains the same, but the short run constraints no longer exist; hence the constraint set changes.
Thus consider the following two increases of the parameter vector 
\[ t = (\theta, S), \]
consisting of the cost parameter \( \theta \) and the constraint set \( S \). In the short run, \( t \) increases to \( t' = (\theta', S) \geq t \) as the cost parameter changes. As the constraint set changes in the long run, the parameter vector increases again from \( t' \) to \( t'' = (\theta', S') \).

The firm’s objective function can be written as \( \pi(x, \theta) = V(F(x)) - C(x, \theta) \), where \( x \) is the firm’s input vector, \( F(x) \) its production function and \( V \) the revenue from output. Quah (2007) specifies the firm’s cost function as \( C(x, \theta) = p \cdot x - \theta x_1 \), where \( p \) denotes the input price vector and \( \theta > 0 \) is the considered price reduction for input 1.

In the short run, the firm faces constraints, since not all inputs can be varied without costs. Thus, this constraint set \( S \) needs to include the pre-price change optimal input vector \( x^* \). So, \( S = \{ x \in \mathbb{R}^l_+ : x^* \in S \} \). Thus the short run adjustment from \( x^* \) to \( x^{**} \) is a comparative statics problem where the parameter \( \theta \) in the objective function changes, but the constraint set remains unchanged at \( S \). In the long run, the change of the constraint set from \( S \) to \( S' \) leads to the change in input demand from \( x^{**} \) to \( x^{***} \).

The proof of the following proposition is in parts similar to the proof of Proposition 8 in Quah (2007).

**Proposition 2.** Let \( x^* \) maximize \( \pi(x, \theta) \) subject to \( x \in \mathbb{R}^l_+ \). Suppose also, that solutions \( x^{**} \) and \( x^{***} \) to the problems

1. maximize \( \pi(x, \theta') \) subject to \( x \in S \) and
2. maximize \( \pi(x, \theta') \) subject to \( x \in S' = \mathbb{R}^l_+ \)

where \( \theta' \geq \theta \) exist.
Then $x_i^{***} \geq x_i^{**} \geq x_i^*$ if $\pi$ is $C_i$-quasisupermodular, satisfies the $i$-Single Crossing Property and $X_\pi = \{ x \in \mathbb{R}^s_+ : x \geq s \text{ for some } s \in S \} \geq_i S$.

**Proof.** First consider the increase from $t = (\theta, S)$ to $t' = (\theta', S)$. Since $\pi$ is $C_i$-quasisupermodular, satisfies the $i$-Single Crossing Property and the constraint set remains unchanged, $x_i^{**} \geq x_i^*$ by Theorem 1.

Now $t' \text{ increases to } t'' = (\theta'', S')$. Given that $\pi$ is $C_i$-quasisupermodular, has the $i$-SCP and $\text{argmax}_{x \in S'} \pi(x, \theta')$ exists, there is $\bar{x} \in \text{argmax}_{x \in S'} \pi(x, \theta')$ such that $\bar{x} \geq x^*$. Since $x^* \in S$, $\bar{x} \in \text{argmax}_{x \in X_S} \pi(x, \theta')$.

Moreover, I know that $x^{**} \in \text{argmax}_{x \in S} \pi(x, \theta')$. Since $X_S \geq_i S$, by Theorem 1 there is $x^{***} \in \text{argmax}_{x \in X_S} \pi(x, \theta')$ such that $x_i^{***} \geq x_i^{**}$.

Lastly, $x^{***} \in \text{argmax}_{x \in S'} \pi(x, \theta')$, because $\bar{x} \in X_S$ and $\bar{x} \in \text{argmax}_{x \in S'} \pi(x, \theta')$.

We can easily check that the above objective function is $C_i$-quasisupermodular if $V \circ F$ is $C_i$-quasisupermodular. In the case of a perfectly competitive output market, this is the case when all inputs are complementary to each other and the production function displays decreasing marginal product for each input.\(^{17}\)

Moreover, the objective function needs to satisfy the $i$-SCP. This is the case whenever marginal costs of production are nonincreasing in $\theta$.\(^{18}\)

\(^{17}\)For supermodularity, I need $\frac{\partial^2 \pi}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j} (\frac{\partial V}{\partial F} \frac{\partial F}{\partial x_i}) \geq 0$. Assuming a perfectly competitive output market, $\frac{\partial V}{\partial F} = P$, where $P$ is the market price of output. Then $\frac{\partial^2 \pi}{\partial x_i \partial x_j} = P \cdot \frac{\partial^2 F}{\partial x_i \partial x_j}$, which is nonnegative if $\frac{\partial^2 F}{\partial x_i \partial x_j} \geq 0$.

For $i$-concavity in the simple two goods case with a linear cost function, I need $\frac{\partial^2 \pi}{\partial x_i \partial x_j} = 0$, for all $j \neq i$. In a perfectly competitive environment, $\frac{\partial^2 \pi}{\partial x_i \partial x_j} = P \cdot \frac{\partial^2 C}{\partial x_i \partial x_j}$, which is nonpositive if $\frac{\partial^2 C}{\partial x_i \partial x_j} \leq 0$.

In the general case, I need to verify that the Hessian of the production function, $H_{F,i}$, is negative semi-definite and that the Hessian of the cost function, $H_{C,i}$, is positive semi-definite.

\(^{18}\)The $i$-SCP is satisfied if $\frac{\partial^2 \pi}{\partial x_i \partial \theta} = -\frac{\partial^2 C(x, \theta)}{\partial x_i \partial \theta} \geq 0$. This is the case whenever $\frac{\partial^2 C(x, \theta)}{\partial x_i \partial \theta} \leq 0$. 

\(^{40}\)
The advantages in this version of the LeChatelier principle is that like Milgrom and Roberts (1996), it allows for more general parameter changes than Quah (2007) (who only considers the very specific case described above) while also permitting larger classes of constraint sets than Milgrom and Roberts (1996) like Quah (2007). The result in Milgrom and Roberts (1996) is limited to constraint sets $S$ and $X_S$ that can be ranked in the strong set order, which is the case, for example, when certain inputs are held fixed in the short run. Quah (2007) however allows for more general, economically plausible constraint sets. For example, suppose there is one good that serves different roles in the production process and is therefore considered as several inputs. If in the short run, the total quantity used of these inputs cannot be changed, the constraint set can be written as $S = \{x \in \mathbb{R}_{+}^l : \sum_{i=m}^{l} x_i = \sum_{i=m}^{l} x_i^*\}$. In this case, $X_S$ and $S$ cannot be ranked by the strong set order; however, $X_S$ dominates $S$ in the $C$-flexible set order.
Appendix

Proof of Theorem 2

Proof. ($\Rightarrow$) Suppose $S' \succeq_{(\Delta, \nabla)} S$ for $t' \geq t$ and let $x \in M(t, S)$, $y \in M(t', S')$. Since $x \in M(t, S)$ and $S \leq_{(\Delta, \nabla)} S'$, $f(x, t) \geq f(x \Delta y, t)$. By $(\Delta, \nabla)$-quasisupermodularity of $f$, this implies $f(x \nabla y, t) \geq f(y, t)$. As $f$ also has the $(\Delta, \nabla)$-Single Crossing Property in $(x, t)$, $f(x \nabla y, t) \geq f(y, t)$ implies $f(x \nabla y, t') \geq f(y, t')$ for $t' \geq t$. Since $y \in M(t', S')$ it follows that $x \nabla y \in M(t', S')$.

Now suppose $x \Delta y \notin M(t, S)$ and hence $f(x, t) > f(x \Delta y, t)$. $(\Delta, \nabla)$-quasisupermodularity of $f$ implies $f(x \nabla y, t) > f(y, t)$ and by the $(\Delta, \nabla)$-Single Crossing Property it follows that $f(x \nabla y, t') > f(y, t')$ for any $t' \geq t$. This contradicts the assumption that $y \in M(t', S')$. Therefore, $x \Delta y \in M(t, S)$ and $M(t, S) \leq_{(\Delta, \nabla)} M(t', S')$.

($\Leftarrow$) Fix $t$. Let $x$ and $y$ be two elements in $X$ and suppose that $f$ is not $(\Delta, \nabla)$-quasisupermodular. The only case I need to look at is when $x$ and $y$ are unordered. Also, $x \Delta y \neq x$ and $x \nabla y \neq y$. Let $S = \{x, x \Delta y\}$ and $S' = \{y, x \nabla y\}$.

Then $S' \succeq_{(\Delta, \nabla)} S$.

$(\Delta, \nabla)$-quasisupermodularity of $f$ can be violated in the following two ways.

First, suppose $f(x, t) \geq f(x \Delta y, t)$, but $f(x \nabla y, t) < f(y, t)$. In this case $x$ is a maximizer of $f$ in $S$ and $y$ maximizes $f$ uniquely in $S'$, which violates $M(t, S') \succeq_{(\Delta, \nabla)} M(t, S)$ (since $x$ and $y$ are unordered).

Alternatively, suppose $f(x, t) > f(x \Delta y, t)$, but $f(x \nabla y, t) = f(y, t)$. Now $y$ maximizes $f$ in $S'$ while $x$ is the unique maximizer in $S$. This again contradicts $M(t, S') \succeq_{(\Delta, \nabla)} M(t, S)$. So $f$ is $(\Delta, \nabla)$-quasisupermodular.
Now let \( S \equiv \{x, \bar{x}\} \) with \( x \leq_{(\Delta, \nabla)} \bar{x} \). Then \( f(\bar{x}, t) - f(x, t) \geq 0 \) implies \( \bar{x} \in M(t, S) \).

Since \( M(t, S) \leq_{(\Delta, \nabla)} M(\bar{t}, S) \) for \( \bar{t} \geq t \) it follows that \( f(\bar{x}, \bar{t}) - f(x, \bar{t}) \geq 0 \) for all \( \bar{t} \geq t \). Thus \( f \) has the \((\Delta, \nabla)\)-Single Crossing Property. \( \square \)

\( C_i \)-Quasisupermodularity and the \( i \)-SCP of Profit Functions

The objective function is \( C_i \)-quasisupermodular if it is supermodular and \( i \)-concave (Proposition 2, Quah (2007)). \( \pi \) is supermodular if \( \frac{\partial^2 \pi}{\partial x_i \partial x_j} \geq 0 \) for all \( i, j \) and \( i \neq j \). This is equivalent to \( \frac{\partial^2 V}{\partial x_i \partial x_j} - \frac{\partial^2 C}{\partial x_i \partial x_j} \geq 0 \). Thus the profit function is supermodular if marginal revenue of good \( i \) is nondecreasing in \( x_j \) and marginal cost of good \( i \) is nonincreasing in \( x_j \). We can decompose the total revenue in two parts, the revenue from regulated goods and revenue from unregulated goods.

So, \( V(x_R, x_{UR}, \phi) = p_R(x_R, x_{UR}, \phi_R) \cdot x_R + p_{UR}(x_R, x_{UR}, \phi_{UR}) \cdot x_{UR} \).

Hence,

\[
\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial^2 p_i}{\partial x_i \partial x_j} \cdot x_i + \frac{\partial p_i}{\partial x_j} + \frac{\partial^2 p_j}{\partial x_i \partial x_j} \cdot x_j + \frac{\partial p_j}{\partial x_i}.
\]

For linear demand, \( \frac{\partial^2 p_i}{\partial x_i \partial x_j} = \frac{\partial^2 p_j}{\partial x_i \partial x_j} = 0 \). Therefore, marginal revenue of good \( i \) is nondecreasing in \( x_j \) for linear demand if either demand of good \( i \) and good \( j \) are unrelated or if \( \frac{\partial p_i}{\partial x_j} \geq 0 \) and \( \frac{\partial p_j}{\partial x_i} \geq 0 \). By Definition 4, the objective function is \( i \)-concave if it is concave in all variables other than \( i \).

So for the firm’s profit function, the Hessian,
$H_{-i} = \begin{bmatrix}
\pi_{11} & \cdots & \pi_{1,i-1} & \pi_{1,i+1} & \cdots & \pi_{1,n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\pi_{i-1,1} & \cdots & \pi_{i-1,i-1} & \pi_{i-1,i+1} & \cdots & \pi_{i-1,n} \\
\pi_{i+1,1} & \cdots & \pi_{i+1,i-1} & \pi_{i+1,i+1} & \cdots & \pi_{i+1,n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\pi_{n,1} & \cdots & \pi_{i-1,n} & \pi_{i+1,n} & \cdots & \pi_{n,n}
\end{bmatrix}
$

needs to be negative semi-definite.

In the simple case of only 2 commodities and linear cost and demand functions, we only need to check that $\frac{\partial^2 \pi}{\partial x_j^2} \leq 0$ for all $j \neq i$.

For linear cost and demand functions, this expression becomes

$$\frac{\partial^2 \pi}{\partial x_j^2} = \frac{\partial^2 V}{\partial x_j^2} - \frac{\partial^2 C}{\partial x_j^2} = \frac{\partial^2 V}{\partial x_j^2} \cdot x_j + 2\frac{\partial p_i}{\partial x_j} \cdot x_i + 2\frac{\partial p_j}{\partial x_j} \leq 0.$$ 

Hence in this simple case, $\pi$ is $C_i$-quasisupermodular if the demand and cost function are linear and if either demand of good $i$ and good $j$ are unrelated or if $\frac{\partial p_i}{\partial x_j} \geq 0$ and $\frac{\partial p_j}{\partial x_i} \geq 0$.

The $i$-Single Crossing Property is satisfied if $\frac{\partial^2 \pi}{\partial x_i \partial \phi_i} \geq 0$, $\frac{\partial^2 \pi}{\partial x_i \partial \omega} \geq 0$ and $\frac{\partial^2 \pi}{\partial x_i \partial \theta} \geq 0$.

Again assuming a linear demand function,

$$\frac{\partial^2 \pi}{\partial x_i \partial \phi_i} = \frac{\partial^2 V}{\partial x_i \partial \phi_i} = \frac{\partial^2 p_i}{\partial x_i \partial \phi_i} \cdot x_i + \frac{\partial p_i}{\partial \phi_i} = \frac{\partial p_i}{\partial \phi_i},$$
$$\frac{\partial^2 \pi}{\partial x_i \partial \omega} = -\frac{\partial^2 C}{\partial x_i \partial \omega} = -\frac{\partial MC_i}{\partial \omega} \text{ and }$$
$$\frac{\partial^2 \pi}{\partial x_i \partial \theta} = 0.$$

Thus, we need $\frac{\partial p_i}{\partial \phi_i} \geq 0$ and marginal cost of good $i$ to be nonincreasing in $\omega$ for the $i$-Single Crossing Property to hold.
References


